Balanced presentations for fundamental groups of curves over finite fields

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We show that the algebraic fundamental group of a smooth projective curve over a finite field admits a finite topological presentation where the number of relations does not exceed the number of generators.

1. Introduction

Let p be a prime number, let k be a finite field of characteristic p, and let X be a smooth projective curve defined over k. Associated to X is its algebraic (or étale) fundamental group $\pi_1(X)$. This profinite group fits into the short exact sequence

$$(1.1) \qquad 1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

where $\bar{k}$ is an algebraic closure of k, and $X_{\bar{k}}$ is the base change of X to $\bar{k}$.

Grothendieck has shown in [SGA1, exp. XIII] that $\pi_1(X_{\bar{k}})$, and thus also $\pi_1(X)$, is topologically finitely generated. Moreover, he shows that for every prime $\ell \neq p$, the maximal pro-ℓ quotient of $\pi_1(X_{\bar{k}})$ admits the pro-ℓ presentation

$$(1.2) \qquad \langle x_1, \ldots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] = 1 \rangle$$

where g is the genus of X, and $[a, b] = aba^{-1}b^{-1}$. This result is complemented by the work [Sha47] of Shafarevich, showing that the maximal pro-p quotient of $\pi_1(X_{\bar{k}})$ is free pro-p.

The structure of $\pi_1(X_{\bar{k}})$, and therefore also of $\pi_1(X)$, is well-understood once $g \leq 1$. On the other hand, in case $g \geq 2$, the group $\pi_1(X)$ encompasses the structure of X in a non-trivial way. One evidence for this is the work [Tam04] of Tamagawa which shows that there are only finitely many smooth projective curves Y over $\bar{k}$ with $\pi_1(Y) \cong \pi_1(X_{\bar{k}})$. Another result in this vein, obtained by Mochizuki in [Moch07], says (roughly speaking) that X can be (functorially) reconstructed from $\pi_1(X)$.

In this work, we shed further light on the structure of $\pi_1(X)$. To state our result in full generality, we need to introduce some notation. For a profinite group G we denote by $d(G)$ the least cardinality of a topological generating set of G, and by $r(G)$ the least number of relations needed to present G (with any number of generators). In other words, the number $r(G)$ is the minimum over all short exact sequences

$$(1.3) \qquad\qquad 1 \to K \to F \to G \to 1$$

of the number $d_F(K)$ of topological generators of K as a normal subgroup of F, where F is a free profinite group.

Theorem 1.1. *The étale fundamental group $G = \pi_1(X)$ of a smooth projective curve X over a finite field is topologically finitely presented. Moreover, if X is not of genus zero, then $r(G) = d(G)$.*

We also define the deficiency of a profinite group G by

$$(1.4) \qquad\qquad \mathrm{def}(G) := d(G) - r(G).$$

Corollary 1.2. *Let X be a smooth projective curve over a finite field. Suppose that the genus of X is not zero, and set $G = \pi_1(X)$. Then for any presentation*

$$(1.5) \qquad\qquad 1 \to K \to F \to G \to 1$$

of G, we have

$$(1.6) \qquad\qquad d_F(K) = d(F)$$

namely the number of generators equals the number of relations in the presentation. In other words, the deficiency of any presentation of G equals $\mathrm{def}(G)$ which is 0. In particular, taking a presentation with $d(F) = d(G)$ we get a balanced presentation (a presentation where the number of relations does not exceed the number of generators).

Recall that if K is the function field of X, then $\pi_1(X)$ can be identified with $\mathrm{Gal}(K^{\mathrm{ur}}/K)$, where K^{ur} is the maximal unramified extension of K (inside a fixed separable closure). As a result, the group $\pi_1(X)$ is also studied as a function field analog of the generalized class group $\mathrm{Gal}(L^{\mathrm{ur}}/L)$ of a number field L. Examples include the works [BW17, W17] by Boston and

Wood. These works suggest, vaguely speaking, that for a randomly chosen X, the group $\pi_1(X)$ is given by a random balanced presentation. Motivated by this, Liu and Wood in [LW17] study random balanced presentations and their variants. Theorem 1.1 may thus be viewed as a deterministic counterpart of the works by Boston, Liu, and Wood.

Arithmetic topology, as studied for instance in [Mor11], postulates that the group $\mathrm{Gal}(L^{\mathrm{ur}}/L)$ should exhibit properties similar to those of the fundamental group of a 3-manifold. Indeed, balanced presentations are attributes of 3-manifold groups (see [AFW15]), and Pardon asks in [Par11] whether an analog of this can be established for $\mathrm{Gal}(L^{\mathrm{ur}}/L)$. This serves as an additional motivation for our theorem.

Our proof of Theorem 1.1 starts from a formula proved by Lubotzky in [Lub01]. The latter, once combined with Grothendieck's finite generation result mentioned above, reduces our task to an estimation of the dimensions of several cohomology groups. We perform these estimations using the Lyndon-Hochschild-Serre spectral sequence. An important input is duality in cohomology, deduced from the aforementioned results on maximal pro-ℓ and pro-p quotients by Grothendieck and Shafarevich. Consequently, we determine the deficiency of (any presentation of) $\pi_1(X)$, obtaining a result that mirrors Epstein's work [Eps61] on the deficiency of 3-manifold groups. Furthermore, our arguments apply not only to $\pi_1(X)$ itself, but to any closed topologically finitely generated subgroup of it.

Corollary 1.3. *The group $\pi_1(X)$ is topologically coherent.*

Coherence means that all the closed topologically finitely generated subgroups of $\pi_1(X)$ are topologically finitely presented. Once again, we have an analogy to a result on 3-manifolds, namely the coherence of their fundamental groups, established by Scott in [Sco73a, Sco73b].

2. Proof of Theorem 1.1

We set $G := \pi_1(X)$, and $N := \pi_1(X_{\bar{k}})$, so equation (1.1) reads

$$(2.1) \qquad\qquad 1 \to N \to G \to \widehat{\mathbb{Z}} \to 1.$$

If $g = 0$, we get from [NSW13, Theorem 10.1.2 (i) a] that $N = 1$, so $G \cong \widehat{\mathbb{Z}}$ by (2.1). Hence $\mathrm{def}(G) = d(G) - r(G) = 1 - 0 = 1$. We assume henceforth that $g \geq 1$, and the core of the proof is devoted to showing that $\mathrm{def}(G) = 0$, or equivalently that $r(G) = d(G)$.

By [Lub01, Theorem 0.2], we know that $r(G)$ equals

$$(2.2) \quad \sup_{\lambda} \sup_{M} \left\lceil \frac{\dim_{\mathbb{F}_{\lambda}} H^2(G, M) - \dim_{\mathbb{F}_{\lambda}} H^1(G, M)}{\dim_{\mathbb{F}_{\lambda}} M} \right\rceil + d(G) - \mathbf{1}_{M \not\cong \mathbb{F}_{\lambda}}$$

where λ ranges over the prime numbers, $\mathbb{F}_{\lambda}$ is the finite field with λ elements (and a trivial G-action), and M ranges over all simple $\mathbb{F}_{\lambda}[\![G]\!]$-modules.

Let us first see that $r(G) \geq d(G)$. For that, take any $\lambda \neq p$ and let M be the (trivial) G-module $\mathbb{F}_{\lambda}$. Grothendieck's aforementioned result on the maximal pro-λ quotient of (open subgroups of) N implies that N is a Poincaré group of dimension 2 at λ (for an alternative proof, see [NSW13, Theorem 10.1.2 (i) b]). As $\widehat{\mathbb{Z}}$ is a Poincaré group of dimension 1 at λ, it follows from [NSW13, Theorem 3.7.4] applied to (2.1) that G is a Poincaré group of dimension 3 at λ (see also [NSW13, Corollary 10.1.3 (ii)]). The latter implies that

$$(2.3) \quad \dim_{\mathbb{F}_{\lambda}} H^2(G, M) = \dim_{\mathbb{F}_{\lambda}} H^1(G, \mathrm{Hom}(M, \mathbb{F}_{\lambda})) = \dim_{\mathbb{F}_{\lambda}} H^1(G, M)$$

as $M = \mathbb{F}_{\lambda}$. It follows from (2.2) that indeed $r(G) \geq d(G)$.

We shall now start proving that $r(G) \leq d(G)$. For that, we fix a prime λ and a simple (and thus finite) $\mathbb{F}_{\lambda}[\![G]\!]$-module M. By (2.2), it suffices to show that

$$(2.4) \qquad \dim_{\mathbb{F}_{\lambda}} H^2(G, M) - \dim_{\mathbb{F}_{\lambda}} H^1(G, M) \leq \mathbf{1}_{M \not\cong \mathbb{F}_{\lambda}} \cdot \dim_{\mathbb{F}_{\lambda}} M.$$

The Lyndon-Hochschild-Serre spectral sequence, associated to (2.1) and to M, gives rise (see [KHW12]) to the seven-term exact sequence

$$0 \rightarrow H^1\big(\widehat{\mathbb{Z}}, M^N\big) \rightarrow H^1(G, M) \rightarrow H^1(N, M)^{\widehat{\mathbb{Z}}} \rightarrow H^2\big(\widehat{\mathbb{Z}}, M^N\big)$$
$$\rightarrow \mathrm{Ker}\big(H^2(G, M) \rightarrow H^2(N, M)\big) \rightarrow H^1\big(\widehat{\mathbb{Z}}, H^1(N, M)\big) \rightarrow H^3\big(\widehat{\mathbb{Z}}, M^N\big).$$

Since the cohomological dimension of $\widehat{\mathbb{Z}}$ is 1, the rightmost term in each row of the exact sequence above vanishes, so we obtain the exact sequence

$$(2.5) \qquad 0 \rightarrow H^1\big(\widehat{\mathbb{Z}}, M^N\big) \rightarrow H^1(G, M) \rightarrow H^1(N, M)^{\widehat{\mathbb{Z}}} \rightarrow 0$$

and the exact sequence

$$(2.6) \qquad 0 \rightarrow H^1\big(\widehat{\mathbb{Z}}, H^1(N, M)\big) \rightarrow H^2(G, M) \rightarrow H^2(N, M).$$

It follows from (2.5) that

$$\dim_{\mathbb{F}_\lambda} H^1\big(G, M\big) \geq \dim_{\mathbb{F}_\lambda} H^1\big(N, M\big)^{\widehat{\mathbb{Z}}} \tag{2.7}$$

and from (2.6) (combined with [NSW13, Proposition 1.7.7 (i)]) that

$$\begin{aligned}
\dim_{\mathbb{F}_\lambda} H^2\big(G, M\big) &\leq \dim_{\mathbb{F}_\lambda} H^1\big(\widehat{\mathbb{Z}}, H^1(N, M)\big) + \dim_{\mathbb{F}_\lambda} H^2\big(N, M\big) \\
&\leq \dim_{\mathbb{F}_\lambda} H^1\big(N, M\big)_{\widehat{\mathbb{Z}}} + \dim_{\mathbb{F}_\lambda} H^2\big(N, M\big).
\end{aligned} \tag{2.8}$$

In order to establish (2.4), we want to compare (2.7) with (2.8), so we now show that

$$\dim_{\mathbb{F}_\lambda} H^1\big(N, M\big)^{\widehat{\mathbb{Z}}} = \dim_{\mathbb{F}_\lambda} H^1\big(N, M\big)_{\widehat{\mathbb{Z}}}. \tag{2.9}$$

From Grothendieck's result, we know that N is finitely generated, so $V := H^1(N, M)$ is a finite dimensional vector space over $\mathbb{F}_\lambda$. Therefore, taking a generator σ of $\widehat{\mathbb{Z}}$, and considering the linear map it induces on V, we find that

$$\begin{aligned}
\dim_{\mathbb{F}_\lambda} V_{\widehat{\mathbb{Z}}} &= \dim_{\mathbb{F}_\lambda} V/\mathrm{Im}(\sigma - 1) = \dim_{\mathbb{F}_\lambda} V - \dim_{\mathbb{F}_\lambda} \mathrm{Im}(\sigma - 1) \\
&= \dim_{\mathbb{F}_\lambda} \mathrm{Ker}(\sigma - 1) = \dim_{\mathbb{F}_\lambda} V^{\widehat{\mathbb{Z}}}
\end{aligned} \tag{2.10}$$

so we have arrived at (2.9).

Subtracting (2.7) from (2.8), we can therefore use (2.9) to get that

$$\dim_{\mathbb{F}_\lambda} H^2\big(G, M\big) - \dim_{\mathbb{F}_\lambda} H^1\big(G, M\big) \leq \dim_{\mathbb{F}_\lambda} H^2\big(N, M\big). \tag{2.11}$$

We are now ready to verify (2.4), distinguishing between several cases.

Suppose that $\lambda \neq p$. We have already treated the case $M = \mathbb{F}_\lambda$ while showing that $r(G) \geq d(G)$, so let us assume that $\mathbf{1}_{M \not\cong \mathbb{F}_\lambda} = 1$. As already mentioned before, N is a Poincaré group of dimension 2 at λ, so we have

$$\begin{aligned}
\dim_{\mathbb{F}_\lambda} H^2\big(N, M\big) &= \dim_{\mathbb{F}_\lambda} H^0\big(N, \mathrm{Hom}(M, \mathbb{F}_\lambda)\big) \\
&\leq \dim_{\mathbb{F}_\lambda} \mathrm{Hom}\big(M, \mathbb{F}_\lambda\big) = \dim_{\mathbb{F}_\lambda} M.
\end{aligned} \tag{2.12}$$

Hence, in this case, (2.4) is a consequence of (2.11) and (2.12).

Suppose now that $\lambda = p$. Any open subgroup U of N is the étale fundamental group of a smooth projective curve (which is an étale cover of $X_{\overline{k}}$) over $\overline{k}$. It follows from [Sha47] that the maximal pro-λ quotient $U^{(\lambda)}$ of U is a free pro-λ group. Pick a λ-Sylow subgroup N_λ of N, and recall from

[NSW13, page 209] that

$$(2.13) \qquad N_\lambda = \varprojlim_{N_\lambda \leq U \leq_o N} U^{(\lambda)}$$

an inverse limit of free pro-λ groups. It follows from [RZ10, Theorem 3.5.15] that N_λ is a free pro-λ group, or in other words, that the cohomological dimension of N_λ is less than 2.

We conclude from [NSW13, Corollary 3.3.6] that the λ-cohomological dimension of N is less than 2, and thus

$$(2.14) \qquad \dim_{\mathbb{F}_\lambda} H^2(N, M) = 0.$$

Therefore, in this case, (2.4) follows from (2.11) and (2.14). We have thus finished showing that $r(G) \leq d(G)$, hence $r(G) = d(G)$ as required.

3. Deducing Corollary 1.2

We have seen that N is not procyclic, therefore

$$(3.1) \qquad r(G) = d(G) > 1$$

so in the language of [Lub01, Theorem 0.2], G is not $d(G)$-abelian-indexed. Hence, it follows from [Lub01, Theorem 0.1] that the deficiency of every presentation of G equals $\mathrm{def}(G)$.

4. Deducing Corollary 1.3

An extension of the above proof that $r(G) \leq d(G)$ shows that any finitely generated $G_0 \leq_c G$ is finitely presented. Indeed, setting $N_0 := G_0 \cap N$ we get the exact sequence

$$(4.1) \qquad 1 \to N_0 \to G_0 \to C \to 1$$

where C is a procyclic group. Fixing a prime number λ, and a simple $\mathbb{F}_\lambda[\![G_0]\!]$-module M, we arrive (arguing as in (2.7) and (2.8)) at the bound

$$\dim H^2(G_0, M) - \dim H^1(G_0, M) \leq \dim H^2(N_0, M) + \dim V_C - \dim V^C$$

where (as before) $V := H^1(N_0, M)$, and dimensions are taken over $\mathbb{F}_\lambda$.

Let W be a finite dimensional quotient of V on which C acts trivially. As V is a direct limit of finite C-modules, there exists a finite dimensional C-submodule U of V that surjects onto W. Arguing as in (2.10), we get

$$(4.2) \qquad \dim W \leq \dim U_C = \dim U^C \leq \dim V^C$$

so it follows that

$$(4.3) \qquad \dim V_C \leq \dim V^C.$$

Since G_0 is finitely generated, it follows from (the analog of) (2.7) that $\dim V^C$ is finite, so combining the first inequality of this section with (4.3), we obtain

$$(4.4) \qquad \dim H^2(G_0, M) - \dim H^1(G_0, M) \leq \dim H^2(N_0, M)$$

which is the analog of (2.11).

In (2.12) and (2.14), it is shown that

$$(4.5) \qquad \dim H^2(L, M) \leq \dim M$$

for every open subgroup L of N (to which the action of N_0 on M extends). Writing N_0 as the inverse limit of a family of open subgroups of N, we conclude from (4.5) (using [NSW13, Proposition 1.5.1]) that

$$(4.6) \qquad \dim H^2(N_0, M) \leq \dim M.$$

Combining (4.6) with (4.4), we conclude from (2.2) (or from [Lub01, Theorem 0.3]) that G_0 is finitely presented.

Acknowledgments

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