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Real-analytic coordinates for smooth strictly pseudoconvex CR-structures

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For a smooth strictly pseudoconvex hypersurface in a complex manifold, we give a necessary and sufficient condition for being CR-diffeomorphic to a real-analytic CR manifold. Our condition amounts to a holomorphic extension property for the canonically associated function expressing 2-jets of the formal Segre varieties in terms of their 1-jets. We also express this condition in equivalent terms for a Fefferman type determinant [Fe76].

1. Introduction

In this paper, we address the following problem:

Problem 1. Let M be a (C^∞) smooth real hypersurface in a complex manifold X. Find necessary and sufficient conditions on M to be CR-diffeomorphic to a real-analytic CR manifold.

If M is CR-diffeomorphic to a real-analytic CR manifold, we shall call it *analytically regularizable*. Problem 1 is of interest because of the case of real-analytic CR manifolds being much better studied with more results and tools available, such as complexification and Segre varieties (see e.g. [BER99]). On the other hand, the problem seems to be widely open even for strictly pseudoconvex hypersurfaces, where it is non-trivial, i.e. there exist smooth non-analytic hypersurfaces that are analytically regularizable and there exist those that are not, see the end of Section 2 for respective examples. (The latter phenomenon is in contrast with the case of hypersurfaces of mixed Levi form signature, where any CR-diffeomorphism to a real-analytic hypersurface extends holomorphically to both sides, hence any analytically regularizable hypersurface must be already real-analytic in its ambient complex manifold X.)

The goal of this paper is to provide a nontrivial *necessary and sufficient* condition (Condition E below) giving a solution to Problem 1 in the case when M is *strictly pseudoconvex*. Our condition is formulated in terms of holomorphic extension of certain functions invariantly associated to M

(these functions can be viewed as Fefferman type determinants [Fe76], as shown in Section 2). Thus our results provide a *two-way bridge* for Problem 1 with the much better studied questions of holomorphic extension of smooth functions from real submanifolds in complex manifolds.

We proceed by giving a high level non-technical formulation of our result, while more precise details can be found in Section 2. Given a smooth real hypersurface M in a complex manifold X of dimension $n+1$, consider smooth local defining equations of the kind $\rho(z,\bar z)=0$ for M with $d\rho\neq 0$. Then ρ can be formally complexified at each point, i.e. there exist formal power series $\rho(z,\bar w)$ giving for $w=z$ the Taylor series of ρ. This allows to invariantly define formal Segre varieties

$$Q_p=\{z:\rho(z,\bar p)=0\}$$

for $p\in M$, as well as their k-jets $j^k_pQ_p$ for every $k\geq 1$. In particular (see [We78]), if M is strictly pseudoconvex, then j^kQ_p, $p\in M$, defines a canonical embedding of M into the space $J^{k,n}(X)$ of all k-jets of complex-analytic hypersurfaces for every $k\geq 1$, given by

$$p\mapsto j^kQ_p,$$

and the image of M appears to be totally real.

Furthermore, for any $k,l\geq 1$, we obtain canonical smooth maps $s^{k,l}$ between respective images of those embeddings of M, sending j^kQ_p to j^lQ_p for every p. To formulate our main result, denote by $M_J\subset J^{1,n}$ the image of the embedding $p\mapsto j^1Q_p$ of M as above, and consider the map

$$s^{1,2}\colon M_J\to J^{2,n}$$

sending $j^1_pQ_p$ to $j^2_pQ_p$ for every $p\in M$. Let $\widehat{M}\subset J^{1,n}$ be the (smooth) real hypersurface consisting of all 1-jets with base points in M.

Theorem 1. *Let M be a smooth strictly pseudoconvex hypersurface in a complex manifold X. Then M is analytically regularizable (i.e. CR-diffeomorphic to a real-analytic CR manifold) if and only if the map $s^{1,2}$ admits a $J^{2,n}$–valued holomorphic extension to a neighborhood of $\widehat{M}$ in the pseudoconvex side of it in $J^{1,n}$, that is smooth up to $\widehat{M}$.*

Holomorphic extension property of the kind required in Theorem 1 is well studied in Complex Analysis: see Remark 2.3 below. Further, note that since M_J is generic in $J^{1,n}$, if the desired holomorphic extension of $s^{1,2}$ exists, it is necessarily unique. Also, the CR-diffeomorphism to a real-analytic CR manifold is unique if exists, up to a real-analytic CR-diffeomorphism, as follows directly from the reflection principle for real-analytic strictly pseudoconvex hypersurfaces.

We shall particularly emphasize that, while the initial Problem 1 asks on the existence of a CR-diffeomorphism onto an *unknown* real-analytic target, Theorem 1 reduces the problem to studying the holomorphic extension problem for a *concrete* smooth function invariantly associated with a hypersurface and computable in local coordinates by an elementary calculus (see Section 2).

As an application of our effective procedure, we provide an answer to a question by Professor Takeo Ohsawa about a parametric version of the main result. For simplicity, we formulate a local version here. We consider a smooth family $\{M^t\}$ of pseudoconvex hypersurfaces $M^t \subset \mathbb{C}^{n+1}$ depending smoothly on a parameter $t \in D$ (D is a domain in $\mathbb{R}^m$), i.e. $\{M^t\} \subset \mathbb{C}^{n+1} \times D$ is given near its point (z_0, t_0) by $\rho(z, \bar{z}, t) = 0$, $z \in \mathbb{C}^{n+1}$, $t \in D$, where ρ is a smooth function with $\partial\rho(\cdot, t_0) \neq 0$. We assume that, for each t, the pseudoconvex side of M^t is given by $\rho(z, t) < 0$. The intersection of the one-sided neighborhood $\{\rho(z, \bar{z}, t) < 0\} \subset \mathbb{C}^{n+1} \times D$ with a fixed open neighborhood of (z_0, t_0) is called a *uniform pseudoconvex neighborhood* of (z_0, t_0). Next, a family $\{M^t\}$ *is uniformly analytic* at a point (z_0, t_0), if in its neighborhood, $\rho(z, \bar{z}, t)$ can be chosen as the restriction of a smooth function $\tilde{\rho}(z, \bar{w}, t)$ in a neighborhood of $(z_0, \bar{z}_0, t)$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{R}^m$ that is holomorphic in $(z, \bar{w})$. We say that a family $\{M^t\}$ is *uniformly analytically regularizable*, if there is a uniformly analytic family $\{\widetilde{M}^t\}$ of hypersurfaces in $\mathbb{C}^{n+1}$ (as defined above) and a smooth family $F(z, \bar{z}, t) = F^t(z, \bar{z})$ of CR-diffeomorphisms of M^t onto the respective $\widetilde{M}^t$ in a neighborhood of (z_0, t_0). We then have the following parametric version of Theorem 1:

Theorem 2. *A smooth family $\{M^t\}$ of strictly pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ is uniformly analytically regularizable at a point (z_0, t_0) if and only if the respective smooth family of canonical maps $s^t_{1,2}$ admits a smooth family of holomorphic and smooth up to $\widehat{M}^t \subset J^{1,n}$ extensions $S^t(z)$ to a uniform pseudoconvex neighborhood of $(\pi^{-1}(z_0), t_0)$ in $J^{1,n} \times \mathbb{R}^m$. Here $\pi\colon J^{1,n} \to \mathbb{C}^{n+1}$ is the canonical projection.*

We conclude by mentioning that, while analyticity problems have been studied for other geometric structures, e.g. Riemannian structures (see e.g. [DK81]), no similar nontrivial necessary and sufficient conditions seem to be known. At the same time, we would like to point out results on obstructions for the *algebraizability* of real-analytic hypersurfaces (see e.g. the work of Forstneric [Fo04], Huang, Ji and Yau [HJY01], and the recent survey [HX17] by Huang and Xiao).

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2. Condition E for smoothly embedded real hypersurfaces

We now describe in details the holomorphic extension condition in Theorem 1 that we shall call *Condition E*. We shall give below both an invariant and a coordinate-based formulations of it. For the basic concepts in CR-geometry (such as Segre varieties and formal submanifolds) we refer to [BER99], and for *jet bundles* and related concepts to [CS09].

Let

$$\pi\colon J^{1,n} \to \mathbb{C}^{n+1}$$

be the bundle of 1-jets of complex hypersurfaces of $\mathbb{C}^{n+1}$, which is a projective holomorphic bundle over $\mathbb{C}^{n+1}$ with the fiber dimension n, and $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, be a smooth strictly pseudoconvex real hypersurface. Then the complex tangent bundle $T^{\mathbb{C}}M$ induces the natural embedding

$$\varphi\colon M \to J^{1,n}, \quad x \mapsto \big(x, [T^{\mathbb{C}}_x M]\big).$$

The image

$$\varphi(M) =: M_J \subset J^{1,n}$$

of it is a smooth $(2n+1)$-dimensional real submanifold in the $(2n+1)$-dimensional complex manifold $J^{1,n}$. Webster in [We78] observed that $M_J \subset J^{1,n}$ is *totally real* whenever M is Levi-nondegenerate. Next, associated with

M is the smooth (weakly) pseudoconvex real hypersurface

$$\widehat{M} := \pi^{-1}(M) \subset J^{1,n}.$$

The manifold M_J is a smooth submanifold in $\widehat{M}$. Note that $\widehat{M}$ itself is locally CR-equivalent to $M \times \mathbb{C}^n$ (and thus is *holomorphically degenerate*, see [BER99]). In what follows we denote by U^+ the pseudoconvex side of M and by

$$\widehat{U}^+ := \pi^{-1}(U^+)$$

that of $\widehat{M}$ ($\widehat{U}^+$ is locally biholomorphic to $U^+ \times \mathbb{C}$ at a point in $\widehat{M}$).

We next fix a point $p \in M$. Since M is smooth, we may consider at each point $q \in M$ near p, its formal complexfication at q as a formal complex hypersurface in $\mathbb{C}^{n+1} \times \overline{\mathbb{C}^{n+1}}$ obtained by complexifying the formal Taylor series of its defining function at q. In this way, the formal Segre variety Q_q of M at q is well defined. Then the 2-jets

$$j^2_q Q_q,\ q \in M \tag{2.1}$$

of such formal Segre varieties induce a smooth embedding of M (and hence $M_J \subset J^{1,n}$) into the bundle

$$J^{2,n} = J^{2,n}(\mathbb{C}^{n+1})$$

of 2-jets of complex hypersurfaces in $\mathbb{C}^{n+1}$. The space $J^{2,n}$ is canonically a fiber bundle

$$\pi^2_1 \colon J^{2,n} \to J^{1,n}.$$

The above 2-jet embedding defines a canonical section of π^2_1,

$$s \colon M_J \to J^{2,n}.$$

Now our analyticity condition for a smooth strictly pseudoconvex submanifold looks as follows.

Definition 2.1. We say that M *satisfies Condition E at* p, if for some choice of a neighborhood U of p, the section s extends as a smooth section of π^2_1 over the pseudoconvex side $\widehat{U}^+ \cup \widehat{M}$, which is furthermore holomorphic in $\widehat{U}^+$.

We next give an (equivalent to the above) coodinate formulation of Condition E. If $M \subset \mathbb{C}^{n+1}$ is a smooth hypersurface with the defining equation

$$\rho(Z, \bar{Z}) = 0, \quad Z = (z, w) = (z_1, ..., z_n, w) \in \mathbb{C}^{n+1}, \tag{2.2}$$

$p \in M$ the distinguished point and $\rho_w(p, \bar{p}) \neq 0$, then its formal Segre variety at a point $q = (\tilde{q}, q_{n+1}) \in M$ nearby p is a graph of a function $w(z)$ (considered as a formal power series in $(z - \tilde{q})$). Then the 2-jets (2.1) amount to either the scalar function Φ defined pointwise as $w''(z)$ for $z = \tilde{q}$ (case $n = 1$), or to the symmetric matrix function $\Phi = (\Phi_{ij})$, $i, j = 1, ..., n$, defined pointwise as the collection of $w_{z_i z_j}$ for $z = \tilde{q}$ (case $n > 1$). It is possible to verify that, in turn, for $n = 1$ we have

$$\Phi = \frac{1}{(\rho_w)^3} \begin{vmatrix} \rho & \rho_z & \rho_w \\ \rho_z & \rho_{zz} & \rho_{zw} \\ \rho_w & \rho_{zw} & \rho_{ww} \end{vmatrix}, \tag{2.3}$$

and for $n > 1$ we have

$$\Phi_{ij} = \frac{1}{(\rho_w)^3} \begin{vmatrix} \rho & \rho_{z_j} & \rho_w \\ \rho_{z_i} & \rho_{z_i z_j} & \rho_{z_i w} \\ \rho_w & \rho_{z_j w} & \rho_{ww}, \end{vmatrix} \quad i, j = 1, ..., n. \tag{2.4}$$

(To obtain (2.3),(2.4), one has to differentiate the identity (2.2) once, assuming w to be a function of z, and obtain all the w_{z_j} in terms of the 1-jet of ρ; then, one has to differentiate (2.2) once more to obtain $w_{z_i z_j} = \Phi_{ij}$ in terms of the 2-jet of ρ). Both the scalar function (2.3) and the matrix valued function (2.4) can be considered as either smooth functions on the strictly pseudoconvex hypersurface M or as that on the totally real manifold M_J introduced above.

We shall remark that the invariant determinants in (2.3),(2.4) were used also by Ebenfelt and the second author [EZ16] as well as by Ebenfelt, Duong and the second author [EDZ18] for characterizing the Cartan tensor of a Levi-degenerate hypersurface in terms of its defining function. They can be seen as certain generalizations of the determinants used by Fefferman in [Fe76] for studying asymptotics of the Bergman metric in a smoothly bounded strictly pseudoconvex domain at a boundary point.

In terms of the Φ-function, Condition E reads as follows.

Definition 2.2. We say that M *satisfies Condition E at* p, if for some choice of a neighborhood U of p, the function Φ defined on M_J by either (2.3) or (2.4) extends to the pseudoconvex side $\widehat{U}^+ \cup \widehat{M}$ holomorphically and smoothly up to the boundary.

It is obvious that Definition 2.2 is equivalent to Definition 2.1.

We give now the more precise local version of our main result, from which the global result in Theorem 1 follows directly in view of the uniqueness of the extension and the real-analytic CR-structure.

Theorem 3. *A smooth strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, is locally analytically regularizable (i.e. CR-equivalent near a point $p \in M$ to a real-analytic hypersurface $\widetilde{M} \subset \mathbb{C}^{n+1}$) if and only it satisfies Condition E at p.*

Remark 2.3. In order to explain how Condition E can be checked in practice, recall that this condition asks for a holomorphic extension of a smooth function given on a totally real submanifold of maximal dimension to an open set in the ambient space. Extension problems of this kind are well studied in Complex Analysis, e.g. in Carleman's type formulas (see Aizenberg [Ai93, Chapter V]).

We finish this section by providing two simple examples of smooth non-analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$, one of which admits and the other does not a CR-diffeomorphism onto a real-analytic hypersurface, i.e. one is analytically regularizable while the other isn't.

Example 2.4. Let $f(z, w)$ be a holomorphic function in the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ which is smooth up to $\partial\mathbb{B}^2 = S^3$ but does not extend holomorphically across S^3, e.g. one can take a branch of $e^{(w-1)^{-1/3}}$. Then, for ϵ sufficiently small, the map (small perturbation of identity)

$$(z, w) \mapsto (z + \epsilon f(z, w), w)$$

defines a CR-diffeomorphism from S^3 onto a smooth but not analytic strictly pseudoconvex hypersurface $M \subset \mathbb{C}^2$ that is obviously analytically regularizable.

Example 2.5. The real hypersurface $M \subset \mathbb{C}^2$ (a flat perturbation at the origin of the standard hyperquadric) given by

$$\operatorname{Im} w = |z^2| + e^{-1/|z|^2}$$

is smooth and strictly pseudoconvex near the origin, but is not CR-diffeomorphic to a real-analytic hypersurface $\tilde{M} \subset \mathbb{C}^2$, not even locally at 0. Indeed, since M is *formally spherical* at 0 (i.e. spherical up to infinite order), the existence of a CR-diffeomorphism H onto $\tilde{M}$ with, say, $H(0) = 0$ would mean that $\tilde{M}$ is also formally and hence biholomorphically spherical at 0. (The formal expansion of H at 0 yields a formal transformation of the hyperquadric $\operatorname{Im} w = |z|^2$ onto $\tilde{M}$ which is locally biholomorphic in view of the Chern-Moser theory [CM74].) Hence M must itself be spherical in a neighborhood of 0. On the other hand, one can see by computing the Chern-Moser's curvature (e.g. following [Lo98] or using the determinant expression [EZ16]), that M is not spherical in any neighborhood of 0. Thus M is not analytically regularizable in any neighborhood of 0.

3. Associated differential equations and the necessity of Condition E

In this section, we show that the necessity of Condition E follows rather easily from the construction of *holomorphic differential equations associated with a real-analytic hypersurface*. On the other hand, the sufficiency of Condition E is already quite nontrivial. It is addressed in Section 4.

3.1. The method of associated differential equations

It was observed by Cartan [Car32] and Segre [Se32] (see also Webster [We77]) that the geometry of a real hypersurface in $\mathbb{C}^2$ parallels that of a second order ODE

$$w'' = \Phi(z, w, w'). \tag{3.1}$$

More generally, the geometry of a real hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$, parallels that of a complete second order system of PDEs

$$w_{z_k z_l} = \Phi_{kl}(z_1, ..., z_n, w, w_{z_1}, ..., w_{z_n}), \quad \Phi_{kl} = \Phi_{lk}, \quad k, l = 1, ..., n. \tag{3.2}$$

Moreover, *in the real-analytic case* this parallel becomes algorithmic by using the Segre family of a real hypersurface. With any real-analytic Levi-nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ one can uniquely associate a holomorphic ODE (3.1) ($n = 1$) or a holomorphic PDE system (3.2) ($n \geq 2$). The Segre family of M plays a role of a mediator between the hypersurface and the associated differential equations. A more recent exposition of this method was given in the work [Su01, Su03] of Sukhov. For recent work on associated differential equations in the degenerate setting, see e.g. the papers [KS16, KS17, KL18] of the first author with Lamel and Shafikov.

The associated differential equation procedure is particularly clear in the case of a Levi-nondegenerate hypersurface in $\mathbb{C}^2$. In this case the Segre family is a 2-parameter anti-holomorphic family of holomorphic curves. It then follows from standard ODE theory that there exists an unique ODE (3.1), for which the Segre varieties are precisely the graphs of solutions. This ODE is called *the associated ODE.*

In the general case, both right hand sides in (3.1),(3.2) appear as functions determining the 2-jet of a Segre variety as an analytic function of the 1-jet. More explicitly, we denote the coordinates in $\mathbb{C}^{n+1}$ by

$$(z,w) = (z_1,\dots,z_n,w).$$

Let then fix $M \subset \mathbb{C}^{n+1}$ to be a smooth real-analytic hypersurface, passing through the origin, and choose a small neighborhood U of the origin. In this case we associate a complete second order system of holomorphic PDEs to M, which is uniquely determined by the condition that the differential equations are satisfied by all the graphing functions $h(z,\zeta) = w(z)$ of the Segre family $\{Q_\zeta\}_{\zeta\in U}$ of M in a neighbourhood of the origin. To be more explicit we consider the so-called *complex defining equation* (see, e.g., [BER99]) $w = \rho(z,\bar z,\bar w)$ of M near the origin, which one obtains by substituting $u = \frac{1}{2}(w+\bar w)$, $v = \frac{1}{2i}(w-\bar w)$ into the real defining equation and applying the holomorphic implicit function theorem. The Segre variety Q_p of a point

$$x = (a,b) \in U,\ a \in \mathbb{C}^n,\ b \in \mathbb{C}$$

is now given as the graph

$$w(z) = \rho(z,\bar a,\bar b). \tag{3.3}$$

Differentiating (3.3) we obtain

$$w_{z_j} = \rho_{z_j}(z,\bar a,\bar b), \quad j = 1,\dots,n. \tag{3.4}$$

Considering (3.3) and (3.4) as a holomorphic system of equations with the unknowns $\bar a, \bar b$, in view the Levi-nondegeneracy of M, an application of the implicit function theorem yields holomorphic functions $A_1,...,A_n,B$ such that (3.3) and (3.4) are solved by

$$\bar a_j = A_j(z,w,w'), \quad \bar b = B(z,w,w'),$$

where we write

$$w' = (w_{z_1},\dots,w_{z_n}).$$

The implicit function theorem applies here because the Jacobian of the system coincides with the Levi determinant of M for $(z,w)\in M$ ([BER99]).

Differentiating (3.3) twice and substituting the above solution for $\bar{a}, \bar{b}$ finally yields

$$
\begin{aligned}
w_{z_k z_l} &= \rho_{z_k z_l}(z, A(z,w,w'), B(z,w,w')) \\
&=: \Phi_{kl}(z,w,w'), \quad k,l = 1,\dots,n,
\end{aligned} \tag{3.5}
$$

or, more invariantly,

$$
j^2_{(z,w)} Q_x = \Phi(x, j^1_{(z,w)} Q_x). \tag{3.6}
$$

Now (3.5) is the desired complete system of holomorphic second order PDEs denoted by $\mathcal{E} = \mathcal{E}(M)$.

Definition 3.1. We call $\mathcal{E} = \mathcal{E}(M)$ *the system of PDEs associated with* M. We also regard the collection $\{\Phi_{ij}\}_{i,j=1}^n$ as *the PDE system defining the CR structure* of a Levi-nondegenerate hypersurface M.

3.2. The necessity of Condition E

We now explain the necessity of Condition E for the existence of a smooth CR-diffeomorphism F of $(M,0)$ onto a real-analytic germ $(\tilde{M},0)$. Indeed, given such a CR-diffeomorphism F of $(M,0)$ onto $(\tilde{M},0)$, we may consider the section $\tilde{\Phi}$, as in (3.6), associated with $(\tilde{M},0)$. Clearly, Condition E is satisfied by $(\tilde{M},0)$ since $\tilde{\Phi}$, considered as a function on the 1-jet bundle $J^{1,n}$, already gives the holomorphic extension required in Condition E. Further, we note that the CR-diffeomorphism F extends holomorphically to the pseudoconvex side U^+. The latter extension lifts naturally to a fiber-preserving map $\widehat{F}$ of the pseudoconvex neighborhood $\widehat{U}^+$ of $\widehat{M}$ into $J^{1,n}$ which is smooth up to $\widehat{M}_J$ ($\widehat{F}$ is the 1*-jet prolongation* of the extension of F, see e.g. [CS09]). Now, since F transforms formal complexifications of $M, \tilde{M}$ respectively onto each other, we conclude that the 2-jet prolongation of the extension of F^{-1} transforms $\tilde{\Phi}$ into the desired holomorphic extension Φ, as required. □

4. The sufficiency of Condition E

In this section, we consider a smooth strictly pseudoconvex hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, defined near the point $0 \in M$ and satisfying condition E. We shall prove that M is CR-diffeomorphic (locally near 0) to a real-analytic hypersurface $\tilde{M} \subset \mathbb{C}^{n+1}$.

4.1. Segre foliation in the space of 1-jets

We start by recalling that the affine subset

$$E \simeq \mathbb{C}^{n+1} \times \mathbb{C}^n$$

of the bundle $J^{1,n} \to \mathbb{C}^{n+1}$ of 1-jets of complex hypersurfaces is endowed with the canonical (up to a scalar function multiple) 1-form

$$\omega_0 := dw - \sum_1^n \xi_j dz_j.$$

Here $(z_1, ..., z_n, w) = (z, w)$ denote the coordinates in $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$, and $\xi = (\xi_1, ..., \xi_n)$ are the respective "jet"-variables corresponding to the derivatives $w_{z_1}, ..., w_{z_n}$ respectively. The restriction to E of the canonical projection $\pi : \; J^{1,n} \mapsto \mathbb{C}^{n+1}$ then becomes

$$\pi : \; (z, w, \xi) \mapsto (z, w) \tag{4.1}$$

and E consists precisely of the 1-jets of hypersurfaces that project submersively onto $\mathbb{C}^n \times \{0\} \subset \mathbb{C}^{n+1}$.

The main use of the canonical form ω_0 here is the following. The (complex) tangent bundle TS of a complex hypersurface $S \subset \mathbb{C}^{n+1}$ given as a graph of a function $w = w(z)$ allows to naturally lift S to a complex n-dimensional submanifold of E. Then an n-dimensional submanifold $\tilde{S} \subset E$ of the kind

$$w = w(z), \; \xi = \xi(z)$$

is a lifting in the above sense of a complex hypersurface $S \subset \mathbb{C}^{n+1}$ if and only if $\omega_0|_{\tilde{S}} = 0$.

Further, we observe that, in the case of a *real-analytic* hypersurface $M \subset \mathbb{C}^{n+1}$, the associated system (3.2) amounts to an integrable holomorphic n-distribution in E given by the condition:

$$\omega = (\omega_0, \omega_1, ..., \omega_n) = 0, \tag{4.2}$$

where

$$\omega_k := d\xi_k - \sum_1^n \Phi_{kl} dz_l, \; k = 1, ..., n \tag{4.3}$$

(in the sense that the leaves of the foliation $\mathcal{F}$ determined by (4.2) are precisely the lifts to E of graphs of solutions for (3.2)).

In accordance with the latter observation, let us denote the smooth symmetric matrix function (2.4) (or respectively (2.3)) associated with a smooth hypersurface M satisfying Condition E by Φ, and consider the associated complex valued differential 1-forms $\omega_1, ..., \omega_n$, defined by (4.3). In view of Condition E, the function Φ and hence all the 1-forms $\omega_0, \omega_1, ..., \omega_n$ extend holomorphically to the pseudoconvex side $\widehat{U}^+$ of $\widehat{M}$, defining there a holomorphic n-distribution D. Alternatively, D is spanned by the n holomorphic vector fields

$$L_j := \frac{\partial}{\partial z_j} + \xi_j \frac{\partial}{\partial w} + \sum_1^n \Phi_{sj} \frac{\partial}{\partial \xi_s}. \tag{4.4}$$

We then have

Proposition 4.1. *The distribution D in $\widehat{U}^+$ is integrable.*

Proof. Integrability of the distribution D amounts to the conditions

$$L_j \Phi_{kl} - L_k \Phi_{jl} = 0, \quad j, k, l = 1, ..., n, \tag{4.5}$$

where L_j are as in (4.4). (In terms of the system (3.2), conditions (4.5) mean simply the symmetry of third order derivatives of w in all indices). In view of the Condition E, the left hand side in (4.5) extends smoothly to the real hypersurface $\widehat{M} \subset E$. We claim that the latter extension vanishes on the totally real manifold M_J. Indeed, the fact of vanishing of the left hand side in (4.5) when restricted on M_J is nothing but the symmetry of the third order jet of formal Segre varieties of M in all indices, which proves the claim. Since M_J is totally real of dimension $2n + 1$, this implies that the left hand side in (4.5) vanishes identically in $\widehat{U}^+ \cup \widehat{M}$, as required. □

Proposition 4.1 implies the existence of an n-dimensional holomorphic foliation $\mathcal{F}$ in $\widehat{U}^+$ generated by D. We specify that by *leaves* of the foliation $\mathcal{F}$ we mean maximal connected components of integral submanifolds of $\mathcal{F}$.

Definition 4.2. In what follows we call $\mathcal{F}$ *the Segre type foliation in* $\widehat{U}^+$.

4.2. Changing the complex structure on the pseudoconcave side of M

In this section, we show that the pseudoconcave side U^- of M can be interpreted as the space of leaves for the Segre foliation $\mathcal{F}$ constructed above, and this endows U^- with a *different* (integrable) complex structure, which is smooth up to M and which induces on M a CR-structure coincident with the initially given CR-structure on M induced from $\mathbb{C}^{n+1}$. We construct the desired complex structures in multiple steps discussed in detail below.

Step I. We recall that the distribution D above can be, according to Condition E, smoothly extended to $\widehat{U}^+ \cup \widehat{M}$ as a function valued in the complex Grassmannian $\mathrm{Gr}(n, E)$. (Note though that this extension is *not* everywhere tangent to $\widehat{M}$!). Furthermore, we note that M_J is precisely the locus of points in $\widehat{M}$, for which the value of the extension of D *is tangent* (and hence complex tangent) to $\widehat{M}$. This follows directly from the construction of $M_J, \widehat{M}$.

Step II. Our next goal is to show that the space of leaves of the Segre foliation $\mathcal{F}$ is a smooth manifold in its natural (quotient) topology, which can be furthermore extended to a smooth manifold with boundary M_J.

We will make use of the following

Proposition 4.3. *Let $U \subset \mathbb{R}^m$ be a neighborhood of the origin and $M \subset U$ a smooth strictly convex hypersurface through the origin and, furtheremore, one has $T_0M = \{x_m = 0\}$ and the second fundamental form of M at 0 equals to*

$$dx_1^2 + ... + dx_k^2 \tag{4.6}$$

for some $1 \le k < m$. Let U^+ be the convex side of M, U^- the concave side of M, and D a smooth k-dimensional integrable distribution in U^+. Assume that

(i) D extends to M smoothly (as a function valued in $Gr(k, \mathbb{R}^m)$);

(ii) The k-plane D_0 at 0 is spanned by $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_k}$ (in particular, $D_0 \subset T_0M$) and, moreover, the $\frac{\partial}{\partial x_{k+1}}, ..., \frac{\partial}{\partial x_m}$ components of some collection of vector fields spanning D have zero linear parts at the origin.

Then, after possibly changing the neighborhood U, the distribution D extends to a smooth integrable distribution in U. Furthermore, if D is given by smooth up to M, (pointwise) linearly independent and commuting vector fields $X_1, .., X_k$ in U^+, then these vector fields can be extended smoothly to U in such a way that the extensions still commute. The foliation $\mathcal{F}$ in U^+ generated by D extends therefore to a smooth k-dimensional foliation $\mathcal{F}'$ in U, in the sense that each leaf of $\mathcal{F}$ is an open subset of a unique leaf of $\mathcal{F}'$ and each intersection of a leaf of $\mathcal{F}'$ with U^+ is connected.

Proof. We prove the proposition by induction.

For $k=1$, we choose a vector field X generating $\mathcal{F}$. We split $x=(x_1,\tilde{x})$. In view of (ii), we may assume

$$X=p\frac{\partial}{\partial x_1}+q\frac{\partial}{\partial \tilde{x}},\quad p(0)=1,\ q(0)=0,\ \frac{\partial q}{\partial x_1}=0. \tag{4.7}$$

Consider a smooth extension X' of X to U (possible by (i)). Then (4.7) immediately implies that the orbit of X' at 0 has the form

$$x_1=t,\quad \tilde{x}=O(t^3),\quad t\in(-\epsilon,\epsilon).$$

Now if, for example, $m=2$ (so that $\tilde{x}=x_2$), we consider the defining equation $x_2=\psi(x_1)$ of M as well as the definining equation $x_2=\phi_0(x_1)$ of the orbit. Then (4.6) implies that $(\psi-\phi)''>0$ on $(-\epsilon,\epsilon)$ for small elough ϵ, that is why the part of the orbit lying inside $U^+=\{x_2>\psi\}$ is the set of negative values of a convex function, i.e. the intersection of the orbit with U^+ is *connected*. By continuity, after shrinking possibly U, the same argument applies for all points nearby 0, and this proves the proposition for $k=1$ and $m=2$. The case $k=1$ and $m>2$ can be reduced to the previous one by considering the intersection with U^+ of the 2-dimensional surface S containing the orbit and the x_m coordinate axis. Such a surface can be endowed with the local coordinates (x_1,x_m). Arguing then by contradiction and repeating the above argument (restricted to S) we obtain the connectness of the intersection of the orbit with U^+. By continuity, after shrinking possibly U, the same argument applies for all points nearby 0.

We now proceed with the induction step. We choose k linearly independent smooth vector fields $X_1,...,X_k$ spanning the distribution D in U^+ and extending smoothly to M. Following a proof of the Frobenius theorem (e.g. [Mo01]), it is not difficult to show that we can choose $X_1,...,X_k$ to be furthermore *commuting*. Indeed, we have:

$$X_i=\sum_{j=1}^{m}\alpha_{ij}\frac{\partial}{\partial x_j}$$

for smooth up to M functions α_{ij} in U^+ ($(x_1,...,x_m)$ are the coordinates in $\mathbb{R}^m$). Since X_i are linearly independent, we can assume without loss of generality that the matrix

$$(\alpha_{ij})_{i,j=1}^{k}$$

is invertible in $U^+ \cup M$, and consider the inverse smooth matrix (β_{ij}). Then it is straightforward to check that the vector fields

$$\sum_{j=1}^{k} \beta_{ij} X_j, \quad 1 \leq i \leq k,$$

obviously spanning the same distribution in U^+ and smooth up to M, in fact in addition *commute.*

The latter allows us to consider the *integrable* distribution generated by the commuting vector fields $X_1, ..., X_{k-1}$. Applying then the induction assumption, we obtain smooth *commuting* extensions $X'_1, .., X'_{k-1}$ of $X_1, .., X_{k-1}$ and hence a $(k-1)$-dimensional integrable distribution in U. The foliation $\mathcal{X}$ given by this distribution has the property that its leaves can have only connected intersections with U^+. After that, let us perform (after possibly shrinking U) a smooth in U diffeomorphism, tangent to the identity at 0 and transforming the vector fields $X'_1, ..., X'_{k-1}$ to $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{k-1}}$ respectively (the latter is possible since the vector fields commute). Now all the leaves of $\mathcal{X}$ become parallel to the $(x_1, ..., x_{k-1})$-plane. We keep the same notation for $U, U^\pm$ in the new coordinates. Then the vector field X_k (defined so far in the closure of U^+) have the form

$$X_k = a_1(x_k, ..., x_m) \frac{\partial}{\partial x_1} + \cdots + a_m(x_k, ..., x_m) \frac{\partial}{\partial x_m} \tag{4.8}$$

where we use the commutativity $[X_j, X_k] = 0$, $j = 1, ..., k-1$).

Consider the orthogonal projection Ω of the closure of U^+ onto the $(x_k, ..., x_m)$-plane. Then the leaves of $\mathcal{X}$ intersecting the closure of U^+ project onto a single point in Ω. As follows from (4.8), the vector field X_k is *constant* on each leaf of $\mathcal{X}$, which allows to extend X_k constantly along each leaf intersecting the closure of U^+. In view of the above, this gives a smooth function on Ω, which we first extend smoothly to a neighborhood of the origin in the $(x_k, ..., x_m)$-plane, and then again constantly along each leaf of $\mathcal{X}$. Since the intersection of each leaf of $\mathcal{X}$ with the closure of U^+ is connected, the extension obtained is well defined. We thus are able to extend X_k smoothly to a full neighborhood of the origin still being constant on each leaf of $\mathcal{X}$. In view of the latter property, the extended vector field X'_k also satisfies $[X'_j, X'_k] = 0$, $1 \leq j \leq k-1$.

In summary, we obtain an integrable distribution in U spanned by $X'_1, ..., X'_k$. For the respective foliation $\mathcal{F}'$, each leaf of $\mathcal{F}$ is clearly contained in that of $\mathcal{F}'$. It remains to show that intersections of leaves of $\mathcal{F}'$

with U^+ are connected. This however can be seen from (ii) by an argument identical to the one in the 1-dimensional case. The proof is complete. □

We now apply Proposition 4.3 to the situation of the strictly pseudo-convex hypersurface $\widehat{M} \subset E$, its neighborhood $\widehat{U}$ and the distribution D (considered as a real distribution). For that, we perform a biholomorphic (in fact polynomial) coordinate change mapping the origin and the tangent plane $\operatorname{Im} w = 0$ onto themselves, and removing holomorphic quadratic terms from the formal Taylor expansion of M in the origin such that property (4.6) holds for $\widehat{M}$. In fact, in such coordinates M becomes approximated by a quadric to order 2 at the origin:

$$\operatorname{Im} w = Q(z, \bar{z}) + O(3), \tag{4.9}$$

where Q is a positive definite Hermitian form (we assume it to be simply sum of squares) and $O(3)$ stand for terms of degree 3 and higher. In case M is already this quadric itself, D is spanned by the (real parts of) the vector fields

$$L_j := \frac{\partial}{\partial z_j} + \xi_j \frac{\partial}{\partial w},$$

so that (ii) is satisfied. However, terms of order 3 and higher in (4.9) do not effect (ii), and this finally shows that $\widehat{M}$ satisfies all the conditions of Proposition 4.3.

We end up with a smooth extension of the distribution D to a full neighborhood of the origin in E in such a way that it is still integrable and defines a foliation $\mathcal{F}'$ extending $\mathcal{F}$, in the sense that each leaf of $\mathcal{F}$ is an open subset of some leaf of $\mathcal{F}'$ and, furthermore, leaves of $\mathcal{F}'$ have only connected intersections with $\widehat{U}^+$ (so that each leaf of $\mathcal{F}$ is contained in exactly one leaf of $\mathcal{F}'$).

Step III. Based of the outcome of Step II, we are finally able to endow the space of leaves of $\mathcal{F}$ with the structure of a smooth $(2n+2)$-manifold with boundary M_J in the natural (quotient) topology. Indeed, first note that the tangent plane at 0 to the leaf of $\mathcal{F}'$ through 0 is $w = 0$, $\xi_j = 0$ (as follows from the definition and the initial normalization of M). Hence, the $(2n+2)$-plane

$$\{z = 0\} \tag{4.10}$$

has the property that all the leaves of (the ambient foliation) $\mathcal{F}'$ intersect it transversally at single points (after possibly shrinking $\widehat{U}$). Thus the space of leaves of $\mathcal{F}'$ can be identified with a domain in (4.10) (viewed as $\mathbb{R}^{2n+2}$). Accordingly, the space of leaves of $\mathcal{F}$ is an open connected subset G of the latter domain (since it is given by the condition of having nonempty intersection with the domain $\widehat{U}^+$). This gives a structure of a smooth manifold on the space of leaves of $\mathcal{F}$ (in its natural quotient topology)!

It remains to show that the space of leaves of $\mathcal{F}$ has, furthermore, a structure of a smooth $(2n+2)$-manifold with a boundary. Indeed, as follows from the discussion in Steps I and II, any leaf of $\mathcal{F}'$ intersecting the closure of $\widehat{U}^+$ either intersects $\widehat{M}$ transversally and consequently intersects the open part $\widehat{U}^+$, or intersects the boundary $\widehat{M}$ only at a point in M_J, or is contained in $\widehat{U}^+$ (the last possibility in fact does not occur, but we do not need this fact). In this way, *M_J can be identified with the subset of leaves of $\mathcal{F}'$ intersecting the boundary $\widehat{M}$ but not the open part $\widehat{U}^+$*. Thus, we are looking for the set of leaves of $\mathcal{F}'$ which are tangent at some point to $\widehat{M}$. The latter set is a smooth $(2n+1)$-submanifold in $\mathbb{R}^{2n+2}$ (which is in fact the boundary of the above open set G of leaves of $\mathcal{F}'$ intersecting $\widehat{U}^+$). Indeed, perform a local diffeomorphism (near the origin in $E \sim \mathbb{R}^{4n+2}$) with the identity linear part in such a way that in the new local coordinates the leaves become "horizontal" $2n$-planes (that is, they are given by

$$x_j = c_j,\ j = 2n+1, ..., 4n+2,$$

where c_j are constant), and the hypersurface $\widehat{M}$ becomes

$$x_{4n+2} = \varphi(x_1, \dots, x_{4n+1})$$

for a smooth function φ. Note that, in particular, φ has nondegenerate at the origin Hessian in $x_1, ..., x_{2n}$ (as follows, for example, from (4.9)). Now the condition that a leaf is tangent to $\widehat{M}$ at some point is:

$$\begin{aligned} c_{4n+2} &= \varphi(x_1, ..., x_{2n}, c_{2n+1}, ..., c_{4n+1}), \\ \varphi_{x_j}(x_1, ..., x_{2n}, c_{2n+1}, ..., c_{4n+1}) &= 0,\ j = 1, .., 2n. \end{aligned} \tag{4.11}$$

Solving the last $2n$ equations of (4.11) in $x_1, ..., x_{2n}$ by the implicit function theorem (note that, in view of (4.9), $\varphi_{x_i x_j}(0) = 2\delta_{ij}$, $i, j = 1, ..., 2n$, so that the implicit function theorem is applicable), and substituting the result into the first equation of (4.11), we obtain a smooth hypersurface

$$c_{4n+2} = \psi(c_{2n+1}, \dots, c_{4n+1})$$

in $\mathbb{R}^{2n+2}$ (endowed with the coordinates $c_{2n+1},...,c_{4n+1}$) for an appropriate smooth function ψ with $\psi(0) = d\psi(0) = 0$, as desired.

We thus have proved the following

Proposition 4.4. *The leaf space of the foliation $\mathcal{F}$ is a smooth $(2n+2)$-manifold in its natural (quotient) topology. Furthermore, it can be regarded as a smooth $(2n+2)$-manifold with boundary M_J. The (germ at the origin of) the upper half space*

$$\bar{H} = \{(x_{2n+1},...,x_{4n+2}) : \ x_{4n+2} \geq 0\}$$

serves as a coordinate chart for it.

Step IV. We denote the leave space from Proposition 4.4 by $\mathcal{U}$, and the respective manifold with boundary by $\overline{\mathcal{U}}$. As was mentioned above, $\mathcal{U}$ represents, in a certain sense, the pseudoconcave side U^- endowed however with a *different* (still integrable) complex structure.

We next note that the leave space $\mathcal{U}$ is, on the other hand, the quotient topological space $\widehat{U}^+/\mathcal{F}$, and this means (if read together with Proposition 4.4) that $\mathcal{U}$ has also a structure of a *complex* manifold with boundary. We emphasize at this point that, exclusively for the purpose of obtaining the right complex structure on $\widehat{U}^+/\mathcal{F}$,

we shall change the complex structure on $\mathcal{U}$ to its conjugate structure.

This is related to the antiholomorphic dependence of Segre varieties on their parameters, for a real-analytic hypersurface. Thus we obtain a smooth integrable complex structure on $\mathcal{U}$, which extends further to a smooth CR-structure on $\partial\mathcal{U}$. The boundary $\partial\mathcal{U}$ is naturally diffeomorphic to M_J and hence to M. We will show later that the two CR-structures on M (the one coming from the quotient space and the one induced from the embedding into $\mathbb{C}^{n+1}$) in fact agree with each other together with all higher order derivatives.

Step V. We now have to take into consideration the "one sided Segre varieties of M", that is, images of leaves of the foliation $\mathcal{F}$ under the projection map π, as in (4.1). This gives us the family

$$\mathcal{S}^+ := \Big\{\pi(T)\Big\}_{T\in\mathcal{F}}.$$

Note that, as the extension construction in Step II above shows, all the leaves in $\mathcal{F}$ (after possibly shrinking the basic neighborhood $\widehat{U}$) are open subsets of graphs of smooth functions of the kind $w = w(z)$, $\xi = \xi(z)$ with connected intersections with $\widehat{U}^+$. In this way, all elements of $\mathcal{S}^+$ are n-dimensional complex submanifolds in U^+, and we conclude that $\mathcal{S}^+$ *is an* $(n+1)$*-dimensional anti-holomorphic family of pairwise transverse complex* n*-dimensional submanifolds in* U^+ *of the form* $w = w(z)$. (Transversality here means that no two elements of $\mathcal{S}^+$ are tangent at a point $p \in U^+$). The anti-holomorphic parametrization of $\mathcal{S}^+$ here comes from the integrable complex structure on the space of leaves $\mathcal{U}$.

We note that $\mathcal{U}$ itself is endowed with a natural $(n+1)$-dimensional anti-holomorphic family of n-dimensional complex submanifolds as follows. We fix a point $p \in U^+$ and consider the set Q_p of all the manifolds from $\mathcal{S}^+$ passing through p as a subset of $\mathcal{U}$. Following the geometric interpretation in the real-analytic case, we call Q_p *the Segre variety of* p. The structure of Q_p becomes particularly clear when considering the foliation $\mathcal{F}$: then the set of all elements of $\mathcal{S}^+$ passing through p lifts to the set of all fibers in $\mathcal{F}$ intersecting the fiber $\pi^{-1}(p)$ of the bundle E. In this way, we easily see, from the construction of the manifold $\mathcal{U}$, that each Q_p can be identified via the 1-jet map with the fiber $\pi^{-1}(p)$ and thus is a complex n-dimensional submanifold in $\mathcal{U}$ (with respect to the above described complex structure on $\mathcal{U}$), as required. We denote the resulting family of submanifolds in $\mathcal{U}$ by $\mathcal{S}^-$ (it becomes an anti-holomorphic family parameterized by U^+).

For completeness of the picture, we also call, for each $p \in \mathcal{U}$, the respective leaf $T \in \mathcal{S}^+$ its *Segre variety* and denote the latter one by Q_p. We then obtain the following familiar symmetry property:

$$p \in Q_q \Leftrightarrow q \in Q_p, \quad p \in U^+,\ q \in \mathcal{U}.$$

Step VI. We recall that the boundary manifold $\partial\mathcal{U}$ is naturally diffeomorphic to the initial CR-manfold M, as follows from the construction of $\overline{\mathcal{U}}$. We further extend (locally near 0) the latter diffeomorphism smoothly to a diffeomorphism between the manidold with boundary $\overline{\mathcal{U}}$ and the pseudoconcave side U^- of M (which is possible since both are manifolds of equal dimension with boundary). We end up with a smooth manifold U decomposed as a union of two manifolds with boundary:

$$U = (U^- \cup M) \cup (U^+ \cup M),$$

where both U^- and U^+ are endowed with their individual complex structures (for U^- this is the integrable structure induced from $\mathcal{U}$ and for U^+ this is the standard complex structure induced from $\mathbb{C}^{n+1}$). Moreover, both structures admit a smooth extension to the boundary and induce boundary CR-structures on M. Our goal is now to show that the two structures (considered for the moment as $(2n+2)\times(2n+2)$ matrices) agree on M (together with all derivatives). In particular, they define a smooth structure in a full neighborhood of the origin, and the two induced CR-structures on M coincide.

For doing so, let us fix $p \in M$ and the respective point $\tilde{p} \in \partial\mathcal{U}$. We observe the following: all data required for computing the boundary value at $\tilde{p}$ of the complex structure on $\mathcal{U}$ comes from the 2-jet of M at p (as follows from our construction). Similarly, for computing the k-jet at p of the limit of the complex structure we just need to know the $(k+2)$-jet of M at p. Since M can be approximated to any order by a real-analytic hypersurface, we conclude that it suffies to show that for a *real-analytic* hypersurface M the two above structures coincidence and define a real-analytic (in particular smooth) structure in a neighborhood of p.

If now M is real-analytic, then, as a well know fact (e.g. [BER99]), after choosing an appropriate neighborhood U of p, we have the property that a Segre variety of a point $q \in U$ intersects the pseudoconvex side U^+ iff q lies in the pseudoconcave side U^- of M. Thus, if $M = \{\rho(Z,\bar{Z}) = 0\}$ near p, then the above manifold $\mathcal{U}$ consists of Segre varieties $\{\rho(Z,\bar{q}) = 0\}$ with $q \in U^-$, and thus can be identified with U^- with the standard complex structure on it, while its boundary can be identified with $M = \{\rho(q,\bar{q}) = 0\}$ with the standard CR-structure on it. This immediately yields the desired property.

4.3. End of proof of the main result

In this section, we complete the proof of the main result.

Recall that, as an outcome of Step VI, we obtain a smooth manifold U endowed with an *integrable* smooth complex structure J (the integrability follows from that on both U^- and U^+ and thus, by continuity, at points in M as well). By the Newlander-Nirenberg theorem [NN57], there is a smooth diffeomorphism χ of (U,J) (preserving the origin), mapping U onto a neighborhood of the origin and transforming the above complex structure J on U into the standard complex structure J_{st} in $\mathbb{C}^{n+1}$ (that is, χ is a (J,J_{st})-biholomorphism). The resulting smooth strictly pseudoconvex image of M we still denote by M, and the pseudoconvex and the pseudoconcave sides of it respectively we still denote by $U^{\pm}$.

We shall consider now the above families $\mathcal{S}^{\pm}$ on $U^{\pm}$ respectively, after applying the diffeomorphism χ. We recall that elements of both families

$\mathcal{S}^{\pm}$ are J-invariant and have the transversality property, hence they become *families of holomorphic curves* on $U^{\pm}$ respectively. This allows us to consider, in the same fashion as in Section 2, their lifting to the space of 1-jets $J^{1,n}$, and this results in two foliations defined in some domains $\widehat{U}^{\pm} \subset E$ with $\pi(\widehat{U}^{\pm}) = U^{\pm}$ respectively (here E is again the affine subset of the bundle of 1-jets of complex hypersurfaces). In particular, we may consider the respective holomorphic direction fields defined in the same domains. As follows from the above, these two directions fields extend smoothly to M_J.

Importantly, we can *not* conclude at this step that the domain $\widehat{U}^{-}$ has the form $U^{-} \times \Omega$, where Ω is an open neighborhood of the origin in $\mathbb{C}^n$. This is because 1-jets of elements of $\mathcal{S}^{-}$ passing through a point $p^{-} \in U^{-}$ do not cover a full neighborhood of the origin of a uniform size. To see the latter, we note that the "pencil" of Segre varieties passing through $p^{-} \in U^{-}$ is the union of "pencils" of Segre varieties from $\mathcal{S}^{+}$ at points belonging to the Segre variety $Q_{p^-} \in \mathcal{S}^{+}$. Unlike the situation in the real-analytic case, Q_{p-} is defined so far *only* as a variety in U^{+}, which is not a full neighborhood of 0. (In the real-analytic case, such "one-sided" Segre varieties extend analytically across M and become varieties in a uniform neighborhood of 0). That is why possible jets of Segre varieties through p^{-} in our construction form a "smaller" set in the space of 1-jets (compared to the real-analytic case) and thus do not give a uniform neighborhood of the origin. (However, the domain $\widehat{U}^{+}$ does have the desired form $U^{+} \times \Omega$ for an open neighborhood Ω of the origin).

To overcome the latter difficulty, we use Webster's considerations from [We78] to conclude that, after possibly shrinking the neighborhood U:

(i) There exists a wedge $W^{+} \subset \widehat{U}^{+}$ with the totally real edge M_J;

(ii) The *reflection map* τ defined as

$$\tau(z, T_z Q_\zeta) = (\zeta, T_\zeta Q_z), \quad z \in \widehat{U}^{\pm},\ \zeta \in Q_z \tag{4.12}$$

is anti-holomorphic, well defined in $\widehat{U}^{\pm}$, and smooth up to M_J;

(iii) The image $\tau(W^{+})$ contains a wedge $W^{-} \subset \widehat{U}^{-}$, and τ satisfies

$$\tau \circ \tau = \mathrm{Id}$$

(thus τ is an anti-holomorphic involution);

(iv) $\tau|_{M_J} = \mathrm{Id}$.

To prove (i)-(iii) we note that, even though [We78] deals with the real-analytic case, the proof of (i)-(iii) in [We78] is based solely on the approximation of M by a quadric to order 3 and the implicit function theorem subsequently, that is why it can be essentially word-by-word repeated in our situation. We leave the details here to the reader. Statement (iv) follows directly from the construction in Section 4.2 (alternatively, in can be viewed from (i)-(iii) read together).

We now use the edge-of-the-wedge theorem (e.g. [Ro86]) to conclude that τ extends anti-holomorphically to a full neighborhood of the origin in E (still being an involution, by uniqueness). For simplicity, we denote the latter neighborhood by $\widehat{U}$.

Let us consider finally (locally near the origin) the fixed point set $M' \subset U$ of τ. In view of the fact that τ is an anti-holomorphic involution, M' is a *real-analytic* totally real submanifold in $\widehat{U}$ of dimension $2n+1$. In view of (iv), we conclude that $M' = M_J$, i.e. both M_J and M are real-analytic. The proof of Theorem 3 is complete. □

Proof of Theorem 2. The proof of Theorem 2 follows directly from the above effective procedure for finding real-analytic coordinates for a smooth strictly pseudoconvex hypersurface satisfying Condition E, and the parametric version of the Newlander-Nirenberg Theorem (see [NN57] and also [Go16] and references therein). □

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