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Pólya enumeration theorems in algebraic geometry

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We generalize a formula due to Macdonald that relates the singular Betti numbers of X^n/G to those of X, where X is a compact manifold and G is any subgroup of the symmetric group S_n acting on X^n by permuting coordinates. Our result is completely axiomatic: in a general setting, given an endomorphism on the cohomology $H^\bullet(X)$, it explains how we can explicitly relate the Lefschetz series of the induced endomorphism on $H^\bullet(X^n)^G$ to that of the given endomorphism on $H^\bullet(X)$ in the presence of the Künneth formula with respect to a cup product. For example, when X is a compact manifold, we take the Lefschetz series given by the singular cohomology with rational coefficients. On the other hand, when X is a projective variety over a finite field $\mathbb{F}_q$, we use the l-adic étale cohomology with a suitable choice of prime number l. We also explain how our formula generalizes the Pólya enumeration theorem, a classical theorem in combinatorics that counts colorings of a graph up to given symmetries, where X is taken to be a finite set of colors. When X is a smooth projective variety over $\mathbb{C}$, our formula also generalizes a result of Cheah that relates the Hodge numbers of X^n/G to those of X. We also discuss how the generating function for the Lefschetz series of the endomorphisms on $H^\bullet(X^n)^{S_n}$ is rational, and this generalizes the following facts: 1. the generating function of the Poincaré polynomials of symmetric powers of a compact manifold X is rational; 2. the generating function of the Hodge-Deligne polynomials of symmetric powers of a smooth projective variety X over $\mathbb{C}$ is rational; 3. the zeta series of a projective variety X over $\mathbb{F}_q$ is rational. We also prove analogous rationality results when we replace S_n by the alternating groups A_n.

1. Introduction

1.1. Motivation

Let X be a compact complex manifold of (complex) dimension d and consider the n-th symmetric power $\mathrm{Sym}^n(X) = X^n/S_n$ for each $n \in \mathbb{Z}_{\geq 0}$. One may ask how to compute the singular Betti numbers $h^0(\mathrm{Sym}^n(X))$, $h^1(\mathrm{Sym}^n(X)), \ldots$ for various n with respect to those of X. In his influential paper [Mac1962A], Macdonald settled this question: he proved

$$\sum_{n=0}^{\infty} \chi_u(\mathrm{Sym}^n(X))t^n = \frac{(1-ut)^{h^1(X)} \cdots (1-u^{2d-1}t)^{h^{2d-1}(X)}}{(1-t)^{h^0(X)} \cdots (1-u^{2d}t)^{h^{2d}(X)}},$$

where

$$\chi_u(Y) := \sum_{i=0}^{\infty} (-u)^i h^i(Y),$$

a power series[1] in u with integer coefficients, defined for any topological space Y with finite singular Betti numbers. Note that the right-hand side of the above identity is rational in t.

[1]In this paper, we call $\chi_u(Y)$ the **Poincaré series** of Y although it is more common to use the terminology for $\chi_{(-u)}(Y)$, the generating function for $h^i(Y)$. If $h^i(Y) = 0$ for large enough i, we have $\chi_1(Y) = \chi(Y)$, the Euler characteristic of Y.

There is an analogous result when X is a projective variety of dimension d over a finite field $\mathbb{F}_q$ due to Grothendieck:

$$Z_X(t) = \frac{\det(\mathrm{id}_{H^1(X)} - \mathrm{Fr}^*_{q,1}t)\cdots\det(\mathrm{id}_{H^{2d-1}(X)} - \mathrm{Fr}^*_{q,2d-1}t)}{\det(\mathrm{id}_{H^0(X)} - \mathrm{Fr}^*_{q,0}t)\cdots\det(\mathrm{id}_{H^{2d}(X)} - \mathrm{Fr}^*_{q,2d}t)},$$

where

$$Z_X(t) := \exp\left(\sum_{r=1}^{\infty}\frac{|X(\mathbb{F}_{q^r})|t^r}{r}\right),$$

is the **zeta series** of X. In the above result, the notation $H^i(X)$ now denotes the i-th l-adic étale cohomology[2]

$$H^i_{\text{ét}}(X,\mathbb{Q}_l) := H^i_{\text{ét}}(X_{/\overline{\mathbb{F}}_q},\mathbb{Z}_l)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$$

of $X_{/\overline{\mathbb{F}}_q} := X\times_{\mathrm{Spec}(\mathbb{F}_q)}\mathrm{Spec}(\overline{\mathbb{F}}_q)$, which is a finite-dimensional vector space over the field $\mathbb{Q}_l$ of l-adic rational numbers for any fixed prime number l not dividing q. We write Fr_q, which we call the **Frobenius** endomorphism on X, to mean the map from X to itself given by the identity on the underlying topological space and the q-th power map on the structure sheaf $\mathscr{O}_X$, giving an endomorphism on $X_{/\overline{\mathbb{F}}_q}$, inducing the $\mathbb{Q}_l$-linear endomorphism $\mathrm{Fr}^*_{q,i}$ on $H^i(X)$. In particular, this shows that $Z_X(t)$ is rational in t, which was first shown by Dwork [Dwo1960]. It is well-known that

$$Z_X(t) = \sum_{n=0}^{\infty}|\mathrm{Sym}^n(X)(\mathbb{F}_q)|t^n,$$

so writing

$$\sum_{n=0}^{\infty}|\mathrm{Sym}^n(X)(\mathbb{F}_q)|t^n = \frac{\det(\mathrm{id}_{H^1(X)} - \mathrm{Fr}^*_{q,1}t)\cdots\det(\mathrm{id}_{H^{2d-1}(X)} - \mathrm{Fr}^*_{q,2d-1}t)}{\det(\mathrm{id}_{H^0(X)} - \mathrm{Fr}^*_{q,0}t)\cdots\det(\mathrm{id}_{H^{2d}(X)} - \mathrm{Fr}^*_{q,2d}t)},$$

one may visibly find a similarity between Grothendieck's formula and Macdonald's formula. When we take $u = 1$ in Macdonald's formula, we have

$$\sum_{n=0}^{\infty}\chi(\mathrm{Sym}^n(X))t^n = \left(\frac{1}{1-t}\right)^{\chi(X)}.$$

By making analogies between taking the Euler characteristic and counting $\mathbb{F}_q$-points, Vakil [Vak2015] explained how to interpret Grothendieck's

[2] In this paper, we say the "étale cohomology with $\mathbb{Q}_l$-coefficients" to mean this tensor product.

formula as the specialization $u = 1$ of Macdonald's formula in the l-adic setting. In this paper, we take this analogy one step further by generalizing both Macdonald's formula and Grothendieck's formula.

Our main theorem (i.e., Theorem 1.4) is too formal to state without providing a concrete consequence:

Theorem 1.1. Let X be either a compact complex manifold of dimension d or a projective variety of dimension d over a finite field $\mathbb{F}_q$. For any endomorphism F on X, we have

$$\sum_{n=0}^{\infty} L_u(\mathrm{Sym}^n(F)^*)t^n = \frac{\det(\mathrm{id}_{H^1(X)} - F_1^* ut) \cdots \det(\mathrm{id}_{H^{2d-1}(X)} - F_{2d-1}^* u^{2d-1}t)}{\det(\mathrm{id}_{H^0(X)} - F_0^* t) \cdots \det(\mathrm{id}_{H^{2d}(X)} - F_{2d}^* u^{2d}t)},$$

where

- $H^i(X)$ is the singular cohomology of X with $\mathbb{Q}$-coefficients when X is a compact complex manifold,
- $H^i(X)$ is the étale cohomology of $X_{/\overline{\mathbb{F}}_q}$ with $\mathbb{Q}_l$-coefficients when X is a projective variety over $\mathbb{F}_q$ for some prime number l,
- $\mathrm{Sym}^n(F)$ is the endomorphism on $\mathrm{Sym}^n(X)$ induced by F,
- F_i^* is the endomorphism on $H^i(X)$ induced by F, and
- $L_u(F^*) := \sum_{i \geq 0} (-u)^i \mathrm{Tr}(F_i^*)$.

Indeed, taking $F = \mathrm{id}_X$ in the singular setting of Theorem 1.1, we obtain Macdonald's formula. Taking $F = \mathrm{Fr}_q$ in the l-adic setting for choosing primes $l \nmid q$ with $u = 1$, we obtain Grothendieck's formula thanks to the Grothendieck-Lefschetz trace formula (e.g., [Mil1980], VI, Theorem 13.4), which implies that

$$|\mathrm{Sym}^n(X)(\mathbb{F}_q)| = L_1(\mathrm{Sym}^n(\mathrm{Fr}_q)^*),$$

noting that $\mathrm{Sym}^n(\mathrm{Fr}_q)$ is equal to the Frobenius endomorphism on $\mathrm{Sym}^n(X)$. Note that Theorem 1.1 is more general than the two formulas in either setting, and we still get the rational generating function in t. Our main theorem, which we introduce in the next subsection, is much more general than Theorem 1.1, and yet it is a simple representation-theoretic observation.

We hope that experts in various cohomology theories may find our general formulation clear and useful.

Remark 1.2. Theorem 1.1 holds more generally, although we do not seek its maximum generality in this paper. For the singular setting, one may take X to be any compact smooth manifold of (real) dimension $2d$ or any finite CW complex such that $h^i(X) = 0$ for all $i > 2d$. In the l-adic setting, we must require $l > n$ whenever we deal with $H^i(\mathrm{Sym}^n(X)) = H^i_{\text{ét}}(\mathrm{Sym}^n(X), \mathbb{Q}_l)$ because the result depends on the isomorphism $H^i(\mathrm{Sym}^n(X)) \simeq H^i(X^n)^{S_n}$ ([HN1975], Proposition 3.2.1) that uses the fact that l does not divide $|S_n| = n!$. Moreover, since $H^i(X) = H^i_{\text{ét}}(X, \mathbb{Q}_l) = 0$ for $i > 2d$ ([Mil1980], VI, Theorem 1.1), we have

$$H^i(\mathrm{Sym}^n(X)) \simeq H^i(X^n)^{S_n} \simeq \left(\bigoplus_{i_1+\cdots+i_n=i} H^{i_1}(X) \otimes \cdots \otimes H^{i_n}(X) \right)^{S_n} = 0$$

if $i > 2dn$ with any choice of $l > n$, so each $L_u(\mathrm{Sym}^n(F)^*)$ is a polynomial in u for any such l. In the l-adic setting, one can instead use the compactly supported l-adic étale cohomology $H^\bullet_{\text{ét,c}}(X, \mathbb{Q}_l)$, which allows us to consider X to be any quasi-projective variety over $\mathbb{F}_q$. This is particularly useful for revisiting a previously known point-counting result over $\mathbb{F}_q$, discussed in Section 5.

Later in this paper, we run the same story, replacing the full symmetric groups S_n with their alternating subgroups A_n. In particular, we obtain the following analogue of Theorem 1.1. This is restated as Theorem 4.4, and more concrete consequences of this can be found in Section 4:

Theorem 1.3. Let X be either a compact complex manifold of dimension d or a projective variety of dimension d over a finite field $\mathbb{F}_q$. For any endomorphism F on X, we have

$$\begin{aligned}\sum_{n=0}^{\infty} L_u(\mathrm{Alt}^n(F)^*)t^n = &\prod_{i=0}^{2d} \left(\frac{1}{\det(\mathrm{id}_{H^i(X)} - F_i^* u^i t)} \right)^{(-1)^i} \\ &+ \prod_{i=0}^{2d} \left(\frac{1}{\det(\mathrm{id}_{H^i(X)} + F_i^* u^i t)} \right)^{(-1)^{i+1}} - 1 - L_u(F^*)t,\end{aligned}$$

where we use the same notation as in Theorem 1.1 except $\mathrm{Alt}^n(F)$, the endomorphism on the n-th alternating power $\mathrm{Alt}^n(X) = X^n/A_n$ of X induced by F.

1.2. Main result and its applications

In this subsection, we formulate our main result. Let $\mathcal{C}$ be a category where any finite products exist. Fix a field k, and suppose that we have a functor

$$H^\bullet : \mathcal{C}^{\mathrm{op}} \to \mathbf{GrVec}_k$$

from the opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ to the category $\mathbf{GrVec}_k$ of $\mathbb{Z}_{\geq 0}$-graded vector spaces over k whose morphisms are k-linear graded maps (of degree 0). Given any object X in $\mathcal{C}$, we may write

$$H^\bullet(X) = \bigoplus_{i=0}^{\infty} H^i(X),$$

where each $H^i(X)$ is a vector space over k. Given any morphism $f : X \to Y$ in $\mathcal{C}$, the induced k-linear map $f^* : H^\bullet(Y) \to H^\bullet(X)$ can be decomposed into $f_i^* : H^i(Y) \to H^i(X)$ for each $i \in \mathbb{Z}_{\geq 0}$ by definition. In addition, we assume the following axioms:

Axiom 1. Given any object X in $\mathcal{C}$, we assume that there is a **cup product**, namely a k-bilinear map $\cup : H^i(X) \times H^j(X) \to H^{i+j}(X)$ defined for each $i, j \in \mathbb{Z}_{\geq 0}$ such that

$$a \cup b = (-1)^{ij} b \cup a$$

for all $a \in H^i(X)$ and $b \in H^j(X)$.

Axiom 2. Assuming Axiom 1, given any objects X and Y in $\mathcal{C}$, we assume the **Künneth formula**:

$$H^\bullet(X) \otimes_k H^\bullet(Y) \simeq H^\bullet(X \times Y)$$

given by $a \otimes b \mapsto p_X^*(a) \cup p_Y^*(b)$ for any **homogeneous** elements $a \in H^\bullet(X)$ and $b \in H^\bullet(Y)$, meaning $a \in H^i(X)$ and $b \in H^j(Y)$ for some $i, j \in \mathbb{Z}_{\geq 0}$. (In such a case, we write $i = \deg(a)$ and $j = \deg(b)$ and call them the **degree** of a and that of b, respectively, for the rest of this paper.)

Axiom 3. Given any object X in $\mathcal{C}$ and $i \in \mathbb{Z}_{\geq 0}$, the k-vector space $H^i(X)$ is finite-dimensional.

The reader may immediately note that Axiom 1 is only meaningful due to Axiom 2 since otherwise one can always give trivial bilinear maps for a cup product of $H^\bullet(X)$. Note that these two axioms give

$$H^\bullet(X)^{\otimes n} \simeq H^\bullet(X^n)$$

defined by

$$v_1 \otimes \cdots \otimes v_n \mapsto p_1^*(v_1) \cup \cdots \cup p_n^*(v_n),$$

for any homogeneous $v_1, \ldots, v_n \in H^\bullet(X)$, where $p_1, \ldots, p_n$ are the projection maps $X^n \to X$. If G is any subgroup of S_n, then G acts on X^n by permuting coordinates. The induced action of G on $H^\bullet(X^n)$ is precisely given by

$$g \cdot (p_1^*(v_1) \cup \cdots \cup p_n^*(a_n)) = p_{g(1)}^*(v_1) \cup \cdots \cup p_{g(n)}^*(v_n).$$

for $g \in G$. If $\phi = \bigoplus_{i=0}^\infty \phi_i : H^\bullet(X) \to H^\bullet(X)$ is any k-linear graded endomorphism, then it induces a k-linear graded map $\phi_{X^n} : H^\bullet(X^n) \to H^\bullet(X^n)$ given by

$$p_1^*(v_1) \cup \cdots \cup p_n^*(v_n) \mapsto p_1^*(\phi(v_1)) \cup \cdots \cup p_n^*(\phi(v_n)).$$

This map is compatible with the G-action we discussed above, so ϕ induces a k-linear graded map $\phi_{X^n}|_{H^\bullet(X^n)^G} : H^\bullet(X^n)^G \to H^\bullet(X^n)^G$ on the G-invariant subspaces. Note that if $F : X \to X$ is an endomorphism in $\mathcal{C}$, then

$$\begin{aligned} F_{X^n}^*(p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)) &= p_1^*(F^*(v_1)) \cup \cdots \cup p_n^*(F^*(v_n)) \\ &= (F \circ p_1)^*(v_1) \cup \cdots \cup (F \circ p_n)^*(v_n) \\ &= (p_1 \circ F^n)^*(v_1) \cup \cdots \cup (p_n \circ F^n)^*(v_n) \\ &= (F^n)^*(p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)), \end{aligned}$$

so $F_{X^n}^* = (F^n)^*$, where $F^n : X^n \to X^n$ is induced by $F : X \to X$. Using Axiom 3, we can define the **Lefschetz series** of ϕ as

$$L_u(\phi) := \sum_{i=0}^\infty (-u)^i \mathrm{Tr}(\phi_i) \in k[\![u]\!].$$

We are now ready to state our main theorem:

Theorem 1.4. Keeping all the notation above, suppose that $H^\bullet : \mathcal{C}^{\mathrm{op}} \to$ $\mathbf{GrVec}_k$ satisfies Axiom 1, Axiom 2, and Axiom 3. If the characteristic of k does not divide $|G|$, then for any object X of $\mathcal{C}$, we have

$$L_u(\phi_{X^n}|_{H^\bullet(X^n)^G}) = Z_G(L_u(\phi), L_{u^2}(\phi^2), \dots, L_{u^n}(\phi^n)),$$

where

$$Z_G(x_1, \dots, x_n) := \frac{1}{|G|} \sum_{g \in G} x_1^{m_1(g)} \cdots x_n^{m_n(g)} \in k[x_1, \dots, x_n]$$

denoting by $m_i(g)$ the number of i-cycles in the cycle decomposition of g in S_n.

In combinatorics, the polynomial $Z_G(x_1, \dots, x_n)$, often defined over $\mathbb{Q}$, is called the **cycle index** of G in S_n. Much is known about cycle indices. For instance (e.g., [Sta1999], p.20), we have

$$\sum_{n=0}^{\infty} Z_{S_n}(x_1, \dots, x_n) t^n = \exp\left(\sum_{r=1}^{\infty} \frac{x_r t^r}{r}\right).$$

This immediately provides the following:

Corollary 1.5. Assume the same hypotheses as in Theorem 1.4. If $\dim_k(H^\bullet(X))$ is finite so that $H^i(X) = 0$ for all $i > 2d$ for some d, then

$$\sum_{n=0}^{\infty} L_u(\phi_{X^n}|_{H^\bullet(X^n)^{S_n}}) t^n = \frac{\det(\mathrm{id}_{H^1(X)} - \phi_1 u t) \cdots \det(\mathrm{id}_{H^{2d-1}(X)} - \phi_{2d-1} u^{2d-1} t)}{\det(\mathrm{id}_{H^0(X)} - \phi_0 t) \cdots \det(\mathrm{id}_{H^{2d}(X)} - \phi_{2d} u^{2d} t)}.$$

Proof. Both sides are invariant under taking any field extension of k, so we may assume that k is algebraically closed. In particular, the field k we work with is now infinite, so we may assume that u is an element of k. By

Theorem 1.4, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_u(\phi_{X^n}|_{H^\bullet(X^n)^{S_n}})t^n &= \sum_{n=0}^{\infty} Z_{S_n}(L_u(\phi), L_{u^2}(\phi^2), \ldots, L_{u^n}(\phi^n))t^n \\
&= \exp\left(\sum_{r=1}^{\infty} \frac{L_{u^r}(\phi^r)t^r}{r}\right) \\
&= \exp\left(\sum_{r=1}^{\infty}\sum_{i=0}^{2d} \frac{(-u^r)^i \mathrm{Tr}(\phi_i^r)t^r}{r}\right) \\
&= \prod_{i=0}^{2d} \exp\left(\sum_{r=1}^{\infty} \frac{(-1)^i \mathrm{Tr}((\phi_i u^i)^r)t^r}{r}\right) \\
&= \prod_{i=0}^{2d} \exp\left(\sum_{r=1}^{\infty} \frac{\mathrm{Tr}((\phi_i u^i)^r)t^r}{r}\right)^{(-1)^i}.
\end{aligned}$$

Hence, the result follows from the fact that

$$\frac{1}{\det(\mathrm{id} - tA)} = \exp\left(\sum_{r=1}^{\infty} \frac{\mathrm{Tr}(A^r)t^r}{r}\right)$$

for any linear map A on a finite-dimensional vector space (e.g., [Mus], Lemma 4.12). □

Theorem 1.1 is an immediate corollary of Corollary 1.5. This is because, in either the singular or the l-adic setting, we have the quotient map $X^n \to X^n/S_n = \mathrm{Sym}^n(X)$ either in the category of topological spaces or the category of varieties over $\mathbb{F}_q$, and the map induces an isomorphism

$$H^\bullet(\mathrm{Sym}^n(X)) \simeq H^\bullet(X^n)^{S_n}$$

in either setting, whose proofs can be found in [Mac1962J] and [HN1975] (Proposition 3.2.1) as long as we choose $l > n$ in the l-adic setting.

Over the course of proving Theorem 1.4, we will show that

$$L_u(g\phi_{X^n}) = L_u(\phi)^{m_1(g)} L_{u^2}(\phi^2)^{m_2(g)} \cdots L_{u^n}(\phi^n)^{m_n(g)} \in k[\![u]\!]$$

without any assumption on the characteristic of the base field k for the cohomology. When $\dim_k(H^\bullet(X))$ is finite (i.e., $H^i(X) = 0$ for $i \gg 0$), taking

$u = 1$ and $\phi = \mathrm{id}_{H^\bullet(X)}$ in the above identity gives us

$$\sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}(g \circlearrowleft H^i(X^n)) = \chi(X)^{m_1(g)+2m_2(g)+\cdots+nm_n(g)} = \chi(X)^n$$

for any $g \in G$, where $\chi(X) = \sum_{i\geq 0}(-1)^i \dim_k(H^i(X))$. In the l-adic setting, the expression on the left-hand side is generally known to be an integer independent to the choice of l, due to Deligne and Lusztig ([DL1976], Proposition 3.3) for $l \nmid q$. If X is a smooth projective variety over $\mathbb{F}_q$, this follows from the fact that $\chi(X) = \sum_{i\geq 0}(-1)^i \dim_k(H^i(X, \mathbb{Q}_l))$ is independent to the choice of l as a consequence of a theorem of Deligne, which states that the size of the eigenvalues of the Frobenius action on the i-th l-adic étale cohomology of X is $q^{i/2}$ (e.g., [Mil1980], VI, Remark 12.5.(b)). It is worth noting that in our case, the number on the left-hand side is also independent of the choice of $g \in G$, which must be due to the simplicity of the group action we are dealing with. Answering this question for our specific case does not require anything more than merely applying the proof of Macdonald's formula in the l-adic setting on top of Deligne's result.

If X is a smooth projective variety over $\mathbb{C}$ with dimension d, then i-th singular cohomology $H^i(X)$ of (the analytification of) X with $\mathbb{C}$-coefficients has the Hodge decomposition:

$$H^i(X) = \bigoplus_{p+q=i} H^{p,q}(X).$$

In general, the variety $\mathrm{Sym}^n(X)$ is not smooth, but its singular cohomology still admits the Hodge decomposition as $H^i(\mathrm{Sym}^n(X)) \hookrightarrow H^i(X^n)$ so that we can use the Hodge decomposition of $H^i(X^n)$. In fact, it turns out that this structure on $H^i(\mathrm{Sym}^n(X))$ is identical to its **pure Hodge structure**, in the sense of Deligne's mixed Hodge structure introduced in [Del1971], although we do not need this language for the sake of this paper. In this setting, if we take

$$\phi = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} x^p y^q \mathrm{id}_{H^{p,q}(X)}$$

for fixed $x, y \in \mathbb{C}$ in Corollary 1.5, using $H^\bullet(\mathrm{Sym}^n(X)) \simeq H^\bullet(X^n)^{S_n}$ (over $\mathbb{C}$), we have

$$\sum_{n=0}^{\infty}\sum_{i=0}^{2d}\sum_{p+q=i} h^{p,q}(\mathrm{Sym}^n(X))x^p y^q(-u)^i t^n = \prod_{i=0}^{2d}\prod_{p+q=i}\left(\frac{1}{1-x^p y^q u^i t}\right)^{(-1)^i h^{p,q}(X)},$$

where

$$h^{p,q}(\mathrm{Sym}^n(X)) := \dim_{\mathbb{C}}(H^i(\mathrm{Sym}^n(X)) \cap H^{p,q}(X^n)),$$

whenever $p+q=i$. Since x, y are arbitrary, we may treat them as formal variables, and this identity is a result of Cheah ([Che1994], p.119). When we take $u=1$, this shows that the generating function for the Hodge-Deligne polynomials of $\mathrm{Sym}^n(X)$ is rational in t. This generating function is hence analogous to the zeta series of a projective variety over a finite field. Moreover, the two settings for the specialization $u=1$ can be studied at once using the motivic zeta series of a variety defined over the Grothendieck ring of varieties as explained in [Vak2015] and [VW2015]. However, it is unclear whether the Grothendieck ring is the right general setting to study these phenomena when we do not specialize the variable u.

1.3. Pólya enumeration theorems

Let G be a subgroup of S_n. Keeping in mind that $H^\bullet(X^n/G) \simeq H^\bullet(X^n)^G$, if we directly apply Theorem 1.4 to a situation where X is a smooth projective variety over $\mathbb{C}$ without specifying G to be the full symmetric group S_n, we have

$$\chi_u(X^n/G) = Z_G(\chi_u(X), \chi_{u^2}(X), \ldots, \chi_{u^n}(X))$$

and

$$\chi_u(X^n/G, x, y) = Z_G(\chi_u(X, x, y), \chi_{u^2}(X, x^2, y^2), \ldots, \chi_{u^n}(X, x^n, y^n))$$

where

$$\chi_u(Z, x, y) := \sum_{i=0}^{\infty}\sum_{p+q=i} h^{p,q}(Z)x^p y^q(-u)^i,$$

and these are also results of Macdonald [Mac1962A] and Cheah [Che1994]. Taking $u=1$ in the l-adic setting, Theorem 1.4 also implies the following result regarding the $\mathbb{F}_q$-point counting:

$$|(X^n/G)(\mathbb{F}_q)| = Z_G(|X(\mathbb{F}_q)|, |X(\mathbb{F}_{q^2})|, \ldots, |X(\mathbb{F}_{q^n})|).$$

One can even take X to be a finite set. Then giving X the discrete topology, we have $\chi(X) = |X|$, and thus

$$|X^n/G| = Z_G(|X|, |X|, \ldots, |X|) = \frac{1}{|G|}\sum_{g \in G} |X|^{m(g)},$$

where $m(g)$ is the number of cycles in the cycle decomposition of g in S_n. Since $|(X^n)^g| = |X|^{m(g)}$, where $(X^n)^g$ is the set of elements in X^n fixed by g, the last statement also follows from Burnside's lemma. This statement is a special case of the **Pólya enumeration theorem** in combinatorics, which we discuss in Section 2, so it makes sense to use the same name for the preceding results including Theorem 1.4, even though they seem to be in the realm of algebraic geometry. This is the rationale behind the title of this paper.

1.4. Structure of the rest of the paper

In Section 2, we explain a more general version of the Pólya enumeration theorem in combinatorics and show how Theorem 1.4 generalizes this as well. In Section 3, we give a proof of Theorem 1.4, the main theorem of this paper. In Section 4, we show how to compute various cohomological information about the alternating powers $\mathrm{Alt}^n(X) = X^n/A_n$ of given X analogous to computations for the symmetric powers $\mathrm{Sym}^n(X) = X^n/S_n$ in the introduction. In Section 5, we point out that our formula for $|(X^n/G)(\mathbb{F}_q)|$ holds even when X is a quasi-projective variety over $\mathbb{F}_q$. We give an example to explain why this generalization is interesting.

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2. The Pólya enumeration theorem in combinatorics

Let $X = \{x_1, \dots, x_r\}$ be a finite set of colors. A common problem in combinatorics is to count the number of ways to color n vertices (which we write as $1, 2, \dots, n$) of a graph with colors in X. The graph may have symmetries, so we want to count the colorings of n vertices modulo the action of the group G of symmetries of the graph. This group G is a subgroup of S_n, and each coloring corresponds to an element $\boldsymbol{x} \in X^n/G$, and we denote by $e_i = e_i(\boldsymbol{x})$ the number of x_i appearing in $\boldsymbol{x}$. For example, we have $e_2([x_1, x_2, x_2]) = 2$. Note that $e_1 + \cdots + e_r = n$. Given any $(k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 0})^r$ such that $\sum_{i=1}^r k_i = n$, we write $N_{(k_1,\dots,k_r)}$ to mean the number of $\boldsymbol{x} \in X^n/G$ such that $e_i(\boldsymbol{x}) = k_i$ for all $1 \leq i \leq r$. We note that our counting problem is equivalent to computing the following degree n homogeneous polynomial:

$$\begin{aligned} P_{X^n/G}(\boldsymbol{t}) &= P_{X^n/G}(t_1, \dots, t_r) \\ &:= \sum_{\substack{(k_1,\dots,k_r)\in(\mathbb{Z}_{\geq 0})^r, \\ k_1+\cdots+k_r=n}} N_{(k_1,\dots,k_r)} t_1^{k_1} \cdots t_r^{k_r} \in \mathbb{Z}[t_1, \dots, t_n]. \end{aligned}$$

A classical theorem of Redfield [Red1927], which was also independently discovered by Pólya [Pol1937], computes the polynomial $P_{X^n/G}(\boldsymbol{x})$ in terms of the subgroup $G \leqslant S_n$. This theorem is often called the **Pólya enumeration theorem**:

Proposition 2.1 (Pólya enumeration). Given the notation above, we have

$$P_{X^n/G}(\boldsymbol{t}) = Z_G(\boldsymbol{t}, \boldsymbol{t}^2, \dots, \boldsymbol{t}^n),$$

where $\boldsymbol{t}^j := t_1^j + \cdots + t_r^j$.

Our main theorem, Theorem 1.4, generalizes this classical result. Namely, we may consider $X = \{x_1, \dots, x_r\}$ as a topological space with the discrete topology and ϕ the diagonal matrix on the singular cohomology

$$H^\bullet(X) = H^0(X) = \mathbb{Q}x_1 \oplus \cdots \oplus \mathbb{Q}x_r = \mathbb{Q}X$$

whose entries are given by $t_1, \dots, t_r$. We have $(\mathbb{Q}X^n)^G \simeq \mathbb{Q}X^n/G$ given by $(x_{i_1}, \dots, x_{i_n}) \mapsto [x_{i_1}, \dots, x_{i_n}]$, whose inverse is given by

$$[x_{i_1}, \dots, x_{i_n}] \mapsto \frac{1}{|G|} \sum_{g \in G} (x_{i_{g(1)}}, \dots, x_{i_{g(n)}}).$$

Thus, we have

$$H^\bullet(X^n/G) = H^0(X^n/G) = \mathbb{Q}X^n/G \simeq (\mathbb{Q}X^n)^G = H^0(X^n)^G = H^\bullet(X^n)^G,$$

and the induced endomorphism ϕ_{X^n} satisfies

$$\phi_{X^n} : (x_{i_1}, \dots, x_{i_n}) \mapsto t_{i_1} \cdots t_{i_n} (x_{i_1}, \dots, x_{i_n}),$$

so on $\mathbb{Q}X^n/G$, it satisfies

$$\boldsymbol{x} \mapsto t_1^{e_1(\boldsymbol{x})} \cdots t_r^{e_r(\boldsymbol{x})} \boldsymbol{x}.$$

Therefore, Theorem 1.4 with $u = 1$ implies Proposition 2.1. That is, the classical Pólya enumeration theorem is a special case of Theorem 1.4, which deals with more than degree 0 piece of the cohomology with more diverse choices for X.

3. Proof of main theorem

3.1. Motivation for general set-up

In this section, we prove our main theorem, Theorem 1.4. In fact, we prove Theorem 3.2, a more general statement about a particular permutation representation on the n-fold tensor product of a graded vector space, which immediately implies Theorem 1.4. The representation we work with is not the usual permutation representation on the pure tensors, and it involves a sign depending on the grading. The reason is that for any $G \leqslant S_n$, we are interested in the G-action on $H^\bullet(X^n)$, induced by the G-action on X^n, which can be seen as

$$g \cdot (p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)) = p_{g(1)}^*(v_1) \cup \cdots \cup p_{g(n)}^*(v_n)$$

where $v_1, \ldots, v_n \in H^\bullet(X)$ are homogeneous elements and $g \in G$, denoting by $p_1, \ldots, p_n : X^n \to X$ the projection maps. Following the Künneth formula

$$H^\bullet(X)^{\otimes n} \simeq H^\bullet(X^n),$$

given by $v_1 \otimes \cdots \otimes v_n \mapsto p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)$, if $g = (1\ 2)$, the transposition switching 1 and 2, then the corresponding action of g on $H^\bullet(X)^{\otimes n}$ is given by

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^{\deg(v_1)\deg(v_2)} v_2 \otimes v_1 \otimes v_3 \otimes \cdots \otimes v_n,$$

which is not equal to $v_2 \otimes v_1 \otimes v_3 \otimes \cdots \otimes v_n$ unless one of v_1 or v_2 has even degree. (This is often called the **Koszul sign rule**.) In general, one may check that the formula

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^{Q_g(\deg(v_1),\ldots,\deg(v_n))} v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)},$$

where $Q_g(x_1, \ldots, x_n) = \sum_{1 \le i < j \le n} \epsilon_{ij}(g) x_i x_j \in \mathbb{Z}[x_1, \ldots, x_n]$ is defined by

$$\epsilon_{ij}(g) := \begin{cases} 1 & \text{if } g(i) > g(j) \text{ and} \\ 0 & \text{if } g(i) < g(j), \end{cases}$$

defines the G-action on $H^\bullet(X)^{\otimes n}$ that is compatible with the G-action on $H^\bullet(X^n)$. The above formula for the G-action on $H^\bullet(X)^{\otimes n}$ is the most crucial observation in Macdonald's work [Mac1962A], which we fully use for the proof of Theorem 1.4. In what follows, we use this formula as a definition, not a result so that we can merely work on graded vector spaces instead of cohomology groups.

3.2. General set-up

Throughout this section, we fix a ground field k. Let $V = \bigoplus_{i \ge 0} V_i$ be a graded vector space over k. Given $n \in \mathbb{Z}_{\ge 0}$, consider the n-fold tensor product $V^{\otimes n}$ of V over k, where $V^{\otimes 0} = k$. We have

$$V^{\otimes n} = \bigoplus_{r \ge 0} (V^{\otimes n})_r,$$

where

$$(V^{\otimes n})_r = \bigoplus_{i_1+\cdots+i_n=r} V_{i_1} \otimes \cdots \otimes V_{i_n}.$$

This makes $V^{\otimes n}$ a graded vector space over k. Given any subgroup $G \leqslant S_n$, we consider the action of G on $V^{\otimes n}$ according to the Koszul rule. That is, we define

$$g \cdot (v_1 \otimes \cdots \otimes v_n) := (-1)^{Q_g(\deg(v_1),\ldots,\deg(v_n))} v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)},$$

for homogeneous $v_1, \ldots, v_n \in V$ (i.e., $v_i \in V_{\deg(v_i)}$) and $g \in G$, where $Q_g(x_1, \ldots, x_n)$ is defined right before this subsection. It is important to note that this action respects the grading of $V^{\otimes n}$. In particular, it can be thought of as a family of k-linear maps $\{G \to \mathrm{GL}_k((V^{\otimes n})_r)\}_{r \in \mathbb{Z}_{\geq 0}}$.

3.3. Trace formulas on tensor products

To discuss traces of linear endomorphisms, assume that each homogeneous piece V_i of V is finite-dimensional. Let $\phi \in \mathrm{End}_k(V)$ be graded (with degree 0) meaning that $\phi = \bigoplus_{i \geq 0} \phi_i$, where $\phi_i \in \mathrm{End}_k(V_i)$. This means that if $v \in V$ is a homogeneous element, then $\phi(v) \in V$ is a homogeneous element of degree $\deg(v)$ so that $\phi(v) = \phi_{\deg(v)}(v)$. Consider the Lefschetz series

$$L_u(\phi) = \sum_{i \geq 0} (-u)^i \mathrm{Tr}(\phi_i) \in k[\![u]\!]$$

of ϕ in u. It is important to note that when we have another graded endomorphism $\psi = \bigoplus_{i \geq 0} \psi_i$ on V and a constant $c \in k$, we have

$$L_u(\phi + c\psi) = L_u(\phi) + cL_u(\psi).$$

We also get the induced endomorphism $\phi^{\otimes n} \in \mathrm{End}_k(V^{\otimes n})$ given by

$$\phi^{\otimes n}(v_1 \otimes \cdots \otimes v_n) := \phi(v_1) \otimes \cdots \otimes \phi(v_n)$$

for homogeneous $v_1, \ldots, v_n \in V$, which hence respects the grading of $V^{\otimes n}$ so that we can write

$$\phi^{\otimes n} = \bigoplus_{r \geq 0} (\phi^{\otimes n})_r,$$

where

$$(\phi^{\otimes n})_r := \bigoplus_{i_1+\cdots+i_n=r} \phi_{i_1} \otimes \cdots \otimes \phi_{i_n}$$
$$\in \mathrm{End}_k((V^{\otimes n})_r) = \mathrm{End}_k\left(\bigoplus_{i_1+\cdots+i_n=r} V_{i_1} \otimes \cdots \otimes V_{i_n}\right).$$

Given any $g \in G \leqslant S_n$ and homogeneous $v_1, \ldots, v_n \in V$, we define

$$\begin{aligned}
&(g \cdot \phi^{\otimes n})(v_1 \otimes \cdots \otimes v_n) \\
&\quad := g(\phi(v_1) \otimes \cdots \otimes \phi(v_n)) \\
&\quad = (-1)^{Q_g(\deg(v_1),\ldots,\deg(v_n))} \phi(v_{g^{-1}(1)}) \otimes \cdots \otimes \phi(v_{g^{-1}(n)}) \\
&\quad = (-1)^{Q_g(\deg(v_1),\ldots,\deg(v_n))} \phi_{\deg(v_{g^{-1}(1)})}(v_{g^{-1}(1)}) \otimes \cdots \otimes \phi_{\deg(v_{g^{-1}(n)})}(v_{g^{-1}(n)}).
\end{aligned}$$

This extends to a k-linear endomorphism $g\phi^{\otimes n}$ on $V^{\otimes n}$. It is important to note that we have the following commutativity even though it is immediate from definitions:

Lemma 3.1. Keeping the notation above, we have

$$(g\phi^{\otimes n})(v_1 \otimes \cdots \otimes v_n) = \phi^{\otimes n}(g(v_1 \otimes \cdots \otimes v_n)).$$

The following is the core of the proof of Theorem 1.4:

Theorem 3.2 (Trace formula on $V^{\otimes n}$). Let $\phi = \bigoplus_{i\geq 0} \phi_i$ be a graded endomorphism on a graded vector space $V = \bigoplus_{i\geq 0} V_i$ over k, where each V_i is finite-dimensional. For any $g \in S_n$, we have

$$L_u(g\phi^{\otimes n}) = L_u(\phi)^{m_1(g)} L_{u^2}(\phi^2)^{m_2(g)} \cdots L_{u^n}(\phi^n)^{m_n(g)} \in k[\![u]\!].$$

Remark 3.3. In the setting of Section 1.2 (assuming Axioms 1, 2, an 3), Theorem 3.2 gives

$$L_u(g\phi_{X^n}) = L_u(\phi)^{m_1(g)} L_{u^2}(\phi^2)^{m_2(g)} \cdots L_{u^n}(\phi^n)^{m_n(g)} \in k[\![u]\!],$$

for any $g \in G \leqslant S_n$ as mentioned in the introduction. We note that until now there is no extra condition on the field k.

In our proof of Theorem 3.2, we will make use of the following properties about the quadratic forms $Q_g(x_1, \ldots, x_n)$ defined above that we learned from [Mac1962A]. Both properties are immediate from definition:

Lemma 3.4. For any disjoint $\sigma, \tau \in S_n$, we have

$$Q_{\sigma\tau}(\boldsymbol{x}) = Q_\sigma(\boldsymbol{x}) + Q_\tau(\boldsymbol{x}).$$

If σ is a cycle of the form $\sigma = (\lambda+1\ \lambda+2\ \cdots\ \lambda+r)$ with $1 \leq r \leq n$ (and $0 \leq \lambda \leq n-r$), then

$$Q_\sigma(\boldsymbol{x}) = (x_{\lambda+1} + x_{\lambda+2} + \cdots + x_{\lambda+r-1})x_{\lambda+r}.$$

Proof of Theorem 3.2. Since the identity is only regarding traces of (homogeneous parts of) endomorphisms $g\phi^{\otimes n}$ and $\phi, \phi^2, \ldots, \phi^n$, we may assume that k is algebraically closed. Both sides of the identity are power series in $k[\![u]\!]$, so it is enough to show that for any $r \in \mathbb{Z}_{\geq 0}$, their coefficients of u^r match. This lets us reduce the problem to the case $V = V_0 \oplus \cdots \oplus V_r$ and $\phi = \phi_1 \oplus \cdots \oplus \phi_r$ essentially because

$$(V^{\otimes n})_r = \bigoplus_{i_1+\cdots+i_n=r} V_{i_1} \otimes \cdots \otimes V_{i_r},$$

where the right-hand side only consists of tensor products of $V_0, \ldots, V_r$. In particular, we are now dealing with the case where $d = \dim_k(V) = \dim_k(V_0) + \cdots + \dim_k(V_r)$ is finite.

Considering $\phi \in \mathrm{Mat}_d(k) = \mathbb{A}^{d^2}(k)$, where $d = \dim_k(V)$, we note that the desired equality for the coefficients of u^r cuts out a closed subset in $\mathbb{A}^{d^2}(k)$, with respect to the Zariski topology (on the set of closed points in $\mathbb{A}^{d^2}$ over k) as we can use the Kronecker product for the matrix form of $\phi^{\otimes n}$. The matrices with distinct eigenvalues form a Zariski open subset in $\mathrm{Mat}_d(k) = \mathbb{A}^{d^2}(k)$ because we can understand them as points of the locus whose discriminant of the characteristic polynomial is nonzero. This open locus is nonempty because k has at least d elements as it is infinite now that we are in the setting where k is algebraically closed. Thus, such matrices are dense in $\mathrm{Mat}_d(k) = \mathbb{A}^{d^2}(k)$, as the affine space is irreducible. This means that it is enough to show the desired statement for ϕ with distinct eigenvalues, and this means that each ϕ_i is diagonalizable.

Thus, we may find $\eta_i \in \mathrm{GL}_{d_i}(k) = \mathrm{GL}(V_i)$ such that $\eta_i\phi_i\eta_i^{-1}$ is a diagonal matrix whose diagonal entries are eigenvalues of ϕ_i, where $d_i = \dim_k(V_i)$. Then $\eta_i\phi_i^m\eta_i^{-1}$ for any $m \geq 1$ is a diagonal matrix whose diagonal entries consists of m-th powers of the full list eigenvalues of ϕ_i counting with multiplicity. Writing $\eta = \eta_1 \oplus \cdots \oplus \eta_r \in \mathrm{GL}_d(k)$, we see $\eta\phi\eta^{-1} = \eta_1\phi_1\eta_1^{-1} \oplus \cdots \oplus$

$\eta_r\phi_r\eta_r^{-1}$ is a diagonal matrix, and so is

$$(\eta\phi\eta^{-1})^m = \eta\phi^m\eta^{-1} = \eta_1\phi_1^m\eta_1^{-1} \oplus \cdots \oplus \eta_r\phi_r^m\eta_r^{-1}.$$

Note that η respects the grading of V and $\eta^{\otimes n}$ commutes with the action of g by Lemma 3.1. Since $(\eta\phi\eta^{-1})^{\otimes n} = \eta^{\otimes n}\phi^{\otimes n}(\eta^{-1})^{\otimes n}$, by Lemma 3.1, we have

$$(g(\eta\phi\eta^{-1})^{\otimes n})_r = (\eta^{\otimes n}g\phi^{\otimes n}(\eta^{-1})^{\otimes n})_r = (\eta^{\otimes n})_r(g\phi^{\otimes n})_r((\eta^{-1})^{\otimes n})_r.$$

Since

$$\eta^{\otimes n}(\eta^{-1})^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = \eta\eta^{-1}v_1 \otimes \cdots \otimes \eta\eta^{-1}v_n = v_1 \otimes \cdots \otimes v_n$$

for any homogeneous $v_1, \ldots, v_n \in V$, we see that $(\eta^{\otimes n})_r$ and $((\eta^{-1})^{\otimes n})_r$ are k-linear endomorphisms on $(V^{\otimes n})_r$ that are inverses to each other. Thus, replacing ϕ with $\eta\phi\eta^{-1}$, or equivalently ϕ_i with $\eta_i\phi_i\eta_i^{-1}$ for each i, does not affect the desired identity, so our problem is reduced to the case where each ϕ_i is diagonal.

Let $v_{i,1}, \ldots, v_{i,d_i} \in V_i$ be homogeneous elements of V forming an eigenbasis of V_i for ϕ_i as we vary $i \geq 0$. We shall denote the corresponding eigenvalues as $\alpha_{i,j} \in k$ so that $\phi(v_{i,j}) = \phi_i(v_{i,j}) = \alpha_{i,j}v_{i,j}$. To compute the coefficient of u^r on the left-hand side of the desired statement, fix pure tensor

$$w_1 \otimes \cdots \otimes w_n \in (V^{\otimes n})_r = \bigoplus_{i_1+\cdots+i_n=r} V_{i_1} \otimes \cdots \otimes V_{i_n},$$

where $w_j = v_{i_j,h_j}$ for some h_j so that $\deg(w_j) = i_j$ and $\phi(w_j) = \phi_{i_j}(w_j) = \alpha_{i_j,h_j}w_j$. We have

$$\begin{aligned}
&(g\phi^{\otimes n})(w_1 \otimes \cdots \otimes w_n) \\
&\quad = \phi^{\otimes n}(g(w_1 \otimes \cdots \otimes w_n)) \\
&\quad = (-1)^{Q_g(i_1,\ldots,i_n)}\phi(w_{g^{-1}(1)}) \otimes \cdots \otimes \phi(w_{g^{-1}(n)}) \\
&\quad = (-1)^{Q_g(i_1,\ldots,i_n)}\alpha_{i_{g^{-1}(1)},h_{g^{-1}(1)}}w_{g^{-1}(1)} \otimes \cdots \otimes \alpha_{i_{g^{-1}(n)},h_{g^{-1}(n)}}w_{g^{-1}(n)} \\
&\quad = \alpha_{i_1,h_1} \cdots \alpha_{i_n,h_n}(-1)^{Q_g(i_1,\ldots,i_n)}w_{g^{-1}(1)} \otimes \cdots \otimes w_{g^{-1}(n)},
\end{aligned}$$

so the vector $w_1 \otimes \cdots \otimes w_n$ can possibly contribute a nonzero amount to $\mathrm{Tr}(g\phi^{\otimes n})_r$ only when $w_j = w_{g^{-1}(j)}$ for all $1 \leq j \leq n$. Now, the key is to note that the statement only depends on the cycle type of g in S_n because any other $h \in S_n$ with the same cycle type is conjugate to g in S_n so that $h = \omega g \omega^{-1}$ for some $\omega \in S_n$, which gives us

$$\begin{aligned}\mathrm{Tr}(h\phi^{\otimes n})_r &= \mathrm{Tr}(\omega g \omega^{-1}\phi^{\otimes n})_r = \mathrm{Tr}(\omega g \phi^{\otimes n}\omega^{-1})_r \\ &= \mathrm{Tr}(\omega_r (g\phi^{\otimes n})_r (\omega^{-1})_r) = \mathrm{Tr}(g\phi^{\otimes n})_r.\end{aligned}$$

Thus, we have reduced the problem to the case where we have the following cycle decomposition for g:

$$g = (1 \ \cdots \ \lambda_1)(\lambda_1 + 1 \ \cdots \lambda_1 + \lambda_2) \cdots (\lambda_1 + \cdots + \lambda_{l-1} + 1 \ \cdots \ \lambda_1 + \cdots + \lambda_l),$$

where $\lambda_1 + \cdots + \lambda_l = n$. In this situation, saying that $w_j = w_{g^{-1}(j)}$ for all $1 \leq j \leq n$ is equivalent to saying

- $y_1 := w_1 = \cdots = w_{\lambda_1}$,
- $y_2 := w_{\lambda_1+1} = \cdots = w_{\lambda_1+\lambda_2}$,

  $\vdots$
- $y_l := w_{\lambda_1+\cdots+\lambda_{l-1}+1} = \cdots = w_{\lambda_1+\cdots+\lambda_l}$,

while $y_1, \ldots, y_l$ may or may not be distinct. This also guarantees that

- $e_1 := \deg(y_1) = i_1 = \cdots = i_{\lambda_1}$,
- $e_2 := \deg(y_2) = i_{\lambda_1+1} = \cdots = i_{\lambda_1+\lambda_2}$,

  $\vdots$
- $e_l := \deg(y_l) = i_{1+\cdots+\lambda_{l-1}+1} = \cdots = i_{\lambda_1+\cdots+\lambda_l}$.

Thus, we also have

- $\alpha_1 := \alpha_{i_1,h_1} = \cdots = \alpha_{i_{\lambda_1},h_{\lambda_1}}$,
- $\alpha_2 := \alpha_{i_{\lambda_1+1},h_{\lambda_1+1}} = \cdots = \alpha_{i_{\lambda_1+\lambda_2},h_{\lambda_1+\lambda_2}}$,

  $\vdots$
- $\alpha_l := \alpha_{i_{\lambda_1+\cdots+\lambda_{l-1}+1},h_{\lambda_1+\cdots+\lambda_{l-1}+1}} = \cdots = \alpha_{i_n,h_n}$.

Note that $y_j \in V_{e_j}$ and $\phi(y_j) = \phi_{e_j}(y_j) = \alpha_j y_j$. We also note that $\lambda_1 e_1 + \cdots + \lambda_l e_l = r$ because $(V_{e_1})^{\otimes \lambda_1} \otimes \cdots \otimes (V_{e_l})^{\otimes \lambda_l}$ is a direct summand of $(V^{\otimes n})_r$ in the decomposition of $V^{\otimes n}$ that gives the grading for the tensor product.

Thus, for this particular $g \in S_n$, applying Lemma 3.4, we have

$$\begin{aligned}
&Q_g(i_1, \ldots, i_n) \\
&\quad = Q_{(1\ \cdots\ \lambda_1)}(i_1, \ldots, i_n) + \cdots + Q_{(\lambda_1+\cdots+\lambda_{l-1}+1\ \cdots\ \lambda_1+\cdots+\lambda_l)}(i_1, \ldots, i_n) \\
&\quad = (i_1 + \cdots + i_{\lambda_1-1})i_{\lambda_1} + \cdots + (i_{\lambda_1+\cdots+\lambda_{l-1}+1} + \cdots + i_{\lambda_1+\cdots+\lambda_l-1})i_{\lambda_1+\cdots+\lambda_l} \\
&\quad = (\lambda_1 - 1)e_1 \cdot e_1 + \cdots + (\lambda_l - 1)e_l \cdot e_l \\
&\quad = (\lambda_1 - 1)e_1^2 + \cdots + (\lambda_l - 1)e_l^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
Q_g(i_1, \ldots, i_n) &\equiv (\lambda_1 + 1)e_1 + \cdots + (\lambda_l + 1)e_l \\
&= r + e_1 + \cdots + e_l,
\end{aligned}$$

where the congruence is taken modulo 2. Hence, we have computed the sign:

$$(-1)^{Q_g(i_1,\ldots,i_n)} = (-1)^{r+e_1+\cdots+e_l}.$$

This implies that the vector $w_1 \otimes \cdots \otimes w_n = y_1^{\otimes\lambda_1} \otimes \cdots \otimes y_l^{\otimes\lambda_l}$ contributes

$$(-1)^{r+e_1+\cdots+e_l}\alpha_{i_1,h_1} \cdots \alpha_{i_n,h_n} = (-1)^{r+e_1+\cdots+e_l}\alpha_1^{\lambda_1} \cdots \alpha_l^{\lambda_l}$$

to $\mathrm{Tr}(g\phi^{\otimes n})_r$. We keep fixing the partition $[\lambda_1, \ldots, \lambda_l] \vdash n$, which is the equivalent datum to the cycle decomposition of g in S_n. Write $B_i := \{v_{i,1}, \ldots, v_{i,d_i}\}$, the chosen eigenbasis for V_i. So far, we have seen that

$$\mathrm{Tr}(g\phi^{\otimes n})_r = \sum_{\lambda_1 e_1+\cdots+\lambda_l e_l = r}\ \sum_{(y_1,\ldots,y_l)\in B_{e_1}\times\cdots\times B_{e_l}} (-1)^{r+e_1+\cdots+e_l}\alpha_1^{\lambda_1} \cdots \alpha_l^{\lambda_l}.$$

We note that α_i appearing in the computation above is the eigenvalue for $y_i \in B_{e_i}$. (Ideally, it is better use α_{y_i} instead of α_i, but we just keep our notation for the sake of simplicity.) This implies that

$$
\begin{aligned}
L_u(g\phi^{\otimes n}) &= \sum_{r\geq 0}(-u)^r \mathrm{Tr}(g\phi^{\otimes n})_r \\
&= \sum_{r\geq 0}\sum_{\lambda_1 e_1+\cdots+\lambda_l e_l=r}\sum_{(y_1,\ldots,y_l)\in B_{e_1}\times\cdots\times B_{e_l}} (-1)^{e_1+\cdots+e_l}\alpha_1^{\lambda_1}\cdots\alpha_l^{\lambda_l}u^r \\
&= \sum_{r\geq 0}\sum_{\lambda_1 e_1+\ldots+\lambda_l e_l=r}\sum_{(y_1,\ldots,y_l)\in B_{e_1}\times\cdots\times B_{e_l}} (-1)^{e_1+\ldots+e_l}\alpha_1^{\lambda_1}\cdots\alpha_l^{\lambda_l}u^{\lambda_1 e_1+\cdots+\lambda_l e_l} \\
&= \sum_{r\geq 0}\sum_{\lambda_1 e_1+\cdots+\lambda_l e_l=r}\sum_{(y_1,\ldots,y_l)\in B_{e_1}\times\cdots\times B_{e_l}} \alpha_1^{\lambda_1}\cdots\alpha_l^{\lambda_l}(-u^{\lambda_1})^{e_1}\cdots(-u^{\lambda_l})^{e_l} \\
&= \sum_{r\geq 0}\sum_{\lambda_1 e_1+\cdots+\lambda_l e_l=r}\left(\sum_{y_1\in B_{e_1}}\alpha_1^{\lambda_1}(-u^{\lambda_1})^{e_1}\right)\cdots\left(\sum_{y_l\in B_{e_l}}\alpha_l^{\lambda_l}(-u^{\lambda_l})^{e_l}\right) \\
&= \sum_{r\geq 0}\sum_{\lambda_1 e_1+\cdots+\lambda_l e_l=r}\mathrm{Tr}(\phi_{e_1}^{\lambda_1})(-u^{\lambda_1})^{e_1}\cdots\mathrm{Tr}(\phi_{e_l}^{\lambda_l})(-u^{\lambda_l})^{e_l} \\
&= \sum_{e_1,\ldots,e_l\geq 0}\mathrm{Tr}(\phi_{e_1}^{\lambda_1})(-u^{\lambda_1})^{e_1}\cdots\mathrm{Tr}(\phi_{e_l}^{\lambda_l})(-u^{\lambda_l})^{e_l} \\
&= \left(\sum_{e_1\geq 0}\mathrm{Tr}(\phi_{e_1}^{\lambda_1})(-u^{\lambda_1})^{e_1}\right)\cdots\left(\sum_{e_l\geq 0}\mathrm{Tr}(\phi_{e_l}^{\lambda_l})(-u^{\lambda_l})^{e_l}\right) \\
&= \left(\sum_{i\geq 0}\mathrm{Tr}(\phi_i^{\lambda_1})(-u^{\lambda_1})^{i}\right)\cdots\left(\sum_{i\geq 0}\mathrm{Tr}(\phi_i^{\lambda_l})(-u^{\lambda_l})^{i}\right) \\
&= \left(\sum_{i\geq 0}\mathrm{Tr}(\phi_i)(-u)^{i}\right)^{m_1(g)}\left(\sum_{i\geq 0}\mathrm{Tr}(\phi_i^2)(-u^2)^{i}\right)^{m_2(g)} \\
&\quad \cdots\left(\sum_{i\geq 0}\mathrm{Tr}(\phi_i^n)(-u^n)^{i}\right)^{m_n(g)} \\
&= L_u(\phi)^{m_1(g)}L_{u^2}(\phi^2)^{m_2(g)}\cdots L_{u^n}(\phi^n)^{m_n(g)},
\end{aligned}
$$

as desired. □

3.4. Proof of Theorem 1.4

Keeping all the notation in the previous subsection, the following immediately proves Theorem 1.4:

Theorem 3.5 (Trace formula on $(V^{\otimes n})^G$). Assume the notation in Theorem 3.2. Let $G \leqslant S_n$ such that $|G| \neq 0$ in k. Then

$$L_u(\phi^{\otimes n}|_{(V^{\otimes n})^G}) = \frac{1}{|G|}\sum_{g\in G} L_u(\phi)^{m_1(g)} L_{u^2}(\phi^2)^{m_2(g)} \cdots L_{u^n}(\phi^n)^{m_n(g)}.$$

Proof. Since $|G|\neq 0$ in k, we can consider the averaging operator $e_G : V^{\otimes n} \to V^{\otimes n}$ given by

$$e_G(\alpha) := \frac{1}{|G|}\sum_{g\in G} g\alpha,$$

where again, we use the representation of G on $V^{\otimes n}$ introduced in the beginning of this section according to the Koszul sign rule. Note that we have $(V^{\otimes n})^G = e_G(V^{\otimes n})$, so any element of $(V^{\otimes n})^G$ can be written as $e_G(\alpha)$ with $\alpha \in V^{\otimes n}$. Using Lemma 3.1, we have

$$\phi^{\otimes n}(e_G(\alpha)) = \phi^{\otimes n}\left(\frac{1}{|G|}\sum_{g\in G} g\alpha\right) = \frac{1}{|G|}\sum_{g\in G} g\phi^{\otimes n}(\alpha).$$

Thus, we have shown that

$$\phi^{\otimes n} \circ e_G = \frac{1}{|G|}\sum_{g\in G} g\phi^{\otimes n} \in \mathrm{End}_k(V^{\otimes n}).$$

Note that both sides restrict to $(V^{\otimes n})^G$, and since e_G is the identity on $(V^{\otimes n})^G$, we get

$$\phi^{\otimes n}|_{(V^{\otimes n})^G} = \frac{1}{|G|}\sum_{g\in G} g\phi^{\otimes n} \in \mathrm{End}_k((V^{\otimes n})^G).$$

Applying L_u both sides, we get

$$L_u(\phi^{\otimes n}|_{(V^{\otimes n})^G}) = \frac{1}{|G|}\sum_{g\in G} L_u(g\phi^{\otimes n}),$$

so applying Theorem 3.2, we are done. □

4. Alternating powers

In the introduction, say for Corollary 1.5, we only cared about the full symmetric groups $(S_n)_{n\in\mathbb{Z}_{\geq 0}}$. It is natural to consider other sequences of subgroups of S_n for $n \in \mathbb{Z}_{\geq 0}$. In this section, we consider the alternating subgroups $A_n \leqslant S_n$, using a well-known lemma in combinatorics:

Lemma 4.1 ([HP1973], p.36, (2.2.6)). For any $n \in \mathbb{Z}_{\geq 2}$, we have the following identity relating cycle indices of A_n and S_n:

$$Z_{A_n}(x_1, x_2, \dots, x_n) = Z_{S_n}(x_1, x_2 \dots, x_n) + Z_{S_n}(x_1, -x_2, \dots, (-1)^{n+1}x_n).$$

Given a sequence $G_n \leqslant S_n$ of subgroups for $n \in \mathbb{Z}_{\geq 0}$, we write

$$Z_{G_\bullet}(\boldsymbol{x}, t) := \sum_{n=0}^{\infty} Z_{G_n}(\boldsymbol{x})t^n \in \mathbb{Q}[\boldsymbol{x}][\![t]\!].$$

Corollary 4.2. We have

$$Z_{A_\bullet}(\boldsymbol{x}, t) = Z_{S_\bullet}(\boldsymbol{x}, t) + \frac{1}{Z_{S_\bullet}(\boldsymbol{x}, -t)} - 1 - x_1 t.$$

Proof. Recall from the introduction that

$$\sum_{n=0}^{\infty} Z_{S_n}(x_1, \dots, x_n)t^n = \exp\left(\sum_{r=1}^{\infty} \frac{x_r}{r} t^r\right),$$

which implies that

$$\begin{aligned} \sum_{n=0}^{\infty} Z_{S_n}(x_1, -x_2 \dots, (-1)^{n+1}x_n)t^n &= \exp\left(\sum_{r=1}^{\infty} \frac{(-1)^{r+1}x_r}{r} t^r\right) \\ &= \exp\left(-\sum_{r=1}^{\infty} \frac{x_r}{r} (-t)^r\right) \\ &= \exp\left(\sum_{r=1}^{\infty} \frac{x_r}{r} (-t)^r\right)^{-1} \\ &= Z_{S_\bullet}(\boldsymbol{x}, -t)^{-1}. \end{aligned}$$

Therefore, applying Lemma 4.1, we are done. ∎

Applying Corollary 4.2, our main theorem (Theorem 1.4) and its corollary immediately gives the following:

Corollary 4.3. Assume the same hypotheses as in Theorem 1.4. Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_u(\phi_{X^n}|_{H^\bullet(X^n)^{A_n}})t^n \\
&= \sum_{n=0}^{\infty} L_u(\phi_{X^n}|_{H^\bullet(X^n)^{S_n}})t^n + \frac{1}{\sum_{n=0}^{\infty} L_u(\phi_{X^n}|_{H^\bullet(X^n)^{S_n}})(-t)^n} - 1 - L_u(\phi)t \\
&= \prod_{i=0}^{2d}\left(\frac{1}{\det(\mathrm{id}_{H^i(X)} - \phi_i u^i t)}\right)^{(-1)^i} + \prod_{i=0}^{2d}\left(\frac{1}{\det(\mathrm{id}_{H^i(X)} + \phi_i u^i t)}\right)^{(-1)^{i+1}} \\
&\quad - 1 - L_u(\phi)t.
\end{aligned}$$

Just as Corollary 1.5 implies Theorem 1.1, Corollary 4.3 implies the following concrete theorem:

Theorem 4.4. Let X be either a compact complex manifold of dimension d or a projective variety of dimension d over a finite field $\mathbb{F}_q$. For any endomorphism F on X, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_u(\mathrm{Alt}^n(F)^*)t^n &= \prod_{i=0}^{2d}\left(\frac{1}{\det(\mathrm{id}_{H^i(X)} - F_i^* u^i t)}\right)^{(-1)^i} \\
&\quad + \prod_{i=0}^{2d}\left(\frac{1}{\det(\mathrm{id}_{H^i(X)} + F_i^* u^i t)}\right)^{(-1)^{i+1}} - 1 - L_u(F^*)t,
\end{aligned}$$

where we use the same notation as in Theorem 1.1 except $\mathrm{Alt}^n(F)$, the endomorphism on the n-th alternating power $\mathrm{Alt}^n(X) = X^n/A_n$ of X induced by F.

Theorem 4.4 is interesting in its own right. For instance, in the singular setting, if we take $F = \mathrm{id}_X$, we have the following identity that computes the singular Betti numbers of $\mathrm{Alt}^n(X)$ in terms of those of X:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \chi_u(\mathrm{Alt}^n(X))t^n \\
&\quad = \prod_{i=0}^{2d}\left(\frac{1}{1-u^i t}\right)^{(-1)^i h^i(X)} + \prod_{i=0}^{2d}\left(\frac{1}{1+u^i t}\right)^{(-1)^{i+1} h^i(X)} - 1 - \chi_u(X)t.
\end{aligned}$$

In particular, taking $u = 1$, we have the following formula for the Euler characteristics:

$$\sum_{n=0}^{\infty} \chi(\mathrm{Alt}^n(X))t^n = \left(\frac{1}{1-t}\right)^{\chi(X)} + \left(\frac{1}{1+t}\right)^{-\chi(X)} - 1 - \chi(X)t.$$

If X is a smooth projective variety over $\mathbb{C}$, then we also get the alternating power analogue of Cheah's result:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{i\geq 0}\sum_{p+q=i} h^{p,q}(\mathrm{Alt}^n(X))x^p y^q(-u)^i t^n \\ &\quad= \prod_{i=0}^{2d}\prod_{p+q=i}\left(\frac{1}{1-x^p y^q u^i t}\right)^{(-1)^i h^{p,q}(X)} \\ &\quad\quad+ \prod_{i=0}^{2d}\prod_{p+q=i}\left(\frac{1}{1+x^p y^q u^i t}\right)^{(-1)^{i+1} h^{p,q}(X)} \\ &\quad\quad- 1 - \sum_{i=0}^{2d}\sum_{p+q=i} h^{p,q}(X)x^p y^q(-u)^i t.\end{aligned}$$

In particular, the right-hand side is rational in t. We can also obtain an alternating analogue of the formula

$$Z_X(t) = \sum_{n=0}^{\infty} |\mathrm{Sym}^n(X)(\mathbb{F}_q)|t^n,$$

where $Z_X(t)$ is the zeta series of a projective variety X over $\mathbb{F}_q$. That is, taking $u = 1$ in the l-adic setting of Theorem 4.4, we can deduce

$$Z_X(t) + \frac{1}{Z_X(-t)} - 1 - |X(\mathbb{F}_q)|t = \sum_{n=0}^{\infty} |\mathrm{Alt}^n(X)(\mathbb{F}_q)|t^n.$$

Remark 4.5. In a collaboration with Yinan Nancy Wang, we started to question if the above identity holds in the Grothendieck ring of varieties over a field, where taking the $\mathbb{F}_q$-point counting is replaced by taking the class in the Grothendieck ring. The problem seems nontrivial even when X is $\mathbb{A}^1$ or $\mathbb{P}^1$ over any field.

Remark 4.6. The upshot of this section is that because Theorem 1.4 is formulated in terms of cycle indices, we can use combinatorial knowledge about the cycle indices $Z_{A_n}(\boldsymbol{x})$ of alternating subgroups A_n to understand cohomological information about alternating powers $\mathrm{Alt}^n(X) = X^n/A_n$. We consider another example here. The cycle index for the cyclic subgroup C_n generated by $(1\ 2 \cdots\ n)$ in S_n is given by

$$Z_{C_n}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{d|n} \varphi(d) x_d^{n/d},$$

where φ is the Euler's totient function. Applying Theorem 1.4, we get

$$|(X^n/C_n)(\mathbb{F}_q)| = \frac{1}{n} \sum_{d|n} \varphi(d) |X(\mathbb{F}_{q^d})|^{n/d}$$

for a projective variety X over $\mathbb{F}_q$ and

$$\chi_u(X^n/C_n) = \frac{1}{n} \sum_{d|n} \varphi(d) \chi_{u^d}(X)^{n/d},$$

for any finite CW-complex X, the second of which appears in Macdonald's paper ([Mac1962A], p.568, (8.4)). We believe that there are more sequences of subgroups G_n of S_n such that the generating function for certain cohomological information (e.g., singular Betti numbers, $\mathbb{F}_q$-point counts, or Hodge numbers) of X^n/G_n is rational. Namely, whenever the generating function for $Z_{G_n}(x_1, \ldots, x_n)$ has a formula that involves exponentiation, we should be able to get such a rationality for cohomological information of X^n/G_n by applying Theorem 1.4. Classifying the list of such sequences $(G_n)_{n \in \mathbb{Z}_{\geq 0}}$ may be an interesting combinatorial problem.

5. More on point counting over finite fields

Let X be a projective variety over a finite field $\mathbb{F}_q$, and consider any subgroup $G \leqslant S_n$ acting on X^n by permuting coordinates. An immediate consequence of Theorem 1.4 in the l-adic setting, for a prime l not dividing q nor $|G|$, by taking $u = 1$ and applying the Grothendieck-Lefschetz trace formula is that

$$\begin{aligned}|(X^n/G)(\mathbb{F}_q)| &= Z_G(|X(\mathbb{F}_q)|, |X(\mathbb{F}_{q^2})|, \ldots, |X(\mathbb{F}_{q^n})|) \\ &= \frac{1}{|G|} \sum_{g \in G} |X(\mathbb{F}_q)|^{m_1(g)} |X(\mathbb{F}_{q^2})|^{m_2(g)} \cdots |X(\mathbb{F}_{q^n})|^{m_n(g)}.\end{aligned}$$

It turns out that the formula even holds when X is a quasi-projective variety over $\mathbb{F}_q$ by using Theorem 1.4 with the compactly supported l-adic étale cohomology, noting that all the results we use for the l-adic étale cohomology when X is projective over $\mathbb{F}_q$ generalize to the compactly supported l-adic étale cohomology when X is quasi-projective over $\mathbb{F}_q$ as long as $l \nmid q, |G|$. In particular, taking $X = \mathbb{A}^1$ over $\mathbb{F}_q$, we have

$$|(\mathbb{A}^n/G)(\mathbb{F}_q)| = q^n.$$

When $G = A_n$ and q is odd, we have

$$\mathbb{A}^n/A_n \simeq \operatorname{Spec}\left(\frac{\mathbb{F}_q[t_1, \dots, t_n, y]}{(y^2 - \Delta_n(t_1, \dots, t_n))}\right),$$

where $\Delta_n(t_1, \dots, t_n)$ is the discriminant of the monic polynomial

$$x^n + t_1 x^{n-1} + \cdots + t_{n-1}x + t_n.$$

Thus, we see that for $n \geq 2$, the polynomial function $\Delta_n : \mathbb{F}_q^n \to \mathbb{F}_q$ given by the discriminant satisfies

$$\begin{aligned}&|\Delta_n^{-1}(\{\text{quadratic residues in } \mathbb{F}_q^\times\})|\\ &\quad = |\Delta_n^{-1}(\{\text{quadratic non-residues in } \mathbb{F}_q^\times\})|,\end{aligned}$$

because $|\Delta_n^{-1}(0)| = q^{n-1}$, as there are precisely $q^n - q^{n-1}$ degree n monic square-free polynomials in $\mathbb{F}_q[x]$. The above equality was also observed by Chan, Kwon, and Seaman using more direct computations ([CKS2018], Corollary 3.3).

6. Further directions

The conjecture with Yinan Nancy Wang mentioned in Remark 4.5 is extremely challenging. For the case $X = \mathbb{A}^1$ over $\mathbb{C}$, it says $[\mathbb{A}^n/A_n] = [\mathbb{A}^n]$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex varieties. In particular, it implies that $\mathrm{Alt}^n(\mathbb{A}^1) = \mathbb{A}^n/A_n$ is stably rational ([CNS2018], Theorem 6.1.5), which seems to be an open problem for any $n \geq 6$. Thus, any significant progress of this conjecture would require innovative approaches to deal with the relations defining $K_0(\mathrm{Var}_{\mathbb{C}})$, which is likely to go beyond the approaches introduced in this paper. Nevertheless, we are still hopeful that extending approaches in this paper will be able to reach more results on various specializations of this conjecture in the near future.

References

[Che1994] J. Cheah, *The cohomology of smooth nested Hilbert schemes of points*, Ph. D. dissertation, the University of Chicago, (1994).

[CKS2018] J. Chan, S. Kwon, and M. Seaman, *On the distribution of discriminants over a finite field*, preprint: `arXiv:1812.06231`, (2018).

[CNS2018] A. Chambert-Loir, J. Nicaise, and J. Sebag *Motivic Integration*, Progress in Mathematics **325**, Birkhäuser (2018).

[Del1971] P. Deligne, *Théorie de Hodge II*, Publications Mathématiques de l'I.H.E.S **40** (1971), 5–58.

[DL1976] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Annals of Mathematics **103** (1976), 631–648.

[Dwo1960] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, American Journal of Mathematics **82** (1960), 631–648.

[FK1988] E. Freitag and R. Kiehl, *Etale Cohomology and the Weil Conjecture*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3**. Folge / A Series of Modern Surveys in Mathematics.

[Gro1957] A. Grothendieck, *Sur quelques points d'Algébre homologique*, Tohoku Mathematical Journal **9** (1957), 119–221.

[HN1975] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Mathematische Annalen **212** (1975), 215–248.

[HP1973] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press (1973).

[Hat2002] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[IZ2013] L. Illusie and W. Zheng, *Odds and Ends on Finite Group Actions and Traces*, International Mathematics Research Notices **1** (2013), 1–62.

[Kap2000] M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, preprint, `arXiv:math/0001005`.

[Mac1962A] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Mathematical Proceedings of the Cambridge Philosophical Society **58** (1962), 563–568.

[Mac1962J] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.

[Mus] M. Mustaţă, *Zeta functions in algebraic geometry*, lecture notes, available at `http://www-personal.umich.edu/~mmustata/zeta_book.pdf`

[Mus2] M. Mustaţă, *Singular cohomology as sheaf cohomology with constant coefficients*, lecture notes, available at `http://www-personal.umich.edu/~mmustata/SingSheafcoho.pdf`

[Mil1980] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series **33** (1980).

[Pol1937] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen* (2nd ed.), Oxford (1995).

[PR1987] G. Pólya and R. C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Spring-Verlag (1987).

[Red1927] J. H. Redfield, *The theory of group-reduced distributions*, American Journal of Mathematics **49** (1927), 433–455.

[Sta1999] R. Stanley, *Enumerative Combinatorics: Volume 2*, Cambridge University Press (1999).

[Vak2015] R. Vakil, Lecture notes for Arizona Winter School 2015.

[VW2015] R. Vakil and M. M. Wood, *Discriminants in the Grothendieck ring*, Duke Mathematical Journal **164** (2015), No. 6.

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