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Extending vector bundles on curves

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Given a curve in a (smooth) projective variety $C \subset X$ over an algebraically closed field k, we show that a vector bundle V on C can be extended to a (μ-stable) vector bundle on X if $\mathrm{rank}(V) \geq \dim(X)$ and $\det(V)$ extends to X.

1. Introduction

Understanding which vector bundles on a subvariety extend to an ambient variety is a well studied problem in algebraic geometry. A famous example is the Grothendieck-Lefschetz theorems which considers the case of a complete intersection in a projective variety. A particularly striking consequence of this work is that complete intersections $X \subset \mathbf{P}^n$ with dimension $\dim(X) \geq 3$ always have a Picard group which is freely generated by $\mathscr{O}_{\mathbf{P}^n}(1)|_X$. However, there are many counterexamples to this statement when the hypothesis on the dimension is dropped: indeed, all elliptic curves can be realized as ample divisors in $\mathbf{P}^2$ but they have a non-finitely generated Picard group. The purpose of this paper is to show that this is the only obstruction when the subvariety is a curve and the rank of the vector bundle is sufficiently large.

Theorem 1 (Main Theorem). *Let $(X, \mathscr{O}_X(1))$ be a projective scheme over an algebraically closed field k, $C \subset X$ a 1-dimensional closed subscheme and V a vector bundle on C with $\mathrm{rank}(V) \geq \dim(X)$. Then V extends to X if and only if det(V) extends to X. If X is assumed to be integral, smooth and det(V) extends to X, then V may be extended to a μ-stable vector bundle on X.*

The proof involves two main ingredients:

1) Classical avoidance lemmas, Bertini-type arguments, and
2) the theory of elementary transformations due to Maruyama (see [5] for a gentle introduction to the technique).

Example 2. The main theorem has some surprising consequences. For instance, if $C \subset X$ is as in the statement of the theorem and L is *any* line bundle on C, the vector bundle $V = (L \oplus L^\vee)^{\oplus n}$ extends to a vector bundle on X for sufficiently large n. Similarly, if V is *any* vector bundle on C with sufficiently large rank, then $V \otimes V^\vee$ and $V \oplus V^\vee$ both extend to vector bundles on X. In general, the main theorem shows that (smooth) projective schemes have plenty of (μ-stable) vector bundles.

2. Some Lemmas

Let k denote a fixed algebraically closed field.

The first step of the proof of Theorem 1 boils down to a simple observation: the dual of any globally generated vector bundle V on a curve C can be realized as an elementary transformation of the trivial bundle along a Cartier divisor $D \subset C$. This is the content of the following:

Lemma 3. *Let C be a 1-dimensional scheme which is proper over k and suppose $Z \subset C$ is a finite set containing the associated points of C. Moreover, let V denote a rank r vector bundle on C which is globally generated, then there exists an effective Cartier divisor D (which can be viewed as a closed subscheme $i : D \to C$ in a natural way) which doesn't meet Z, as well as an exact sequence*

$$0 \to V^\vee \to \mathscr{O}_C^{\oplus r} \to i_*\mathscr{O}_D \to 0.$$

Proof. By [7, Lemma 27] there is a Cartier divisor $i : D \subset C$ not meeting Z, a line bundle L on D and an exact sequence

$$0 \to \mathscr{O}_C^{\oplus r} \to V \to i_*L \to 0.$$

Since D is a finite scheme, L is trivial and so we have

$$0 \to \mathscr{O}_C^{\oplus r} \to V \to i_*\mathscr{O}_D \to 0.$$

Dualizing this sequence we obtain:

$$0 \to V^\vee \to \mathscr{O}_C^{\oplus r} \to \underline{\mathrm{Ext}}^1(i_*\mathscr{O}_D, \mathscr{O}_C) \to 0.$$

Indeed, the first term $\underline{\mathrm{Hom}}(i_*\mathscr{O}_D, \mathscr{O}_C)$ vanishes because D contains no associated points of C and the term $\underline{\mathrm{Ext}}^1(V, \mathscr{O}_C)$ vanishes since V is locally free. To determine

$$\underline{\mathrm{Ext}}^1(i_*\mathscr{O}_D, \mathscr{O}_C)$$

one can dualize the exact sequence

$$0 \to \mathscr{O}_C(-D) \to \mathscr{O}_C \to i_*\mathscr{O}_D \to 0$$

to see that

$$\underline{\mathrm{Ext}}^1(i_*\mathscr{O}_D, \mathscr{O}_C) \simeq \mathrm{coker}(s_D : \mathscr{O}_C \to \mathscr{O}_Y(D)) \simeq i_*\mathscr{O}_D \otimes \mathscr{O}_C(D)$$

where s_D is the section corresponding to D. Since D is a finite scheme this is just isomorphic to $i_*\mathscr{O}_D$. □

The following lemma is a refinement of an argument in [3, Theorem 5.2.5] which, itself, is adapted from the appendix of [6]. It will allow us to guarantee our construction produces a μ-stable vector bundle.

Lemma 4. *Fix an integral, smooth, projective k-scheme $(X, \mathscr{O}_X(1))$ of dimension n and an effective Cartier divisor $H \subset X$ which is irreducible and generically smooth. Let E be a vector bundle of rank $r \geq 2$ on X which fits into an exact sequence*

$$0 \to E \to \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N) \to 0$$

for some integer $N > 0$. Moreover, suppose the surjection $\mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N)$ does not factor through any torsion-free sheaf F with $\mu(F) \leq \frac{r-1}{r}\deg(\mathscr{O}_X(H))$ and $\mathrm{rk}(F) < r$. Then E is a μ-stable sheaf on X.

Proof. Let $0 \neq E' \subset E$ be a saturated proper subsheaf and set F' equal to the saturation of E' in $\mathscr{O}_X^{\oplus r}$. Since E' is saturated in E, it follows that the natural map $j : F'/E' \to \mathscr{O}_H(N)$ is injective. First suppose that F'/E' is nonzero. It follows that the injection j induces an isomorphism $(F'/E')_\eta \cong \mathscr{O}_H(N)_\eta$ at the (unique) generic point η of H in X. Indeed, $\mathscr{O}_H(N)$ is a line bundle on an irreducible and generically smooth scheme so $\mathscr{O}_H(N)_\eta \cong \mathscr{O}_{H,\eta} = k(H)$ for a field $k(H)$. Moreover, since H is Cohen-Maucalay, it has no embedded associated points which implies the localization $(F'/E')_\eta$ is a nonzero vector space over $k(H)$ (see, for instance, [8, Tag 0B3L]). It follows that the induced map $(F'/E')_\eta \cong \mathscr{O}_H(N)_\eta$ is an isomorphism. Therefore we have an exact sequence

$$0 \to F'/E' \to \mathscr{O}_H(N) \to B \to 0$$

where the support of B has codimension ≥ 2 in X, so the determinant of B in X is trivial. From this, it follows that $\det(F'/E') \cong \det(\mathscr{O}_H(N)) \cong \mathscr{O}_X(H)$ and $\det(E') \cong \det(F') \otimes \mathscr{O}_X(-H)$. Therefore:

$$\begin{aligned}\mu(E') &= \frac{\deg(\det(F')) + \deg(\mathscr{O}_X(-H))}{\operatorname{rk}(E')} \\ &= \mu(F') - \frac{\deg(\mathscr{O}_X(H))}{\operatorname{rk}(E')} < 0 - \frac{\deg(\mathscr{O}_X(H))}{\operatorname{rk}(E)} = \mu(E)\end{aligned}$$

where the second equality follows because $\operatorname{rk}(E') = \operatorname{rk}(F')$, the inequality follows because $\mu(F') \leq \mu(\mathscr{O}_X^{\oplus r}) \leq 0$ and $\operatorname{rk}(E') < \operatorname{rk}(E)$, as desired.

On the other hand, if $F'/E' = 0$, then $F = \mathscr{O}_X^{\oplus r}/E'$ is torsion-free and $\phi : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N)$ factors through the quotient $\mathscr{O}_X^{\oplus r} \to F$. By hypothesis, this implies $\mu(F) > \frac{r-1}{r}\deg(\mathscr{O}_X(H))$ and so

$$\begin{aligned}\mu(E') = \frac{\deg(\det(E'))}{\operatorname{rk}(E')} &= \frac{-\deg(\det(F))}{\operatorname{rk}(E')} = \frac{-(r - \operatorname{rk}(E'))}{\operatorname{rk}(E')}\mu(F) \\ &< \frac{-(r - \operatorname{rk}(E'))}{\operatorname{rk}(E')}\frac{(r-1)}{r}\deg(\mathscr{O}_X(H)) \\ &\leq \frac{-\deg(\mathscr{O}_X(H))}{r} \\ &= \mu(E)\end{aligned}$$

The first inequality follows because $\mu(F) > \frac{r-1}{r}\deg(\mathscr{O}_X(H))$ and the second follows because E' is a saturated *proper* subsheaf of E, so $\frac{(r-\operatorname{rk}(E'))(r-1)}{\operatorname{rk}(E')} \geq 1$. It follows that E is μ-stable. □

To be able to apply the previous lemma, we will need to find a Cartier divisor $H \subset X$ which is irreducible and generically smooth. This can be arranged by a standard Bertini-type result. However, the statement we require could not be found in the literature and so we include a proof for the sake of completeness.

Recall that if X is a proper k-scheme, L a line bundle on X and $N \subset \mathrm{H}^0(X, L)$ a linear system, then one says N *separates tangent vectors on* U

if the natural map

$$\mathrm{Ker}[N \to L_p/\mathrm{m}_p L_p] \to \mathrm{m}_p L_p/\mathrm{m}_p^2 L_p$$

is surjective for every $p \in U$. As the term suggests, if N separates tangent vectors on $U \subset X$ and has no base points along U, then the associated map $f : U \to \mathbf{P}^n$ induces injective differentials

$$\mathrm{d}f_p : T_{U/k,p} \hookrightarrow T_{\mathbf{P}^n/k,f(p)}$$

and so by [8, Tag 0B2G], the map f is unramified.

Lemma 5. *Let $(X, \mathscr{O}_X(1))$ be an integral, smooth, and projective k-scheme and fix proper closed subschemes $D \subset C \subset X$ with $\mathrm{codim}_X D \geq 2$ and C a 1-dimensional subscheme. If a linear system $N \subset \mathrm{H}^0(X, L)$ is base point free on $X \backslash D$ and separates tangent vectors on $X \backslash C$, then the vanishing of the general member $s \in N$ is an irreducible Cartier divisor which is smooth away from C.*

Proof. Since N is base point free away from D and separates tangent vectors away from C, a choice of basis of N defines a morphism $f : X \backslash D \to \mathbf{P}^n$ which is unramified over $X \backslash C$. Since $X \backslash D$ is geometrically integral, it follows that the general hyperplane section in $X \backslash D$ is geometrically irreducible by [4, Corollaire 6.11.3]. Indeed, the fact that $X \backslash C \to \mathbf{P}^n$ is unramified implies $\mathrm{Im}(f|_{X \backslash C}) \geq 2$ and so we may apply the cited Corollary. Similarly, since $X \backslash C$ is smooth and N induces an unramified morphism on $X \backslash C$, the general hyperplane section in $X \backslash C$ is smooth by [4, Corollaire 6.11.2]. Since the general hyperplane section over $X \backslash D$ (respectively, $X \backslash C$) corresponds to the intersection of the vanishing of the general member of N in X with $X \backslash D$ (respectively, $X \backslash C$), viewing the vanishing of the general member in X yields the result. The only thing to check is that the vanishing of such a general section remains irreducible, but the only components the vanishing over X could gain are those which lie entirely in D. This is not possible because a Cartier divisor cannot have components of codimension ≥ 2. □

Recall that the dual of any globally generated vector bundle on a curve C can be written as an elementary transformation of the trivial vector bundle along a Cartier divisor $D \subset C$. In the proof of Theorem 1, we will show that one may extend this elementary transformation to one over all of X. As such, we need to show that the Cartier divisor $D \subset C$ must be the intersection of a Cartier divisor, H, on X, with C. This is the primary purpose of the following lemma.

Lemma 6. *Let $C \subset X$ be the inclusion of a proper 1-dimensional closed subscheme in a projective k-scheme X. Suppose that E is a rank $r \geq 2$ vector bundle on C whose determinant extends to X, then there is an ample line bundle L on X with the following properties*

1) *$E \otimes L|_C$ is globally generated,*
2) *$\det(E \otimes L|_C)$ is ample,*
3) *if $\det(E \otimes L|_C) \cong \mathscr{O}_C(D)$ for some effective Cartier divisor $D \subset C$ not containing any associated point of X, then there is an ample Cartier divisor $H \subset X$ with $H \cap C = D$ scheme-theoretically, and*
4) *if X is integral, smooth and projective we may also take H in (3) to be irreducible and generically smooth.*

Proof. The existence of L satisfying the conditions (1)-(3) follows from [7, Lemma 28]. Assume X is integral and smooth, we will show that L can be chosen so that (4) is satisfied. Let L' denote a line bundle on X with $L'|_C \cong \det(E)$. Begin by choosing an ample line bundle L on X so that

1) $E \otimes L|_C$ is globally generated,
2) $L' \otimes L^{\otimes r}$ is very ample,
3) $\mathrm{H}^1(X, L' \otimes L^{\otimes r} \otimes I_C) = 0$, and
4) The linear system $M = \mathrm{H}^0(X, L' \otimes L^{\otimes r} \otimes I_C) \subset \mathrm{H}^0(X, L' \otimes L^{\otimes r})$ has no base points away from C and separates tangent vectors on $X \backslash C$.

It is well-known that the first three conditions can be arranged by choosing L to be a high power of an ample line bundle. For the fourth, just note that we can arrange L to be ample enough so that $L' \otimes L \otimes I_C$ is globally generated and $L^{\otimes r-1}$ is very ample (so, in particular, the associated complete linear system separates tangent vectors on X). Then the linear system $\mathrm{H}^0(X, L' \otimes L^{\otimes r} \otimes I_C)$ has no base points on $X \backslash C$ and separates tangent vectors on $X \backslash C$. Note that $\det(E \otimes L|_C) \cong L' \otimes L^{\otimes r}|_C$ and therefore the first two items in the statement of the lemma are now satisfied.

Now suppose $\det(E \otimes L|_C) \cong \mathscr{O}_C(D)$ for some effective Cartier divisor $D \subset C$. Then take cohomology of the exact sequence

$$0 \to L' \otimes L^{\otimes r} \otimes I_C \to L' \otimes L^{\otimes r} \to (L' \otimes L^{\otimes r})|_C \to 0$$

and use the fact that $\mathrm{H}^1(X, L' \otimes L^{\otimes r} \otimes I_C) = 0$ to see that the second map induces a surjection on global sections. In particular, a section

$s_D \in \mathrm{H}^0(C, L' \otimes L^{\otimes r}|_C)$ defining $D \subset C$ can be lifted to a section $s_{H'} \in \mathrm{H}^0(X, L' \otimes L^{\otimes r})$ whose vanishing, $H' \subset X$ intersects C precisely at D. However, it is not clear that H' is irreducible or generically smooth. To remedy this, note that the general member of the linear system

$$N = \langle s_{H'} \rangle + \mathrm{H}^0(X, L' \otimes L^{\otimes r} \otimes I_C) \subset \mathrm{H}^0(X, L' \otimes L^{\otimes r})$$

is smooth away from C and irreducible by lemma 5. Indeed, the linear system N contains $\mathrm{H}^0(X, L' \otimes L^{\otimes r} \otimes I_C)$ and $s_{H'}$ and therefore is base point free away from D and separates tangent vectors away from C. Lastly, observe that for the general $s \in N$, $s|_C$ and s_D cut out the same Cartier divisor since it has the form $s = \lambda s_{H'} + s'$ where $s'|_C = 0$ and $\lambda \neq 0$. Thus a general $s \in N$ defines an effective Cartier divisor H on X which is irreducible, generically smooth and has $H \cap C = D$ scheme-theoretically. □

We are almost ready to construct an elementary transformation of X along H. However, to guarantee μ-stability, we need to show that the hypothesis of lemma 4 is satisfied. The next lemma shows that there are enough points in the space of maps $\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N))$ which satisfy the desired properties.

Before we state the following lemma, we introduce some basic notation. If F is a coherent sheaf on a scheme Y, then define

$$\mathbb{A}(F) = \underline{\mathrm{Spec}}_{\mathscr{O}_Y}(\mathrm{Sym}^* F^\vee)$$

to be the associated linear scheme. However, if $Y = \mathrm{Spec}\, A$ and $V = \tilde{M}$ for an A-module M, then we will write $\mathbb{A}(M)$ instead.

Lemma 7. *Let $(X, \mathscr{O}_X(1))$ denote an integral k-projective variety and let H be an ample Cartier divisor with a finite subscheme $D \subset H$. Fix integers r and ρ, then there exists an N_0 such that for all $N \geq N_0$ and any $\psi \in \mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N))$ there is a nonempty open subscheme*

$$U_N \subset \psi + \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N) \otimes I_D)) \subset \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N)))$$

such that all points of U_N correspond to maps $\phi : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N)$ which do not factor through a torsion-free quotient F of $\mathscr{O}_X^{\oplus r}$ with $\mu(F) \leq \rho$ and rk$(F) < r$.

Proof. Consider the family $\mathscr{F}$ of all torsion-free quotients of $\mathscr{O}_X^{\oplus r}$ with $\mu(F) \leq \rho$ and $\mathrm{rk}(F) < r$. By [3, Lemma 1.7.9], there is a finite type Quot-scheme Q of $\mathscr{O}_X^{\oplus r}$ so that each $F \in \mathscr{F}$ appears as a fiber of the universal quotient on $X \times Q$:

$$u : \mathscr{O}_{X\times Q}^{\oplus r} \to F_{\mathrm{univ}}.$$

Since a Quot-scheme parametrizes flat families of quotients, the universal object F_{univ} is Q-flat. Now, throw out all the connected components of Q which do not contain a quotient belonging to $\mathscr{F}$. Since F_{univ} and $\mathrm{p}_1^*\mathscr{O}_H(N)$ are flat over Q for every $N \geq 0$ (here $\mathrm{p}_1 : X \times Q \to X$ is the projection), all the fibers $(F_{\mathrm{univ}})_q$ have rank less than r and [2, Corollaire 7.7.8] implies the functors $G_N, H_N : \underline{\mathrm{Aff}}/Q \to \underline{\mathrm{Set}}$ defined by

$$G_N(T) = \mathrm{Hom}_{\mathscr{O}_{X_T}}(F_{\mathrm{univ}}|_T, \mathrm{p}_1^*\mathscr{O}_H(N)|_T)$$

$$H_N(T) = \mathrm{Hom}_{\mathscr{O}_{X_T}}(\mathrm{p}_1^*\mathscr{O}_X^{\oplus r}|_T, \mathrm{p}_1^*\mathscr{O}_H(N)|_T)$$

for any affine Q-scheme T, are representable by linear schemes. Let Y_N denote the linear Q-scheme representing G_N and note that, by cohomology and base change, for all sufficiently large N, H_N is naturally represented by the geometric vector bundle $\mathbb{A}(\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N))) \times_k Q$. Thus, the quotient u induces a monomorphism

$$g : Y_N \hookrightarrow \mathbb{A}(\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N))) \times_k Q$$

which can be interpreted as follows: at a closed point $q \in Q$, the morphism g is the inclusion of homomorphisms $\psi : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N)$ which factor through $(F_{\mathrm{univ}})_q$. Thus, to prove the lemma it suffices to show there is a N_0 so that for every $N \geq N_0$

$$\mathrm{dim} Y_N < \dim(\mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N) \otimes I_D))).$$

Indeed, in that case the composition $\pi_1 \circ g$:

$$Y_N \hookrightarrow \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N))) \times_k Q \to \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N)))$$

cannot have an image whose closure contains a coset of $\mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N) \otimes I_D))$ (here π_1 denotes the projection onto the first factor). Thus, the complement of the closure of the image of $\pi_1 \circ g$ can be intersected with any such coset to define the nonempty open:

$$U_N \subset \psi + \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N) \otimes I_D)) \subset \mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N)))$$

so all $\phi \in U_N$ do not factor through any $F \in \mathscr{F}$.

For all $N \gg 0$:

$$\dim Y_N \leq \dim(Q) + \max_{q\in Q}\{\chi(\underline{\mathrm{Hom}}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H(N))_q))\}$$

because there is a canonical identification of fibers

$$(Y_N)_q = \mathbb{A}(\mathrm{Hom}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H(N))_q))$$

for any point $q \in Q$. Moreover, for every $q \in Q$ and all $N \geq 0$, we have:

$$\chi(\mathscr{O}_H^{\oplus r}(N)) = \chi(\underline{\mathrm{Hom}}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H(N))_q)) + p_q(N)$$

for some nonconstant polynomial $p_q(t)$ since $(F_{\mathrm{univ}})_q$ has rank $< r$. Indeed, $p_q(t)$ is the Hilbert polynomial of the cokernel of

$$0 \to \underline{\mathrm{Hom}}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H)_q)) \to \underline{\mathrm{Hom}}((\mathscr{O}_{X\times Q}^{\oplus r})_q, (\mathrm{p}_1^*\mathscr{O}_H)_q)$$

and because this cokernel is supported on the positive-dimensional projective scheme H, $p_q(t) \to \infty$ as t gets large. Next, we claim that the set $\{p_q(t)\}_{q\in Q}$ is finite. Indeed, this is a consequence of the finiteness of the Hilbert polynomials associated to the sheaves $\{\underline{\mathrm{Hom}}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H)_q)\}_{q\in Q}$. This follows by Noetherian induction on Q, generic Q-flatness of $\underline{\mathrm{Hom}}(F_{\mathrm{univ}}, \mathrm{p}_1^*\mathscr{O}_H)$ (see [8, Tag 052A]) and [1, Lemma 6.8]. Lastly, since D is a finite subscheme of H, there is a fixed constant $d > 0$ with

$$\chi(\mathscr{O}_H^{\oplus r}(N) \otimes I_D) + d = \chi(\mathscr{O}_H^{\oplus r}(N))$$

for all large N. Thus, by the finiteness of the set $\{p_q(t)\}_{q\in Q}$, there is a $N_0 \gg 0$ so that

$$\begin{aligned}\dim Y_N &\leq \dim(Q) + \max_{q\in Q}\{\chi(\underline{\mathrm{Hom}}((F_{\mathrm{univ}})_q, (\mathrm{p}_1^*\mathscr{O}_H(N))_q))\}\\ &\leq \dim(Q) + \chi(\mathscr{O}_H(N)^{\oplus r}) - \min_{q\in Q}\{p_q(N)\}\\ &= \dim(Q) + \chi(\mathscr{O}_H(N)^{\oplus r} \otimes I_D) + d - \min_{q\in Q}\{p_q(N)\}\\ &< \dim(\mathbb{A}(\mathrm{Hom}(\mathscr{O}_X^{\oplus r}, \mathscr{O}_H(N) \otimes I_D)))\end{aligned}$$

for all $N \geq N_0$, as desired. □

3. Proof of the main theorem

Proof of Theorem 1. Let $\mathscr{O}_X(1)$ denote an ample line bundle on X. By replacing V with $V(m) = V \otimes \mathscr{O}_X(1)^{\otimes m}|_C$ for $m \gg 0$ we may suppose that the conclusion of lemma 6 holds. In particular, V is globally generated. Thus, by lemma 3 there exists a Cartier divisor $i : D \to C$ which misses the associated points of X and the associated points of C with the property that there is an exact sequence

$$0 \to V^\vee \to \mathscr{O}_C^{\oplus r} \to i_*\mathscr{O}_D \to 0.$$

By adjunction, the surjection on the right, call it ϕ'_D, is determined by the induced map of sheaves on D:

$$\phi_D : \mathscr{O}_C^{\oplus r}|_D = \mathscr{O}_D^{\oplus r} \to \mathscr{O}_D.$$

To prove the theorem it suffices to show $V^\vee$ extends to X. Let L denote a fixed line bundle on X with $L|_C \cong \det(V)$. Observe that $L|_C \cong \mathscr{O}_C(D) \cong \det(\mathscr{O}_D)$ and that $D \subset C$ is a Cartier divisor in C missing the associated points of X. Thus, we may apply the full conclusion of lemma 6. In particular, there is an effective ample Cartier divisor $H \subset X$ with $H \cap C = D$ scheme-theoretically. Moreover, if X is integral and smooth, we may take H to be smooth away from C and irreducible.

The idea will be to extend the elementary transformation of $\mathscr{O}_C^{\oplus r}$ on C along D to an elementary transformation of $\mathscr{O}_X^{\oplus r}$ on X along H. To make this precise, first fix an isomorphism $g_1 : \mathscr{O}_X(1)|_D \to \mathscr{O}_D$ and note that this induces isomorphisms $g_N : \mathscr{O}_X(N)|_D \cong \mathscr{O}_D$ for every $N > 0$. Our goal is to find a surjective morphism $\psi : \mathscr{O}_H^{\oplus r} \to \mathscr{O}_X(N)|_H$ for some $N > 0$ so that the following diagram commutes:

$$\begin{array}{ccccc}
\mathscr{O}_H^{\oplus r}|_D & \xrightarrow{\psi|_D} & \mathscr{O}_X(N)|_D & \longrightarrow & 0 \\
\downarrow \text{id} & & \downarrow g_N & & \\
\mathscr{O}_D^{\oplus r} & \xrightarrow{\phi_D} & \mathscr{O}_D & \longrightarrow & 0.
\end{array}$$

Once we have found such a ψ, compose it with the natural adjunction morphism to obtain $\psi' : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H^{\oplus r} \to \mathscr{O}_X(N)|_H$. Now consider the associated elementary transformation on X:

$$0 \to W \to \mathscr{O}_X^{\oplus r} \to \mathscr{O}_X(N)|_H \to 0$$

and observe that because H is a Cartier divisor, W must be locally free. Moreover, note that $\underline{\mathrm{Tor}}_1^{\mathscr{O}_X}(\mathscr{O}_X(N)|_H, \mathscr{O}_C) = 0$ since it injects into the vector bundle $W|_C$ and is supported on D (which contains no associated points of C). Thus, upon restriction to C, the isomorphism $g_N : \mathscr{O}_X(N)|_D \to \mathscr{O}_D$ induces a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W|_C & \longrightarrow & \mathscr{O}_X^{\oplus r}|_C & \xrightarrow{\psi'|_C} & \mathscr{O}_X(N)|_D & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \mathrm{id} & & \downarrow g_N & & \\ 0 & \longrightarrow & V^\vee & \longrightarrow & \mathscr{O}_C^{\oplus r} & \xrightarrow{\phi'_D} & \mathscr{O}_D & \longrightarrow & 0 \end{array}$$

thereby producing an isomorphism $W|_C \simeq V^\vee$, as desired. Note that the isomorphisms $g_N : \mathscr{O}_X(N)|_D \simeq \mathscr{O}_D$ we have fixed determine isomorphisms

$$\underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes \mathscr{O}_D \simeq \underline{\mathrm{Hom}}(\mathscr{O}_D^{\oplus r}, \mathscr{O}_D).$$

As such, the rest of the proof is devoted to finding a ψ which restricts to ϕ_D via the isomorphism g_N (for some large $N > 0$).

Next, take N to be large enough so that the short exact sequence on H

$$\begin{aligned} 0 &\to \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes I_D \\ &\to \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \to \underline{\mathrm{Hom}}(\mathscr{O}_D^{\oplus r}, \mathscr{O}_D) \to 0 \end{aligned}$$

remains exact after taking global sections so that we may lift $\phi_D \in \mathrm{H}^0(H, \underline{\mathrm{Hom}}(\mathscr{O}_D^{\oplus r}, \mathscr{O}_D))$ to a section $\psi_D \in \mathrm{H}^0(H, \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H))$. The issue is that

$$\psi_D : \mathscr{O}_H^{\oplus r} \to \mathscr{O}_X(N)|_H$$

may not be surjective away from D. We will rectify this by adding a factor from

$$\mathrm{H}^0(H, \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes I_D)$$

which doesn't change the behavior of ψ_D along D.

For all sufficiently large N, we may fix a basis

$$\psi_1, ..., \psi_n \in \mathrm{H}^0(H, \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes I_D)$$

so that at any point $p \in H \backslash D$, there is a collection of r sections among the $\psi_1, ..., \psi_n$ which form a basis for the vector space $\underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes k(p)$. Viewing the sections $\psi_D, \psi_1, ..., \psi_n$ in $\mathrm{H}^0(H, \underline{\mathrm{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H))$ we set $\mathbf{A}_k^n = \operatorname{Spec} k[x_1, ..., x_n]$ and consider the universal section

$$\psi_{\text{univ}} = \psi_D + \Sigma_{i=1}^n x_i \psi_i$$

of $\psi_D + \underline{\operatorname{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes I_D$ pulled back to $\mathbf{A}_k^n \times_k H$. Thus, by construction, the universal section restricts to the section $\psi_{\underline{a}} = \psi_D + \Sigma_{i=1}^n a_i \psi_i$ over $\underline{a} = (a_1, ..., a_n) \in \mathbf{A}_k^n(k)$.

Over the complement $U = H \backslash D \subset H$, consider the closed locus of non-surjective maps

$$Z = \{(\underline{a}, u) | \psi_{\underline{a}} \otimes k(u) \text{ is not surjective}\} \subset \mathbf{A}_k^n \times U.$$

For any $u \in U$ the fiber Z_u has codimension r since the $\psi_1, ..., \psi_n$ generate

$$\underline{\operatorname{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H)$$

at all $u \in U$. Indeed, there is a surjective map

$$\pi : \mathbf{A}_{k(u)}^n \to \operatorname{Hom}_{k(u)}(k(u)^r, k(u)) = \underline{\operatorname{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes k(u)$$

sending

$$\underline{a} = (a_1, ..., a_n) \mapsto (\psi_D)_{k(u)} + \Sigma_{i=1}^n a_i (\psi_i)_{k(u)}$$

and only the zero map doesn't have full rank. Therefore the fiber $\pi^{-1}(0) = Z_u$ has dimension $n - r$ so the dimension of Z (and the closure of its image $\overline{\mathrm{p}_1(Z)}$ in $\mathbf{A}_k^n$) is at most

$$n - r + \dim(H) < n$$

because $r = \operatorname{rank}(V) \geq \dim(X) > \dim(H)$.

Thus, there is a point $\underline{c} = (c_1, ..., c_n) \in \mathbf{A}_k^n(k)$ avoiding $\overline{\mathrm{p}_1(Z)}$, and we claim that the corresponding section of $\underline{\operatorname{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H)$ works as desired. Indeed, a point avoiding $\overline{\mathrm{p}_1(Z)}$ corresponds to a section

$$\psi_{\underline{c}} = \psi_D + \Sigma_{i=1}^n c_i \psi_i \in \mathrm{H}^0(H, \underline{\operatorname{Hom}}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H))$$

which is a surjective linear map for every $u \in U$ (since $(\underline{c}, u)$ is not in Z). Moreover, on D, we have $\psi_{\underline{c}}|_D = \psi_D|_D = \phi_D$. Also, ϕ_D is surjective so Nakayama's lemma implies $\psi_{\underline{c}}$ is surjective over all of H. Thus, the kernel of $\psi'_{\underline{c}} : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H^{\oplus r} \to \mathscr{O}_H(N)$ is a vector bundle W on X extending $V^\vee$.

If X is smooth, lemma 7 says that after perhaps enlarging N, there is a nonempty open subset

$$U_N \subset \psi_D + \mathrm{Hom}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H) \otimes I_D$$

so that the corresponding composition $\mathscr{O}_X^{\oplus r} \to \mathscr{O}_H^{\oplus r} \to \mathscr{O}_X(N)|_H$ does not factor through a torsion-free sheaf F on X with $\mathrm{rk}(F) < r$ and $\mu(F) \leq \frac{r}{r-1}\deg(\mathscr{O}_X(H))$. Therefore, the general section

$$\underline{c} \in \mathbf{A}_k^n \cong \psi_D + \mathrm{Hom}(\mathscr{O}_H^{\oplus r}, \mathscr{O}_X(N)|_H \otimes I_D)$$

induces a surjective map $\psi'_{\underline{c}} : \mathscr{O}_X^{\oplus r} \to \mathscr{O}_H(N)$ (because it misses $\overline{\mathrm{p}_1(Z)}$) and the resulting kernel satisfies the hypothesis of lemma 4 (because $\underline{c} \in U_N$). It follows that the kernel of $\psi'_{\underline{c}}$ is a μ-stable vector bundle on X extending $V^\vee$. □

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