Math. Res. Lett.
Volume 29, Number 5, 1551–1570, 2022

Counting twisted Higgs bundles

Sergey Mozgovoy and Ronan O'Gorman

We prove an explicit formula, conjectured earlier by the first author, counting semistable twisted Higgs bundles over a smooth projective curve.

1. Introduction

Let X be a smooth projective curve of genus g defined over a finite field $\mathbb{F}_q$. Let L be a line bundle of degree ℓ over X and let $\mathcal{M}_L(r,d)$ be the moduli space of semistable L-twisted Higgs bundles over X. It parametrizes pairs (E,ϕ), where E is a vector bundle of rank r and degree d over X and $\phi : E \to E \otimes L$ is a homomorphism. A formula for the computation of the number of points of $\mathcal{M}_L(r,d)$ for coprime r,d was conjectured in [18] and is proved in this note.

The above conjecture was obtained as a solution of a recursive formula, called an ADHM recursion, conjectured by Chuang, Diaconescu, and Pan [3]. The ADHM recursion was itself based on a conjectural wall-crossing formula for the refined Donaldson-Thomas invariants on a noncompact 3CY variety $Y = L \oplus (\omega_X \otimes L^{-1})$, where ω_X is the canonical bundle of X, as well as a conjectural formula for the asymptotic ADHM invariants. The latter invariants can be interpreted as Pandharipande-Thomas invariants of Y [21]. The formula counting them was derived in [3] by string theoretic methods, hence remains conjectural from the mathematical point of view.

On the other hand, the formula for $\mathcal{M}_L(r,d)$ conjectured in [18] can be considered as a generalization of the conjecture by Hausel and Rodriguez-Villegas [11] in the case of usual Higgs bundles, where the twisting line bundle L is equal to ω_X. A breakthrough for the counting of usual Higgs bundles was made by Schiffmann [23] who proved an explicit, albeit rather complicated formula for these invariants, quite different from the conjecture of [11]. An equivalence between these formulas was proved recently by purely combinatorial methods in a series of papers by Mellit [14–16].

Results on the invariants of moduli spaces of Higgs bundles for small rank and degree were obtained in [3, 7, 9–12, 22]. The conjecture of Hausel and

Rodriguez-Villegas was proved for the y-genus in [6]. An alternative general formula for twisted Higgs bundles on $\mathbb{P}^1$ – in terms of quiver representations – was obtained in [19]. Other interesting results related to counting of Higgs bundles can be found in [1, 2, 4, 5].

In this paper we will apply Mellit's methods in order to prove a formula for general L-twisted Higgs bundles. This task will be rather straightforward as Schiffmann's computation was generalized earlier for twisted Higgs bundles in [20] (see §3). More precisely, let $\mathfrak{M}^{\mathrm{ss}}_L(r,d)$ be the moduli stack of semistable L-twisted Higgs bundles over X. Given a finite type algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$, we define its volume (see §2.4 for more details on volumes)

$$[\mathcal{X}] = (\#\mathcal{X}(\mathbb{F}_{q^n}))_{n\ge1}, \qquad \#\mathcal{X}(\mathbb{F}_{q^n}) = \sum_{x\in\mathcal{X}(\mathbb{F}_{q^n})/\sim} \frac{1}{\#\operatorname{Aut}(x)}. \tag{1}$$

Define (integral) Donaldson-Thomas (DT) invariants $\Omega_{r,d}$ using the plethystic logarithm (see §2.2)

$$\sum_{d/r=\tau} \Omega_{r,d} T^r z^d = (q-1)\operatorname{Log}\left(\sum_{d/r=\tau} (-q^{\frac12})^{-\ell r^2}[\mathfrak{M}^{\mathrm{ss}}_L(r,d)]T^r z^d\right), \quad \tau\in\mathbb{Q}. \tag{2}$$

Note that if r and d are coprime, then every $E\in\mathfrak{M}^{\mathrm{ss}}_L(r,d)$ is stable and $\operatorname{End}(E)=\mathbb{F}_q$ (see Remark 3.1). Therefore

$$\frac{[\mathcal{M}_L(r,d)]}{q-1} = [\mathfrak{M}^{\mathrm{ss}}_L(r,d)] = (-q^{\frac12})^{\ell r^2}\frac{\Omega_{r,d}}{q-1}, \tag{3}$$

hence we can recover $[\mathcal{M}_L(r,d)]$ from the DT invariant $\Omega_{r,d}$. In the case of non-coprime r and d we can recover only the volumes $[\mathfrak{M}^{\mathrm{ss}}_L(r,d)]$ from the DT invariants. Consider the zeta function of the curve X

$$Z_X(t) = \exp\left(\sum_{n\ge1}\frac{\#X(\mathbb{F}_{q^n})}{n}t^n\right) = \frac{\prod_{i=1}^g(1-\alpha_i t)(1-\alpha_i^{-1}qt)}{(1-t)(1-qt)},$$

where α_i are the Weil numbers of X (see §2.4). The following result was conjectured in [18] (cf. §4.3). We formulate it in the case $\deg L > 2g-2$ (see Remark 4.5 for the case $L=\omega_X$).

Theorem 1.1. *Assume that $p=\ell-(2g-2)>0$. Given a partition λ and a box $s\in\lambda$, let $a(s)$ and $l(s)$ denote its arm and leg lengths respectively* *(see §2). Define*

$$\text{(4)}\quad \hat{\Omega}^\circ(T,q,z) = \sum_\lambda T^{|\lambda|} \prod_{s\in\lambda} (-q^{a(s)}z^{l(s)})^p \frac{\prod_{i=1}^g (q^{a(s)} - \alpha_i^{-1} z^{l(s)+1})(q^{a(s)+1} - \alpha_i z^{l(s)})}{(q^{a(s)} - z^{l(s)+1})(q^{a(s)+1} - z^{l(s)})},$$

$$\text{(5)}\qquad \sum_{r\geq 1} \Omega_r^\circ(q,z)T^r = (q-1)(1-z)\operatorname{Log}\hat{\Omega}^\circ(T,q,z).$$

Then $\Omega_r^\circ(q,z) \in \mathbb{Z}[q,z,\alpha_1^{\pm 1},\dots,\alpha_g^{\pm 1}]$ *and* $\Omega_{r,d} = q^{pr/2}\Omega_r^\circ(q,1)$ *for all* $d\in\mathbb{Z}$. *In particular, if* r,d *are coprime, then*

$$\text{(6)}\qquad [\mathcal{M}_L(r,d)] = (-1)^{pr} q^{(g-1)r^2 + p\binom{r+1}{2}} \Omega_r^\circ(q,1).$$

Note that, by the Weil conjectures, the formula for the Poincaré polynomials of the moduli spaces $\mathcal{M}_L(r,d)$ is obtained from the previous result by considering $\alpha_i = q^{\frac{1}{2}}$. It is unclear how to compute DT invariants when the twisting line bundle has degree $0 < \ell < 2g-2$.

2. Preliminaries

2.1. Partitions

A partition is a sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ such that $\lambda_n = 0$ for $n \gg 0$. We define its length $l(\lambda) = \#\{i \mid \lambda_i \neq 0\}$ and its weight $|\lambda| = \sum_i \lambda_i$. Define its Young diagram (also denoted by λ)

$$\text{(7)}\qquad d(\lambda) = \left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, i \geq 1, 1 \leq j \leq \lambda_i \right\}.$$

An element $s = (i,j) \in \lambda$ is called a box of the Young diagram located at the i-th row and j-th column. Define the conjugate partition λ' with λ'_j equal the number of boxes in the j-th column of λ. Given a box $s=(i,j)\in\lambda$, define its arm and leg lengths respectively

$$\text{(8)}\qquad a(s) = \lambda_i - j, \qquad l(s) = \lambda'_j - i.$$

Define the hook length $h(s) = a(s) + l(s) + 1$.

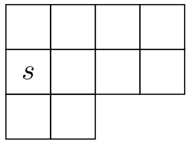

Figure 1: Young diagram for $\lambda=(4,4,2)$. Here $\lambda'=(3,3,2,2)$, $s=(2,1)$, $a(s)=3$, $l(s)=1$, $h(s)=5$.

Define

$$n(\lambda)=\sum_{s\in\lambda}l(s)=\sum_{i\geq 1}\binom{\lambda'_i}{2}=\sum_{i\geq 1}(i-1)\lambda_i, \tag{9}$$

$$\langle\lambda,\lambda\rangle=\sum_{i\geq 1}(\lambda'_i)^2=2n(\lambda)+|\lambda|\,. \tag{10}$$

Define

$$N_\lambda(u,q,t)=\prod_{s\in\lambda}(q^{a(s)}-ut^{l(s)+1})(q^{a(s)+1}-u^{-1}t^{l(s)}). \tag{11}$$

One can show that

$$N_\lambda(u,q,t)=N_{\lambda'}(u,t,q). \tag{12}$$

2.2. λ-rings and symmetric functions

For simplicity we will introduce only λ-rings without $\mathbb{Z}$-torsion. To make things even simpler we can assume that our rings are algebras over $\mathbb{Q}$. Then the axioms of a λ-ring can be formulated just in terms of Adams operations.

We define the (graded) ring of symmetric polynomials

$$\Lambda_n=\mathbb{Z}[x_1,\dots,x_n]^{S_n},$$

where $\deg x_i=1$. We define the (graded) ring of symmetric functions $\Lambda=\varprojlim\Lambda_n$, where the limit is taken in the category of graded rings. For any commutative ring A, we define $\Lambda_A=\Lambda\otimes_{\mathbb{Z}}A$. As in [13], we define generators of Λ (complete symmetric and elementary symmetric functions)

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \dots x_{i_n}, \qquad e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n},$$

and generators of $\Lambda_{\mathbb{Q}}$ (power sums)

$$p_n = \sum_i x_i^n.$$

The elements h_n, e_n, p_n have degree n. We also define $h_0 = e_0 = p_0 = 1$ for convenience. For any partition λ of length $\le n$, define monomial symmetric polynomials $m_\lambda = \sum x^\alpha \in \Lambda_n$, where the sum runs over all distinct permutations $\alpha = (\alpha_1, \dots, \alpha_n)$ of $(\lambda_1, \dots, \lambda_n)$. They induce monomial symmetric functions $m_\lambda \in \Lambda$ which form a $\mathbb{Z}$-basis of Λ.

We define a λ-ring R to be a commutative ring (without $\mathbb{Z}$-torsion) equipped with a map

$$\Lambda \times R \to R, \qquad (f, a) \mapsto f[a],$$

called a plethysm, such that, with $\psi_n = p_n[-] : R \to R$, called Adams operations, we have

(1) The map $\Lambda \to R$, $f \mapsto f[a]$, is a ring homomorphism, for all $a \in R$.

(2) $\psi_1 : R \to R$ is the identity map.

(3) The map $\psi_n : R \to R$ is a ring homomorphism, for all $n \ge 1$.

(4) $\psi_m \psi_n = \psi_{mn}$, for all $m, n \ge 1$.

The first axiom implies that it is enough to specify just Adams operations $\psi_n = p_n[-]$ or σ-operations $\sigma_n = h_n[-]$ or λ-operations $\lambda_n = e_n[-]$. It also implies that $1[a] = 1$, for all $a \in R$.

We equip algebras of the form $\mathbb{Q}[x_1, \dots, x_k]$, $\mathbb{Q}(x_1, \dots, x_k)$, $\mathbb{Q}[\![x_1, \dots, x_k]\!]$ with the λ-ring structure given by

$$\psi_n(f) = p_n[f] = f(x_1^n, \dots, x_k^n). \tag{13}$$

Similarly, we equip the ring Λ of symmetric functions with the λ-ring structure given by

$$\psi_n(f) = p_n[f] = f(x_1^n, x_2^n, \dots), \qquad f \in \Lambda. \tag{14}$$

Note that $p_m[p_n] = p_{mn}$. If R is a λ-ring, then we have

$$f[g[a]] = (f[g])[a], \qquad f, g \in \Lambda,\ a \in R. \tag{15}$$

Given two λ-rings R and R', we equip the ring $R \otimes_{\mathbb{Z}} R'$ with the λ-ring structure given by

$$\psi_n(a \otimes b) = \psi_n(a) \otimes \psi_n(b), \qquad a \in R,\ b \in R'. \tag{16}$$

The ring Λ can be considered as a free λ-ring with one generator in the following sense. Consider the category Ring_λ of λ-rings (with morphisms that respect plethystic operations). The forgetful functor $F : \mathrm{Ring}_\lambda \to \mathrm{Set}$ has a left adjoint

$$\mathrm{Sym} : \mathrm{Set} \to \mathrm{Ring}_\lambda\,.$$

Given a finite set $\{X_1, \dots, X_n\}$, we denote $\mathrm{Sym}\ \{X_1, \dots, X_n\}$ by $\mathrm{Sym}[X_1, \dots, X_n]$. Then, for a one-point set $\{X\}$, there is a unique isomorphism of λ-rings

$$\mathrm{Sym}[X] \xrightarrow{\sim} \Lambda$$

that maps X to p_1. We will usually identify Λ and $\mathrm{Sym}[X]$ using this isomorphism.

Let us define a filtered λ-ring R to be a λ-ring equipped with a filtration

$$R = F^0R \supset F^1R \supset \dots$$

such that $F^iR \cdot F^jR \subset F^{i+j}R$ and $\psi_n(F^iR) \subset F^{ni}R$. It is called complete if the natural homomorphism $R \to \varprojlim R/F^iR$ is an isomorphism. For example, the ring Λ is graded, hence we have a decomposition $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$ into graded components. We equip Λ with the filtration $F^k\Lambda = \bigoplus_{i \geq k} \Lambda^i$ and define the completion

$$\hat{\Lambda} = \varprojlim \Lambda/F^k\Lambda \simeq \mathbb{Z}[\![h_1, h_2, \dots]\!]. \tag{17}$$

This ring can be considered as a free complete λ-ring with one generator. One can see that if R is a complete λ-ring, then the plethystic pairing extends to

$$\hat{\Lambda} \times F^1R \to R.$$

In particular, the element

$$\text{Exp}[X] = \sum_{n\geq 0} h_n[X] = \exp\left(\sum_{n\geq 1} \frac{p_n[X]}{n}\right) = \prod_{i\geq 1} \frac{1}{1-x_i} \in \hat{\Lambda}, \tag{18}$$

called a plethystic exponential, induces a map $\text{Exp} : F^1R \to 1 + F^1R$ which satisfies

$$\text{Exp}[a+b] = \text{Exp}[a]\,\text{Exp}[b]. \tag{19}$$

This map has an inverse, called a plethystic logarithm,

$$\text{Log} : 1 + F^1R \to F^1R, \qquad \text{Log}[1+a] = \sum_{n\geq 1} \frac{\mu(n)}{n} p_n[\log(1+a)]. \tag{20}$$

2.3. Modified Macdonald polynomials

For an introduction to modified Macdonald polynomials see [8] or [15]. Let $\mathcal{P}_n$ denote the set of partitions λ with $|\lambda| = n$. Define the natural partial order on $\mathcal{P}_n$ by

$$\lambda \leq \mu \iff \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \forall k \geq 1.$$

One can show that $\lambda \leq \mu \iff \mu' \leq \lambda'$ [13, 1.1.11]. Let $\Lambda^{\leq\lambda} \subset \Lambda$ be the subspace spanned by monomial symmetric functions $m_\mu \in \Lambda$ with $\mu \leq \lambda$.

Let us define

$$F = \mathbb{Q}(q,t), \qquad \Lambda_F = \Lambda \otimes_{\mathbb{Z}} F. \tag{21}$$

We equip F and Λ with the λ-ring structures using (13) and (14), and we equip Λ_F with the λ-ring structure using (16). For any symmetric function $f \in \Lambda_F$, we will sometimes denote $f[X]$ by $f[X;q,t]$ to indicate dependence on q, t. Let $P_\lambda[X; q, t] \in \Lambda_F$ be Macdonald polynomials [13, §6]. Define modified Macdonald polynomials $\widetilde{H}_\lambda[X; q, t] \in \Lambda_F$ [8, I.8–I.11]

$$\begin{aligned} \widetilde{H}_\lambda[X; q, t] &= H_\lambda\left[X; q, t^{-1}\right] \cdot t^{n(\lambda)}, \\ H_\lambda[X] &= P_\lambda\left[\frac{X}{1-t}\right] \cdot \prod_{s\in\lambda}(1 - q^{a(s)} t^{l(s)+1}). \end{aligned} \tag{22}$$

Alternatively, one can uniquely determine $\widetilde{H}_\lambda[X; q, t] \in \Lambda_F$ by the properties

(1) $\widetilde{H}_\lambda[(1-t)X] \in \Lambda_F^{\leq\lambda}$.

(2) Cauchy identity:

$$\sum_\lambda \frac{\widetilde{H}_\lambda[X]\widetilde{H}_\lambda[Y]}{\prod_{s\in\lambda}(q^{a(s)} - t^{l(s)+1})(q^{a(s)+1} - t^{l(s)})} = \operatorname{Exp}\left[\frac{XY}{(q-1)(1-t)}\right].$$

We have by [8, Cor. 2.1] (see also [13, 6.6.17])

$$\widetilde{H}_\lambda[1-u; q, t] = \prod_{s\in\lambda}(1 - q^{a'(s)} t^{l'(s)} u), \tag{23}$$

where $a'(s) = j-1$, $l'(s) = i-1$ for $s = (i, j) \in \lambda$. This implies $\widetilde{H}_\lambda[1; q, t] = 1$. The symmetric function $\widetilde{H}_\lambda$ has degree $|\lambda|$, hence, applying it to $z \in F[z]$, we obtain

$$\widetilde{H}_\lambda[z; q, t] = z^{|\lambda|}. \tag{24}$$

Finally, we have by [8, Cor.2.2]

$$\widetilde{H}_\lambda[X; q, t] = \widetilde{H}_{\lambda'}[X; t, q]. \tag{25}$$

2.4. Volume ring

Following [17], we will introduce in this section a λ-ring which is an analogue of the Grothendieck ring of algebraic varieties or the ring of motives. We define it to be the ring $\mathcal{V} = \prod_{n\geq 1}\mathbb{Q}$ with Adams operations

$$\psi_m(a) = (a_{mn})_{n\geq 1}, \qquad a = (a_n)_{n\geq 1} \in \mathcal{V}, \tag{26}$$

and call it the volume ring or the ring of counting sequences [17].

Given an algebraic variety X over a finite field $\mathbb{F}_q$, we define its volume

$$[X] = (\#X(\mathbb{F}_{q^n}))_{n\geq 1} \in \mathcal{V}. \tag{27}$$

More generally, given a finite type algebraic stack $\mathcal{X}$ over $\mathbb{F}_q$, we define its volume

$$[\mathcal{X}] = (\#\mathcal{X}(\mathbb{F}_{q^n}))_{n\geq 1} \in \mathcal{V}, \tag{28}$$

where, for the finite groupoid $\mathcal{G} = \mathcal{X}(\mathbb{F}_{q^n})$, we define

$$\#\mathcal{G} = \sum_{x\in\mathcal{G}/\sim} \frac{1}{\#\operatorname{Aut}(x)}. \tag{29}$$

Let us fix a projective genus g curve X over the field $\mathbb{F}_q$ and consider its zeta function

$$Z_X(t) = \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right) = \frac{\prod_{i=1}^g (1-\alpha_i t)(1-\alpha_i^{-1} qt)}{(1-t)(1-qt)}, \tag{30}$$

$$\#X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^g \alpha_i^n - q^n \sum_{i=1}^g \alpha_i^{-n} \qquad \forall n \geq 1. \tag{31}$$

Note that the zeta function $Z_X(t)$ (and the elements $\alpha_i \in \mathbb{C}$) is uniquely determined by the volume $[X]$ and, conversely, the volume $[X]$ is uniquely determined by the zeta function.

Let us consider the algebra

$$R_g = \mathbb{Q}[\boldsymbol{q}^{\pm 1}, \boldsymbol{\alpha}_1^{\pm 1}, \dots \boldsymbol{\alpha}_g^{\pm 1}, (\boldsymbol{q}^n - 1)^{-1} : n \geq 1], \tag{32}$$

equipped with the usual λ-ring structure

$$\psi_n(f) = f(\boldsymbol{q}^n, \boldsymbol{\alpha}_1^n, \dots, \boldsymbol{\alpha}_g^n) \qquad \forall f \in R_g.$$

We consider the λ-ring homomorphism

$$\sigma : R_g \to \mathcal{V}_{\mathbb{C}} = \prod_{n\geq 1} \mathbb{C}, \qquad \boldsymbol{q} \mapsto (q^n)_{n\geq 1},\ \boldsymbol{\alpha}_i \mapsto (\alpha_i^n)_{n\geq 1}. \tag{33}$$

It restricts to

$$\sigma : R_g^{S_g \ltimes S_2^g} \to \mathcal{V}, \tag{34}$$

where S_g permutes variables $\boldsymbol{\alpha}_i$ and the i-th copy of S_2 permutes $\boldsymbol{\alpha}_i$ and $\boldsymbol{q}\boldsymbol{\alpha}_i^{-1}$.

Note that $\sigma(\boldsymbol{q}) = [\mathbb{A}^1]$, the volume of the affine line. Applying (31), we obtain

$$\sigma\left(1 + \boldsymbol{q} - \sum_{i=1}^{g} \boldsymbol{\alpha}_i - \boldsymbol{q}\sum_{i=1}^{g} \boldsymbol{\alpha}_i^{-1}\right) = [X]. \tag{35}$$

In this paper we will express volumes of stacks as images under σ of some elements in R_g. For simplicity, we will write $[\mathcal{X}] = f$, whenever $[\mathcal{X}] \in \mathcal{V}$ and $f \in R_g$ satisfy $[\mathcal{X}] = \sigma(f)$. Also, we will write q and α_i instead of $\boldsymbol{q}$ and $\boldsymbol{\alpha}_i$ respectively, hoping it will not lead to any confusion.

3. Positive Higgs bundles

In this section we will review the formula from [20] counting positive Higgs bundles. Then we will simplify it using an approach from [16]. Let X be a smooth projective curve of genus g over a field $\Bbbk$ and let L be a line bundle of degree ℓ over X. Given a coherent sheaf $E \in \operatorname{Coh} X$, we define its slope $\mu(E) = \deg E / \operatorname{rk} E$ and we call E semistable if $\mu(F) \le \mu(E)$ for all $F \subset E$.

Remark 3.1. We call E stable if $\mu(F) < \mu(E)$ for all proper $F \subset E$. In this case $K = \operatorname{End}(E)$ is a finite-dimensional division algebra over $\Bbbk$ by Schur's lemma. In particular, $K = \Bbbk$ if $\Bbbk$ is algebraically closed. If $\operatorname{rk} E$ and $\deg E$ are coprime and E is semistable, then E is automatically stable. Let us show that if rank and degree are coprime and the field $\Bbbk$ is finite, then $K = \Bbbk$. First of all, K is a finite (Galois) field extension of $\Bbbk$ by Wedderburn's little theorem. We can decompose $E_K = E \otimes_{\Bbbk} K$ over $X_K = X \times_{\operatorname{Spec}\Bbbk} \operatorname{Spec} K$ as a direct sum $\bigoplus_{\sigma \in \operatorname{Gal}(K/\Bbbk)} F^\sigma$, where F^σ have the same rank and degree [17]. If $[K : \Bbbk] > 1$, this would imply that $\operatorname{rk} E$ and $\deg E$ are not coprime, a contradiction.

Every coherent sheaf $E \in \operatorname{Coh} X$ has a unique filtration, called a Harder-Narasimhan filtration,

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that E_i/E_{i-1} are semistable and $\mu(E_1/E_0) > \cdots > \mu(E_n/E_{n-1})$. We will say that E is positive if $\mu(E_n/E_{n-1}) \ge 0$. Equivalenly, for any semistable sheaf F with $\mu(F) < 0$, we have $\operatorname{Hom}(E, F) = 0$.

Recall that an L-twisted Higgs sheaf is a pair (E, ϕ), where E is a coherent sheaf over X and $\phi : E \to E \otimes L$ is a homomorphism. We will say that (E, ϕ) is positive if E is positive. Let $\mathrm{Higgs}_L(X)$ be the category of L-twisted Higgs sheaves and $\mathrm{Higgs}_L^+(X)$ be the category of positive L-twisted Higgs sheaves. We will say that $(E, \phi) \in \mathrm{Higgs}_L(X)$ is semistable if $\mu(F) \le \mu(E)$ for every $(F, \phi') \subset (E, \phi)$.

Let $\mathfrak{M}_L(r, d)$ denote the stack of all Higgs bundles having rank r and degree d, $\mathfrak{M}_L^{\mathrm{ss}}(r, d) \subset \mathfrak{M}_L(r, d)$ denote the stack of semistable Higgs bundles and $\mathfrak{M}_L^+(r, d) \subset \mathfrak{M}_L(r, d)$ denote the stack of positive Higgs bundles (not necessarily semistable). Assuming that $\Bbbk$ is a finite field $\mathbb{F}_q$, we define (exponential) DT invariants

$$\hat{\Omega}_{r,d} = (-q^{\frac{1}{2}})^{-\ell r^2} [\mathfrak{M}_L^{\mathrm{ss}}(r, d)] \tag{36}$$

and define (integral) DT invariants by the formula

$$\sum_{d/r=\tau} \Omega_{r,d} T^r z^d = (q-1) \operatorname{Log} \left(\sum_{d/r=\tau} \hat{\Omega}_{r,d} T^r z^d \right), \qquad \tau \in \mathbb{Q}, \tag{37}$$

Ideally, one would like to define DT invariants by taking the plethystic logarithm of the series that counts volumes of the stacks $\mathfrak{M}_L(r, d)$ (of all Higgs bundles) of arbitrary slope, instead of the above formula, where the stacks $\mathfrak{M}_L^{\mathrm{ss}}(r, d)$ of semistable Higgs bundles of a fixed slope are considered. The problem with this approach is that the stacks $\mathfrak{M}_L(r, d)$ have infinite volume in general. To resolve this issue, it was suggested in [20] to use the stacks $\mathfrak{M}_L^+(r, d)$ of positive Higgs bundles as an approximation of the stacks $\mathfrak{M}_L(r, d)$. Let us consider the series

$$\hat{\Omega}^+(T, q, z) = \sum_{r,d} (-q^{\frac{1}{2}})^{-\ell r^2} [\mathfrak{M}_L^+(r, d)] T^r z^d \tag{38}$$

and define positive (integral) DT invariants by the formula

$$\sum_{r,d} \Omega_{r,d}^+ T^r z^d = (q-1) \operatorname{Log} \hat{\Omega}^+(T, q, z). \tag{39}$$

The following result was proved in [20]:

Theorem 3.2. *For every $r \ge 1$, we have*

(1) $\hat{\Omega}_{r,d+r} = \hat{\Omega}_{r,d}$.

(2) $\Omega_{r,d+r} = \Omega_{r,d}$.

(3) $\Omega_{r,d} = \Omega^+_{r,d}$ *for* $d \gg 0$.

The last result implies that it is enough to find the positive DT invariants $\Omega^+_{r,d}$ in order to determine the usual DT invariants $\Omega_{r,d}$. The following explicit formula for the series $\hat{\Omega}^+(T, q, z)$ was proved in [20].

Theorem 3.3. *Assuming that* $p = \ell - (2g - 2) > 0$, *we have*

$$\hat{\Omega}^+(T, q, z) = \sum_\lambda (-q^{\frac{1}{2}})^{\ell\langle\lambda,\lambda\rangle} z^{pn(\lambda')} J_\lambda(q, z) H_\lambda(q, z) T^{|\lambda|},$$

where the sum runs over all partitions λ *and* $J_\lambda(q, z), H_\lambda(q, z)$ *are certain expressions (independent of* ℓ*) defined in [20].*

The following simplification of the above expression was obtained in [16, Prop. 3.1].

Proposition 3.4. *Given a partition* λ *of length* n, *let us define*

$$(40)\quad f(z_1, \dots, z_n; q, \bar{\alpha}) = \prod_{i=1}^n \prod_{k=1}^g \frac{1 - \alpha_k^{-1}}{1 - \alpha_k^{-1} z_i}$$
$$\times \sum_{\sigma \in S_n} \sigma \left(\prod_{i>j} \left(\frac{1}{1 - z_i/z_j} \prod_{k=1}^g \frac{1 - \alpha_k^{-1} z_i/z_j}{1 - q\alpha_k^{-1} z_i/z_j} \right) \prod_{i>j+1} (1 - qz_i/z_j) \prod_{i \geq 2} (1 - z_i) \right),$$

$$(41)\qquad f_\lambda(q, z) = f(z_1, \dots, z_n; q, \bar{\alpha}), \qquad z_i = q^{i-n} z^{\lambda_i},\ i = 1, \dots, n,$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_g)$. *Then (see* (11) *for the definition of* N_λ*)*

$$(42)\qquad q^{(g-1)\langle\lambda,\lambda\rangle} J_\lambda(q, z) H_\lambda(q, z) = \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1}, z, q)}{N_\lambda(1, z, q)} f_\lambda(q, z).$$

The last two results imply

Corollary 3.5. *Assuming that* $p = \ell - (2g - 2) > 0$, *we have*

$$(43)\quad \hat{\Omega}^+(q^{-p/2} T, q, z)$$
$$= \sum_\lambda \left((-1)^{|\lambda|} q^{n(\lambda')} z^{n(\lambda)} \right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1}, q, z)}{N_\lambda(1, q, z)} f_{\lambda'}(q, z) \cdot T^{|\lambda|}.$$

Proof. Using the fact that $\langle\lambda,\lambda\rangle = 2n(\lambda)+|\lambda|$ (see (10)), we obtain

$$\hat{\Omega}^+(T,q,z) = \sum_\lambda \left((-1)^{|\lambda|}q^{n(\lambda)}z^{n(\lambda')}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},z,q)}{N_\lambda(1,z,q)} f_\lambda(q,z)\cdot(q^{p/2}T)^{|\lambda|}.$$

Now we sum up over conjugate partitions and apply (12). □

Lemma 3.6. *We have*

$$f\in\mathbb{Q}[z_1^{\pm1},\dots,z_n^{\pm1};q^{\pm1}][\![\alpha_1^{-1},\dots,\alpha_g^{-1}]\!].$$

Proof. The factors $(1-z_i/z_j)$ disappear from the denominator of f when we sum over S_n, so looking at the remaining factors we see that

$$f(z_1,\dots,z_n)\cdot\prod_{k=1}^g\left(\prod_{i=1}^n\left(1-\alpha_k^{-1}z_i\right)\prod_{i\neq j}\left(1-q\alpha_k^{-1}z_i/z_j\right)\right)$$

is a Laurent polynomial. The result follows on observing that every factor in the brackets is invertible in $\mathbb{Q}[z_1^{\pm1},\dots,z_n^{\pm1};q^{\pm1}][\![\alpha_1^{-1},\dots,\alpha_g^{-1}]\!]$. □

Proposition 3.7 (see [16, §4.2]). *We have*

$$f(1,z_1,\dots,z_n)=f(qz_1,\dots,qz_n).$$

4. Main result

4.1. Admissibility

Let R be a λ-ring flat over $\mathbb{Q}(q)[t^{\pm1}]$ and let $R^*=R\otimes_{\mathbb{Q}(q)[t^{\pm1}]}\mathbb{Q}(q,t)$. We will say that $F\in R^*$ is admissible if $(1-t)\operatorname{Log}F$ is contained in R (usually R will be clear from the context). In view of Proposition 3.7, we introduce the following concept.

Definition 4.1. Let us consider the rings

$$\bar{\Lambda}_n = R[z_1^{\pm1},\dots,z_n^{\pm1}]^{S_n},\qquad n\ge 0,$$

and the ring homomorphisms

$$\pi_n:\bar{\Lambda}_{n+1}\to\bar{\Lambda}_n,\qquad (\pi_n f)(z_1,\dots,z_n)=f(1,q^{-1}z_1,\dots,q^{-1}z_n).$$

We define a q-twisted symmetric function $f = (f_n)_{n\ge 0}$ to be an element of $\bar\Lambda = \varprojlim \bar\Lambda_n$.

Given a q-twisted symmetric function f, we define, for any partition λ (cf. (41)),

$$f_\lambda(q,t) = f_n(z_1,\dots,z_n), \qquad z_i = q^{i-n}t^{\lambda_i},\ n \ge l(\lambda). \tag{44}$$

Note that this expression is independent of the choice of $n \ge l(\lambda)$.

Remark 4.2. The following result is a reformulation of [16, Lemma 5.1]. Here we exchange the roles of q, t and use conjugate partitions. We also add an invertible factor $(q-1)$.

Theorem 4.3. *Let $f(u) = \sum_{i\ge 0} f^{(i)}u^i \in \bar\Lambda[\![u]\!]$ be a power series with $f^{(0)} = 1$ and let*

$$\hat\Omega[X;u] = \sum_\lambda c_\lambda \widetilde{H}_\lambda[X;q,t] f_{\lambda'}(u,q,t),$$
$$\Omega[X;u] = (q-1)(1-t)\operatorname{Log}\hat\Omega[X;u],$$

where $c_\lambda \in R^*$ and $c_\varnothing = 1$. If $\hat\Omega[X;0]$ is admissible, then $\Omega[X;u] - \Omega[X;0]$ has coefficients in $(t-1)R$. In particular, $\Omega[X;u]$ is independent of u at $t=1$.

4.2. Proof of Theorem 1.1

In this section we will use the variable t in place of z as it is customary in the theory of orthogonal symmetric polynomials.

Theorem 4.4 (cf. Theorem 1.1). *Assume that $p = \ell - (2g-2) > 0$. Define (see (11) for the definition of N_λ)*

$$\hat\Omega^\circ(T,q,t) = \sum_\lambda \left((-1)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)} T^{|\lambda|}, \tag{45}$$

$$\Omega^\circ(T,q,t) = \sum_{r\ge 1}\Omega_r^\circ(q,t)T^r = (q-1)(1-t)\operatorname{Log}\hat\Omega^\circ(T,q,t). \tag{46}$$

Then $\Omega_r^\circ(q,t) \in \mathbb{Z}[q,t,\alpha_1^{\pm 1},\dots,\alpha_g^{\pm 1}]$ and

$$\Omega_{r,d} = q^{pr/2}\Omega_r^\circ(q,1) \qquad \forall d \in \mathbb{Z}.$$

Proof. According to Theorem 3.2 it is enough to show that $\Omega^+_{r,d} = q^{pr/2}\Omega^\circ_r(q,1)$ for $d \gg 0$, where $\Omega^+_{r,d}$ are determined by (39) and Corollary 3.5:

$$\begin{aligned}(47)\quad & \hat\Omega^+(q^{-p/2}T,q,t)\\ &= \sum_\lambda \left((-1)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)} f_{\lambda'}(q,t)\cdot T^{|\lambda|},\end{aligned}$$

$$(48)\quad \Omega^+(T,q,t) = \sum_r \Omega^+_r(q,t)T^r = \sum_{r,d}\Omega^+_{r,d}T^r t^d = (q-1)\operatorname{Log}\hat\Omega^+(T,q,t).$$

We will compare the series $\hat\Omega^+(q^{-p/2}T,q,t)$ and the series $\hat\Omega^\circ(T,q,t)$ using Theorem 4.3. Consider the ring of Laurent series

$$(49)\qquad R = \mathbb{Q}(q)[t^{\pm1}](\!(\alpha_1^{-1},\dots,\alpha_g^{-1})\!)$$

and the series $\tilde f(u) = \sum_{i\ge0}\tilde f^{(i)}u^i$ which is a deformation of f (40) defined by

$$\tilde f^{(i)} = (\tilde f^{(i)}_n)_{n\ge0},\qquad \tilde f_n(z_1,\dots,z_n;u) = \sum_{i\ge0}\tilde f^{(i)}_n u^i = f(z_1,\dots,z_n;q,u^{-1}\bar\alpha),$$

where every α_i is substituted by $u^{-1}\alpha_i$. It follows from Lemma 3.6 that

$$\tilde f_n \in \mathbb{Q}[q^{\pm1},\alpha_1^{\pm1},\dots,\alpha_g^{\pm1}][z_1^{\pm1},\dots,z_n^{\pm1}]^{S_n}[\![u]\!],$$

hence by Proposition 3.7 the coefficients $\tilde f^{(i)}$ are q-twisted symmetric functions over R. It follows from [16, Theorem 5.2] that $\tilde f_n|_{u=0}=1$, hence $\tilde f(0)=1$.

As before, we define

$$\tilde f_\lambda(u,q,t) = \tilde f_n(z_1,\dots,z_n;u),\qquad z_i = q^{i-n}t^{\lambda_i},\ n\ge l(\lambda),$$

and consider the series of symmetric functions

$$\begin{aligned}(50)\quad & \hat\Omega[X;u,q,t]\\ &= \sum_\lambda \left((-1)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)} \widetilde H_\lambda[X;q,t]\tilde f_{\lambda'}(u,q,t),\end{aligned}$$

$$\Omega[X;u,q,t] = (q-1)(1-t)\operatorname{Log}\hat{\Omega}[X;u,q,t]. \tag{51}$$

Then (45) and (47) translate to

$$\hat{\Omega}[T;0,q,t] = \hat{\Omega}^\circ(T,q,t), \qquad \Omega[T;0,q,t] = \Omega^\circ(T,q,t),$$
$$\hat{\Omega}[T;1,q,t] = \hat{\Omega}^+(q^{-p/2}T,q,t), \quad \Omega[T;1,q,t] = (1-t)\Omega^+(q^{-p/2}T,q,t).$$

In order to apply Theorem 4.3 we need to show that

$$\hat{\Omega}[X;0,q,t] = \sum_\lambda \left((-1)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)}\widetilde{H}_\lambda[X;q,t]$$

is admissible. The series

$$\sum_\lambda \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)}\widetilde{H}_\lambda[X;q,t]$$

is admissible according to [14]. The operator ∇ defined by

$$\widetilde{H}_\lambda \mapsto (-1)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}\widetilde{H}_\lambda$$

preserves admissibility by [14, Cor. 6.3]. Therefore the series $\hat{\Omega}[X;0,q,t]$ is also admissible (one actually obtains from [14] that the coefficients of $\Omega[X;0,q,t]$ are in $\mathbb{Z}[q,t,\alpha_1^{\pm1},\dots,\alpha_g^{\pm1}]$, hence the same is true for $\Omega^\circ(T,q,t)$).

We conclude from Theorem 4.3 that

$$\Omega[T;u,q,t] - \Omega[T;0,q,t] \in (1-t)R[\![T,u]\!]. \tag{52}$$

By Lemma 3.6 we can consider $\hat{\Omega}[T;u,q,t]$ (50) as a series with polynomial coefficients in u

$$\hat{\Omega}[T;u,q,t] \in \mathbb{Q}(q,t)[u](\!(\alpha_1^{-1},\dots,\alpha_g^{-1})\!)[\![T]\!].$$

The same then applies to $\Omega[T;u,q,t]$ and we can set $u=1$ in (52). We obtain

$$(1-t)\Omega^+(q^{-p/2}T,q,t) - \Omega^\circ(T,q,t) \in (1-t)R[\![T]\!].$$

This implies that $(1-t)q^{-pr/2}\Omega_r^+(q,t) - \Omega_r^\circ(q,t) = (1-t)h$ for some $h\in R$. Therefore

$$q^{-pr/2}\sum_{d\ge0}\Omega_{r,d}^+t^d = \frac{\Omega_r^\circ(q,t)}{1-t} + h.$$

Comparing the coefficients of the monomials in $\alpha_1, \ldots, \alpha_g$ and using the fact that $\Omega^+_{r,d+r} = \Omega^+_{r,d}$ for $d \gg 0$, we conclude that $q^{-pr/2}\Omega^+_{r,d} = \Omega^\circ_r(q,1)$ for $d \gg 0$. □

Remark 4.5. Let us also formulate the result in the case $L = \omega_X$ (the canonical bundle) for completeness [16]. In this case we have $\ell = 2g-2$ and $p = \ell - (2g-2) = 0$. Define as before

$$\hat{\Omega}^\circ(T,q,t) = \sum_\lambda \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)} T^{|\lambda|} \tag{53}$$

$$\Omega^\circ(T,q,t) = \sum_{r\ge1} \Omega^\circ_r(q,t)T^r = (q-1)(1-t)\operatorname{Log}\hat{\Omega}^\circ(T,q,t). \tag{54}$$

Using results of [20] and the same proof as before, we obtain the formula for integral Donaldson-Thomas invariants $\Omega_{r,d} = q\Omega^\circ_r(q,1)$ (note the additional factor q). These invariants are related to the invariants $A_{r,d}$ counting absolutely indecomposable vector bundles of rank r and degree d over X: $\Omega_{r,d} = qA_{r,d}$ [20]. This implies that $A_{r,d} = \Omega^\circ_r(q,1)$, as was proved by Mellit in [16].

4.3. Alternative formulation

The following result was conjectured in [18, Conj. 3].

Theorem 4.6. *Assume that $p = \ell - (2g-2) > 0$. Consider the series*

$$\mathcal{H}(T,q,t) = \sum_\lambda T^{|\lambda|} \prod_{s\in\lambda} (-t^{a(s)-l(s)}q^{a(s)})^p t^{(1-g)(2l(s)+1)} Z_X(t^{h(s)}q^{a(s)}),$$

$$\mathbf{H}^\circ(T,q,t) = \sum_{r\ge1} \mathbf{H}^\circ_r(q,t)T^r = (1-t)(1-qt)\operatorname{Log}\mathcal{H}(T,q,t).$$

Then $\mathbf{H}^\circ_r(q,t) \in \mathbb{Z}[q, t^{\pm1}, \alpha_1^{\pm1}, \ldots, \alpha_g^{\pm1}]$ and $\Omega_{r,d} = q^{pr/2}\mathbf{H}^\circ_r(q,1)$.

Proof. Using the substitution $t \mapsto t^{-1}$, we obtain

$$\begin{aligned}\mathcal{H}(T,q,t^{-1}) &= \sum_\lambda T^{|\lambda|} \prod_{s\in\lambda} (-t^{l-a}q^a)^p t^{(g-1)(2l+1)} Z_X(t^{-h}q^a) \\ &= \sum_\lambda T^{|\lambda|} \prod_{s\in\lambda} (-t^{l-a}q^a)^p \frac{\prod_{i=1}^g (t^{l+1} - \alpha_i t^{-a}q^a)(t^l - \alpha_i^{-1}t^{-a-1}q^{a+1})}{(t^{l+1} - t^{-a}q^a)(t^l - t^{-a-1}q^{a+1})},\end{aligned}$$

while

$$t\mathbf{H}^{\circ}(T,q,t^{-1}) = (1-t)(t^{-1}q-1)\operatorname{Log}\mathcal{H}(T,q,t^{-1}).$$

Using the substitution $q \mapsto qt$, we obtain

$$\begin{aligned}\mathcal{H}(T,qt,t^{-1}) &= \sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} (-t^l q^a)^p \frac{\prod_{i=1}^g (t^{l+1}-\alpha_i q^a)(t^l - \alpha_i^{-1} q^{a+1})}{(t^{l+1}-q^a)(t^l - q^{a+1})} \\ &= \sum_{\lambda} T^{|\lambda|} \left((-1)^{|\lambda|} q^{n(\lambda')} t^{n(\lambda)}\right)^p \frac{\prod_{i=1}^g N_\lambda(\alpha_i^{-1},q,t)}{N_\lambda(1,q,t)}.\end{aligned}$$

Now the result follows from Theorem 4.4. □

References

[1] P.-H. Chaudouard, *Sur le comptage des fibrés de Hitchin*, Astérisque (2015), no. 369, 223–284.

[2] P.-H. Chaudouard and G. Laumon, *Sur le comptage des fibrés de Hitchin nilpotents*, J. Inst. Math. Jussieu **15** (2016), no. 1, 91–164.

[3] W.-y. Chuang, D.-E. Diaconescu, and G. Pan, *Wallcrossing and cohomology of the moduli space of Hitchin pairs*, Commun. Number Theory Phys. **5** (2011), no. 1, 1–56.

[4] G. Dobrovolska, V. Ginzburg, and R. Travkin, *Moduli spaces, indecomposable objects and potentials over a finite field* (2016).

[5] R. Fedorov, A. Soibelman, and Y. Soibelman, *Motivic classes of moduli of Higgs bundles and moduli of bundles with connections*, Commun. Number Theory Phys. **12** (2018), no. 4, 687–766.

[6] O. García-Prada and J. Heinloth, *The y-genus of the moduli space of* PGL_n*-Higgs bundles on a curve (for degree coprime to n)*, Duke Math. J. **162** (2013), no. 14, 2731–2749.

[7] O. García-Prada, J. Heinloth, and A. Schmitt, *On the motives of moduli of chains and Higgs bundles*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 12, 2617–2668.

[8] A. M. Garsia and M. Haiman, *A remarkable* q,t*-Catalan sequence and* q*-Lagrange inversion*, J. Algebraic Combin. **5** (1996), no. 3, 191–244.

[9] P. B. Gothen, *The Betti numbers of the moduli space of stable rank* 3 *Higgs bundles on a Riemann surface*, Internat. J. Math. **5** (1994), no. 6, 861–875.

[10] T. Hausel, *Mirror symmetry and Langlands duality in the non-abelian Hodge theory of a curve*, in Geometric methods in algebra and number theory, Vol. 235 of *Progr. Math.*, 193–217, Birkhäuser Boston, Boston, MA (2005).

[11] T. Hausel and F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties*, Invent. Math. **174** (2008), no. 3, 555–624. With an appendix by Nicholas M. Katz.

[12] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126.

[13] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, Oxford University Press, second edition (1995), `ISBN 0-19-853489-2`. With contributions by A. Zelevinsky.

[14] A. Mellit, *Integrality of Hausel-Letellier-Villegas kernels*, Duke Math. J. **167** (2018), no. 17, 3171–3205.

[15] ———, *Poincaré polynomials of character varieties, Macdonald polynomials and affine Springer fibers*, Ann. of Math. (2) **192** (2020), no. 1, 165–228.

[16] ———, *Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures)*, Invent. Math. **221** (2020), no. 1, 301–327.

[17] S. Mozgovoy, *Poincaré polynomials of moduli spaces of stable bundles over curves*, Manuscripta Math. **131** (2010), no. 1-2, 63–86.

[18] ———, *Solutions of the motivic ADHM recursion formula*, Int. Math. Res. Not. (2012), no. 18, 4218–4244.

[19] ———, *Higgs bundles over* $\mathbb{P}^1$ *and quiver representations* (2016).

[20] S. Mozgovoy and O. Schiffmann, *Counting Higgs bundles and type A quiver bundles*, Compos. Math. **156** (2020), no. 4, 744–769.

[21] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447.

[22] S. Rayan, *Co-Higgs bundles on* $\mathbb{P}^1$, New York J. Math. **19** (2013) 925–945.

[23] O. Schiffmann, *Indecomposable vector bundles and stable Higgs bundles over smooth projective curves*, Ann. of Math. (2) **183** (2016), no. 1, 297–362.

School of Mathematics, Trinity College Dublin
College Green, Dublin 2, Ireland
E-mail address: mozgovoy@maths.tcd.ie

Deptartment of Mathematics, University of California, Berkeley
CA 94720-3840, USA
E-mail address: ronan_ogorman@berkeley.edu

Received August 3, 2020
Accepted December 21, 2021