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On links with Khovanov homology of small ranks

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We classify all links whose Khovanov homology have ranks no greater than 8, and all three-component links whose Khovanov homology have ranks no greater than 12, where the coefficient ring is $\mathbb{Z}/2$. The classification is based on the previous results of Kronheimer-Mrowka [8], Batson-Seed [3], Baldwin-Sivek [1], and the authors [19].

1. Introduction

Khovanov homology [7] is a combinatorially defined invariant for oriented links in S^3. For a commutative ring R and an oriented link L, the Khovanov homology assigns a bi-graded R–module $\mathrm{Kh}(L;R)$.

The detection properties of Khovanov homology have been studied intensively in the past decade. In 2011, Kronheimer and Mrowka [8] proved that Khovanov homology detects the unknot. Since then, many other detection results of Khovanov homology have been obtained. It is now known that Khovanov homology detects the unlink [3, 6], the trefoil [1], the Hopf link [2], the forest of unknots [19], the splitting of links [9], and the torus link $T(2,6)$ [10].

In this paper, we classify all the links L such that $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2) \leq 8$, and all the 3-component links L such that $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2) \leq 12$. Since the rank of the Khovanov homology does not depend on the orientation, it makes sense to refer to $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Kh}(L;\mathbb{Z}/2)$ without orienting L.

Let $\mathrm{Khr}(L;R)$ be the reduced Khovanov homology of L with coefficient ring R. If R is a PID, then the graded Euler characteristics of $\mathrm{Khr}(L;R)$ recover the Jones polynomial $J_L(t)$ of L. Therefore, the parity of $\mathrm{rank}_{\mathbb{Z}/2}\mathrm{Khr}(L;\mathbb{Z}/2)$ is the same as the parity of $J_L(1)$, which is odd if L is a

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knot and is even if L has at least two components. By [16, Corollary 3.2.C], we have

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 2\,\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Khr}(L;\mathbb{Z}/2).$$

Therefore, $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2)$ has the form $4k+2$ ($k \in \mathbb{Z}$) if L is a knot, and is a multiple of 4 if L has at least two components.

On the other hand, it is well-known that if L is a link with n components, then $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) \geq 2^n$ (see, for example, [19, Equation (1)]). As a consequence, if L is a knot such that $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) \leq 8$, then

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 2 \text{ or } 6,$$

and hence L is an unknot or a trefoil by [1, 8]. If L is a 2-component link such that $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) \leq 8$, then

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 4 \text{ or } 8.$$

If L is a 3-component link such that $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) \leq 12$, then

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 8 \text{ or } 12.$$

If L has at least 4 components, then $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) \geq 16$. In [19], the authors have classified all the n–component links L with $\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 2^n$. Therefore, the essential content of this paper is given by the following two results:

Theorem 1.1. *Suppose L is a 2-component link in S^3, then*

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 8$$

if and only if L is isotopic to the link L4a1 in the Thistlethwaite link table, which is the link given by Figure 1, or its mirror image.

Remark 1.2. In [19, Corollary 1.4], the authors proved that Khovanov homology (together with the bi-grading) distinguishes an oriented link whose underlying un-oriented link is isotopic to L4a1. Theorem 1.1 is a stronger version of that result.

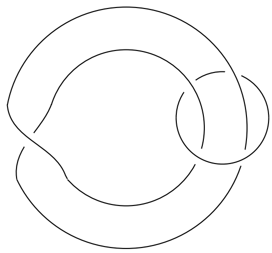

Figure 1. The link L4a1.

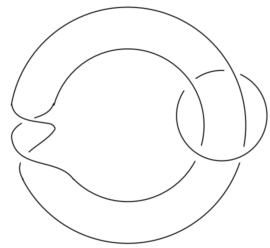

Figure 2. The link L6n1.

Theorem 1.3. *Suppose L is a three-component link in S^3, then*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 12$$

if and only if L is isotopic to the link L6n1 in the Thistlethwaite link table, which is the link given by Figure 2, or its mirror image.

Combining Theorem 1.1, Theorem 1.3, and the results in [1, 8, 19], we have the following two corollaries.

Corollary 1.4. *Suppose $L \subset S^3$ is a link such that $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) \leq 8$, then L is isotopic to one of the following:*

- *the unlink with at most 3 components;*
- *the left-handed or right-handed trefoil;*
- *the Hopf link;*
- *the connected sum of two Hopf links;*
- *the disjoint union of a Hopf link and an unknot;*
- *the link L4a1 or its mirror image.* □

Corollary 1.5. *Suppose $L \subset S^3$ is a link with three components such that*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) \leq 12,$$

then L is isotopic to one of the following:

- *the unlink with 3 components;*
- *the connected sum of two Hopf links;*

- *the disjoint union of a Hopf link and an unknot;*
- *the link L6n1 or its mirror image.* □

2. Preliminaries

Let L be a link in the (framed) solid torus $S^1 \times D^2$, its annular instanton Floer homology $\mathrm{AHI}(L)$ is defined in [18], and the theory is further developed by [19, 20]. This section reviews several results from [18–20] that will be used later.

The annular instanton Floer homology is a $\mathbb{Z}$-graded complex vector space, and the grading is called the f-grading. We use $\mathrm{AHI}(L, i)$ to denote the component of $\mathrm{AHI}(L)$ with f-degree i. For each $i \in \mathbb{Z}$, we have

$$\mathrm{AHI}(L, i) \cong \mathrm{AHI}(L, -i). \tag{2.1}$$

We recall the following definition from [20, Definition 1.5].

Definition 2.1. *A properly embedded, connected surface $S \subset S^1 \times D^2$ is called a* meridional surface *if ∂S is a meridian of $S^1 \times D^2$.*

We recall the following two results from [20].

Theorem 2.2 ([20, Theorem 8.2]). *Given a link L in $S^1 \times D^2$, let S be a meridional surface that intersects L transversely. Let g be the genus of S, and let $n = |S \cap L|$. Suppose S minimizes the value of $(2g + n)$ among meridional surfaces that intersect L transversely, then we have*

$$\mathrm{AHI}(L, \pm(2g + n)) \neq 0,$$

and

$$\mathrm{AHI}(L, i) = 0$$

for all $|i| > 2g + n$.

Proposition 2.3 ([20, Corollary 8.4]). *Let L be a link in $S^1 \times D^2$, let n be a positive integer. Then L is isotopic to the closure of a braid with n strands if and only if the top f-grading of* $\mathrm{AHI}(L)$ *is n and* $\mathrm{AHI}(L, n) \cong \mathbb{C}$.

If $K \subset S^3$ is a knot, we will use $N(K)$ to denote the open tubular neighborhood of K. Suppose K is an unknot, then $S^3 - N(K)$ is a solid torus. Choose the framing of $S^3 - N(K)$ such that the preferred longitude of $S^3 - N(K)$ is a meridian of K. Then for every link L that is disjoint from $N(K)$, we can take the annular instanton Floer homology $\mathrm{AHI}(L)$ by viewing L as a link in $S^3 - N(K)$. Notice that in this case, a meridional surface of $S^3 - N(K)$ induces a Seifert surface of K and vice versa.

The following proposition establishes a relation between the annular instanton Floer homology and the reduced Khovanov homology.

Proposition 2.4. *Suppose $L \subset S^3$ is a link, $U \subset L$ is a component of L that is an unknot, and let $p \in U$ be a base point on U. Let $L_0 = L - U$, then L_0 is a link in the solid torus $S^3 - N(U)$. We have*

$$\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Khr}(L; \mathbb{Z}/2) \geq \dim_{\mathbb{C}} \mathrm{AHI}(L_0).$$

Proof. By Kronheimer-Mrowka's spectral sequence [8, Theorem 8.2], we have

$$\dim_{\mathbb{C}} \mathrm{Khr}(\bar{L}, \bar{p}; \mathbb{C}) \geq \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p; \mathbb{C}),$$

where $(\bar{L}, \bar{p})$ is the mirror image of (L, p), and $\mathrm{I}^{\natural}$ is the reduced singular instanton Floer homology introduced in [8]. By [7, Corollary 11], we have

$$\dim_{\mathbb{C}} \mathrm{Khr}(\bar{L}, \bar{p}; \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Khr}(L, p; \mathbb{C}).$$

By the universal coefficient theorem,

$$\dim_{\mathbb{C}} \mathrm{Khr}(L; \mathbb{C}) \leq \operatorname{rank}_{\mathbb{Z}/2} \mathrm{Khr}(L; \mathbb{Z}/2).$$

By [19, Proposition 2.6], $\mathrm{I}^{\natural}(L, p; \mathbb{C}) \cong \mathrm{AHI}(L_0)$. Therefore the proposition is proved. □

Let $\mathcal{U}_k \subset S^1 \times D^2$ be the unlink with k components, and let $\mathcal{K}_l \subset S^1 \times D^2$ be the link given by $S^1 \times \{p_1, \cdots, p_l\}$. Let $\mathcal{U}_k \sqcup \mathcal{K}_l$ be the disjoint union of $\mathcal{U}_k$ and $\mathcal{K}_l$ such that $\mathcal{U}_k$ is included in a 3-ball disjoint from $\mathcal{K}_l$. By [18, Example 4.2, Proposition 4.3],

$$\mathrm{AHI}(\mathcal{U}_k \sqcup \mathcal{K}_l) \cong \mathbb{C}_{(0)}^{2^k} \otimes (\mathbb{C}_{(1)} \oplus \mathbb{C}_{(-1)})^{\otimes l},$$

where the subscripts represent the f-gradings.

Proposition 2.5 ([19, Proposition 4.3]). *Let $L \subset S^1 \times D^2$ be an oriented link such that every component of L has winding number* 0 *or* ± 1.

Suppose L has k components with winding number 0, and l components with winding number ± 1, then

$$\dim_{\mathbb{C}} \mathrm{AHI}(L,i) \geq \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \sqcup \mathcal{K}_l, i)$$

for all $i \in \mathbb{Z}$.

Proposition 2.6 ([18, Section 4.4]). *Suppose L_1, L_2 are two links in $S^1 \times D^2$. If L_1 and L_2 are* homotopic *to each other in $S^1 \times D^2$, then*

$$\dim_{\mathbb{C}} \mathrm{AHI}(L_1,i) \equiv \dim_{\mathbb{C}} \mathrm{AHI}(L_2,i) \mod 2$$

for all $i \in \mathbb{Z}$.

3. The Multi-variable ALexander polynomial

In this section, we prove several results on the multi-variable Alexander polynomial, which will be used later in the proof of Theorem 1.1.

Let $l \geq 2$ be an integer, recall that the l–strand braid group B_l has the following presentation:

$$B_l = \langle \sigma_1, \cdots, \sigma_{l-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (j-i \geq 2)\ \rangle$$

The reduced Burau representation (see [4]) is a group homomorphism

$$\rho : B_l \to GL(l-1, \mathbb{Z}[t,t^{-1}])$$

which maps σ_i to

$$\begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{l-i-2} \end{pmatrix},$$

where the matrix is truncated appropriately when $i = 1$ or $l-1$. Notice that

$$\det(\rho(\sigma_i)) = -t \quad \text{for all } i,$$

and hence $\det(\rho(\beta)) = \pm t^a$ for all $\beta \in B_l$.

Definition 3.1. *Suppose $\beta \in B_l$ is a braid and $U \subset S^3$ is an unknot, let $\hat{\beta} \subset S^1 \times D^2 \cong S^3 - N(U)$ be the braid closure of β. Define $U \cup \hat{\beta}$ to be the union of U and $\hat{\beta}$ under the standard framing of $S^3 - N(U)$.*

Remark 3.2. The link L4a1 is isotopic to $U \cup \hat{\sigma}_1$, where σ_1 is a generator of B_2.

Theorem 3.3 ([11, Theorem 3]). *Let $\beta \in B_l$, and let $L = U \cup \widehat{\beta}$. Suppose $\hat{\beta}$ is connected, then the multi-variable Alexander polynomial $\Delta_L(x, t)$ of L is given by*

$$\Delta(x, t) \doteq \det\big(xI - \rho(\beta)(t)\big), \tag{3.1}$$

where x and t are the variables corresponding to U and $\hat{\beta}$ respectively, and the sign "$\doteq$" means that the two sides are equal up to a multiplication by $\pm x^a t^b$.

Remark 3.4. The ambiguity in the notation "$\doteq$" is necessary because the multi-variable Alexander polynomial (before normalization) is only well-defined up to a multiplication by $\pm x^a t^b$.

We also need the following result:

Theorem 3.5 ([17]). *Suppose $L = K_1 \cup K_2$ is a 2-component link, and let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of L where x and y are the variables corresponding to K_1 and K_2 respectively. Then we have*

$$\Delta_L(x, 1) \doteq \frac{1 - x^l}{1 - x} \Delta_{K_1}(x),$$

where $\Delta_{K_1}(x)$ is the Alexander polynomial of K_1, and $l = |\operatorname{lk}(K_1, K_2)|$ is the absolute value of the linking number of K_1 and K_2.

The next lemma is an immediate corollary of the results in [5, 13], and is essentially contained in the proof of [19, Lemma 6.1]. We state it here as a separate lemma for future reference.

Lemma 3.6. *Suppose L is a link with n components, let $\Delta_L(x_1, \cdots, x_n)$ be the (multi-variable) Alexander polynomial of L. Let $p \in L$ be a base point.*

1) *If $n = 1$, then the sum of the absolute values of the coefficients of $\Delta_L(x_1)$ is less than or equal to $\operatorname{rank}_{\mathbb{Q}} \operatorname{Khr}(L, p; \mathbb{Q})$.*
2) *If $n \geq 2$, then the sum of the absolute values of the coefficients of*

$$(x_1 - 1) \cdots (x_n - 1) \Delta_L(x_1, \cdots, x_n)$$

is less than or equal to $2^{n-1} \operatorname{rank}_{\mathbb{Q}} \operatorname{Khr}(L, p; \mathbb{Q})$.

Proof. We use $\widehat{\mathrm{HFK}}$ and $\widehat{\mathrm{HFL}}$ to denote the Heegaard knot Floer homology [12, 14] and the link Floer homology [13] respectively. The link Floer homology was originally defined for $\mathbb{Z}/2$-coefficients, and was generalized to $\mathbb{Z}$-coefficients in [15]. It is known that

$$\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFL}}(L;\mathbb{Q}), \tag{3.2}$$

but $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ carries more refined gradings than $\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q})$.

By [5, Corollary 1.7], we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L;\mathbb{Q}) \le 2^{n-1} \operatorname{rank}_{\mathbb{Q}} \mathrm{Khr}(L;\mathbb{Q}). \tag{3.3}$$

By [13, Equation (1)], the multi-graded Euler characteristics of $\widehat{\mathrm{HFL}}(L;\mathbb{Q})$ satisfy

$$\chi\big(\widehat{\mathrm{HFL}}(L;\mathbb{Q})\big) \doteq \begin{cases} \Delta_L(x_1) & \text{if } n=1, \\ (x_1-1)\cdots(x_n-1)\Delta_L(x_1,\cdots,x_n) & \text{if } n\ge 2, \end{cases} \tag{3.4}$$

therefore the result is proved. □

Now let $l \ge 2$ be an integer, let $\beta \in B_l$, $L = U \cup \hat{\beta}$, and let $\Delta_L(x,y)$ be the multi-variable Alexander polynomial of L such that x and y are the variables corresponding to U and $\hat{\beta}$ respectively. By (3.1), we have

$$\begin{aligned} \Delta_L(x,y) &\doteq (-1)^{l-1}\det(\rho(\beta_2)(y)) + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \\ &= \pm y^a + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \end{aligned} \tag{3.5}$$

for $a \in \mathbb{Z}$, $f_i \in \mathbb{Z}[y,y^{-1}]$. By Theorem 3.5,

$$\Delta_L(x,1) \doteq (1+x+x^2+\cdots+x^{l-1})\Delta_U(x) = 1+x+x^2+\cdots+x^{l-1}.$$

Therefore in Equation (3.5), we must have $f_i(1)=1$ for all i, and the sign in front of the term y^a is positive.

A 2-component link $K_1 \cup K_2$ is called *exchangeably braided*, if both K_1, K_2 are unknots and for each $(i,j) \in \{(1,2),(2,1)\}$ the knot K_i is a braid closure with axis K_j. The concept of exchangeably braided links was introduced and studied by Morton in [11]. If we further assume that L is exchangeably braided, then by symmetry and (3.5), we have

$$\Delta_L(x,y) \doteq x^b + g_1(x)y + \cdots + g_{l-2}(x)y^{l-2} + y^{l-1} \tag{3.6}$$

for $b \in \mathbb{Z}$, and $g_i(x) \in \mathbb{Z}[x,x^{-1}]$ with $g_i(1)=1$.

Lemma 3.7. *Let L be a mutually braided link with linking number $l \geq 3$, let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of L. Then the sum of the absolute values of the coefficients of $(x-1)(y-1)\Delta_L(x, y)$ is at least 12.*

Proof. Let $f_i(y)$ be as in (3.5) for $i = 1, \cdots, l-2$, and set

$$f_0(y) = y^a, \quad f_{l-1}(y) = 1.$$

Then we have

$$\begin{aligned}
&(x-1)(y-1)\Delta_L(x, y) \\
\doteq&(y-1)\Big(-y^a + \big(y^a - f_1(y)\big)x + \cdots + \big(f_{l-3}(y) - f_{l-2}(y)\big)x^{l-2} \\
&\quad + \big(f_{l-2}(y) - 1\big)x^{l-1} + x^l\Big) \\
=& -(y-1)y^a + (y-1)x^l + \sum_{i=1}^{l-1}(y-1)\big(f_{i-1}(y) - f_i(y)\big)x^i. \qquad (3.7)
\end{aligned}$$

We discuss three cases depending on how many values of i have $f_i \neq f_{i-1}$. If $f_{i-1} = f_i$ for all $i \in \{1, \cdots, l-1\}$, then $a = 0$, and (3.5) gives

$$\Delta_L(x, y) \doteq 1 + x + \cdots + x^{l-1},$$

which contradicts (3.6) and the assumption that $l \geq 3$.

If there is exactly one element $i \in \{1, \cdots, l-1\}$ such that $f_{i-1}(y) \neq f_i(y)$, then by (3.5), we have

$$\Delta_L(x, y) \doteq y^a + y^a x + \cdots + y^a x^{i-1} + x^i + \cdots + x^{l-1},$$

which also contradicts (3.6) and the assumption that $l \geq 3$.

If there exist at least two elements $i \in \{1, \cdots, l-1\}$ such that $f_{i-1}(y) \neq f_i(y)$, then

$$f_{i-1}(1) - f_i(1) = 1 - 1 = 0$$

implies that $y - 1$ is a factor of $f_{i-1}(y) - f_i(y)$. Therefore for every i such that $f_{i-1}(y) \neq f_i(y)$, the sum of the absolute values of the coefficients of

$$(y-1)(f_{i-1}(y) - f_i(y))$$

is even and strictly greater than 2, therefore it is at least 4. The desired result then follows from (3.7). □

The following lemma refines the proof of Lemma 3.7 and obtains a necessary condition for attaining the lower bound. This result will not be used in the proof of Theorem 1.1.

Lemma 3.8. *Suppose L is an exchangeably braided link with linking number $l \geq 3$, let $\Delta_L(x,y)$ be the multi-variable Alexander polynomial of L. If the sum of the absolute values of the coefficients of $(x-1)(y-1)\Delta_L(x,y)$ is equal to 12, then $l=3$.*

Proof. We use the same notation as in the proof of Lemma 3.7. If the sum of the absolute values of the coefficients of $(x-1)(y-1)\Delta_L(x,y)$ is equal to 12, then the proof of Lemma 3.7 indicates that there are exactly two elements $i \in \{1, \cdots, l-1\}$ such that $f_{i-1}(y) \neq f_i(y)$. Therefore by (3.5), there exists $f(y) \in \mathbb{Z}[y, y^{-1}]$ and $0 \leq k_1 < k_2 \leq l-2$, such that $f(y) \neq 1$, $f(y) \neq y^a$, and

$$\Delta_L(x,y) \doteq y^a(1+\cdots+x^{k_1}) + f(y)(x^{k_1+1}+\cdots+x^{k_2}) + x^{k_2+1}+\cdots+x^{l-1},$$

therefore

$$\Delta_L(1,y) \doteq (1+k_1)y^a + (k_2-k_1)f(y) + (l-1-k_2). \tag{3.8}$$

On the other hand, by Theorem 3.5, we have

$$\Delta_L(1,y) \doteq 1+y+\cdots+y^{l-1}. \tag{3.9}$$

Since $l \geq 3$, Equation (3.9) implies that the coefficients of $\Delta_L(1,y)$ are non-zero for at least 3 different powers of y. Therefore the polynomial $f(y)$ in (3.8) must contain a non-zero term of the form $c \cdot y^b$, where $b \neq 0$ or a. As a result, $\Delta_L(1,y)$ has a term whose coefficient is a non-zero multiple of $k_2 - k_1$. Since (3.9) shows that the coefficients of all the non-zero terms of $\Delta_L(1,y)$ are 1, we must have $k_2 - k_1 = 1$. Hence

$$\Delta_L(x,y) \doteq y^a(1+\cdots+x^{k_1}) + f(y)x^{k_1+1} + x^{k_1+2}+\cdots+x^{l-1}.$$

Since $\Delta_L(x,y) \doteq \Delta_L(x^{-1},y^{-1})$, and recall that $f(y) \neq 1$, $f(y) \neq y^a$, we have $l = 2m+1$ for $m \in \mathbb{Z}$, and

$$\Delta_L(x,y) \doteq y^a(1+\cdots+x^{m-1}) + f(y)x^m + x^{m+1}+\cdots+x^{2m}. \tag{3.10}$$

View $\Delta_L(x,y)$ as a Laurent polynomial of x with coefficients in $\mathbb{Z}[y,y^{-1}]$, the equation above shows that there is only one power of x (namely x^m)

that may have a coefficient which is not a monomial of y. Switching the roles of x and y and repeating the same argument, we conclude that there is at most one power of y in $\Delta_L(x, y)$ whose coefficient is not a monomial of x.

Now we assume $l > 3$ and deduce a contradiction. Since $l > 3$, we must have $a = 0$, because otherwise neither the coefficient of y^a nor the coefficient of y^0 of $\Delta_L(x, y)$ are monomials in $\mathbb{Z}[x, x^{-1}]$, contradicting the previous argument. Therefore by (3.10), we have

$$\Delta_L(x, y) \doteq 1 + \cdots + x^{m-1} + f(y)x^m + x^{m+1} + \cdots + x^{2m}.$$

Flipping the roles of x and y, we have

$$\Delta_L(x, y) \doteq 1 + \cdots + y^{m-1} + g(x)y^m + y^{m+1} + \cdots + y^{2m}$$

for some $g \in \mathbb{Z}[x, x^{-1}]$. By (3.9), we have $f(1) = g(1) = 1$. Therefore

$$\Delta_L(x, y) \doteq -(2m + 1) + \sum_{i=-m}^{m} x^i + \sum_{i=-m}^{m} y^i,$$

and hence

$$\begin{aligned}&(1 - x)(1 - y)\Delta_L(x, y)\\ \doteq &- (2m + 1)(1 - x)(1 - y) + (x^{-m} - x^{m+1})(1 - y) + (1 - x)(y^{-m} - y^{m+1}).\end{aligned}$$

Since $l > 3$, we have $m \geq 2$, therefore the sum of the absolute values of the coefficients of $(1 - x)(1 - y)\Delta_L(x, y)$ is strictly greater than 12, which contradicts the assumption.

In conclusion, we have $l = 3$, and the lemma is proved. □

Combining (3.2), (3.4), Lemma 3.7, and Lemma 3.8, we obtain the following corollary, which may be of independent interest.

Corollary 3.9. *Suppose L is an exchangeably braided link with linking number $l \geq 3$, then $\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L; \mathbb{Q}) \geq 12$. Moreover, if $\operatorname{rank}_{\mathbb{Q}} \widehat{\mathrm{HFK}}(L; \mathbb{Q}) = 12$, then $l = 3$.* □

4. Proof of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1. The "if" part of the theorem follows from a straightforward computation. Now suppose L is a 2-component link such that

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 8, \tag{4.1}$$

we prove that L is isotopic to L4a1 or its mirror image.

Denote the two components of L by K_1 and K_2. Batson-Seed's inequality [3, Corollary 1.6] gives

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) \geq \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_1; \mathbb{Z}/2) \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_2; \mathbb{Z}/2).$$

Since $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i; \mathbb{Z}/2) \geq 2$, we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i; \mathbb{Z}/2) \leq 4.$$

By [16, Corollary 3.2.C],

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i; \mathbb{Z}/2) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i; \mathbb{Z}/2) \leq 2.$$

The parity of $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i; \mathbb{Z}/2)$ is always odd, thus it has to be 1. Kronheimer-Mrowka's unknot detection theorem [8] then implies that both K_1 and K_2 are unknots.

Pick a base point $p \in K_2$. We have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L, p; \mathbb{Z}/2) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 4.$$

By Proposition 2.4,

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1) \leq \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L, p; \mathbb{Z}/2) = 4, \tag{4.2}$$

where K_1 is viewed as a knot in the solid torus $S^3 - N(K_2)$.

If $\operatorname{lk}(K_1, K_2) = 0$, then K_1 is *homotopic* to the unknot in the solid torus. By Proposition 2.5 and Proposition 2.6, we have

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1; 0) \geq \dim_{\mathbb{C}} \operatorname{AHI}(\mathcal{U}_1; 0) = 2,$$

and

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1; i) \equiv \dim_{\mathbb{C}} \operatorname{AHI}(\mathcal{U}_1; i) \equiv 0 \pmod 2, \quad \text{for all } i.$$

Therefore, by (2.1) and (4.2), $\operatorname{AHI}(K_1)$ must be supported at f-degree 0. By Theorem 2.2, this implies K_2 bounds a disk that is disjoint from K_1, and hence L is the disjoint union of K_1 and K_2, which implies L is the unlink. However, the unlink does not satisfy (4.1), which is a contradiction.

Therefore, we have $l = |\operatorname{lk}(K_1, K_2)| > 0$, and hence K_1 is *homotopic* to the closure $\hat{\beta}$ of an l-braid β in the solid torus $S^3 - N(K_2)$. By Proposition 2.6, we have

$$\dim_{\mathbb{C}} \mathrm{AHI}(K_1; i) \equiv \dim_{\mathbb{C}} \mathrm{AHI}(\hat{\beta}; i) \pmod 2.$$

and by (the easy direction of) Proposition 2.3, we have

$$\dim_{\mathbb{C}} \mathrm{AHI}(\hat{\beta}; \pm l) = 1,$$

$$\dim_{\mathbb{C}} \mathrm{AHI}(\hat{\beta}; \pm i) = 0 \text{ for all } i > l.$$

Therefore (2.1) and (4.2) yield that

$$\dim \mathrm{AHI}(K_1; \pm l) = 1$$

and

$$\dim \mathrm{AHI}(K_1; \pm i) = 0 \text{ for all } i > l.$$

By Proposition 2.3, this implies K_1 is the closure of an l-braid in $S^3 - N(K_2)$. A similar argument shows that K_2 is the closure of an l-braid in $S^3 - N(K_1)$. Therefore L is exchangeably braided.

By the universal coefficient theorem,

$$\operatorname{rank}_{\mathbb{Q}} \mathrm{Khr}(L, p; \mathbb{Q}) \leq \operatorname{rank}_{\mathbb{Z}/2} \mathrm{Khr}(L, p; \mathbb{Z}/2) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 4.$$

Therefore by Lemma 3.6 and Lemma 3.7, we have $l \leq 2$.

If $l = 1$, then L is the Hopf link, which does not satisfy (4.1).

If $l = 2$, then $L = U \cup \hat{\sigma}_1^m$, where $\sigma_1 \in B_2$ is a generator of the braid group with 2 strands and $m \in \mathbb{Z}$. Since both components of L are unknots, we have $m = \pm 1$, therefore L is isotopic to L4a1 or its mirror image. □

Proof of Theorem 1.3. The "if" part of the theorem follows from a straightforward calculation. Now suppose L is a 3-component link with $\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 12$, we prove that L isotopic to $L6n1$ or its mirror image.

Denote the three components of L by K_1, K_2, K_3. By Batson-Seed's inequality [3, Corollary 1.6], we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i \cup K_j; \mathbb{Z}/2) \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_k; \mathbb{Z}/2) \le \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 12$$

for all triples (i, j, k) with $\{i, j, k\} = \{1, 2, 3\}$. Since $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_k; \mathbb{Z}/2) \ge 2$, we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i \cup K_j; \mathbb{Z}/2) \le 6.$$

We apply a similar parity argument as before. By [16, Corollary 3.2.C], we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i \cup K_j; \mathbb{Z}/2) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i \cup K_j; \mathbb{Z}/2) \le 3.$$

Since $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i \cup K_j; \mathbb{Z}/2)$ is always even, we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i \cup K_j; \mathbb{Z}/2) \le 2,$$

and hence

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K_i \cup K_j; \mathbb{Z}/2) = 2 \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(K_i \cup K_j; \mathbb{Z}/2) \le 4.$$

By [19, Theorem 1.2], $K_i \cup K_j$ is either a Hopf link or an unlink. In particular, $|\operatorname{lk}(K_i, K_j)| = 0$ or 1. Hence after permuting the labels of the components, we may assume that $|\operatorname{lk}(K_3, K_1)| = |\operatorname{lk}(K_3, K_2)|$.

Pick a base point $p \in K_3$. By [16, Corollary 3.2.C], we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L, p; \mathbb{Z}/2) = \frac{1}{2} \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 6.$$

View $K_1 \cup K_2$ as a link in the solid torus $S^3 - N(K_3)$, Proposition 2.4 gives

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1 \cup K_2) \le \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L, p; \mathbb{Z}/2) = 6. \tag{4.3}$$

We discuss two cases.

Case 1. $|\operatorname{lk}(K_3, K_1)| = |\operatorname{lk}(K_3, K_2)| = 0$. Then $K_1 \cup K_2$ is *homotopic* to the unlink in the solid torus $S^3 - N(K_3)$. Hence by Proposition 2.5 and

Proposition 2.6, we have

$$\dim_{\mathbb{C}} \mathrm{AHI}(K_1 \cup K_2, 0) \geq \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_2, 0) = 4,$$

and

$$\dim_{\mathbb{C}} \mathrm{AHI}(K_1 \cup K_2, i) \equiv \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_2, i) \equiv 0 \pmod 2, \text{ for all } i.$$

By (2.1) and (4.3), $\mathrm{AHI}(K_1 \cup K_2)$ must be supported at f-degree 0. By Theorem 2.2, this implies $K_1 \cup K_2$ is split from K_3, so L is either the unlink or the disjoint union of a Hopf link and an unknot. In both cases we have $\mathrm{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 8$, which contradicts the assumption.

Case 2. $|\operatorname{lk}(K_3, K_1)| = |\operatorname{lk}(K_3, K_2)| = 1$. Recall that the link $\mathcal{K}_2 \subset S^1 \times D^2$ is defined by $\mathcal{K}_2 = S^1 \times \{p_1, p_2\}$, and it can be viewed as a link in $S^3 - N(K_3)$. The assumption above implies that $K_1 \cup K_2$ is *homotopic* to $\mathcal{K}_2$. By Proposition 2.5 and Proposition 2.6, we have

$$\dim \mathrm{AHI}(K_1 \cup K_2; i) \geq \dim \mathrm{AHI}(\mathcal{K}_2; i) \tag{4.4}$$

and

$$\dim \mathrm{AHI}(K_1 \cup K_2; i) \equiv \dim \mathrm{AHI}(\mathcal{K}_2; i) \pmod 2 \tag{4.5}$$

for all i. Recall that

$$\mathrm{AHI}(\mathcal{K}_2) \cong \mathbb{C}_{(2)} \oplus \mathbb{C}_{(-2)} \oplus \mathbb{C}^2_{(0)}, \tag{4.6}$$

where the subscripts represent the f-gradings. It then follows from (2.1), (4.3), (4.4), (4.5), (4.6) that

$$\mathrm{AHI}(K_1 \cup K_2) \cong \mathbb{C}_{(2)} \oplus \mathbb{C}_{(-2)} \oplus \mathbb{C}^4_{(0)} \quad \text{or} \quad \mathbb{C}_{(2)} \oplus \mathbb{C}_{(-2)} \oplus \mathbb{C}^2_{(0)}.$$

By Proposition 2.3, in both cases $K_1 \cup K_2$ is the closure of a 2-braid $\beta \in B_2$ in the solid torus $S^3 - N(K_3)$. Since $K_1 \cup K_2$ is either the unlink or the Hopf link, β is either trivial or $\sigma_1^{\pm 2}$, where σ_1 is a generator of B_2. If β is trivial, then L is the connected sum of two Hopf links, and $\mathrm{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 8$, which contradicts the assumption. Therefore $\beta = \sigma_1^{\pm 2}$, thus L is isotopic to L6n1 or its mirror image, and hence the result is proved. □

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