Math. Res. Lett.
Volume 29, Number 5, 1571–1594, 2022

Convex hull property for ancient harmonic map heat flows

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For an ancient solution u to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold N with curvature bounded between two negative constants, we show that the image of u is contained in the convex hull of its intersection with the ideal boundary of N together with at most k interior points in N, where k is the dimension of the space of bounded ancient solutions to the heat equation on M. In the case M has nonnegative Ricci curvature and u is of polynomial growth, its image is contained in an ideal polyhedron with estimable number of vertices in terms of the growth order.

1. Introduction

Various geometric flows such as the Ricci flow [11] and the mean curvature flow [13] have been introduced and extensively studied ever since the pioneering work of Eells-Sampson [8] on harmonic map heat flow. It is well known by now that singularity analysis of geometric flows naturally leads to ancient solutions to the flows, that is, solutions which exist for all negative time. In the case of harmonic map heat flow, an ancient solution is simply a smooth map $u(x,t) : M \times (-\infty, 0) \to N$ satisfying

$$\tau(u)(x,t) = \frac{\partial u}{\partial t}(x,t),$$

where (M^m, g) and (N^n, h) are smooth Riemannian manifolds of the indicated dimensions, and $\tau(u)$ is the tension field of map u.

Partially supported by a grant from NSCT of Taiwan.

Recall that $\tau(w) = 0$ is the Euler-Lagrange equation for critical points of the energy functional

$$E(v) = \int_M e(v) d\mu_g$$

with $e(v)$ being the energy density or the trace of the pull-back symmetric quadratic form $v^*(h)$ with respect to metric g, where $d\mu_g$ is the Riemannian volume form of M. Such map w is called a harmonic map. In terms of local coordinates $x^1, \cdots, x^m$ on M and $y^1, \cdots, y^n$ on N, the metrics g and h are given as

$$g = \sum_{i,j=1}^{m} g_{ij}(x) dx^i dx^j, \qquad h = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}(y) dy^\alpha dy^\beta$$

and the map v as $v(x) = (v^1, \cdots, v^n)$. Then the energy density $e(v)$ of v is given by

$$e(v)(x) = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij}(x) \frac{\partial v^\alpha}{\partial x^i}(x) \frac{\partial v^\beta}{\partial x^j}(x) h_{\alpha\beta}(v(x))$$

and the tension field of map v by

$$\tau^\beta(v) = \Delta v^\beta + \sum_{i,j,\gamma,\delta} g^{ij}(x) \Gamma^\beta_{\gamma\delta}(v(x)) \frac{\partial v^\gamma}{\partial x^i} \frac{\partial v^\delta}{\partial x^j},$$

were Δ is the Laplace-Beltrami operator on M and $\Gamma^\beta_{\gamma\delta}$ are the Christoffel symbols of N.

Historically, in the case that M and N are compact, and N is non-positively curved, Eells-Sampson [8] have shown that there is a smooth harmonic map $w : M \to N$ homotopic to any given smooth map $u_0 : M \to N$. This is proven by solving the following harmonic map heat flow

$$\frac{\partial u}{\partial t} = \tau(u)$$

with initial data $u(0) = u_0$. Their result says that the solution exists for all time $t > 0$ and converges to a smooth harmonic map w. In solving the so-called Dirichlet problem at infinity for harmonic maps between hyperbolic spaces, Li-Tam [18] have established a general existence result concerning the harmonic map heat flow for the case when M and N are complete. Their result was further extended by Wang [26] to more general initial data.

Suffice to say, in the case when N is a Cartan-Hadamard manifold, solution to the harmonic map heat flow exists for all positive time for any initial map satisfying some mild growth assumptions.

Our interest here is to analyze ancient solutions to the harmonic map heat flow. First, let us introduce some definitions. For a Cartan-Hadamard manifold N, we denote by $\overline{N}$ its geometric compactification [24] and $\partial_\infty(N) = \overline{N} \setminus N$ its ideal boundary. Cartan-Hadamard manifold N is said to have strongly negative curvature if its sectional curvature satisfies $-b \leq K_N \leq -a$ for some positive constants a and b.

Definition 1.1. *A subset C of $\overline{N}$ is a* convex set *if for every pair of points in C, any geodesic segment joining them is also in C.*

Definition 1.2. *For a subset A in $\overline{N}$, its* convex hull $\mathcal{C}(A)$ *is defined to be the smallest convex subset of $\overline{N}$ containing A.*

The following result may be viewed as a parabolic version of the result in [19] for harmonic maps. We denote by $P_0(M)$ the space of bounded ancient solutions f to the heat equation on M, namely, $\left(\Delta - \frac{\partial}{\partial t}\right) f = 0$ on $M \times (-\infty, 0)$ and f is bounded.

Theorem 1.3. *Suppose that the space $P_0(M)$ is finite dimensional. Then for harmonic map heat flow $u : M \times (-\infty, 0) \to N$ from a complete manifold M into a Cartan-Hadamard manifold N with strongly negative curvature, its image must satisfy $\cup_{t<0}\{u(x,t)|x \in M\} \subset \mathcal{C}\left(A \cup \{y_i\}_{i=1}^k\right)$ with $A = \overline{\cup_{t<0}\{u(x,t)|x \in M\}} \cap \partial_\infty(N)$ and $y_i \in N$, $i = 1, \cdots, k$, and $k \leq \dim P_0(M)$.*

Corollary 1.4. *A bounded ancient solution to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold must be constant if* $\dim P_0(M) = 1$.

Corollary 1.5. *An ancient solution to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold with strongly negative curvature must be constant if its image is inside a horoball of N and* $\dim P_+(M) = 1$, *where $P_+(M)$ is the space spanned by the positive ancient solutions to the heat equation on M.*

Clearly, if M satisfies parabolic Harnack inequality, that is, there exists constant $c > 0$ such that $u(x, t_1) \leq c\, u(y, t_2)$ whenever $t_2 - t_1 \geq d^2(x, y)$ for any positive solution u to the heat equation on M, then $\dim P_+(M) = \dim P_0(M) = 1$. More generally, one has $\dim P_0(M) < \infty$ provided that a parabolic mean value inequality holds on M. Note that recent work [7, 21] provides a description of polynomial growth ancient solutions to the heat equation in terms of harmonic functions under suitable assumption on M. In particular, by [7], if the volume of M is of polynomial growth, then $P_0(M) = H_0(M)$, the space of bounded harmonic functions on M. The following result is a nonlinear analogue for ancient solutions to the harmonic map heat flow.

Theorem 1.6. *Let $u : M \times (-\infty, 0) \to N$ be a bounded solution to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold N. If $\dim P_0(M) = \dim H_0(M) < \infty$, then u must be harmonic.*

If we restrict to polynomial growth ancient solutions to the harmonic map heat flows, then Theorem 1.3 can be strengthened considerably.

Theorem 1.7. *Let M be a complete manifold with nonnegative Ricci curvature and N a Cartan-Hadamard manifold with strongly negative curvature. Then any nonconstant, polynomial growth, harmonic map heat flow solution $u : M \times (-\infty, 0) \to N$ must satisfy $\cup_{t<0}\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^k)}$ with*

$$A = \overline{\cup_{t<0}\{u(x,t)|x \in M\}} \cap \partial_\infty(N) = \{a_i\}_{i=1}^k.$$

Moreover, k can be explicitly estimated in terms of m, the dimension of M, and α, the growth order of u.

Here, u is of polynomial growth of order α if for some fixed points $p \in M$ and $q \in N$

$$d_N(u(x,t), q) \le C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^\alpha$$

for all $x \in M$ and $t \in (-\infty, 0)$, where d_M and d_N are the distance on M and N, respectively.

Our proof relies on the following concept of parabolic massive sets and is strongly influenced by the work of [19], where the corresponding results were established for harmonic maps.

Definition 1.8. *A domain $\Omega \subset M \times (-\infty, 0)$ is said to be parabolic α-massive if there exists a nonnegative, nonconstant, function $f(x,t)$ on $M \times (-\infty, 0)$ with $f = 0$ outside Ω such that $\left(\Delta - \frac{\partial}{\partial t}\right) f \ge 0$ and $f(x,t) \le C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^\alpha$ for $x \in M$ and $t < 0$.*

Such function f is called a potential of Ω. By the maximum principle, if

$$\sup_{x\in M} f(x,T) > 0$$

for some T, so is $\sup_{x\in M} f(x,t)$ for $t < T$. In the case $\alpha = 0$, we will simply call Ω parabolic massive. The elliptic analogue of massive sets were first introduced in [9], where it was shown that the dimension of $H_0(M)$, the space of bounded harmonic functions on M, is the same as the maximal number of disjoint massive subsets of M. It turns out that a parabolic analogue holds as well, that is, the dimension of $P_0(M)$ is equal to the maximal number of disjoint parabolic massive subsets in $M \times (-\infty, 0)$.

The concept of α-massive sets were introduced in [19] to study the image of a polynomial growth harmonic map. Our definition of parabolic α-massive sets is a natural extension and seems to suit well for the study of ancient solutions to harmonic map heat flows.

The paper is organized as follows. In section 2, we recall some basic facts concerning convex hulls in Cartan-Hadamard manifolds and show that the maximal number of disjoint parabolic massive sets is exactly the dimension of $P_0(M)$. In section 3, we deal with general ancient solutions to the harmonic map heat flows and prove Theorem 1.3 and Theorem 1.6. Finally, we take up the polynomial growth solutions and establish Theorem 1.7 in section 4.

2. Preliminaries

We collect some basic facts about convex hulls in a Cartan-Hadamard manifold and establish some results concerning parabolic massive sets in this section. The result will be used to prove Theorem 1.3 in next section.

Recall the following definition from [19].

Definition 2.1. *A Cartan-Hadamard manifold N is said to satisfy the* separation property *if for every closed convex subset A in $\overline{N}$ and point p not in A, there exists a closed convex set C properly containing A and separating p from A, i.e., $A \subset C$, $A \cap \partial_\infty(N)$ is contained in the interior of $C \cap \partial_\infty(N)$ and p is not in C.*

The following lemma is proved in [19].

Lemma 2.2. *A Cartan Hadamard manifold N satisfies the separation properly if and only if for every closed subset A and monotone decreasing sequence of closed subsets $\{A_n\}$ in $\overline{N}$ such that $\cap_{n=1}^{\infty} A_n = A$,*

$$\cap_{n=1}^{\infty} \overline{\mathcal{C}(A_n)} = \overline{\mathcal{C}(A)}.$$

The above Lemma indicates that the separation property is quite natural in the study of convex sets. Using the result in [1, 3], Li-Wang showed the following statement.

Lemma 2.3. *Let N be a Cartan-Hadamard manifold with strongly negative curvature. Denote by $d_H(A, B)$ the Hausdorff distance between sets A and B in $\overline{N}$. Then for every closed subset A and monotone decreasing sequence of closed subsets $\{A_n\}$ in $\overline{N}$ such that $\cap_{n=1}^{\infty} A_n = A$,*

$$d_H(\cap_{n=1}^{\infty} \overline{\mathcal{C}(A_n)}, \overline{\mathcal{C}(A)}) < \infty.$$

The following definition and lemma are also from [19].

Definition 2.4. *Cartan-Hadamard manifold N is said to satisfy the* separation property at infinity *if for any closed subset A of $\partial_\infty(N)$, the ideal boundary of N, and any point $p \in \partial_\infty(N) \setminus A$, there exists a closed convex subset C in $\overline{N}$ such that A is contained in the interior of $C \cap \partial_\infty(N)$ and p not in C.*

Lemma 2.5. *Let N be a Cartan-Hadamard manifold. Then for every closed set K in $\overline{N}$,*

$$\overline{\mathcal{C}(K)} \cap \partial_\infty(N) = K \cap \partial_\infty(N)$$

if and only if N satisfies the separation property at infinity.

Upon improving a result of M. Anderson [1], A. Borbély [3] has shown that Cartan-Hadamard manifold N has separation property at infinity provided that its sectional curvature satisfies $-Ce^{\lambda d(x)} \leq K_N(x) \leq -1$ for some constant $C > 0$ and $0 \leq \lambda < 1/3$, where $d(x)$ is the distance from point x to a fixed point $o \in N$.

We now switch our consideration to massive sets defined in [9].

Definition 2.6. *A domain Γ in M is said to be massive if there exists a bounded, nonnegative, nontrivial, subharmonic function f on M such that $f = 0$ outside Γ.*

Clearly, M has at least one massive set if and only if it is nonparabolic or admits a positive Green's function. Moreover, according to [9], the maximal number of disjoint massive sets in M is always equal to $\dim H_0(M)$, the dimension of the space of bounded harmonic functions on M. So if $\dim H_0(M) = k_0$, then M has exactly k_0 disjoint massive subsets $\Gamma_1, \ldots, \Gamma_{k_0}$.

The following result shows that a parabolic analogue holds as well. Recall a domain $\Omega \subset M \times (-\infty, 0)$ is parabolic massive if there exists a bounded, nonnegative, nontrivial, subsolution f of the heat equation on $M \times (-\infty, 0)$ with $f(x,t) = 0$ outside Ω.

Proposition 2.7. *For complete manifold M, the maximal number of disjoint parabolic massive sets in $M \times (-\infty, 0)$ is always equal to $\dim P_0(M)$, the dimension of the space of bounded ancient solutions to the heat equation on M.*

Proof. Suppose that the maximal number of disjoint parabolic massive sets in $M \times (-\infty, 0)$ is k. Denote by $\Omega_1, \cdots, \Omega_k$ disjoint parabolic massive sets in $M \times (-\infty, 0)$ with corresponding potential $f_1, \cdots, f_k$. We may assume $\sup f_i = 1$ for $i = 1, \cdots, k$. For each f_i, we solve the following heat equation for each $T < 0$ and large R on the geodesic ball $B_p(R) \subset M$, where $p \in M$ is a fixed point.

$$(\Delta - \frac{\partial}{\partial t}) u_{R,T}(x,t) = 0$$

with $u_{R,T}(x,T) = f_i(x,T)$ for $x \in B_p(R)$ and $u_{R,T}(x,t) = f_i(x,t)$ for $x \in \partial B_p(R)$ and $T < t < 0$. Then the maximum principle implies that $f_i(x,t) \leq u_{R,T}(x,t) \leq 1$ on $B_p(R) \times [T, 0)$. In particular, we may pick a subsequence of $R_j \to \infty$ and $T_j \to -\infty$ such that u_{R_j,T_j} converges to $u_i(x,t)$, an ancient solution to the heat equation on $M \times (-\infty, 0)$ satisfying

$$f_i(x,t) \leq u_i(x,t) \leq 1.$$

In conclusion, we obtain ancient solutions $u_1, \cdots, u_k$ to the heat equation on $M \times (-\infty, 0)$ satisfying

$$f_i(x,t) \leq u_i(x,t) \leq 1.$$

Since $\Omega_1, \cdots, \Omega_k$ are disjoint, $\sum_{i=1}^k f_i(x,t) \leq 1$ on $M \times (-\infty, 0)$. Again, by the maximum principle, we have $\sum_{i=1}^k u_i(x,t) \leq 1$ on $M \times (-\infty, 0)$. However, $\sup u_i = \sup f_i = 1$. It then follows that $u_1, \cdots, u_k$ must be linearly independent. Consequently, $k \leq \dim P_0(M)$.

We now claim that $\{u_1, \cdots, u_k\}$ forms a basis for the space $P_0(M)$. First, we have $\sum_{i=1}^k u_i(x,t) = 1$ on $M \times (-\infty, 0)$. Otherwise, the function $v = 1 - \sum_{i=1}^k u_i$ is positive somewhere on $M \times (-\infty, 0)$. So for $\sup v = \epsilon_0 > 0$, there exists sufficiently small $\epsilon > 0$ such that the sets $\widetilde{\Omega} = \{v > \frac{\epsilon_0}{2}\}$ and $\widetilde{\Omega_i} = \{u_i > 1 - \epsilon\}$, $i = 1, \cdots, k$, are mutually disjoint. Note that $\widetilde{\Omega}$ and $\widetilde{\Omega_i}$ are parabolic massive with potential given by the positive part of $v - \frac{\epsilon_0}{2}$ and $u_i - 1 + \epsilon$, respectively. This contradicts with the fact that k is the maximal number of disjoint parabolic massive sets. Now for $u \in P_0(M)$, to express u as a linear combination of $u_1, \cdots, u_k$, we may assume $u \geq 0$ as the constant $1 = \sum_{i=1}^k u_i$. Let $L_i = \{u_i > \frac{1}{2}\}$, $i = 1, \cdots, k$, and $c_i = \sup_{L_i} u$. For $w = u - \sum_{i=1}^k c_i\, u_i$, if $\sup w = \delta_0 > 0$, then we claim that there exists sufficiently small $\epsilon > 0$ such that the sets $\{w > \frac{\delta_0}{2}\}$ and $\{u_i > 1 - \epsilon\}$, $i = 1, \cdots, k$, are mutually disjoint. Indeed, this is obvious so for the sets $\{u_i > 1 - \epsilon\}$, $i = 1, \cdots, k$. Now suppose that there exists a point with $w(z) > \frac{\delta_0}{2}$ and $u_j(z) > 1 - \epsilon$ for some j. Then we have

$$u(z) \geq w(z) + c_j\, u_j(z) > \frac{\delta_0}{2} + c_j\,(1 - \epsilon) > c_j.$$

This is a contradiction as $z \in L_j$. So the claim follows.

Since the sets $\{w > \frac{\delta_0}{2}\}$ and $\{u_i > 1 - \epsilon\}$, $i = 1, \cdots, k$, are all parabolic massive, the claim implies that $M \times (-\infty, 0)$ has more than k disjoint parabolic massive sets. This contradiction shows that $w \leq 0$. Similarly, one shows that $w \geq 0$. In conclusion, $w = 0$ or $u = \sum_{i=1}^k c_i\, u_i$. □

We end this section with the following result.

Lemma 2.8. *Domain $\Gamma \subset M$ is a massive set in M if and only if $\Gamma \times (-\infty, 0)$ is a parabolic massive set in $M \times (-\infty, 0)$.*

Proof. Obviously, $\Gamma \times (-\infty, 0)$ is a parabolic massive set in $M \times (-\infty, 0)$ if $\Gamma \subset M$ is a massive set in M. So we need only to show the other implication. To do so, we claim that

$$v(x) = \sup_{-\infty < t < 0} f(x,t)$$

is subharmonic on M for any bounded subsolution f to the heat equation on $M \times (-\infty, 0)$. Clearly, the result follows from the claim.

To check the claim, for any compact smooth domain $\widetilde{\Gamma} \subset M$, let ω be the harmonic function on $\widetilde{\Gamma}$ with $\omega = v$ on $\partial\widetilde{\Gamma}$. Then we have $\omega \geq f(x,t)$ on $\partial\widetilde{\Gamma}$. Since ω is a solution and f is a subsolution to the heat equation on $\widetilde{\Gamma}$, the maximum principle implies that $\omega(x) \geq f(x,t)$ for $x \in \widetilde{\Gamma}$ and $t < 0$.

Indeed, consider $u(x,t) = (f(x,t) - \omega(x))_+$, the positive part of the function $f(x,t) - \omega(x)$. Then u is a subsolution to the heat equation on $\widetilde{\Gamma}$ with $u = 0$ on $\partial\widetilde{\Gamma}$. Now

$$\begin{aligned} \frac{d}{dt}\int_{\widetilde{\Gamma}} u^2 &\leq 2 \int_{\widetilde{\Gamma}} u\,\Delta u \\ &= -2 \int_{\widetilde{\Gamma}} |\nabla u|^2 \\ &\leq -2\,\lambda_1 \int_{\widetilde{\Gamma}} u^2, \end{aligned}$$

where λ_1 is the first Dirichlet eigenvalue of $\widetilde{\Gamma}$. Integrating in t, we obtain that for $t_2 < t_1 < 0$,

$$\int_{\widetilde{\Gamma}} u^2(x,t_1)\,dx \leq \exp\left(-2\,\lambda_1(t_1 - t_2)\right) \int_{\widetilde{\Gamma}} u^2(x,t_2)\,dx.$$

Since u is bounded and $\widetilde{\Gamma}$ is compact, by letting $t_2 \to -\infty$ we conclude that $\int_{\widetilde{\Gamma}} u^2(x,t_1)\,dx = 0$ or $u(x,t_1) = 0$ for all $x \in \widetilde{\Gamma}$ and t_1. Therefore, $\omega(x) \geq f(x,t)$ for $x \in \widetilde{\Gamma}$ and $t < 0$. It follows that

$$\omega \geq v$$

on $\tilde{\Gamma}$ and v is subharmonic on M. The proof is completed. □

3. General ancient solutions

In this section, we prove Theorem 1.3 and Theorem 1.6 together with some corollaries.

Theorem 3.1. *For ancient solution* $u : M \times (-\infty, 0) \to N$ *to the harmonic map heat flow from* M *into a Cartan-Hadamard manifold* N *with strongly negative curvature, let* $A = \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap \partial_\infty(N)$, *where* $\partial_\infty(N)$ *is the ideal boundary of* $\overline{N}$. *Then there exists a set of points* $\{y_i\}_{i=1}^k \subset \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap N$ *with* $k \leq \dim P_0(M)$ *such that*

$$\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}$$

for all $-\infty < t < 0$.

Proof. The proof follows [19] closely. So we only sketch the main steps. Suppose $\dim P_0(M) = k_0$. Then by Proposition 2.7 M has exactly k_0 disjoint parabolic massive subsets $\Omega_1, \cdots, \Omega_{k_0}$ in $M \times (-\infty, 0)$. In the following, we denote $Z = M \times (-\infty, 0)$ and $\hat{Z}$ its Stone-Cêch compactification. For each $i \in \{1, \cdots, k_0\}$, let

$$S_i = \cap \{\hat{z} \in \hat{Z} \,|\, v(\hat{z}) = \sup v\},$$

where the intersection is taken over all the potential functions v of Ω_i. Then arguing as in [19] implies that each S_i is nonempty. Moreover, for a bounded ancient subsolution v to the heat equation on M, the set

$$S = \{\hat{z} \,|\, v(\hat{z}) = \sup v\}$$

must contain some S_i, and for each j, either $S \cap S_j = \emptyset$ or $S_j \subset S$.

We first show that there exist k points $\{y_1, \ldots, y_k\}$ in N with $k \leq k_0$ such that

$$u(Z) \subset \cap_{\epsilon>0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)}$$

for every $\epsilon > 0$, where A_ϵ is the ϵ-neighborhood of A in $S_\infty(N)$.

Pick a point $y_0 \in \overline{u(Z)} \cap N$. If

$$u(Z) \subset \cap_{\epsilon>0} \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})},$$

then we are done. Hence we may assume that there exists a ϵ-neighborhood A_ϵ of A in $S_\infty(N)$ such that the set

$$u(Z) \setminus \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})} \neq \emptyset.$$

One can easily check that it is bounded in N. Since u is a solution to the harmonic map heat flow and the function $d(y, \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})})$ is convex, the composition function

$$f(x,t) = d(u(x,t), \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})})$$

is a bounded nonconstant ancient subsolution to the heat equation on M. Thus, it attains its maximum at every point of some S_i, say S_1. In particular, there exists $\hat{z}_1 \in S_1$ such that $f(\hat{z}_1) = \sup_Z f$. One may find a net $\{z_\alpha\}$ in

Z converging to $\hat{z}_1$ in $\hat{Z}$ such that $u(z_\alpha)$ converges to $y_1 \in N$. Again, if

$$u(Z) \subset \cap_{\epsilon>0} \overline{\mathcal{C}(A_\epsilon \cup \{y_1\})}$$

then we are done. Otherwise, by choosing a smaller ϵ if necessary, the function

$$g(x,t) = d(u(x,t), \overline{\mathcal{C}(A_\epsilon \cup \{y_1\})})$$

is a bounded nonconstant ancient subsolution to the heat equation on M. If $\sup_Z g$ is achieved on S_1, then $g(\hat{z}) = \sup g$ for $\hat{z} \in S_1$. In particular,

$$\sup g = g(\hat{z}_1) = d(y_1, \overline{\mathcal{C}(A_\epsilon \cup \{y_1\})}) = 0,$$

which is impossible. Hence, we may assume g achieves its maximum on S_2.

For a net $\{z_\alpha\}$ in Z converging to a point $\hat{z}_2$ in S_2, there exists a subnet of $\{u(z_\alpha)\}$ that converges to $y_2 \in N$. Suppose that we have chosen l points $y_1, \ldots, y_l$ described in the above procedure. If

$$u(Z) \subset \cap_{\epsilon>0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^l)},$$

then we are done. Otherwise, by choosing a smaller ϵ if necessary, we define the function

$$h(x,t) = d(u(x,t), \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^l)})$$

which is a bounded nonconstant ancient subsolution to the heat equation on M. Then h cannot achieve its maximum on $\cup_{i=1}^l S_i$. Hence, h achieves its maximum on some S_j with $j > l$. We may assume that $j = l+1$.

Let us pick a point $\hat{x}_{l+1} \in S_{l+1}$ and a net $\{z_\alpha\}$ converging to $\hat{z}_{l+1}$. Suppose y_{l+1} is an accumulation point of the net $\{u(z_\alpha)\}$. It is clear that this process must stop after at most k_0 steps since there are only k_0 parabolic massive sets. In particular, there exist k points $\{y_1, \ldots, y_k\}$ with $k \leq k_0$ such that

$$u(Z) \subset \cap_{\epsilon>0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)}.$$

Moreover, $y_i \in \overline{u(Z)} \cap N$.

Finally, using the fact that N has strongly negative curvature and Lemma 2.3, one can follow the argument of Li-Wang [19] to conclude

$$u(Z) \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}.$$

This completes our proof. □

Note that the assumption that N has strongly negative curvature is not needed if u is bounded or the set $A = \emptyset$.

Corollary 3.2. *For a bounded ancient solution $u : M \times (-\infty, 0) \to N$ to the harmonic map heat flow from M into a Cartan-Hadamard manifold N, there exists a set of points $\{y_i\}_{i=1}^k \subset \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap N$ with $k \leq \dim P_0(M)$ such that*

$$\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{y_i\}_{i=1}^k)}$$

for all $-\infty < t < 0$.

This immediately leads to the following Liouville type result. Note in the case that the Ricci curvature of M is nonnegative, the result can be derived from the gradient estimate established in [14].

Corollary 3.3. *Let u be a bounded ancient solution to the harmonic map heat flow from complete manifold M into a Cartan-Hadamard manifold N. Then u must be a constant map if every bounded ancient solution to the heat equation on M is constant or* $\dim P_0(M) = 1$.

The following result may be interpreted as a strong Liouville property for ancient solutions to harmonic map heat flow. The harmonic map case was proved by Shen [23] and Li-Wang [19].

Corollary 3.4. *An ancient solution to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold with strongly negative curvature must be constant if its image is inside a horoball of N and* $\dim P_+(M) = 1$, *where $P_+(M)$ is the space spanned by the positive ancient solutions to the heat equation on M.*

Proof. Since a horoball of N intersects with the ideal boundary $\partial_\infty(N)$ at one point and $P_0(M) \subset P_+(M)$, it follows from Theorem 3.1 that the image of u lies on a geodesic ray. In particular, u may be viewed as a nonnegative ancient solution to the heat equation on M, namely, $u \in P_+(M)$. Therefore, u is a constant map. □

We now turn to Theorem 1.6 which is restated below.

Theorem 3.5. *Let $u : M \times (-\infty, 0) \to N$ be a bounded solution to the harmonic map heat flow from a complete manifold M into a Cartan-Hadamard manifold N. If* $\dim P_0(M) = \dim H_0(M) < \infty$, *then u must be harmonic.*

Proof. Since $\dim P_0(M) = \dim H_0(M) = k_0 < \infty$, by [9], we conclude that M has disjoint massive sets $\Gamma_1, \cdots, \Gamma_{k_0}$ with potential $u_1, \cdots, u_{k_0}$, respectively. We normalize u_i such that $\sup_M u_i = 1$. An elliptic version of the proof of Proposition 2.7 (see [9]) implies that for each i there is a bounded harmonic function f_i with $u_i \le f_i \le 1$.

Now by Proposition 2.7 $\Omega_i = \Gamma_i \times (-\infty, 0)$, $i = 1, \cdots, k_0$, are maximal disjoint parabolic massive sets in $M \times (-\infty, 0)$. Following the proof of Theorem 3.1, for each Ω_i we have the corresponding subset S_i in the Stone-Cêch compactification $\hat{Z}$ of $Z = M \times (-\infty, 0)$. Pick y_i an accumulation point of $u(z_\alpha)$ for a net of points $z_\alpha \in Z = M \times (-\infty, 0)$ converging to a point $\hat{z} \in S_i$, $i = 1, \cdots, k_0$. Let $(y^1, \cdots, y^n)$ be the normal coordinates of N centered at point p. In this coordinate system, $y_i = (a_{1i}, \cdots, a_{ni})$, $1 \le i \le k_0$. Let $h = (h_1, \cdots, h_n)$ be the vector-valued harmonic function with $h_l = \sum_{i=1}^{k_0} a_{li} f_i$. For a sequence of exhausting smooth compact domains $\Omega_i \subset M$, according to [12], for each i, there exists a harmonic map w_i from Ω_i to N such that $w_i = h$ on $\partial\Omega_i$. Moreover, by Lemma 3.1 in [2], w_i satisfies the following estimate.

$$d(w_i, h) \le C \left(v - \sum_{l=1}^{n} h_l^2 \right)$$

on Ω_i for some constant $C > 0$ independent of Ω_i, where

$$v = \sum_{i=1}^{k_0} \left(\sum_{l=1}^{n} (a_{li})^2 \right) f_i.$$

In particular, w_i remains uniformly bounded from h. We may therefore assume, by taking a subsequence if necessary, that w_i converges to a harmonic map w from M to N. Clearly, the following inequality holds for w on M.

$$d(w, h) \le C \left(v - \sum_{l=1}^{n} h_l^2 \right).$$

Viewing the harmonic function f_i as a function on Z, one has $f_i = 1$ on S_i and $f_i = 0$ on S_j for $j \ne i$. It follows from the definition of h that $w = y_i$ on S_i for $i = 1, \cdots, k_0$ by considering w as a map from Z to N. Now the function $d(u(x,t), w(x))$ is a bounded, nonnegative, ancient subsolution to the heat equation on M. So its maximum value on Z is achieved at every point of some S_i. However, by the choice of y_i, there exists a net z_α in Z such that z_α converges to a point in S_i with $u(z_\alpha)$ converging to y_i. In conclusion, we have $d(u(x,t), w(x)) = 0$ and $u = w$ is a harmonic map. □

4. Polynomial growth ancient solutions

In this section, we deal with polynomial growth ancient solutions to harmonic map heat flow and prove Theorem 1.7. We begin by recalling the definition of parabolic α-massive sets.

Definition 4.1. *An open subset Ω of $M \times (-\infty, 0)$ is said to be parabolic α-massive if there exists a nonnegative, nonconstant, ancient subsolution f of the heat equation on $M \times (-\infty, 0)$ satisfying*

$$f(x,t) \leq C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^\alpha$$

Note that a parabolic α-massive set is parabolic α'-massive if $\alpha \leq \alpha'$.

Lemma 4.2. *Let M be a complete manifold such that the maximum number of disjoint parabolic α-massive sets of $M \times (-\infty, 0)$ is k_α. Suppose that $u : M \times (-\infty, 0) \to N$ is an ancient solution to the harmonic map heat flow from M into Cartan-Hadamard manifold N and that N satisfies the separation property at infinity. Assume that there exists a point $q \in N$ such that*

$$d_N(u(x,t), q) \leq C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^\alpha$$

for some nonnegative constant C and $p \in M$. Then

$$A = \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap \partial_\infty(N) = \{a_i\}_{i=1}^{k'}$$

with $k' \leq k_\alpha - k_0$, where k_0 is the maximum number of disjoint parabolic massive sets of $M \times (-\infty, 0)$. If, in addition, N has strongly negative curvature, then there exist k points

$$\{y_j\}_{j=1}^k \subset \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap N$$

with $k' + k \leq k_\alpha$ such that

$$\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)}$$

for all $t \in -(\infty, 0)$.

Proof. Let k_0 be the maximum number of disjoint parabolic massive sets in $M \times (-\infty, 0)$. If A contains at least k' points, then there exist disjoint open sets $\{U_i\}_{i=1}^{k'}$ in $\overline{N}$ such that $U_i \cap A \neq \emptyset$ for $i = 1, 2, \ldots, k'$. Since N is assumed to satisfy the separation property at infinity, Lemma 2.5 implies that $\cup_{t<0}\{u(x,t)|x \in M\}$ is not a subset of $\mathcal{C}(\overline{N} \setminus U_i)$. In particular, the function

$$f_i(x,t) = d_N\left(u(x,t), \overline{\mathcal{C}((\overline{N} \setminus U_i)}\right)$$

is not identically zero on $u^{-1}(U_i) \subset M \times (-\infty, 0)$ and $\sup f_i = \infty$. Clearly, $f_i = 0$ outside the set $u^{-1}(U_i)$ and

$$f_i(x,t) \leq C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^{\alpha}.$$

This implies that each set $u^{-1}(U_i)$ is a parabolic α-massive but not parabolic massive set. In particular, since they are disjoint, $k' \leq k_\alpha - k_0$. It follows that $A = \{a_i\}_{i=1}^{k'}$ has at most $k_\alpha - k_0$ points, and the first conclusion follows. If, in addition, N has strongly negative curvature, then Theorem 3.1 implies that

$$\cup_{t<0}\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^{k})},$$

where

$$\{y_j\}_{j=1}^{k} \subset \left(\overline{\cup_{t<0}\{u(x,t)|x \in M\}}\right) \cap N$$

and $k \leq k_0$. Therefore, $k' + k \leq k_\alpha$ and the theorem is proved. □

We now turn to estimate k_α. The following parabolic mean value property will play an important role for this purpose. It is well-known that a scaling invariant Sobolev inequality implies such a mean value inequality via Moser iteration argument [15]. For further discussions and results of the mean value property, we refer to [20].

Definition 4.3. *Complete manifold M is said to have parabolic mean value property $(\mathcal{PM})$ if there exists a constant $\lambda > 0$, such that, for $x \in M$ and $r > 0$, any nonnegative subsolution of the heat equation f defined on $B_x(2r) \times [t - r^2, t]$ must satisfy*

$$f(x,t) \leq \frac{\lambda}{r^2 V_x(r)} \int_{t-r^2}^{t} \int_{B(x,r)} f(y,s)\,dyds.$$

Lemma 4.4. *Let M be a complete manifold satisfying parabolic mean value property $(\mathcal{PM})$. Suppose that the volume growth of M satisfies $V_p(r) =$*

$O(r^\nu)$ for some positive constant ν, where $V_p(r)$ is the volume of the geodesic ball $B_p(r)$ centered at point p of radius r. Then $M \times (-\infty, 0)$ has only finitely many disjoint parabolic α-massive sets and $k_\alpha \le \lambda 3^{(2\alpha+\nu)}$. If M is further assumed to satisfy weak volume comparison, that is, there exist constants $C_0 > 0$ and $\nu > 0$ such that

$$V_x(r') \le C_0 \left(\frac{r'}{r}\right)^\nu V_x(r)$$

for all $x \in M$ and $0 < r \le r' < \infty$, then there exists a constant $C > 0$ depending only on C_0 and ν such that the number of disjoint parabolic α-massive sets $k_\alpha \le C\,\lambda\,\alpha^{\nu-1}$.

Proof. Our proof is an adaption from [19]. Let $\Omega_1, \ldots, \Omega_{k_\alpha}$ be k_α disjoint parabolic α-massive sets in $M \times (-\infty, 0)$. Let $u_1, \ldots, u_{k_\alpha}$ be the corresponding potential functions. Then u_i is a nonnegative ancient subsolution of the heat equation on M and each u_i is of polynomial growth of degree at most α, namely,

$$u_i(x,t) \le C\left(d_M(x,p) + \sqrt{|t|} + 1\right)^\alpha.$$

Obviously, for a fixed $t < 0$, there exists $r_0 > 0$ such that

$$\int_{t-r_0^2}^{t} \int_{B_p(r_0)} u_i^2 \, dx\, ds > 0$$

for each $i = 1, \cdots, k_\alpha$. Since Ω_i, $i = 1, \cdots, k_\alpha$, are disjoint, the set of functions $\{u_i\}_{i=1}^{k_\alpha}$ forms an orthogonal basis with respect to the inner product

$$A_r(u,v) = \int_{t-r^2}^{t} \int_{B_p(r)} u\, v\, dx\, ds$$

for all $r \ge r_0$ on the space W spanned by $\{u_i\}_{i=1}^{k_\alpha}$. Using the fact that each u_i is of polynomial growth of order α and the assumption that $V_p(r) \le C\, r^\nu$, one concludes from [16] that for each $\beta > 1$ and $\delta > 0$ there exists $r > r_0$ satisfying

$$\sum_{i=1}^{k_\alpha} \frac{\int_{t-r^2}^{t} \int_{B_p(r)} u_i^2}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_i^2} \ge k_\alpha\, \beta^{-(2\alpha+\nu+\delta)}. \tag{4.1}$$

Note that the function

$$\sum_{i=1}^{k_\alpha} \frac{u_i^2(x,s)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_i^2}$$

is an ancient subsolution of the heat equation. The maximum principle implies that there exists a point $q \in \partial B_p(r)$ such that

$$\begin{aligned} \sum_{i=1}^{k_\alpha} \frac{u_i^2(x,s)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_i^2} &\leq \sum_{i=1}^{k_\alpha} \frac{u_i^2(q,t-r^2)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_i^2} \\ &= \frac{u_j^2(q,t-r^2)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_j^2} \end{aligned}$$

for all $x \in B_p(r)$ and $t - r^2 \leq s \leq t$, where in the last equality we have used the fact that functions $\{u_i\}_{i=1}^{k_\alpha}$ have disjoint support.

Applying the parabolic mean value property $(\mathcal{PM})$ we have

$$\begin{aligned} V_q(2r)\, u_j^2(q,t-r^2) &\leq (2r)^{-2}\, \lambda \int_{t-5r^2}^{t-r^2} \int_{B_q(2r)} u_j^2 \\ &\leq (2r)^{-2}\, \lambda \int_{t-(3r)^2}^{t} \int_{B_p(3r)} u_j^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{k_\alpha} \frac{\int_{t-r^2}^{t} \int_{B_p(r)} u_i^2}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_i^2} &\leq r^2 V_p(r) \frac{u_j^2(q,t-r^2)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_j^2} \\ &\leq r^2 V_q(2r) \frac{u_j^2(q,t-r^2)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_j^2} \\ &\leq \lambda \frac{\int_{t-(3r)^2}^{t} \int_{B_p(3r)} u_j^2(x,t)}{\int_{t-(\beta r)^2}^{t} \int_{B_p(\beta r)} u_j^2}. \end{aligned}$$

Choosing $\beta = 3$ in (4.1) we conclude that

$$k_\alpha 3^{(-2\alpha+\nu+\delta)} \leq \lambda$$

or

$$k_\alpha \leq \lambda 3^{(2\alpha+\nu)}$$

as $\delta > 0$ is arbitrary.

In the case M satisfies the weak volume comparison, a similar argument in [19] can be applied to conclude that $k_\alpha \leq C\,\lambda \alpha^{\nu-1}$ with the choice of $\beta = 1 + (2\alpha)^{-1}$. We omit the details here. $\square$

By combining Lemma 4.1 with Lemma 4.2, we deduce the main structural theorem on polynomial growth ancient solutions of harmonic heat flow.

Theorem 4.5. *Let M be a complete manifold with parabolic mean value property $(\mathcal{PM})$ and volume growth $V_p(r) = O(r^\nu)$ for some point $p \in M$. Let N be a Cartan-Hadamard manifold with strongly negative curvature. Then for $u : M \times (-\infty, 0) \to N$, an ancient solution to the harmonic heat flow satisfying*

$$d_N(u(x,t), q) \leq C \left(d_M(x,p) + \sqrt{|t|} + 1 \right)^\alpha$$

for some constant α, there exists sets of k' points

$$\{a_i\}_{i=1}^{k'} = \left(\overline{\cup_{t<0}\{u(x,t) | x \in M\}} \right) \cap \partial_\infty(N)$$

and of k points $\{y_j\}_{j=1}^{k} \subset \left(\overline{\cup_{t<0}\{u(x,t) | x \in M\}} \right) \cap N$ with $k' + k \leq \lambda 3^{(2\alpha+\nu)}$ such that

$$\cup_{t<0}\{u(x,t) | x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^{k})}.$$

In the case that M satisfies the weak volume comparison

$$V_x(r') \leq C_0 \left(\frac{r'}{r} \right)^\nu V_x(r)$$

for all $x \in M$ and $0 < r \leq r' < \infty$, the same conclusion holds with an improved estimate $k' + k \leq C\alpha^{\nu-1}$.

We now prove Theorem 1.7. The theorem in fact holds under weaker assumptions that for all $x \in M$ and $r > 0$ the volume doubling property $(\mathcal{V})$

$$V_x(2r) \leq c\, V_x(r)$$

holds and that the Neumann Poincaré inequality $(\mathcal{P})$

$$\inf_{a \in \mathbb{R}} \int_{B_x(r)} |\phi - a|^2 \leq c\, r^2 \int_{B_x(r)} |\nabla \phi|^2$$

is valid for all smooth function ϕ. We start with the following lemma.

Lemma 4.6. *Let M be a complete manifold satisfying ($\mathcal{V}$) and ($\mathcal{P}$). Suppose Ω is a parabolic massive set of $M \times (-\infty, 0)$. Then $M \times (-\infty, 0) \setminus \Omega$ does not contain any parabolic α-massive set.*

Proof. First, we claim that for any fixed $t < 0$,

$$\lim_{r\to\infty} \frac{1}{r^2\, V_p(r)} \int_{t-2r^2}^{t-r^2} \int_{B_p(r)} f(x,s)\, dx\, ds = \sup_{M\times(-\infty,0)} f$$

for a bounded ancient subsolution f to the heat equation on M.

Indeed, let $a = \sup_{M\times(-\infty,0)} f$. Then $g = a - f$ is a nonnegative ancient supersolution to the heat equation with $\inf_{M\times(-\infty,0)} g = 0$. So for any $\epsilon > 0$, there exists $x \in M$ and $s < 0$ such that $g(x,s) < \epsilon$. By the maximum principle, we may choose s to be as negative as one likes. In particular, we pick $s < t$. Since M satisfies both ($\mathcal{V}$) and ($\mathcal{P}$), the weak Harnack inequality (see [10, 22]) implies that for some constant $c > 0$,

$$\frac{1}{r^2\, V_p(r)} \int_{t-2r^2}^{t-r^2} \int_{B_p(r)} g(x,s)\, dx\, ds \le c \inf_{B_p(\frac{r}{2})\times(t-\frac{r^2}{2},t)} g$$

for all $r > 0$. So we conclude that

$$\limsup_{r\to\infty} \frac{1}{r^2\, V_p(r)} \int_{t-2r^2}^{t-r^2} \int_{B_p(r)} g(x,s)\, dx\, ds \le c\,\epsilon.$$

But $\epsilon > 0$ is arbitrary, this gives

$$\lim_{r\to\infty} \frac{1}{r^2\, V_p(r)} \int_{t-2r^2}^{t-r^2} \int_{B_p(r)} g(x,s)\, dx\, ds = 0$$

and the claim follows.

From the definition of parabolic massive set, there exists a non-negative, bounded, nonconstant, ancient subsolution f to the heat equation on M with support in $\Omega \subset M \times (-\infty, 0)$. Then by the above claim, for any $\epsilon > 0$, there exists r_0 such that for all $r \ge r_0$, we have

$$\begin{aligned}(1-\epsilon) \sup_{M\times(-\infty,0)} f &\le r^{-2}\, V_p^{-1}(r) \int_{t-2r^2}^{t-r^2} \int_{B_p(r)} f(x,s)\, dx\, ds \\ &\le r^{-2}\, V_p^{-1}(r) \int_{t-2r^2}^{t-r^2} V\left(\Omega(s) \cap B_p(r)\right)\, ds \sup_{M\times(-\infty,0)} f,\end{aligned}$$

where $\Omega(s) = \{x \in M : (x, s) \in \Omega\}$. In conclusion,

$$\epsilon\, r^2\, V_p(r) \geq \int_{t-2r^2}^{t-r^2} V\left(B_p(r) \setminus \Omega(s)\right)\, ds \tag{4.2}$$

for $r \geq r_0$.

Suppose now that there exists a parabolic α-massive set disjoint from Ω with g being its potential. Define

$$M(r) = \sup_{B_p(r)\times(t-2r^2, t-r^2)} g.$$

Then

$$\int_{t-(4r)^2}^{t-(2r)^2} \int_{B_p(2r)} g\, dx\, ds \leq M(2r) \int_{t-(4r)^2}^{t-(2r)^2} V\left(B_p(2r) \setminus \Omega(s)\right)\, ds. \tag{4.3}$$

On the other hand, the mean value inequality implies that there exists a constant $C > 0$ such that

$$\int_{t-(4r)^2}^{t-(2r)^2} \int_{B_p(2r)} g\, dx\, ds \geq C\, M(r)\, r^2\, V_p(2r). \tag{4.4}$$

Therefore, we conclude from (4.3) and (4.4) that

$$C\, M(2r) \int_{t-(4r)^2}^{t-(2r)^2} V(B_p(2r) \setminus \Omega(s))\, ds \geq M(r)\, r^2\, V_p(2r).$$

Combining with (4.2), we have

$$C\, \epsilon\, M(2r) \geq M(r)$$

for all $r \geq r_0$. Setting $r = r_0$ and iterating this inequality k times, we arrive at the inequality

$$(C\, \epsilon)^k\, M(2^k r_0) \geq M(r_0). \tag{4.5}$$

Since g is of polynomial growth of degree α, we have

$$M(r) \leq C_1\, r^{\alpha}.$$

Hence

$$M(2^k r_0) \leq C_1 2^{k\alpha}\, r_0^{\alpha}.$$

This contradicts (4.5) if we choose ϵ with $2^\alpha C \epsilon < 1$. The lemma is proved. $\square$

Theorem 4.7. *Let M be a complete manifold satisfying $(\mathcal{V})$ and $(\mathcal{P})$. Then any nonconstant, polynomial growth, harmonic map heat flow $u : M \times (-\infty, 0) \to N$ from M into a Cartan-Hadamard manifold N with strongly negative curvature must satisfy*

$$\cup_{t<0}\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'})}$$

with $\{a_i\}_{i=1}^{k'} = \overline{\cup_{t<0}\{u(x,t)|x \in M\}} \cap \partial_\infty(N)$.

Proof. Due to the fact that the parabolic mean value inequality $(\mathcal{PM})$ is a consequence of $(\mathcal{P})$ and $(\mathcal{V})$ [10, 15], Theorem 4.5 applies and $A = \{a_i\}_{i=1}^{k'}$. Let us assume the contrary that u is nonconstant and its image is not a subset of $\overline{\mathcal{C}(A)}$. Then Lemma 2.3 implies that either $d(u(x,t), \overline{\mathcal{C}(A)})$ is bounded or there exists a tubular neighborhood A_ϵ of A in $\overline{N}$ with $\epsilon > 0$ and the image of u is not contained in $\overline{\mathcal{C}(A_\epsilon)}$. Since M satisfies $(\mathcal{V})$ and $(\mathcal{P})$, the parabolic Harnack inequality holds on M by [10] and [22]. In particular,

$$\dim \mathcal{P}_0(M) = \dim \mathcal{P}_+(M) = 1. \tag{4.6}$$

Therefore, Theorem 3.1 implies that $\cup_{t<0}\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(A \cup \{y\})}$ for some $y \in \overline{\cup_{t<0}\{u(x,t)x \in M\}} \cap N$. It is then easy to see that the function $d(u(x,t), \overline{\mathcal{C}(A_\epsilon)})$ is bounded. In either case, we conclude that there exists a closed subset W in $\overline{N}$ such that the function

$$f(x,t) = d(u(x,t), \overline{\mathcal{C}(W)})$$

is a bounded, nonnegative, non-constant, subsolution to the heat equation on M. Moreover, the set $C = (\cup_{t<0}\{u(x,t)|x \in M\}) \setminus \mathcal{C}(W)$ is a non-empty bounded set in N. Its convex hull $\mathcal{C}(C)$ is also bounded and

$$(\cup_{t<0}\{u(x,t)|x \in M\}) \setminus \mathcal{C}(C)$$

is non-empty because u is non-constant. The distance function

$$g(x,t) = d(u(x,t), \overline{\mathcal{C}(C)})$$

is a non-negative, non-constant, subsolution to the heat equation of polynomial growth. Also the support of f is in $u^{-1}(C)$ and the support of g is on

$M \setminus u^{-1}(\overline{\mathcal{C}(C)})$. This is impossible because of Lemma 4.6, and the theorem is proved. □

On a complete manifold with nonnegative Ricci curvature, note that the volume doubling property ($\mathcal{V}$) follows from the Bishop-Gromov volume comparison theorem (see [15]), and that the Neumann Poincaré inequality ($\mathcal{P}$) holds by a result of Buser [4] (see [5] for a different proof). Since both the volume doubling property ($\mathcal{V}$) and the Neumann Poincaré inequality ($\mathcal{P}$) are preserved under quasi-isometry, we have the following corollary.

Corollary 4.8. *Let $u : M \times (-\infty, 0) \to N$ be a non-constant harmonic map heat flow of polynomial growth with order α. Suppose N is a Cartan-Hadamard manifold with strongly negative curvature. Assume that M is an m-dimensional manifold quasi-isometric to a manifold with non-negative Ricci curvature. Then there exists a set of k' points*

$$\{a_i\}_{i=1}^{k'} = \overline{\cup_{t<0}\{u(x,t)|x \in M} \cap \partial_\infty(N)$$

with $k' \leq C\alpha^{m-1}$ such that $\cup_{t<0}\{u(x,t)|x \in M\} \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'})}$, where the constant $C > 0$ depends only on m and the quasi-isometric constant.

Acknowledgment: Part of the paper was written while the author was visiting the School of Mathematics at the University of Minnesota. We greatly appreciate its hospitality. The author was partially supported by a grant from the National Science and Technology Council of Taiwan.

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Received July 21, 2020
Accepted March 28, 2021