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Deformation of Hermitian metrics

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In this work, we study the deformation of Hermitian metrics and the respective Chern curvature tensors. By adapting the conformal perturbation method of Aubin and Ehrlich to Hermitian setting, we prove that Hermitian metrics with quasi-positive (resp. quasi-negative) second Chern-Ricci curvature is conformal to a Hermitian metric with positive (resp. negative) second Chern-Ricci curvature.

1. Introduction

In differential geometry, it is important to determine those differentiable manifolds which admit metrics of strictly positive or negative curvature. The existence of metrics with curvature of definite sign will often impose strong restriction on the underlying manifolds. In complex geometry, the holomorphic structures are often characterized by various positivity notions in complex differential geometry and algebraic geometry. For instance, thanks to the Yau's solution [44] to the Calabi conjecture, it is now known that the existence of Kähler metric with positive Ricci curvature on M is equivalent to M being Fano.

On the other hand, the holomorphic sectional curvature also carry much information of the holomorphic structure. Indeed, thanks to the recent breakthrough by Wu-Yau [34], it is now known that Kähler manifolds with negative or quasi-negative holomorphic sectional curvature are projective and have ample canonical line bundle [6, 32, 35]. This settles down a long-standing conjecture of Yau affirmatively. For more related recent works, we refer interested readers to [10–12, 15, 42]. On the positive side, it is also conjectured by Yau [45] that compact Kähler manifolds with positive holomorphic sectional curvature must be projective and rationally connected. This was solved recently by Yang [37] where he introduced the concept of RC-positivity for abstract vector bundles. Many interesting properties and applications using the idea of RC-positivity had also been studied and explored, see [17–19, 37–40]. In the same spirit, Ni-Zheng also introduced [21, 22] various notions of Ricci curvature and scalar curvature to obtain rational connectedness of compact Kähler manifolds.

In Hermitian geometry, it is natural to consider the Chern connection of Hermitian metrics which is the connection compatible with both the metric and the complex structure. When the Hermitian metric g is non-Kähler, there are four notions of Chern-Ricci curvature associated to the Chern connection due to the presence of torsion. The first and second Chern-Ricci curvature are particularly important to the geometric structure. More precisely, the first Ricci curvature $\mathrm{Ric}(g)$ represents the first Chern class of the canonical line bundle while the second Ricci curvature $S(g)$ is elliptic with respect to the Hermitian metric g and deeply related to its differential geometric structure.

We are interested to understand the second Chern-Ricci curvature S of a Hermitian metric. As an analog of the Calabi-Yau theorem, Yang [37] proved that if a compact Kähler manifold M admits a smooth Hermitian metric with positive second Chern-Ricci curvature S, then M is projective and rationally connected. This generalized the earlier works of Campana [4] and Kollár-Miyaoka-Mori [13] which states that Fano manifolds are rationally connected. Recently in [41], the result was generalized by weakening positivity to quasi-positivity. In [39], Yang conjectured that a projective and rationally connected manifold M must admit a Hermitian metric with positive second Chern-Ricci curvature.

In this paper, motivated by the above mentioned works and conjecture, we study the deformation of Hermitian metrics. In Riemannian geometry, the deformation of metrics with quasi-positive Ricci curvature was studied by Aubin [1] and Ehrlich [7]. By adapting the conformal perturbation method in their works to Hermitian setting, we have the the deformation result for Hermitian metrics with quasi-positive second Chern-Ricci curvature. More generally, we have the following.

Theorem 1.1. *Suppose M is a compact complex manifold and $g_0, \tilde{g}$ are two smooth Hermitian metrics (possibly different) on M such that $\operatorname{tr}_{\tilde{g}} R^{(TM,g_0)}$ is quasi-positive (resp. quasi-negative). Then there is another Hermitian metric g_1 conformal to g_0 such that $\operatorname{tr}_{\tilde{g}} R^{(TM,g_1)}$ is positive (resp. negative) on M.*

For the definition of $\operatorname{tr}_g R^{(TM,h)}$ and quasi-positivity, we refer to section 2. As a special case, we have the deformation result for the second Chern-Ricci curvature S. This also provides a supporting evidence to Yang's conjecture on the existence of metrics with positive second Chern-Ricci curvature.

Corollary 1.1. *Let (M, g) be a compact Hermitian manifold with quasi-positive(resp. quasi-negative) $S(g)$. Then there is another Hermitian metric $\tilde{g}$ such that $S(\tilde{g}) > 0$ (resp. < 0).*

Together with [37, Corollary 3.7], this implies the following RC-positivity of vector bundles under quasi-positive condition.

Corollary 1.2. *Let M^n be a compact complex manifold with complex dimension n. Suppose there are two Hermitian metric g and $\tilde{g}$ (possibly different) such that $\operatorname{tr}_{\tilde{g}} R^{(TM,g)}$ is quasi-positive, then there is Hermitian metric h so that*

1) *$(TM^{\otimes k}, h^{\otimes k})$ is RC-positive for every $k \geq 1$;*
2) *$(\Lambda^p TM, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq n$.*

As a corollary of RC-positivity, we have the following implication on complex structure when M is a Kähler using [37, Theorem 1.4], giving an alternative proof to [41, Theorem 1.1].

Corollary 1.3. *Suppose M is a compact Kähler manifold and $g, \tilde{g}$ are two smooth Hermitian metrics on M such that $\operatorname{tr}_{\tilde{g}} R^{(TM,g)}$ is quasi-positive, then M is projective and rationally connected. In particular, M is simply connected.*

The analogous result for the first Chern-Ricci curvature Ric is not true in general. A counter-example was pointed out by Ehrlich in [8, Theorem 4] where the Hirzebruch surface Σ_2 is non-Fano but supports a Kähler metric with quasi-positive Ric by the construction of Yau [43]. This demonstrated a fundamental difference between the first and second Chern-Ricci curvature in the non-Kähler setting. With a stronger curvature assumption, the deformation of the first Ricci curvature is possible. In [14], the first author showed that a Hermitian metric g with quasi-negative Ric can be deformed to one with negative Ric if in addition g has Griffths non-positive Chern curvature by using a choice of Hermitian geometric flow introduced by Streets-Tian [28]. Similar result concerning Griffths non-negativity of Chern curvature was proved by Ustinovskiy [33] which generalized the earlier works by Mok [20] and Bando [2] in the Kähler case. Indeed, deformation using parabolic flows was found to be powerful to study the underlying structure of a manifold. For related works in Hermitian geometry, we refer interested readers to [3, 5, 9, 16, 23–27, 29, 31] and the references therein. The curvature often tends to become positive along the geometric flow. In contrast, the elliptic method employed here seems to be more flexible in the negative case.

The paper is organized as follows: In section 2, we will collect the formulas about the Chern connection. In section 3, we will derive the variational formula for the first, second Chern-Ricci curvature. In section 4, we will consider the deformation inside the injectivity radius. In section 5, we will prove the global deformation theorems.

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2. Chern connection

In this section, we collect some formulas for the Chern connection to avoid notational inconsistency. Let (M, g) be a Hermitian manifold, the *Chern connection* ∇ of g is the connection such that $\nabla g = \nabla J = 0$ and the torsion has no $(1,1)$ component. In local holomorphic coordinates $\{z^i\}$, the coefficients Γ of ∇ is given by

$$\Gamma_{ij}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}}.$$

The *torsion* of g is given by $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$. We say that g is Kähler if the torsion vanishes, $T \equiv 0$.

The *Chern curvature tensor* of g is defined by

$$R_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}} \Gamma_{ik}^l.$$

We raise and lower indices by using metric g. The first Chern-Ricci curvature is defined by

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \det g$$

while the second Ricci curvature is defined by

$$S_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}}.$$

Note that if g is not Kähler, then $R_{i\bar{j}}$ is not necessarily equal to $S_{i\bar{j}}$. And the Chern-scalar curvature R is defined to be $R = g^{i\bar{j}} R_{i\bar{j}} = g^{k\bar{l}} S_{k\bar{l}} = g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$.

In the non-Kähler setting, there are two more notions of Chern-Ricci curvature associated to the Chern connection but they are not elliptic with respect to g in general. In this note, we will only focus on the first and second Chern-Ricci curvature. In general, we can also trace the tangent bundle component using a different metric. For notational convenience, we denote

$$\mathrm{tr}_{\tilde{g}}\, R^{(TM,g)} = \tilde{g}^{k\bar{l}} R_{k\bar{l}i\bar{j}}.$$

In this note, all Hermitian metrics are smooth and the connection∇ will be referring to the Chern connections associated to the Hermitian metrics.

Definition 2.1. *Let M be a complex manifold and $A_{i\bar{j}}$ is a tensor on M. We say that $A_{i\bar{j}}$ is quasi-positive (resp. quasi-negative) if $(A_{i\bar{j}})$ is non-negative (resp. non-positive) everywhere and positive (resp. negative) at some point on M.*

3. Variation formulas for the Chern curvature

In this section, we will collect some variational formulas for the Chern connection along variation of Hermitian metrics.

Lemma 3.1. *Suppose $g(t)$ is a family of Hermitian metrics such that $\partial_t g_{i\bar{j}} = -h_{i\bar{j}}$, then we have*

$$\partial_t R_{k\bar{l}i\bar{j}} = \frac{1}{2}\left(\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k\right) h_{i\bar{j}} - \frac{1}{2}\left(R_{k\bar{l}i}{}^{r} h_{r\bar{j}} + R_{k\bar{l}}{}^{\bar{s}}{}_{\bar{j}} h_{i\bar{s}}\right)$$

where ∇ denotes the connection with respect to $g(t)$.

Proof. The computation is standard, we include here for reader's convenience.

$$\begin{aligned}
\partial_t R_{k\bar{l}i\bar{j}} &= \partial_t \left(R_{k\bar{l}i}{}^{r} g_{r\bar{j}}\right) \\
&= -h_{r\bar{j}} R_{k\bar{l}i}{}^{r} - g_{r\bar{j}} \partial_{\bar{l}} \partial_t \Gamma^r_{ki} \\
&= -h_{r\bar{j}} R_{k\bar{l}i}{}^{r} + \nabla_{\bar{l}} \nabla_k h_{i\bar{j}} \\
&= \frac{1}{2}\left(\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k\right) h_{i\bar{j}} - \frac{1}{2}\left(R_{k\bar{l}i}{}^{r} h_{r\bar{j}} + R_{k\bar{l}}{}^{\bar{s}}{}_{\bar{j}} h_{i\bar{s}}\right).
\end{aligned} \tag{3.1}$$

□

Remark 3.1. By tracing the vector bundle component in Lemma 3.1, we would have

$$\begin{aligned}\partial_t R_{i\bar{j}} = \Delta_g h_{i\bar{j}} + \left(T^k_{pi}\nabla^p h_{k\bar{j}} + T^{\bar{l}}_{\bar{q}\bar{j}}\nabla^{\bar{q}} h_{i\bar{l}} + g^{p\bar{q}} T^k_{pi} T^{\bar{l}}_{\bar{q}\bar{j}} h_{k\bar{l}}\right) \\ + h^{k\bar{l}} R_{i\bar{j}k\bar{l}} - \frac{1}{2}\left(S^k_i h_{k\bar{j}} + S^{\bar{l}}_{\bar{j}} h_{i\bar{l}}\right).\end{aligned} \tag{3.2}$$

Using the trick by Ustinovskiy [33], the true variational component is at $h * \mathrm{Rm}$ which do not have any sign in general. This also explains why the deformation on Ric is false unless we have additional information on Rm, see [14].

4. Local deformation of hermitian metrics

In this section, we discuss the local deformation of Hermitian metrics. To simplify our argument, we will work inside the ball of injectivity radius so that the distance function is smooth there although this is not necessary in the Hermitian content.

Let (M^n, g) be a closed Hermitian manifold with complex dimension n. For $p \in M$, we will use $B_g(p, r)$ to denote the g-geodesic ball of radius r and $A_g(p, r_1, r_0) = B_g(p, r_1) \setminus B_g(p, r_0)$. Note that Hermitian metric g is at the same time a Riemannian metric. For each $p \in M$, we can find $i_0 = \mathbf{inj}_g(p)$ such that the exponential map $\exp|_p : B_{\mathbb{R}^{2n}}(0, i_0) \to B_g(p, i_0)$ is a diffeomorphism. In particular, the square distance function $d_g(p, \cdot)^2$ is smooth on $B_g(p, i_0)$. Let $r < i_0$ and denote $\rho(\cdot) = r^2 - d_g(p, \cdot)^2$. We use $\rho(\cdot)$ instead of $d_g(p, \cdot)$ to avoid the non-smooth issue at $x = p$.

Proposition 4.1. *There is $\mu(n) \in (0, 1/2)$ such that the following holds: Suppose $\tilde{g}$ is a Hermitian metric on M with*

$$\sup_M |\mathrm{Rm}(\tilde{g})| + |T(\tilde{g})|^2 < 2. \tag{4.1}$$

And g_0 is another Hermitian metric on M such that $\mathrm{tr}_{\tilde{g}}\, R^{(TM, g_0)} \geq 0$ (resp. ≤ 0) on $B_{\tilde{g}}(p, r)$ where $r < \min\{\mathbf{inj}_{\tilde{g}}(p), 1\}$. Then for any $\varepsilon > 0$ and $k \in \mathbb{N}$, there is another smooth Hermitian metric g_1 conformal to g_0 such that

(a) *g_1 agrees with g_0 outside $B_{\tilde{g}}(p, r)$;*

(b) *$||g_1 - g_0||_{C^k(M, \tilde{g})} < \varepsilon$;*

(d) *$\mathrm{tr}_{\tilde{g}}\, R^{(TM, g_1)} > 0$ (resp. < 0) on $A_{\tilde{g}}(p, r, (1-\mu) r)$.*

Proof. We first prove the non-negative case. Let f be the smooth function on $\mathbb{R}$ given by

$$f(s) = \begin{cases} e^{-1/s} & \text{for } s \geq 0; \\ 0 & \text{for } s \leq 0. \end{cases} \tag{4.2}$$

Define $F(x) = f(\rho)$ where $\rho(x) = r^2 - d_{\tilde{g}}(x,p)^2$. Let us collect some useful inequalities of ρ. Clearly, we have $|\partial\rho|^2_{\tilde{g}} = 4(r^2 - \rho)$. For the second order, note that Chern connection coincides with the canonical connection in the almost Hermitian geometry. Since the Chern curvature and the torsion of $\tilde{g}$ are bounded uniformly by 2, by [30, Theorem 4.2] there is a dimensional constant C_n such that whenever $\rho \geq 0$, we have

$$\begin{aligned} \Delta_{\tilde{g}}\rho &= -2d_{\tilde{g}} \cdot \Delta_{\tilde{g}} d_{\tilde{g}} - 2|\partial d_{\tilde{g}}|^2_{\tilde{g}} \\ &\geq -2d_{\tilde{g}} \cdot \left(\frac{2n}{d_{\tilde{g}}} + C_0(n) \right) - 2 \\ &\geq -C_n. \end{aligned} \tag{4.3}$$

Define a one parameter family of smooth Hermitian metrics by $g = e^{-tF} g_0$ so that $\partial_t g = -Fg = -h$. Conclusion (a) is trivial. Conclusion (b) can be done by choosing $g_1 = g(t_{\varepsilon,k})$ where $t_{\varepsilon,k}$ is sufficiently small. It remains to establish conclusion (c). First, we note that Lemma 3.1 implies that for $B_{i\bar{j}} = e^{Ft} \tilde{g}^{k\bar{l}} R_{k\bar{l}i\bar{j}}$, we have

$$\begin{aligned} \partial_t B_{i\bar{j}} &= F B_{i\bar{j}} + e^{Ft} \tilde{g}^{k\bar{l}} \partial_t R_{k\bar{l}i\bar{j}} \\ &= F B_{i\bar{j}} + \frac{e^{Ft}}{2} \tilde{g}^{k\bar{l}} \left(\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k \right) h_{i\bar{j}} \\ &\quad - \frac{e^{Ft}}{2} \tilde{g}^{k\bar{l}} \left(R_{k\bar{l}i}{}^{r} h_{r\bar{j}} + R_{k\bar{l}}{}^{\bar{s}}{}_{\bar{j}} h_{i\bar{s}} \right) \\ &= F B_{i\bar{j}} + e^{Ft} \tilde{g}^{k\bar{l}} g_{i\bar{j}} \partial_k \partial_{\bar{l}} F - F e^{Ft} \tilde{g}^{k\bar{l}} R_{k\bar{l}i\bar{j}} \\ &= \Delta_{\tilde{g}} F \cdot (g_0)_{i\bar{j}}. \end{aligned} \tag{4.4}$$

We now claim that $B_{i\bar{j}} > 0$ on $A_{\tilde{g}}(p, r, (1-\mu)r)$ for some $0 < \mu(n) < 1/2$ and $t > 0$. We will specify the choice of $\mu(n)$ later.

For any $z \in A_{\tilde{g}}(p, r, (1-\mu)r)$ and $v \in T^{1,0}_z M$ so that $g_0(v, \bar{v}) = 1$. By (4.4) and (4.3), if $t > 0$, then

$$
\begin{aligned}
B(t)_{v\bar{v}} - B(0)_{v\bar{v}} &= t \cdot \tilde{\Delta} F \\
&= t \cdot \left(f'' |\partial \rho|^2_{\tilde{g}} + f' \Delta_{\tilde{g}} \rho \right) \\
&\geq t \rho^{-4} e^{-1/\rho} \left[4(r^2 - \rho) - 8\rho(r^2 - \rho) - C_n \rho^2 \right] \\
&> 0
\end{aligned}
\tag{4.5}
$$

provided that $\mu(n)$ is sufficiently small. This completes the proof of the non-negative case. The non-positive case can be proved analogously by considering the one parameter family $g(t) = e^{tF} g_0$ instead. □

Remark 4.1. In application, we will work on $B_{\tilde{g}}(p, r)$ where $\mathrm{tr}_{\tilde{g}}\, R^{(TM,g)} > 0$ (resp. < 0) on $B_{\tilde{g}}\,(p, (1-\mu)r)$ and will choose $\varepsilon > 0$ small enough so that $\mathrm{tr}_{\tilde{g}}\, R^{(TM,g(t))} > 0$ (resp. < 0) on $B_{\tilde{g}}(p, r)$ after deformation. Here we will require ε to be sufficiently small depending also on the positivity on the smaller ball.

5. Deformation on compact manifolds

In this section, we will carry out the deformation process on M. We will follow and modify the argument in [7]. We will denote the original Hermitian metric as g_0. Since M is closed, the injectivity radius of $x \in M$ has a uniform positive lower bound. Denote

$$\mathbf{inj}_{\tilde{g}}(M) = \inf_M \left\{ \mathbf{inj}_{\tilde{g}}(x) \right\} > 0.$$

Now we are ready to prove the main deformation theorem.

Theorem 5.1. *Let M be a compact complex manifold and $g_0, \tilde{g}$ are two smooth Hermitian metrics on M such that $\mathrm{tr}_{\tilde{g}}\, R^{(TM,g_0)}$ is quasi-positive (resp. quasi-negative). Then for all $\varepsilon > 0, k \in \mathbb{N}$, there is another Hermitian metric $\hat{g}_0$ conformal to g_0 such that $\mathrm{tr}_{\tilde{g}}\, R^{(TM,\hat{g}_0)}$ is positive (resp. negative) on M and $||\hat{g}_0 - g_0||_{C^k(M,\tilde{g})} < \varepsilon$.*

Proof. We will prove the non-negative case. The non-positive case can be proved using the same argument. By rescaling, we may assume

$$\sup_M |\mathrm{Rm}(\tilde{g})| + |T(\tilde{g})|^2 \leq 1.$$

Let $\mu(n)$ be the constant from Proposition 4.1. Suppose $p \in M$ and let $0 < r_0 < \frac{1}{4}\min\left\{\mathbf{inj}_{\tilde{g}}(M), 1\right\}$ be such that $\operatorname{tr}_{\tilde{g}} R^{(TM,g_0)} > 0$ on $B_{\tilde{g}}(p, r_0)$.

Claim 5.1. *Let $r_m = (1 + m\mu(2-\mu)^{-1})r_0$, then there exists Hermitian metrics $\{g_m\}_{m\in\mathbb{N}}$ conformal to g_0 such that*

1) $\operatorname{tr}_{\tilde{g}} R^{(TM,g_m)} > 0$ *on* $B_{\tilde{g}}(p, r_m)$*;*
2) $\operatorname{tr}_{\tilde{g}} R^{(TM,g_m)} \geq 0$ *on* M*;*
3) $||g_m - g_0||_{C^k(M,\tilde{g})} \leq \varepsilon \sum_{i=0}^{m} 2^{-i-1}$.

Proof of claim. The statement is trivially true when $m = 0$. Suppose it is true for some $m \in \mathbb{N}_{\geq 0}$. Let $\{p_l\}_{l=1}^{N_m} \subset B_{\tilde{g}}(p, r_m - r_0)$ so that

$$A_{\tilde{g}}(p, r_{m+1}, r_m) \subset \bigcup_{l=1}^{N_m} B_{\tilde{g}}\left(p_l, \frac{r_0}{1-\mu}\right). \tag{5.1}$$

Now we work on $B_{\tilde{g}}(p_1, \frac{r_0}{1-\mu})$. By our choice of r_0, we have $\frac{r_0}{1-\mu} < \mathbf{inj}_{\tilde{g}}(p_1)$ and $\operatorname{tr}_{\tilde{g}} R^{(TM,g_m)} > 0$ on $B_{\tilde{g}}(p_1, r_0)$. Hence, we may apply Proposition 4.1 and Remark 4.1 on g_m so that there is a Hermitian metric $g_{m,1}$ conformal to g_m in which

(a) $\operatorname{tr}_{\tilde{g}} R^{(TM,g_{m,1})} > 0$ on $B_{\tilde{g}}\left(p_1, \frac{r_0}{1-\mu}\right) \bigcup B_{\tilde{g}}(p, r_m)$;
(b) $\operatorname{tr}_{\tilde{g}} R^{(TM,g_{m,1})} \geq 0$ on M;
(c) $||g_{m,1} - g_0||_{C^k(M,\tilde{g})} \leq \varepsilon \sum_{i=0}^{m} 2^{-i-1} + \varepsilon 2^{-m-2} N_m^{-1}$.

By repeating the argument on each $B_{\tilde{g}}(p_l, r_0)$, $l \leq N_m$ inductively, we will obtain a sequence of Hermitian metrics $\{g_{m,l}\}_{l=1}^{N_m}$ conformal to g_m so that for each $l \in \{1, ..., N_m\}$,

(a') $\operatorname{tr}_{\tilde{g}} R^{(TM,g_{m,l})} > 0$ on $\bigcup_{i=1}^{l} B_{\tilde{g}}(p_i, \frac{r_0}{1-\mu}) \bigcup B_{\tilde{g}}(p, r_m)$;
(b') $\operatorname{tr}_{\tilde{g}} R^{(TM,g_{m,l})} \geq 0$ on M;
(c') $||g_{m,l} - g||_{C^k(M,\tilde{g})} \leq \varepsilon \sum_{i=0}^{m} 2^{-i-1} + l\varepsilon 2^{-m-2} N_m^{-1}$.

Using (5.1) and the fact that

$$B_{\tilde{g}}(p, r_{m+1}) \subset A_{\tilde{g}}(p, r_{m+1}, r_m) \cup B_{\tilde{g}}(p, r_m),$$

we have the desired Hermitian metric if we choose $g_{m+1} = g_{m,N_m}$. This proves the claim. □

Since $r_m \to +\infty$ and M is compact, the process will be terminated at the N-th step for some finite $N \in \mathbb{N}$. Then $\hat{g}_0 = g_N$ will be the desired Hermitian metric. $\square$

Remark 5.1. It is clear from the proof that an analogous result still hold if M is complete non-compact with bounded Chern curvature, bounded torsion and has a uniform injectivity radius lower bound.

Proof of Corollary 1.1. This follows from Theorem 1.1 by taking $\tilde{g} = g_0$ and the fact that the new metric g_1 is conformal to g_0. $\square$

Remark 5.2. The same method also holds for the Chern-scalar curvature $R = \mathrm{tr}_g\, S = \mathrm{tr}_g \,\mathrm{Ric}$. This was shown earlier by Yang in [36, Lemma 3.2] by solving Poisson's equation.

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