Math. Res. Lett.
Volume 29, Number 6, 1701–1744, 2022

Internal stabilization for KdV-BBM equation on a periodic domain

Melek Jellouli and Moez Khenissi

We consider the nonlinear damped KdV-BBM equation on the torus. We shows the global existence of the solution, as well as its convergence in time towards an analytical function. This analyticity property allows the application of unique continuation results to show that the limit function is a constant.

1. Introduction

In the literature we can find several models of partial differential equations to describe the movement of water in shallow depths following a unidirectional propagation, see for instance [4], [5] and [10]. In this work, we study the stabilization for the damped nonlinear KdV-BBM equation

$$\partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u + u\partial_x u = 0, \tag{1}$$

where a, the damping, is non negative. This equation mixes the KdV equation $\partial_t u + \partial_{xxx} u + u\partial_x u = 0$, with the BBM equation $\partial_t u + \partial_x u - \partial_{xxt} u +$

$u\partial_x u = 0$. To our knowledge the KdV-BBM equation was treated for the first time as a system in [6] to study its properties. This equation has been recently considered in [3], [12] and [25] from numerical analysis point of view. Stabilization was not considered for KdV-BBM equation but was considered for KdV in [2] and [23], and for BBM equation in [1] and [30].

The particularity of the KdV- BBM equation, is that it admits a nonlocal unbounded operator of order 1, that introduces some difficulties in the analysis. Our main result is a stabilization property for the solutions of (1) on the torus. The principal theorem is the following.

Theorem 1. *For all $u_0 \in H^1(\mathbb{T})$, there exists a unique solution $u = u(t,x)$ of* (1) *global in time such that*

$$\lim_{t\to+\infty} u(t,\cdot) = \frac{1}{2\pi}\int_{\mathbb{T}} u_0(x)dx, \quad \textit{in } H^1(\mathbb{T}).$$

Now we give the outline of the proof. In Section 2, we show the existence of the solution of the linear problem. We obtain the exponential decay result on the semigroup by an estimation of the resolvent. This estimate comes from semiclassical measures and the technic was used for the stability of the wave equation. This method can be found for the wave equation, see [8], [9], [15], [14], [24], [26] and [29].

In Section 3, from the fixed point theorem we prove local existence and the uniqueness for the nonlinear problem. By an a priori estimate on the energy, we deduce global existence for the nonlinear problem.

In Section 4, from energy decay dynamical system technics, we prove first that solution converges to a bounded solution for all t in $\mathbb{R}$. Second, from result of [16] we prove that this particular solution is analytic in time. This allows to apply uniqueness results (see [20], [28] and [31]) and we deduce that this particular solution is in fact a constant. This gives Theorem 1.

The result of unique continuation which we will arrive at in Section 4 is therefore a result which says that if the solution is a constant on an open set, then it is constant over the whole domain. The unique continuation gives stabilization at the end in the sense that some solutions will goes to a constant as t goes to infinity. This technique can be found in [22] where the authors consider the wave equation. They show that they can found an analytical solution in time and then use an unique continuation result to prove the stabilization.

In Section 5, we give a quick deduction that the approach we made on KdV-BBM is applicable for the equation of BBM in a particular case. More precisely we say that the unique continuation conjecture that Rosier has stated in [30] is true with a dissipator a which depends on time. This type of dissipator was used in [11] to show controllability results for the Boussinesq equation.

According to our knowledge, the rate of decay of the energy of the KdV-BBM nonlinear problem solution is still an open question.

2. Linear equation: global existence and uniqueness

We consider the linear KdV-BBM equation posed on a periodic domain $\mathbb{T}$

$$(2) \qquad \begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x (a(x)\partial_x) u = 0, & x \in \mathbb{T},\ t > 0, \\ u(\cdot, 0) = u_0 \ \in H^1(\mathbb{T}), & x \in \mathbb{T}, \end{cases}$$

where $a \geq 0$ is assumed to be a bounded function in $\mathcal{C}^\infty(\mathbb{T})$ such that $\{a > 0\} \neq \emptyset$.

We start this first part by studying the linear equation and showing the existence of solutions with the Lumer-Phillips theorem.

2.1. Linear equation

We define for $s \in \mathbb{R}$ the Sobolev spaces

$$H^s(\mathbb{T}) = \left\{ u = \sum_{n\in\mathbb{Z}} u_n e^{inx} : \mathbb{T} \longrightarrow \mathbb{C} \ / \ \sum_{n\in\mathbb{Z}} \left(1+n^2\right)^s |u_n|^2 < \infty \right\},$$

equipped by the inner product $\langle \cdot , \cdot \rangle_s \ : (u,v) \longmapsto \sum_{n\in\mathbb{Z}} \left(1+n^2\right)^s u_n \overline{v_n}$, where u_n and v_n are the Fourier coefficients of u and v respectively. We note $\langle \cdot , \cdot \rangle = \langle \cdot , \cdot \rangle_1$.

We define the unbounded operator $(A, \mathcal{D}(A))$ on $H^1(\mathbb{T})$ by $\mathcal{D}(A) = H^2(\mathbb{T})$ and

$$Au = -(1-\partial_{xx})^{-1}(\partial_x u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u)).$$

Theorem 2. *A is the infinitesimal generator of a $\mathcal{C}_0$-group $\{S(t)\}_{t\in\mathbb{R}}$ on $H^1(\mathbb{T})$.*

Proof. We first prove that A -considered as operator on $H^2(\mathbb{T})$ with values in $H^1(\mathbb{T})$- is a Fredholm operator of index 0 then we use Lumer-Phillips theorem.

Let $\delta > 0$. A can be written as sum of A_0 and B with

$$A_0 = -(1-\partial_{xx})^{-1}(\partial_x + \partial_{xxx} + \delta)$$

and

$$B = -(1-\partial_{xx})^{-1}(-\partial_x(a(x)\partial_x) - \delta).$$

Note that if $u(x,t) = \sum_{n\in\mathbb{Z}} u_n(t)e^{inx}$, then $A_0 u = \sum_{n\in\mathbb{Z}} \frac{-in + in^3 - \delta}{1+n^2} u_n(t)e^{inx}$.
The operator A_0 is bijective from $H^2(\mathbb{T})$ to $H^1(\mathbb{T})$. It follows that A_0 is a Fredholm operator of index zero. The operator B is bounded from $H^2(\mathbb{T})$ to itself, and then, it is compact from $H^2(\mathbb{T})$ to $H^1(\mathbb{T})$. This implies $A_0 + B = A$ is a Fredholm operator of index zero.

Note that $\mathcal{D}(A) = H^2(\mathbb{T})$ is dense in $H^1(\mathbb{T})$. For $\mu \in \mathbb{R}$ and $u \in H^2(\mathbb{T})$, we have

$$(3) \quad \begin{aligned} \langle (A+i\mu)u, u\rangle &= ((A+i\mu)u, (1-\partial_{xx})u) \\ &= -(\partial_x u, u) + (\partial_{xx}u, \partial_x u) - (a\partial_x u, \partial_x u) + i\mu \|u\|_{H^1}^2 \end{aligned}$$

The quantity $(\partial_x u, u)$ is pure imaginary since

$$\mathfrak{Re}(\partial_x u, u) = \frac{1}{2}\int_{\mathbb{T}} \partial_x u\overline{u} + \overline{\overline{u}\partial_x u} dx = \frac{1}{2}\left((\partial_x u, u) + (u, \partial_x u)\right) = 0.$$

By the same way we show that $\mathfrak{Re}(\partial_{xxx}u, u) = 0$. Then

$$(4) \qquad \mathfrak{Re}\,\langle (A+i\mu)u, u\rangle = -\int_{\mathbb{T}} a\,|\partial_x u|^2\,dx \leq 0.$$

So A is dissipative.
Now it suffices to prove that $A-\lambda$ is injective for some $\lambda > 0$. Let $u \in H^2(\mathbb{T})$ such that $Au - \lambda u = 0$. We have

$$0 = \mathfrak{Re}\,\langle Au - \lambda u, u\rangle = -\int_{\mathbb{T}} a\,|\partial_x u|^2\,dx - \lambda\|u\|_{H^1}^2 \geq \left(-\|a\|_{L^\infty} - \lambda\right)\|u\|_{H^1}^2.$$

This give us that $A-\lambda$ is injective.
Since A is Fredholm of index 0, and $u \longmapsto \lambda u$ is an operator compact from $H^2(\mathbb{T})$ in $H^1(\mathbb{T})$, then $A-\lambda$ is also Fredholm of index 0, since it is injective, it is therefore surjective. Thus, the operator A generates a $\mathcal{C}^0$-semigroup $\{S_+(t)\}_{t\geq 0}$ of contraction by Lumer-Phillips theorem.

The operator $-A$ generates a $\mathcal{C}^0$-semigroup $\{S_-(t)\}_{t\geq 0}$. Indeed, let $u \in H^2(\mathbb{T})$. For $\lambda > 1 + \|a\|^2_{L^\infty(\mathbb{T})}$ we have

$$\langle(-A-\lambda)u, u\rangle = (\partial_{xxx}u + \partial_x u - \partial_x(a\partial_x)u - \lambda(1-\partial_{xx})u, u). \tag{5}$$

Thus

$$\mathfrak{Re}\,\langle(-A-\lambda)u, u\rangle = \int_{\mathbb{T}} a|\partial_x u|_{H^1(\mathbb{T})} - \lambda\|u\|^2_{H^1(\mathbb{T})} \leq 0. \tag{6}$$

By the same way as the previous proof, we show that the operator $A+\lambda$ is surjective. This proves that $-A-\lambda$ generates a contraction semigroup that we denote by $\{\tilde{S}_-(t)\}_{t\geq 0}$. The map $S_- : t \longmapsto e^{\lambda t}\tilde{S}_-(t)$ defines a semigroup with infinitesimal generator $-A$. Using a result in [27], we know that if A and $-A$ are infinitesimal generators of $\mathcal{C}_0$ semigroups S_+ and S_-, then A is the generator of a $\mathcal{C}_0$-group $\{S(t)\}_{t\in\mathbb{R}}$ given by

$$S(t) = \begin{cases} S_+(t), & \text{if } t \geq 0 \\ S_-(-t), & \text{if } t \leq 0, \end{cases}$$

which complete the proof. □

We finish the first subsection by giving this remark.

Remark 1. *For u solution of (2), we have that $\|u(t)\|^2_{H^1(\mathbb{T})}$ is nonincreasing. Indeed*

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{H^1}\right) &= (-\partial_x u - \partial_{xxx}u + \partial_x(a\partial_x)u, u)_{L^2} \\ &\quad + (u, -\partial_x u - \partial_{xxx}u + \partial_x(a\partial_x)u)_{L^2} \\ &= -2\int_{\mathbb{T}} a(x)\,|\partial_x u(t,x)|^2\,dx \leq 0. \end{aligned} \tag{7}$$

2.2. Exponential stability

The goal of this subsection is to prove the exponential stability of $\{S(t)\}_{t\geq 0}$. More precisely, we will prove the following result

Theorem 3. *There exists $\delta > 0$ and $M > 0$ such that*

$$|||S(t)|||_{\mathcal{L}(H^1(\mathbb{T}))} \leq Me^{-\delta t},\ \forall t \geq 0. \tag{8}$$

We will use the following result.

Theorem 4. *Let H be a Hilbert space, and $A : \mathcal{D}(A) \subset H \longrightarrow H$ a generator infinitesimal of a semigroup $\{T(t)\}_{t\geq 0}$. Assume that*

1) *There exist $c_1 > 0$ such that $|||T(t)|||_{\mathcal{L}(H)} \leq c_1, \ \forall t \geq 0,$*
2) *$A + i\mu$ is inversible for all μ in $\mathbb{R}$,*
3) *There exist $c_2 > 0$ such that $\left|\left|\left|(A + i\mu)^{-1}\right|\right|\right|_{\mathcal{L}(H)} \leq c_2$ for all μ in $\mathbb{R}$.*

Then, there exist $M > 0$ and $c > 0$ such that

$$|||T(t)|||_{\mathcal{L}(H)} \leq Me^{-ct}, \ \forall t \geq 0. \tag{9}$$

See [21, Theorem 3] for more details.

We cannot apply this theorem to A since it is not injective, the second assumption is not verified when $\mu = 0$. For that, we introduce the closed subspace of $H^1(\mathbb{T})$

$$\dot{H}^1(\mathbb{T}) = \left\{ u \in H^1(\mathbb{T}) \ / \ (u, 1)_{L^2(\mathbb{T})} = 0 \right\}.$$

We equip $\dot{H}^1(\mathbb{T})$ and $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ with the norms of $H^1(\mathbb{T})$ of $H^2(\mathbb{T})$ respectively.

Note that $Au \in \dot{H}^1(\mathbb{T})$ for $u \in H^1(\mathbb{T})$. If $u(t,x) = \displaystyle\sum_{k\in\mathbb{Z}} u^k(t)e^{ikx} \in H^1(\mathbb{T})$, we can write $u(t,x) = u^0(t) + \dot{u}(t,x)$, with $\dot{u} \in \dot{H}^1(\mathbb{T})$. It is easily checked that if $\partial_t u = Au$, then $u^0(t)$ is independent of t since

$$\left(\partial_t u^0(t), 1\right)_{L^2(\mathbb{T})} + \underbrace{\left(\partial_t \dot{u}(t,x), 1\right)_{L^2(\mathbb{T})}}_{=0} = \underbrace{\left(Au(t,x), 1\right)_{L^2(\mathbb{T})}}_{=0}.$$

The space $\dot{H}^1(\mathbb{T})$ is then invariant by $\{S(t)\}_{t\geq 0}$, then it is also a semigroup of contraction on $\dot{H}^1(\mathbb{T})$. We thus define $\dot{A}$ with $\mathcal{D}(\dot{A}) = H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ and $\dot{A}u = Au$ for $u \in \mathcal{D}(\dot{A})$. The operator $\dot{A}$ is injective, indeed, for $u \in \dot{H}^1(\mathbb{T})$ such that $\dot{A}u = 0$, we can adapt the proof of Theorem 2 to obtain $\partial_x u = 0$ on $\mathbb{T}$. As $u \in \dot{H}^1(\mathbb{T})$, then $u = 0$. Furthermore, $\dot{A} : H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T}) \longrightarrow \dot{H}^1(\mathbb{T})$ is a Fredholm operator of index 0, so it is bijective. Then we have the second assumption of Theorem 4 for all $\mu \in \mathbb{R}$. Using bounded inverse theorem, A^{-1} is continuous. The third assumption of Theorem 4 is thus checked for $\mu = 0$.

We define now the resolvent of $\dot{A}$ by $\mathcal{R}(\lambda)f = \dot{u}$, where $\dot{u}$ is the solution of $(\dot{A} - \lambda)\dot{u} = f$. Let us show an estimate of the resolvent.

Proposition 1. *There exists $c > 0$ such that for all $f \in \dot{H}^1(\mathbb{T})$ we have*

$$\forall\ \mu \in \mathbb{R},\ \|\dot{u}(t)\|_{H^1(\mathbb{T})} \leq c\,\|f\|_{H^1(\mathbb{T})}\,,$$

where $\dot{u}$ is solution of $(\dot{A} + i\mu)\dot{u} = f$.

Proof. Without loss of generality, we may consider from now on the operator A in $H^1(\mathbb{T})$ to simplify the notation instead of $\dot{A}$ in $\dot{H}^1(\mathbb{T})$. If we prove that there exist μ_0 such that for all $\mu > \mu_0$, $\left|\left|\left|(A + i\mu)^{-1}\right|\right|\right|_{\mathcal{L}(H^1)} \leq c_1$, then since $\mu \longmapsto (A + i\mu)^{-1}$ is continuous from $\mathbb{R}$ to $\mathcal{L}(\dot{H}^1)$ we have $\left|\left|\left|(A + i\mu)^{-1}\right|\right|\right|_{\mathcal{L}(H^1)} \leq\ c_2$ for all $\mu \in \mathbb{R}$. Note that we already know by an argument of symmetry that $\mathcal{R}(i\mu)\overline{f} = \overline{\mathcal{R}(-i\mu)f}$ for $f \in \dot{H}^1(\mathbb{T})$. So it is sufficient to prove the result for $\mu > \mu_0$.

We will prove by contradiction the resolvente estimate. Assume that

$$\forall\ k \in \mathbb{N},\ \exists\ f^k \in H^1(\mathbb{T}),\ \exists\ \mu_k \geq k\ /\ \left\|u^k\right\|_{H^1(\mathbb{T})} > k\left\|f^k\right\|_{H^1(\mathbb{T})}, \tag{10}$$

where u^k is solution of $(A + i\mu_k)u^k = f^k$. We can assume that $\left\|u^k\right\|_{H^1(\mathbb{T})} = 1$ and $\left(f^k\right)_{k\in\mathbb{N}} \longrightarrow 0$ in $H^1(\mathbb{T})$, when $k \longrightarrow +\infty$.

The resolvent equation is

$$(1 - \partial_{xx})^{-1}\left(\partial_{xxx}u^k + \partial_x u^k - \partial_x(a\partial_x)u^k\right) + i\mu_k u^k = f^k, \tag{11}$$

which can be written for all $n \in \mathbb{Z}$ with Fourier coefficients as

$$\frac{1}{1+n^2}\left(-in^3 + in\right)u_n^k - \tilde{a_n^k} + i\mu_k u_n^k = f_n^k, \tag{12}$$

where $u^k = \sum_{n\in\mathbb{Z}} u_n^k e^{inx}$, $f^k = \sum_{n\in\mathbb{Z}} f_n^k e^{inx}$, $a = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ and

$$\begin{aligned}\tilde{a_n^k} &= \left\{(1 - \partial_{xx})^{-1}\left(\partial_x(a\partial_x u^k)\right)\right\}_n \\ &= \frac{in}{1+n^2}(a\partial_x u^k)_n = \frac{-n}{1+n^2}\sum_{j\in\mathbb{Z}} a_j(n-j)u_{n-j}^k.\end{aligned}$$

We have

$$\frac{1}{1+n^2}\left(-in^3 + in\right) + i\mu_k = i(-n + \mu_k) + \frac{2in}{1+n^2}.$$

From (12) we obtain

$$u_n^k = \frac{-2n}{(-n+\mu_k)(1+n^2)} u_n^k + \frac{\tilde{a_n^k}}{i(-n+\mu_k)} + \frac{f_n^k}{i(-n+\mu_k)}. \tag{13}$$

Let $\delta > 0$. We consider first the frequencies n such that $|n-\mu_k| \geq \delta\mu_k$,. Let us estimate each term of (13). Since $\frac{1}{|n-\mu_k|} \leq \frac{1}{\delta\mu_k}$, we have

$$\begin{aligned}
&\left\| \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{-2n}{(-n+\mu_k)(1+n^2)} u_n^k e^{inx} \right\|_{H^1}^2 \\
&= \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{4(1+n^2)}{(-n+\mu_k)^2} \frac{n^2}{(1+n^2)^2} \left| u_n^k \right|^2 \\
&\leq \frac{4}{(\delta\mu_k)^2} \sum_{|n-\mu_k|\geq\delta\mu_k} (1+n^2) \left| u_n^k \right|^2 \\
&\leq \frac{4}{(\delta\mu_k)^2} \left\| u^k \right\|_{H^1(\mathbb{T})}^2 = \frac{4}{(\delta\mu_k)^2}.
\end{aligned} \tag{14}$$

The operator $(1-\partial_{xx})^{-1}(\partial_x(a\partial_x))$ is bounded on $H^1(\mathbb{T})$, we deduce $\sum_{n\in\mathbb{Z}} (1+n^2) \left| \tilde{a_n^k} \right|^2 \leq c$. Thus

$$\begin{aligned}
&\left\| \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{\tilde{a_n^k}}{i(-n+\mu_k)} e^{inx} \right\|_{H^1(\mathbb{T})}^2 \\
&= \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{1}{(-n+\mu_k)^2} (1+n^2) \left| \tilde{a_n^k} \right|^2 \\
&\leq \frac{1}{(\delta\mu_k)^2} \sum_{n\in\mathbb{N}} (1+n^2) \left| \tilde{a_n^k} \right|^2 \leq \frac{c}{(\delta\mu_k)^2}.
\end{aligned} \tag{15}$$

We have also

$$\begin{aligned}
&\left\| \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{f_n^k}{i(-n+\mu_k)} e^{inx} \right\|_{H^1(\mathbb{T})}^2 \\
&= \sum_{|n-\mu_k|\geq\delta\mu_k} \frac{(1+n^2)}{(-n+\mu_k)^2} \left| f_n^k \right|^2 \leq \frac{c}{(\delta\mu_k)^2},
\end{aligned} \tag{16}$$

since $f \to 0$ in $H^1(\mathbb{T})$. Combining, (13), (14), (15) and (16), we obtain

$$\sum_{|n-\mu_k|\geq\delta\mu_k} (1+n^2)\left|u_n^k\right|^2 \leq \frac{C}{(\delta\mu_k)^2} \longrightarrow 0, \text{ when } k\to\infty. \tag{17}$$

For the part where n is of the order of μ_k, we introduce a smooth fonction Ψ such that $0 \leq \Psi \leq 1$ and

$$\Psi(s) = \begin{cases} 1, \text{ if } |s-1| \leq \delta \\ 0, \text{ if } |s-1| \geq 2\delta \end{cases} \tag{18}$$

Let $h = \dfrac{1}{\mu_k}$. The operator $\Psi(-ih\partial_x)$ is defined by $\Psi(-ih\partial_x)u^k = \sum_{n\in\mathbb{Z}} \Psi(hn)u_n^k e^{inx}$. Note that $\Psi(hn) \neq 0$ for $n \in [(1-\delta)\mu_k, (1+\delta)\mu_k]$. We write $u^k = (u^k - \Psi u^k) + \Psi u^k$. Since $\Psi(hn) = 1$ if $n \in [(1-\delta)\mu_k, (1+\delta)\mu_k]$, we have

$$\begin{aligned} \left\|u^k - \Psi(-ih\partial_x)u^k\right\|_{H^1}^2 &= \sum_{n\in\mathbb{Z}}(1+n^2)(1-\Psi(hn))^2\left|u_n^k\right|^2 \\ &\leq 2\sum_{|n-\mu_k|\geq\delta\mu_k}(1+n^2)\left|u_n^k\right|^2 \leq \frac{2C}{(\delta\mu_k)^2}. \end{aligned} \tag{19}$$

For μ_k large enough, we can assume that $\left\|\Psi(-ih\partial_x)u^k\right\|_{H^1(\mathbb{T})} \geq \dfrac{1}{2}$.
Now let $v^k = \dfrac{\Psi(-ih\partial_x)u^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}$ and we have

$$\begin{aligned} &(1-\partial_{xx})^{-1}\left(\partial_{xxx} + \partial_x - \partial_x(a\partial_x)\right)v^k + i\mu_k v^k \\ &\quad = \frac{\Psi(-ih\partial_x)f^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \\ &\qquad + (1-\partial_{xx})^{-1}\left(\partial_x\left[\Psi(-ih\partial_x), a\right]\frac{\partial_x u^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}\right). \end{aligned} \tag{20}$$

We can check easily that

$$\left\|\Psi(-ih\partial_x)u^k\right\|_{L^2(\mathbb{T})} \approx h\left\|\Psi(-ih\partial_x)u^k\right\|_{H^1(\mathbb{T})} \approx h. \tag{21}$$

The term $\dfrac{\Psi(-ih\partial_x)f^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}$ tends towards 0 in $L^2(\mathbb{T})$ since

$$\frac{\left\|\Psi(-ih\partial_x)f^k\right\|_{L^2(\mathbb{T})}}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \approx \frac{h\left\|f^k\right\|_{H^1(\mathbb{T})}}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \approx \left\|f^k\right\|_{H^1(\mathbb{T})} \longrightarrow 0. \tag{22}$$

We apply $h^3(1-\partial_{xx})$ to (20), we obtain

$$\begin{aligned}(h\partial_x)^3 v^k + h^2 (h\partial_x) v^k - h (h\partial_x) (a (h\partial_x)) v^k - i (h\partial_x)^2 v^k + ih^2 v^k \\ = \frac{h^3(1-\partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}\Psi(-ih\partial_x)f^k \\ + \frac{h (h\partial_x)}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} [\Psi(-ih\partial_x), a] (h\partial_x) u^k.\end{aligned} \tag{23}$$

We will show now with the two following lemmas that we have

$$\begin{aligned}\Bigg\| \frac{h^3(1-\partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2}}\Psi(-ih\partial_x)f^k \\ + \frac{h (h\partial_x)}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} [\Psi(-ih\partial_x), a] (h\partial_x) u^k \Bigg\|_{L^2} = o(h).\end{aligned} \tag{24}$$

We denote by $\mathcal{D} = -i\partial_x$.

Lemma 1. *There exist $c > 0$ such that*

$$\left\| \frac{h^3(1-\partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}\Psi(-ih\partial_x)f^k \right\|_{L^2(\mathbb{T})} \leq ch \left\|f^k\right\|_{H^1(\mathbb{T})}. \tag{25}$$

Proof. Let $\tilde{\Psi} : s \mapsto -s^2\Psi(s)$. From (21) we have

$$\begin{aligned}&\left\| \frac{h^3(1-\partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2}}\Psi(-ih\partial_x)f^k \right\|_{L^2} \\ &\quad\lesssim \left\| \left(h^2 - h^2\partial_{xx}\right) \Psi(-ih\partial_x)f^k \right\|_{L^2} \\ &\quad\leq h^2 \left\|\Psi(-ih\partial_x)f^k\right\|_{L^2(\mathbb{T})} + \left\|\tilde{\Psi}(h\mathcal{D})f^k\right\|_{L^2(\mathbb{T})} \\ &\quad\leq h^2 \left\|f^k\right\|_{L^2(\mathbb{T})} + h\left\|\tilde{\Psi}(h\mathcal{D})f^k\right\|_{H^1(\mathbb{T})} \\ &\quad\leq ch \left\|f^k\right\|_{H^1(\mathbb{T})}.\end{aligned}$$

□

Lemma 2. *There exist $c > 0$ such that for $z \in L^2(\mathbb{T})$*

$$\|[\Psi(-ih\partial_x), a]\, z\|_{H^1(\mathbb{T})} \leq c\|z\|_{L^2(\mathbb{T})}.$$

The proof is given in Annex.

We deduce from Lemma 2 taking $z = \partial_x u^k$ ($\in L^2(\mathbb{T})$) that

$$\left\|\frac{h^3}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}\partial_x [\Psi(-ih\partial_x), a]\, \partial_x u^k\right\|_{L^2(\mathbb{T})} \leq ch^2 \left\|\partial_x u^k\right\|_{L^2(\mathbb{T})} \leq ch^2. \tag{26}$$

Combining (25) and (26), we get (24). Furthermore

$$\left\|ih^2 v^k\right\|_{L^2(\mathbb{T})} = o(h), \text{ and } \left\|h^2(h\partial_x)v^k\right\|_{L^2(\mathbb{T})} = o(h), \tag{27}$$

since $\left\|v^k\right\|_{L^2(\mathbb{T})} = 1$.

We give also the following lemma

Lemma 3. *There exist $c > 0$ such that*

$$\left\|(h\partial_x)\,(a\,(h\partial_x))\, v^k\right\|_{L^2(\mathbb{T})} \leq c. \tag{28}$$

Proof. We recall that

$$\Psi(hn) = \begin{cases} 1, \text{ if } |hn-1| \leq \delta \\ 0, \text{ if } |hn-1| \geq 2\delta. \end{cases}$$

We write $(h\partial_x)\,(a\,(h\partial_x)) = a(h\partial_x)^2 + [h\partial_x, ah\partial_x] = a(h\partial_x)^2 + ha'(h\partial_x)$. Since a is $\mathcal{C}^\infty$ and $hn\Psi(nh) \leq 1$, we can estimate these two terms by wrinting for $j = 1, 2$

$$\begin{aligned}\left\|(h\partial_x)^j v^k\right\|^2_{L^2(\mathbb{T})} &\leq \sum_{n\in\mathbb{Z}} h^{2j-2} n^{2j} \psi^2(hn)(u_n^k)^2 \\ &\leq c\sum_{n\in\mathbb{Z}} n^2 (u_n^k)^2 \leq c\left\|u^k\right\|^2_{H^1(\mathbb{T})} \leq c.\end{aligned}$$

And the proof is complete. □

We can write $\left\|h\left(h\partial_x\right)\left(a\left(h\partial_x\right)\right)v^k\right\|_{L^2(\mathbb{T})} = O(h)$. Now (23) becomes

$$(h\mathcal{D})^3 v^k - (h\mathcal{D})^2 v^k = hg^k, \tag{29}$$

where $(g_k)_{k\in\mathbb{N}}$ is a bounded sequence of functions in $L^2(\mathbb{T})$. We can now give the theorem of existence of semiclassical measure, which is a classical theorem and may be found in [7, Theorem 2] or [13, Proposition 3.1].

Definition 1. *Let* $u(x) = \sum_{j\in\mathbb{Z}} u_j e^{ijx} \in L^2(\mathbb{T})$ *and* $b(x,\xi) = \sum_{j\in\mathbb{Z}} b_j(\xi)e^{ijx} \in \mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R})$ *such that* b *is* 2π*-periodic on* x *and*

$$b_j(\xi) = \int_{\mathbb{T}} b(x,\xi)e^{-ijx}dx.$$

Then the semi-classical pseudo-differential operator of b *is defined by*

$$Op_h(b)u = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} b(x,h\xi)\hat{u}(\xi)d\xi = \frac{1}{2\pi}\sum_{j,l\in\mathbb{Z}} u_j(x) b_{l-j}(hj)e^{ilx}. \tag{30}$$

By an analogous calculation that in [18], we show that $Op_h(b)$ *is well defined. And by applying [18, Lemma 6.4] and [18, Proposition 6.5] we have that* $Op_h(b)$ *is a bounded operator in* $L^2(\mathbb{T})$*. See also [32, Section 5.3.1].*

Theorem 5. *Let* $(h_k)_k$ *be a sequence of reals which converges to* 0 *and* (v^{h_k}) *a bounded sequence in* $L^2(\mathbb{T})$ *which converges weakly to* 0*. There exists a subsequence from* (v^{h_k})*-which we called also* (v^{h_k})*- and a non negative Radon measure* ν *on* $\mathbb{T}\times\mathbb{R}$ *such that for every* $b \in \mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R};\mathbb{C})$

$$\lim_{h_k\to 0}\left(b(x,h_k\mathcal{D})v^{h_k}, v^{h_k}\right)_{L^2(\mathbb{T})} = \int_{\mathbb{T}\times\mathbb{R}} b(x,\xi)d\nu(x,\xi).$$

ν *is called the semiclassical measure associated to the sequence* (v^{h_k})*.*

It is not difficult to prove that the sequence $\left(v^{h_k}\right)$ converges weakly to 0. Before we apply Theorem 5, we give some definitions

Remark 2. *We can prove that the mesure* ν *is bounded using Gårding inequality, more precisely*

$$\int_{\mathbb{T}\times\mathbb{R}} d\nu(x,\xi) \le 1. \tag{31}$$

Let us study where the measure ν is supported.

Proposition 2. *We have*

$$\nu(x,\xi) = \widetilde{\nu}(x) \otimes \delta_{\xi=1}. \tag{32}$$

Proof. We multiply the equation (29) by a function $b \in \mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R};\mathbb{C})$

$$b(x,h\mathcal{D})\left((h\mathcal{D})^3 - (h\mathcal{D})^2\right) v^k = hb(x,h\mathcal{D})g^k = O(h).$$

Note that the symbol of $b(x,h\mathcal{D})\left((h\mathcal{D})^3 - (h\mathcal{D})^2\right)$ is $b(x,\xi)\left(\xi^3 - \xi^2\right) \in \mathcal{C}_0^\infty(\mathbb{T}\times\mathbb{R};\mathbb{C})$. According to Theorem 5, we have on the one hand

$$\Big(b(x,h\mathcal{D})\left((h\mathcal{D})^3 - (h\mathcal{D})^2\right) v^k, v^k\Big)_{L^2(\mathbb{T})} \longrightarrow \int_{\mathbb{T}\times\mathbb{R}} b(x,\xi)\left(\xi^3-\xi^2\right) d\nu(x,\xi),$$

and on the other hand

$$\Big(O(h), v^k\Big)_{L^2(\mathbb{T})} \longrightarrow 0. \tag{33}$$

Then

$$\int_{\mathbb{T}\times\mathbb{R}} b(x,\xi)\left(\xi^3-\xi^2\right) d\nu(x,\xi) = 0. \tag{34}$$

Now We want to show that

$$\operatorname{supp}(\nu) \subset \mathbb{T}\times\{\xi=1\}.$$

Let $b \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{C})$ independent of x and $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\chi(\xi) = \begin{cases} 1, & \text{if } |\xi| \le \delta \\ 0, & \text{if } |\xi| \ge 2\delta, \end{cases}$$

where the constant δ is the same one used in (18).
We want to prove that

$$\int_{\mathbb{T}\times\mathbb{R}} b(\xi)(1-\xi)d\nu(x,\xi) = 0. \tag{35}$$

We write $b = b_1 + b_2$ with $b_1 = \chi b$ and $b_2 = (1-\chi)b$. Clearly

$$\operatorname{supp}(b_1) \subset \{\mathbb{R} \setminus \operatorname{supp}(\Psi)\}. \tag{36}$$

On the one hand

$$(37)\quad \left((1-h\mathcal{D})b_1(h\mathcal{D})v^k, v^k\right)_{L^2} = \left((1-h\mathcal{D})\underbrace{b_1(h\mathcal{D})\Psi(h\mathcal{D})}_{=0}\frac{u^k}{\|\Psi(h\mathcal{D})u^k\|}, \Psi(h\mathcal{D})\frac{u^k}{\|\Psi(h\mathcal{D})u^k\|}\right)_{L^2} = 0.$$

From Theorem 5 we have

$$(38)\quad \int_{\mathbb{T}\times\mathbb{R}} (1-\xi)b_1(\xi)d\nu(x,\xi) = 0,$$

for every b_1 such that (36).
On the other hand
(39)

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} b_2(\xi)(1-\xi)d\nu(x,\xi) &= \int_{\mathbb{T}\times\mathbb{R}} (1-\chi)b(\xi)(1-\xi)d\nu(x,\xi)\\ &= \int_{\mathbb{T}\times\mathbb{R}} \frac{(1-\chi)b(\xi)}{\xi^2}(\xi^3-\xi^2)d\nu(x,\xi) = 0,\end{aligned}$$

from (34).

This give us (35) for every $b \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{C})$. Then $(1-\xi)\nu = 0$ and hence we obtain (32). □

Proposition 3. *The measure $\nu \equiv 0$ everywhere.*

Proof. We first show that ν is an uniform measure on $\{\xi = 1\}$, i.e. $\nu(x,\xi) = \widetilde{\nu}\otimes\delta_{\xi=1}$, that is $\widetilde{\nu}$ does not depend on x. We start by giving the following lemma.

Lemma 4. *Let a_1 and a_2 be two symbols in $\mathcal{S}$, then*

$$(40)\quad [Op_h(a_1), Op_h(a_2)] = \frac{h}{i}Op_h(\{a_1,a_2\}) + h^2O(1).$$

See [32, Theorem 4.12] for a proof.

We come back to the equation (23). From (24) and (27) we have

$$(41)\quad (h\mathcal{D})^3v^k - (h\mathcal{D})^2v^k + ih(h\mathcal{D})\left(a(x)(h\mathcal{D})v^k\right) = o(h),$$

where $o(h)$ is a function going to 0 in $L^2(\mathbb{T})$ when h goes to 0.

We first prove that $\nu = 0$ on $\{a > 0\}$. Let $b \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R})$ such that $b(x, \xi) = b(\xi)$. Since b is real and does not depend on x, we know from [32, Theorem 4.1] that the operator $b(h\mathcal{D})$ is self adjoint, and the operator $h\mathcal{D}$ is also self adjoint. Taking the imaginary part of the inner product of (41) with $b(h\mathcal{D})v^k$ we get

$$h\mathfrak{Im}\left(ib(h\mathcal{D})(h\mathcal{D})\left(a(x)(h\mathcal{D})v^k\right), v^k\right)_{L^2(\mathbb{T})} = \mathfrak{Im}\left(o(h), v^k\right)_{L^2(\mathbb{T})}.$$

Thus

$$\mathfrak{Im}\left(ib(h\mathcal{D})(h\mathcal{D})\left(a(x)(h\mathcal{D})v^k\right), v^k\right)_{L^2(\mathbb{T})} \longrightarrow 0.$$

By Theorem 5, we know that

$$\left(b(h\mathcal{D})(h\mathcal{D})\left(a(x)(h\mathcal{D})v^k\right), v^k\right)_{L^2(\mathbb{T})} \longrightarrow \int_{\mathbb{T}\times\mathbb{R}} a(x)\xi^2 b(\xi) d\nu(x,\xi) = 0.$$

That means $\nu = 0$ in $\{a > 0\}$. Now let $b = b(x, \xi) \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R})$, $P = (h\mathcal{D})^3 - (h\mathcal{D})^2 + ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)$, and $p = \xi^3 - \xi^2$ its principal symbol. We have

(42)
$$\begin{aligned}
&\left((Pb(x,h\mathcal{D}) - b(x,h\mathcal{D})P)v^k, v^k\right) \\
&\quad = \left(b(x,h\mathcal{D})v^k, P^*v^k\right) - \left(b(x,h\mathcal{D})Pv^k, v^k\right) \\
&\quad = \left(b(x,h\mathcal{D})v^k, \left((h\mathcal{D})^3 - (h\mathcal{D})^2 - ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)\right)v^k\right) \\
&\qquad\quad - \left(b(x,h\mathcal{D})Pv^k, v^k\right) \\
&\quad = \left(b(x,h\mathcal{D})v^k, \left((h\mathcal{D})^3 - (h\mathcal{D})^2 + ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)\right)v^k\right) \\
&\qquad\quad - \left(b(x,h\mathcal{D})v^k, 2ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)v^k\right) - \left(b(x,h\mathcal{D})Pv^k, v^k\right).
\end{aligned}$$

On the one hand

$$\frac{1}{h}\left(\left(Pb(x,h\mathcal{D}) - b(x,h\mathcal{D})P\right)u^k, u^k\right) = \frac{1}{i}\left(Op_h\left(\{p,b\}\right)u^k, u^k\right) + o(1),$$

according to (40), and converges to $\frac{1}{i}\int_{\mathbb{T}\times\mathbb{R}} \{p,b\}\, d\nu(x,\xi)$ by Theorem 5. On the other hand

$$(43)\quad \frac{1}{h}\left(b(x,h\mathcal{D})Pu^k,u^k\right)$$
$$= \frac{1}{h}\left(\underbrace{b(x,h\mathcal{D})\left((h\mathcal{D})^3-(h\mathcal{D})^2+ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)\right)u^k}_{=o(h)},u^k\right) \longrightarrow 0.$$

The term

$$\frac{1}{h}\left(b(x,h\mathcal{D})u^k, 2ih(h\mathcal{D})\left(a(x)(h\mathcal{D})\right)u^k\right) \longrightarrow -2i\int_{\mathbb{T}\times\mathbb{R}} a(x)\xi^2 b(x,\xi)d\nu(x,\xi).$$

Then

$$(44)\qquad \frac{1}{i}\int_{\mathbb{T}\times\mathbb{R}} \{p,b\}\, d\nu(x,\xi) = 2i\int_{\mathbb{T}\times\mathbb{R}} a(x)\xi^2 b(x,\xi)d\nu(x,\xi) = 0.$$

Now we have

$$(45)\qquad 0 = \int \{p,b\}\, d\nu = \langle \nu, H_p b\rangle = -\langle H_p\nu, b\rangle$$
$$= -\langle \partial_\xi p\partial_x\nu + \partial_x p\partial_\xi\nu, b\rangle = \left\langle (3\xi^2-2\xi)\partial_x\nu, b\right\rangle.$$

This is true for every b. Thus $(3\xi^2-2\xi)\partial_x\nu = 0$. Note that such a choice of b is sufficient to write (44) thanks to the positivity of the measure ν. Since ν is supported in $\{\xi=1\}$ and $(3\xi^2-2\xi)_{|\xi=1}=1$, we get $\partial_x\nu=0$. So, $\widetilde{\nu}(x)$ does not depend of x. Since $\nu=0$ on the support on a, we deduce that $\nu\equiv 0$ everywhere. □

To finish we verify at the same time that $\nu\not\equiv 0$. For that, we recall that $v_h^k = \dfrac{\Psi(h\mathcal{D})u_h^k}{\left\|\Psi(h\mathcal{D})u_h^k\right\|_{L^2(\mathbb{T})}}$. Let Θ be a function in $\mathcal{C}^\infty$ such that $0\le\Theta\le 1$ and

$$(46)\qquad \Theta(s) = \begin{cases} 1, & \text{if } |s-1|\le 2\delta \\ 0, & \text{if } |s-1|\ge 3\delta. \end{cases}$$

Note that $\Theta\Psi=\Psi$. On the one hand

$$\lim_{h\to 0}\left(Op_h(\Theta)v_h^k, v_h^k\right)_{L^2(\mathbb{T})} = \int_{\mathbb{T}\times\mathbb{R}} \Theta(\xi)d\nu(x,\xi),$$

and on the other hand

$$\left(Op_h(\Theta)v_h^k, v_h^k\right)_{L^2(\mathbb{T})} = \frac{1}{\left\|\Psi(h\mathcal{D})u_h^k\right\|_{L^2}} \left(Op_h(\Psi)u_h^k, v_h^k\right)_{L^2(\mathbb{T})} = \left\|v_h^k\right\|^2_{L^2(\mathbb{T})} = 1.$$

Thus $\displaystyle\int_{\mathbb{T}\times\mathbb{R}} \Theta(\xi)d\nu(x,\xi) = 1$. This means that ν cannot be identically 0. This is absurd, so the assumption (10) is false.

We can now deduce our result for $\dot{A}$. From the above, we have

$$\forall\mu \geq \mu_0,\ \forall f \in H^1(\mathbb{T}),\ \exists u \in \mathcal{D}(A)\ /\ (A+i\mu)u = f,$$

and there exist $c>0$ wich is independent on μ such that $\|u(t)\|_{H^1(\mathbb{T})} \leq c\,\|f\|_{H^1(\mathbb{T})}$. In particular, if $f \in \dot{H}^1(\mathbb{T})$, then $u \in \dot{H}^1(\mathbb{T})$. □

Using Theorem 4, we get (8). We finish this subsection by showing the exponential stability in $H^2(\mathbb{T})$, which will be useful later.

Lemma 5. *There exist $M,\delta > 0$ such that for all $t \geq 0$ we have*

$$\|\!|S(t)|\!\|_{\mathcal{L}(\dot{H}^2)} \leq Me^{-\delta t}. \tag{47}$$

Proof. Let $u(t) \in H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ and $t \geq 0$. We have

$$\begin{aligned}\|S(t)u\|_{H^2(\mathbb{T})} &\simeq \|AS(t)u\|_{H^1(\mathbb{T})} + \|S(t)u\|_{H^1(\mathbb{T})}\\ &= \|S(t)Au\|_{H^1(\mathbb{T})} + \|S(t)u\|_{H^1(\mathbb{T})}\\ &\leq Me^{-\delta t}\,\|Au\|_{H^1(\mathbb{T})} + Me^{-\delta t}\big\|u\big\|_{H^1(\mathbb{T})}\\ &\leq Me^{-\delta t}\big\|u\big\|_{H^2(\mathbb{T})}.\end{aligned}$$

□

3. Nonlinear equation: global existence and uniqueness

We consider now the nonlinear equation

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt}u + \partial_{xxx}u - \partial_x(a(x)\partial_x u) + u\partial_x u = 0, & x \in \mathbb{T},\\ u(.,0) = u_0\ \in H^1(\mathbb{T}), & x \in \mathbb{T}. \end{cases} \tag{48}$$

We show in this paragraph that this equation admits a unique solution defined over all $\mathbb{R}$. The principal theorem of this subsection is the following result.

Theorem 6. *Let $R > 0$. There is a unique solution of (48) that exists on $\mathbb{R}$. Moreover, for $T > 0$, the map*

$$\begin{array}{ccc} H^1(\mathbb{T}) & \longrightarrow & \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right) \\ u_0 & \longmapsto & u, \end{array}$$

where u is the solution of the nonlinear problem with the initial data u_0, is Lipschitz on $\overline{B_{\dot{H}^1(\mathbb{T})}(0,R)}$.

The first equation of (48) is equivalent to $\partial_t u = Au - (1-\partial_{xx})^{-1} u\partial_x u$. The local existence is a consequence of the Picard fixed point theorem. Let $R > 0$ and $0 < T \leq 1$. We define

$$B_{R,T} = \left\{ u \in \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right) \ / \ \sup_{s\in[-T,T]} \|u(s)\|_{H^1(\mathbb{T})} \leq R \right\}.$$

We equip $\mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right)$ with the distance

$$d(u,v) = \sup_{s\in[-T,T]} \|u(s)-v(s)\|_{H^1(\mathbb{T})}.$$

The space $\left(\mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right), d\right)$ is a complete metric space. For $u_0 \in \dot{H}^1(\mathbb{T})$ and $t \in [-T,T]$, we introduce

$$\begin{aligned} \Phi : &\mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right) \longrightarrow \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right) \\ &u \longmapsto S(t)u_0 - \int_0^t S(t-s)(1-\partial_{xx})^{-1} u(s)\partial_x u(s) ds. \end{aligned}$$

It is clear that $\Phi(u) \in \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right)$ when $u \in \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right) \subset \mathcal{C}\big([-T,T], L^\infty(\mathbb{T})\big)$ since $\partial_x u \in \mathcal{C}\left([-T,T], L^2(\mathbb{T})\right)$, then

$$u\partial_x u \in \mathcal{C}\left([-T,T], L^2(\mathbb{T})\right).$$

Which implies $(1-\partial_{xx})^{-1} u\partial_x u \ \in \mathcal{C}\left([-T,T], H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})\right)$. So the quantities $S(t)u_0$ and $S(t-s)(1-\partial_{xx})^{-1} u\partial_x u \ \in \mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right)$.

Lemma 6. *We have*

$$\exists\ C_0 > 0\ /\ \forall M > 0,\ \exists\ R, T > 0\ /\ RT < C_0,$$
$$\forall \|u_0\|_{H^1(\mathbb{T})} \leq M, \Phi : B_{R,T} \longrightarrow B_{R,T}$$

is a contraction.

Proof. It is easy to show that for every $g \in \mathcal{C}\left([-T,T], L^2(\mathbb{T})\right)$ we have for every $t \in [-T,T]$

$$\left\| \int_0^t S(t-s)(1-\partial_{xx})^{-1} g(s) ds \right\|_{H^1(\mathbb{T})} \leq cTe^{\delta} \sup_{s\in[-T,T]} \|g(s)\|_{L^2(\mathbb{T})}.$$

Let $u, v \in B_{R,T}$. We have

$$\begin{aligned}
&\Phi(u)(x,t) - \Phi(v)(x,t) \\
&\quad = -\int_0^t S(t-s)(1-\partial_{xx})^{-1} \left(u(s)\partial_x u(s) - v(s)\partial_x v(s)\right) ds \\
&\quad = -\int_0^t S(t-s)(1-\partial_{xx})^{-1} \left\{\left(u(s)-v(s)\right)\partial_x u(s)\right. \\
&\qquad\qquad\qquad\qquad \left. + \left(\partial_x u(s) - \partial_x v(s)\right) v(s)\right\} ds.
\end{aligned}$$

As $\dot{H}^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ continuously, we have

$$\begin{aligned}
&\|\Phi(u)(t) - \Phi(v)(t)\|_{H^1(\mathbb{T})} \\
&\quad \leq cTe^{\delta} \sup_{s\in[-T,T]} \Big(\|u(s)-v(s)\|_{L^\infty(\mathbb{T})} \|\partial_x u(s)\|_{L^2(\mathbb{T})} \\
&\qquad\qquad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2(\mathbb{T})} \|v(s)\|_{L^\infty(\mathbb{T})} \Big) \\
&\quad \leq cTe^{\delta} \sup_{s\in[-T,T]} \Big(\|u(s)-v(s)\|_{H^1(\mathbb{T})} \|\partial_x u(s)\|_{L^2(\mathbb{T})} \\
&\qquad\qquad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2(\mathbb{T})} \|v(s)\|_{H^1(\mathbb{T})} \Big) \\
&\quad \leq 2cTRe^{\delta} \sup_{s\in[-T,T]} \|u(s)-v(s)\|_{H^1(\mathbb{T})} \\
&\quad = 2cTRe^{\delta} d(u,v).
\end{aligned} \tag{49}$$

Taking the sup on t we get

$$d(\Phi(u), \Phi(v)) \leq 2cTRe^{\delta} d(u,v).$$

So, for $T \leq \min\left\{\frac{e^{-\delta}}{4cR}, 1\right\}$, Φ is a contraction.

Let us now show that if $u \in B_{R,T}$ then $\Phi(u) \in B_{R,T}$. From (49) and taking $v = 0$ we obtain

$$\begin{aligned}\left|\|\Phi(u)(t)\|_{H^1} - \|\Phi(0)(t)\|_{H^1}\right| &\leq \|\Phi(u)(t) - \Phi(0)(t)\|_{H^1} \\ &\leq 2cTRe^{\delta} \sup_{s\in[-T,T]} \|u(s)\|_{H^1}.\end{aligned}$$

Let $M > 0$ and $u_0 \in H^1(\mathbb{T})$ be such that $\|u_0\|_{H^1(\mathbb{T})} \leq M$. We can write

$$\|\Phi(u)(t)\|_{H^1(\mathbb{T})} \leq e^{\delta} \|u_0\|_{H^1(\mathbb{T})} + 2cTR^2 e^{\delta} \leq \left(M + 2cTR^2\right) e^{\delta} \tag{50}$$

Choosing $R \geq 4Me^{\delta} + \frac{e^{-\delta}}{4c}$ we get $\|\Phi(u)(t)\|_{H^1(\mathbb{T})} < R$. Since $R > \frac{e^{-c}}{4c}$, this estimate is valid when $T \leq \frac{e^{-\delta}}{4cR}$, and the proof is complete. □

Before showing the global existence, we give the following result of the semi-group.

Proposition 4. *There exists $c > 0$ which depends only on the function $a(x)$ such that the solution u of the nonlinear problem which exists on $]-T, T[$ verifies*

$$\|u(t)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{\delta|t|}, \ \forall t \in]-T, T[. \tag{51}$$

Proof. Let $t \in]-T, 0[$, we write

$$\begin{aligned}\|u(t)\|^2_{H^1(\mathbb{T})} - \|u(0)\|^2_{H^1(\mathbb{T})} &= \int_0^t \partial_t \|u(\sigma)\|^2_{H^1(\mathbb{T})} d\sigma \\ &= -4\int_0^t \int_{\mathbb{T}} a(x) \left|\partial_x u(\sigma)\right|^2 dx d\sigma \\ &\leq 4\|a\|_{L^\infty(\mathbb{T})} \int_t^0 \|u(\sigma)\|^2_{H^1(\mathbb{T})} d\sigma.\end{aligned}$$

Then

$$\|u(t)\|^2_{H^1(\mathbb{T})} \leq c\int_t^0 \|u(\sigma)\|^2_{H^1(\mathbb{T})} d\sigma + \|u(0)\|^2_{H^1(\mathbb{T})}.$$

Using Gronwall inquality

$$\|u(t)\|^2_{H^1(\mathbb{T})} \leq \|u(0)\|^2_{H^1(\mathbb{T})} e^{-ct}, \ \forall t \in]-T, 0[.$$

We can repeat the same arguments for $t \in [0, T[$. This was to be demonstrated. □

We can now prove Theorem 6.

Proof. Let u and v be two solutions of (48) admitting the same initial data. The function $w = u - v$ verifies

$$\partial_t w + \partial_x w - \partial_{xxt} w + \partial_{xxx} w - \partial_x (a(x)\partial_x) w + w\partial_x u + v\partial_x w = 0.$$

We know that

$$w(t) = \int_0^t S(t-s)(1-\partial_{xx})^{-1} (w(s)\partial_x u(s) + v(s)\partial_x w(s))\, ds.$$

According to (51) we can deduce that $\|\partial_x u\|_{L^2(\mathbb{T})} \leq c_1$ and $\|v(s)\|_{L^2(\mathbb{T})} \leq c_2$. It follows for $t \geq 0$

$$\begin{aligned} &\|w(t)\|_{H^1(\mathbb{T})} \\ &\quad = \left\| \int_0^t S(t-s)(1-\partial_{xx})^{-1} (w(s)\partial_x u(s) + v(s)\partial_x w(s))\, ds \right\|_{H^1(\mathbb{T})} \\ &\quad \leq \int_0^t \max\{c_1, c_2\} \|w(s)\|_{H^1(\mathbb{T})}\, ds. \end{aligned}$$

By Gronwall's inequality, we obtain $\|w(t)\|_{H^1(\mathbb{T})} = 0$. We can do the same calculus for $t \leq 0$, so that we have uniqueness.

To prove that the solution is global, We recall that for the local existence of the solution, by taking R and T such that $RT \leq \dfrac{e^{-\delta}}{4c}$, we found a solution which exists on $[0, T]$.

Now let $T^* = \sup\{t \geq 0 \ / \ u \text{ exists on } [0,t]\}$ and $0 < T_1 < T^*$.

Suppose that $T^* < \infty$. From Proposition 4, if $\|u(T_1)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{\delta T_1} = R_1$, then we have a solution which exists on $[T_1, T]$ as soon as $(T - T_1)R_1 \leq \dfrac{e^{-\delta}}{4c}$. That means

$$T \leq \frac{e^{-\delta}}{4cR_1} + T_1 = \frac{e^{-\delta}}{4c\|u_0\|_{H^1} e^{\delta T_1}} + T_1 = f(T_1).$$

It is clear that $f(T^*) > T^*$, since f is continuous, there exists $T < T^*$ such that $f(T) > T^*$. This means that the solution starting from T_1 will exist beyond T^*, which is absurd. This implies $T^* = +\infty$.

The same argument can be repeat for $t \leq 0$.

Once we know that the solution of the nonlinear problem exists on all $\mathbb{R}$, we can repeat the same proof as that of Proposition 4 with all $T > 0$ to write

$$\|u(t)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{c|t|}, \ \forall t \in \mathbb{R}, \tag{52}$$

where u is the solution of the nonlinear problem which exists on all $\mathbb{R}$. Now we finish the proof by proving that the map $u_0 \longmapsto u$ is locally Lipschitz continuous from $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ to $\mathcal{C}\left([-T,T], \dot{H}^1(\mathbb{T})\right)$. Let u and v be two solutions of the nonlinear problem with initial data u_0 and v_0 respectively, and $T > 0$. We have $\forall t \in [-T,T]$,

$$\begin{aligned} u(t) - v(t) = S(t)(u_0 - v_0) \\ - \int_0^t S(t-s) \underbrace{(1-\partial_{xx})^{-1} (u(s)\partial_x u(s) - v(s)\partial_x v(s))}_{=(Fu-Fv)(s)} ds. \end{aligned}$$

But

$$\begin{aligned} &\|(Fu - Fv)(s)\|_{H^1} \\ &\quad = \|((u(s) - v(s))\partial_x u(s) - (\partial_x u(s) - \partial_x v(s))v(s))\|_{L^2} \\ &\quad \leq c\big(\|u(s) - v(s)\|_{L^\infty} \|\partial_x u(s)\|_{L^2} \\ &\qquad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2} \|v(s)\|_{L^\infty} \big) \\ &\quad \leq c\left(\|u(s)\|_{H^1} + \|v(s)\|_{H^1}\right) \|u(s) - v(s)\|_{H^1}. \end{aligned} \tag{53}$$

Then there exists $c > 0$ which depend on T such that

$$\begin{aligned} \|u(t) - v(t)\|_{H^1} &\leq c\|u_0 - v_0\|_{H^1} \\ &\quad + c\left|\int_0^t \left(\|u(s)\|_{H^1} + \|v(s)\|_{H^1}\right) \|u(s) - v(s)\|_{H^1} ds\right| \\ &\leq c\|u_0 - v_0\|_{H^1} \\ &\quad + c\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right) \left|\int_0^t \|u(s) - v(s)\|_{H^1} ds\right|. \end{aligned} \tag{54}$$

By Gronwall inequality

$$\|u(t) - v(t)\|_{H^1(\mathbb{T})} \leq C\|u_0 - v_0\|_{H^1(\mathbb{T})}, \tag{55}$$

with $C = ce^{\delta T\left(\|u_0\|_{H^1} + \|v_0\|_{H^1}\right)}$. □

4. Proof of main theorem

We give in this section a proof of Theorem 1.

4.1. Convergence of the solutions

We start by introducing a function called u_∞, which is a limit in a certain way of the solution given in the following theorem. The advantage of u_∞ is that it is an analytic function in time, this property will be used later to show that u_∞ is a constant.

Theorem 7. *Let $(t_n)_n$ be a nondecreasing sequence of times which goes to $+\infty$, and u_0 a real valued initial data in $\dot{H}^1(\mathbb{T})$ with u the corresponding solution of (48). Then, there exists a subsequence $(t_{\phi(n)})_n$ and an analytic function in time u_∞ such that*

$$\forall T>0,\ \lim_{n\to+\infty} u(t_{\phi(n)}+\cdot)=u_\infty(\cdot) \text{ in } \mathcal{C}^0\left([-T,T],\dot{H}^1(\mathbb{T})\right). \tag{56}$$

We can easily prove the convergence of $u(t_{\phi(n)}+\cdot)$ towards a function $u_\infty(\cdot)$. We start by giving this remark.

Remark 3. *If u_0 real valued, the corresponding solution u of (48) is bounded for $t\geq 0$. Moreover*

$$\|u(t)\|_{H^1(\mathbb{T})}\leq \|u_0\|_{H^1(\mathbb{T})},\ \forall t\geq 0. \tag{57}$$

Indeed, note that u is real when u_0 is real. We recall that the norm $||\cdot||^2_{H^1(\mathbb{T})}$ of the solution of the linear problem is nonincreasing. Since $\mathfrak{Re}(u\partial_x u,u)=0$, the same calculus as in remark 1 gives us that $\|u(t)\|_{H^1(\mathbb{T})}$ is nonincreasing where $u(t)$ is the solution of nonlinear problem.

Now let $(t_n)_n$ be a sequence which goes to $+\infty$, according to Duhamel's formula

$$u(t_n)=S(t_n)u(0)-\int_0^{t_n} S(s)(1-\partial_{xx})^{-1}u(t_n-s)\partial_x u(t_n-s)ds.$$

When $t_n\to+\infty$, the term $S(t_n)u(0)$ goes to 0 from (8).

Since $u\in L^\infty\left([0,+\infty[,\dot{H}^1(\mathbb{T})\right)\subset L^\infty\left([0,+\infty[\times\mathbb{T}\right)$, and $\partial_x u\in L^\infty\left([0,+\infty[,L^2(\mathbb{T})\right)$, then $u\partial_x u\in L^\infty\left([0,+\infty[,L^2(\mathbb{T})\right)$. Which implies

$(1-\partial_{xx})^{-1}u\partial_x u \in L^\infty\left([0,+\infty[, H^2(\mathbb{T})\cap \dot{H}^1(\mathbb{T})\right)$. From (47) we have thus

$$\left\|\int_0^{t_n} S(s)(1-\partial_{xx})^{-1}u(t_n-s)\partial_x u(t_n-s)ds\right\|_{H^2} \le c\int_0^{t_n} \|\|S(s)\|\|_{\mathcal{L}(\dot{H}^2)}ds$$
$$\le c\int_0^{+\infty} e^{-\delta s}ds < \infty.$$

The sequence $\left(\int_0^{t_n} S(s)(1-\partial_{xx})^{-1}u(t_n-s)\partial_x u(t_n-s)ds\right)_n$ is uniformly bounded in $H^2(\mathbb{T})$, and then, it converges weakly in $H^2(\mathbb{T})$ up to a subsequence $(\phi_n)_n$ and so, strongly in $H^1(\mathbb{T})$ to u_∞.
Now let $\left\{\tilde{S}(t)\right\}_{t\in\mathbb{R}}$ be the nonlinear semigroup, in other words $u(t)=\tilde{S}(t)u_0$. On the one hand, for $s\in\mathbb{R}$ we have

$$\tilde{S}(s)u(t_{\phi_n}) = \tilde{S}\left(t_{\phi_n}+s\right)u_0 = u(t_{\phi_n}+s).$$

On the other hand, we use the continuity property of u seen in Theorem 6 to write

$$\lim_{n\to+\infty}\tilde{S}(s)u(t_{\phi_n}) = \tilde{S}(s)u_\infty = u_\infty(s).$$

This give us the limit (56).

Remark 4. *The function u_∞ given by Theorem 7 will be now considered as initial data. We will write it down $u_\infty(0)$, and we write down u_∞ the corresponding solution.*

Now let us prove the analyticity of u_∞. To do that we shall apply the following theorem, see [16, Theorem 2.20] with assumptions (H3mod) and (H5).

Theorem 8. *Let Y be a complex Banach space. Let $P_n\in\mathcal{L}(Y)$ be a sequence of continuous linear maps and let $Q_n = Id - P_n$. Let $A:\mathcal{D}(A)\to Y$ be the generator of a continuous semigroup $\{e^{tA}\}_{t\ge 0}$ and let $G\in\mathcal{C}^1(Y)$. We assume that V is a complete mild solution in Y of*

$$V'(t) = AV(t) + G(V(t)),\ t\in\mathbb{R}. \tag{58}$$

We further assume that

1) *$\{V(t),\ t\in\mathbb{R}\}$ is contained in a compact set K of Y.*

2) *For any* $y \in Y$, $(P_n y)_n$ *converges to* y *when* $n \to +\infty$ *and* (P_n) *and* $(Q_n)_n$ *are sequences of* $\mathcal{L}(Y)$ *bounded by* K_0.

3) *The operator* A *splits in* $A = A_1 + B_1$ *where* B_1 *is bounded and* A_1 *commutes with* P_n.

4) *There exists* $M, \lambda > 0$ *such that* $\left|\!\left|\!\left| e^{At} \right|\!\right|\!\right|_{\mathcal{L}(Y)} \le M e^{-ct}$ *and* $\left|\!\left|\!\left| e^{(A_1 + Q_n BB^*)t} \right|\!\right|\!\right|_{\mathcal{L}(Q_n Y, Y)} \le M e^{-ct}$ *for all* $t \ge 0$.

5) G *is analytic in the ball* $B_Y(0,r)$, *where* r *is such that* $r \ge 4 \sup_{t\in\mathbb{R}} \|V(t)\|_Y$.

6) $\{DG(V(t))Z \ / \ t \in \mathbb{R}, \|Z\|_Y \le 1\}$ *is relatively compact set of* Y.

Then, the solution $V(t)$ *is analytic from* $t \in \mathbb{R}$ *into* Y.

We use this theorem taking $A : u \mapsto -(1-\partial_{xx})^{-1}(\partial_x u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u)$ and $G : u \mapsto -(1-\partial_{xx})^{-1}(u\partial_x u)$. Note that u_∞ verifies the same equation as u. We deduce from this the existence of u_∞ on all $\mathbb{R}$ as well as the first part of the assumption 4.

We check the rest of the assumptions.

Proposition 5. *Let* $u_\infty(0)$ *be the function obtained by Theorem 7 and* $u_\infty(t)$ *the corresponding solution. Then there exists* $c > 0$ *such that*

$$\sup_{t\in\mathbb{R}} \|u_\infty(t)\|_{H^1(\mathbb{T})} \le c. \tag{59}$$

Proof. Let $t \in \mathbb{R}$ and $(t_{\phi(n)})_n$ the subequence given by Theorem 7. Using Proposition 4, there exists $N_1 > 0$ be such that for all $n \ge N_1$ we have $\left\|u_\infty(t) - u(t_{\phi(n)} + t)\right\|_{H^1} \le 1$. On the other hand, there exist $N_2 > 0$ such that for $n \ge N_2$ we have $t_{\phi(n)} + t > 0$. We deduce from (52) that for $n \ge N_1 + N_2$

$$\begin{aligned} \|u_\infty(t)\|_{H^1} &\le \left\|u_\infty(t) - u(t_{\phi(n)} + t)\right\|_{H^1} + \left\|u(t_{\phi(n)} + t)\right\|_{H^1} \\ &\le 1 + \|u(0)\|_{H^1} \le c. \end{aligned}$$

It follows the estimate (59). □

Proposition 6. *Let* $u_\infty(0)$ *be the function obtained by Theorem 7 and* $u_\infty(t)$ *the corresponding solution. Then there exist* $c > 0$ *such that*

$$\sup_{t\in\mathbb{R}} \|u_\infty(t)\|_{H^2(\mathbb{T})} \le c. \tag{60}$$

Proof. Let $t \in \mathbb{R}$, $n \in \mathbb{N}$ and v a solution of the nonlinear problem such that $v(\cdot) = u_\infty(\cdot + t - n)$. We can write

$$v(\sigma) = S(\sigma)v(0) - \int_0^\sigma S(\sigma - s)(1 - \partial_{xx})^{-1} v(s)\partial_x v(s) ds.$$

Note that $v(n - t) = u_\infty(0)$. Thus, we can use a simple change of variables to get

$$\begin{aligned} u_\infty(t) = {} & S(n)u_\infty(t-n) \\ & + \int_0^n S(\tau)(1 - \partial_{xx})^{-1} u_\infty(t-\tau)\partial_x u_\infty(t-\tau) d\tau. \end{aligned} \tag{61}$$

We know that $\|S(n)u_\infty(t-n)\|_{H^1(\mathbb{T})} \leq ce^{-\delta n} \|u_\infty(t-n)\|_{H^1(\mathbb{T})} \leq Ce^{-\delta n}$. Then, when n goes to $+\infty$, (61) becomes

$$u_\infty(t) = \int_0^{+\infty} S(\tau)(1 - \partial_{xx})^{-1} u_\infty(t-\tau)\partial_x u_\infty(t-\tau) d\tau.$$

Since the operator $u \longmapsto (1 - \partial_{xx})^{-1} u \partial_x u$ is bounded from $\dot{H}^1(\mathbb{T})$ into $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$, using Lemma 5 and Proposition 5 we obtain

$$\|u_\infty(t)\|_{H^2(\mathbb{T})} \leq c \int_0^{+\infty} e^{-\delta\tau} \|u_\infty(t-\tau)\|^2_{H^1(\mathbb{T})} d\tau \leq c \int_0^{+\infty} e^{-\delta\tau} d\tau \leq C$$

$\{u_\infty(t),\ t \in \mathbb{R}\}$ is then a bounded set of $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$, and consequently it is a relatively compact set of $\dot{H}^1(\mathbb{T})$. □

We deduce that u_∞ satisfies the first assumption of Theorem 8 with $K = \overline{\{u_\infty(t),\ t \in \mathbb{R}\}}$.

We define $(P_n)_n$ and $(Q_n)_n$ for $u = \sum_{k \in \mathbb{Z}^*} u_k e^{ikx} \in \dot{H}^1(\mathbb{T})$ with

$$P_n(u) = \sum_{|k| \leq n} u_k e^{ikx} \quad \text{and} \quad Q_n(u) = \sum_{|k| > n} u_k e^{ikx}.$$

Clearly $\|P_n\|_{\mathcal{L}(H^1, H^1)} \leq 1$ and $\|Q_n\|_{\mathcal{L}(H^1, H^1)} \leq 1$, as well as $(P_n(u))_n$ converges towards u for all $u \in \dot{H}^1(\mathbb{T})$.

We denote by $A_1 = -(1 - \partial_{xx})^{-1}(\partial_x + \partial_{xxx})$ and $B = (1 - \partial_{xx})^{-1}\partial_x a^{\frac{1}{2}}(1 - \partial_{xx})^{\frac{1}{2}}$. It is clear too that A_1 commutes with P_n. The assumption 2 and 3 of Theorem 8 are verified.

We can easily check that the adjoint of B in $H^1(\mathbb{T})$ is given by $B^* = -(1-\partial_{xx})^{\frac{-1}{2}} a^{\frac{1}{2}} \partial_x$. Furthermore, $BB^* = -(1-\partial_{xx})^{-1}\partial_x a \partial_x$. We can write $A = A_1 + BB^*$. The operator BB^* is nonnegative and $(BB^*u, u)_{H^1} = \int_{\mathbb{T}} a(x)|\partial_x u(t,x)|^2 dx$. We will now prove the following result.

Theorem 9. *There exists $M > 0$ and $c > 0$ be such that for every $t \geq 0$ and for every $n \in \mathbb{N}$*

$$\left|\left|\left| e^{(A_1 + Q_n BB^*)t} \right|\right|\right|_{\mathcal{L}(Q_n H^1, H^1)} \leq M e^{-ct}. \tag{62}$$

Let us start by giving an abstract result following the same approach as [17]. We consider a complex Hilbert space H, A_1 an unbounded skew-adjoint, m-dissipative linear operator on H and B a bounded linear operator such as $BB^* \geq 0$.

We also consider these two equations

$$\varphi'(t) + A_1 \varphi(t) = 0, \tag{63}$$

and

$$y'(t) + A_1 y(t) + BB^* y(t) = 0. \tag{64}$$

Theorem 10. *The following properties are equivalent.*

1) *There exists $T_0 > 0$ and $c > 0$ such that every solution φ of (63) satisfies*

$$\|\varphi(0)\|_H^2 \leq c \int_0^{T_0} \|B^* \varphi(s)\|_H^2 \, ds. \tag{65}$$

2) *There exists $T_1 > 0$ and $\delta > 0$ such that every solution y of (64) satisfies*

$$\forall y_0 \in \mathcal{D}(A_1), \forall t \geq T_1, \ \|y(t)\|_H = \|S(t)y_0\|_H \leq e^{-\delta t} \|y_0\|_H, \tag{66}$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup generated by $A_1 + BB^*$.

The proof is given in Annex.

Proposition 7. *Let $u \in \dot{H}^1(\mathbb{T})$, and let v be a solution of*

$$\begin{cases} \partial_t v = (A_1 + Q_n BB^*)\, v \\ v(0) = Q_n u \end{cases}$$

and w a solution of

$$\begin{cases} \partial_t w = Q_n\,(A_1 + BB^*)\, Q_n w \\ w(0) = Q_n u. \end{cases}$$

Then $v = w$.

Proof. Let $z = Q_n w$. We have

$$\partial_t z = Q_n \partial_t w = Q_n\,(Q_n(A_1 + BB^*) Q_n w) = Q_n\,(A_1 + BB^*)\, Q_n z.$$

Since $z(0) = w(0)$, by uniqueness $w = z = Q_n w$, and we have

$$\begin{aligned} \partial_t w &= Q_n\,(A_1 + BB^*)\, Q_n w = Q_n A_1 Q_n w + Q_n BB^* Q_n w \\ &= A_1 Q_n Q_n w + Q_n BB^* Q_n w = (A_1 + Q_n BB^*)\, w \end{aligned} \tag{67}$$

Then $w = v$. □

We can now prove Theorem 9.

Proof. According to Proposition 7, it suffices to show that we have an exponential decrease for the semigroup associated with the operator $Q_n\,(A_1 + BB^*)\, Q_n$. For this, we will use Theorem 10. So we must prove that

$$\|\varphi(0)\|^2_{H^1} \le c_1 \int_0^{T_0} \|Q_n B^* \varphi(s)\|^2_{H^1}\, ds, \tag{68}$$

where φ is solution of

$$\begin{cases} \varphi'(t) = Q_n A_1 Q_n \varphi(t) \\ \varphi(0) = Q_n \varphi(0). \end{cases}$$

We consider the problem

$$\begin{cases} \psi'(t) = A_1 \psi(t) \\ \psi(0) = Q_n \varphi(0). \end{cases}$$

From (9) and Theorem 10, ψ satisfies

$$\|\psi(0)\|_{H^1}^2 \leq c_2 \int_0^{T_0} \|B^*\psi(s)\|_{H^1}^2 \, ds.$$

Note that the constant c_2 depends only on the operator A_1. So the estimate is uniform in n. Since Q_n commutes with A_1, we can take $\psi = Q_n\varphi$. Thus we have (68) and then (62). □

The two last assumptions of Theorem 8 are satisfied with the following proposition.

Proposition 8. *The map G is holomorphic from $\dot{H}^1(\mathbb{T})$ to itself. Moreover, the set$\{DG(u_\infty(t))h \ / \ t \in \mathbb{R}, \|h\|_{\dot{H}^1(\mathbb{T})} \leq 1\}$ is a bounded set in $\dot{H}^2(\mathbb{T})$.*

The proof is given in Annex.

All the assymptions of Theorem 8 are verified, the solution u_∞ of the nonlinear problem is an analytical function.

4.2. Unique continuation

We will show in this last part that u_∞ is constant for all $(x,t) \in \mathbb{T} \times \mathbb{R}$.

Proposition 9. *There exists $c^* \geq 0$ such that for all $(x,t) \in \mathbb{T} \times \mathbb{R}$,*

$$u_\infty(x,t) = c^*.$$

Let $-2\pi < \alpha < \beta < 0$ be two real numbers such that $]\alpha, \beta[\subset \{a > 0\}$. We first give the following proposition.

Proposition 10. *There exists $c^* \geq 0$ such that for all $(x,t) \in]\alpha, \beta[\times \mathbb{R}$,*

$$u_\infty(x,t) = c^*.$$

Proof. Let u be a solution of the non linear problem. From Remark 3, $\|u(t)\|_{H^1(\mathbb{T})}^2$ is nonincreasing to a constant $c \geq 0$, and we have with the notation of Theorem 7

$$\lim_{n \to +\infty} \|u(t_n + t) - u_\infty(t)\|_{H^1(\mathbb{T})} \longrightarrow 0, \ \forall \ t \in \mathbb{R},$$

so $\|u_\infty(t)\|_{H^1(\mathbb{T})}^2 = c$ for all $t \in \mathbb{R}$. Now

$$\|u_\infty(t)\|_{H^1(\mathbb{T})}^2 - \|u_\infty(0)\|_{H^1(\mathbb{T})}^2 = -2 \int_0^t a(x) \, |\partial_x u_\infty(x,\sigma)|^2 \, d\sigma, \ \forall \ t \in \mathbb{R}.$$

Then

$$\int_0^t a(x)\,|\partial_x u_\infty(x,\sigma)|^2\,d\sigma = 0,\ \forall\, t\in\mathbb{R}.$$

This implies that $u_\infty(x,t)$ is a constant on x for $x \in\,]\alpha,\beta[$.

Now, since u_∞ satisfies

$$(1-\partial_{xx})\partial_t u_\infty + \partial_{xxx}u_\infty + \partial_x u_\infty + u_\infty\partial_x u_\infty = 0,$$

and $\partial_x u_\infty = 0$, we deduce that we have also $\partial_t u_\infty = 0$. □

The function $v = u_\infty - c^*$ verifies the equation

$$\text{(69)}\quad \begin{cases} (1-\partial_{xx})\partial_t v + \partial_{xxx}v + (1+u_\infty)\partial_x v = 0, & (x,t)\in\mathbb{T}\times\mathbb{R}\\ v(x,0) = v_0 \in H^1(\mathbb{T}), & \\ v(x,t) = 0, & (x,t)\in\,]\alpha,\beta[\times\mathbb{R}. \end{cases}$$

Let $P(x,\partial_x,t,\partial_t) = \partial_{xxx} - \partial_{xx}\partial_t + \partial_t + w(t,x)\partial_x$, where $w = 1+u_\infty$. The principal symbol is given by $p(x,t,\xi,\tau) = \xi^2(\xi-\tau)$.

Let $x^* \in]\alpha,\beta[$. We denote by

$$\psi : (x,t) \longmapsto (x-x^*)^2 - t^2,$$

and

$$\Gamma = \{(x,t,\xi,0)\ /\ x\in\mathbb{T}, t,\xi\in\mathbb{R}\}.$$

We recall the following definition (see [31, Definition 1.2])

Definition 2. *(Pseudo-convex surface) Let P be a differential operator of ordre m, with a principal symbol p, S a level set of a smooth function ψ, and $(x_0,t_0)\in S$ such that $\nabla\psi(x_0,t_0)\neq 0$. We say that S is strongly pseudo-convex in (x_0,t_0) with respect to P on Γ if*

1)

$$\mathfrak{Re}\{\overline{p},\{p,\psi\}\}(x_0,t_0,\xi,0) > 0,$$

on $\{(x_0,t_0,\xi,0)\in\Gamma\ /\ p(x_0,t_0,\xi,0) = \{p,\psi\}(x_0,t_0,\xi,0) = 0,$ with $\xi\neq 0\}$, 2)

$$\frac{1}{\gamma i}\left\{\overline{p}(x,t,\xi-i\gamma\psi'_x,\tau-i\gamma\psi'_t),p(x,t,\xi+i\gamma\psi'_x,\tau+i\gamma\psi'_t)\right\}(x_0,t_0,\xi,0) > c(\xi^2+\gamma^2)^{m-1}$$

on $\{(x_0,t_0,\xi,0)\in\Gamma/\ p(x_0,t_0,\xi+i\gamma\psi'_x,i\gamma\psi'_t)$
$=\{p(x,t,\xi+i\gamma\psi'_x,\tau+i\gamma\psi'_t),\psi\}(x_0,t_0,\xi,0)=0,\ \text{with } \gamma>0\}$.

We will use this theorem given in [31, Theorem 2]

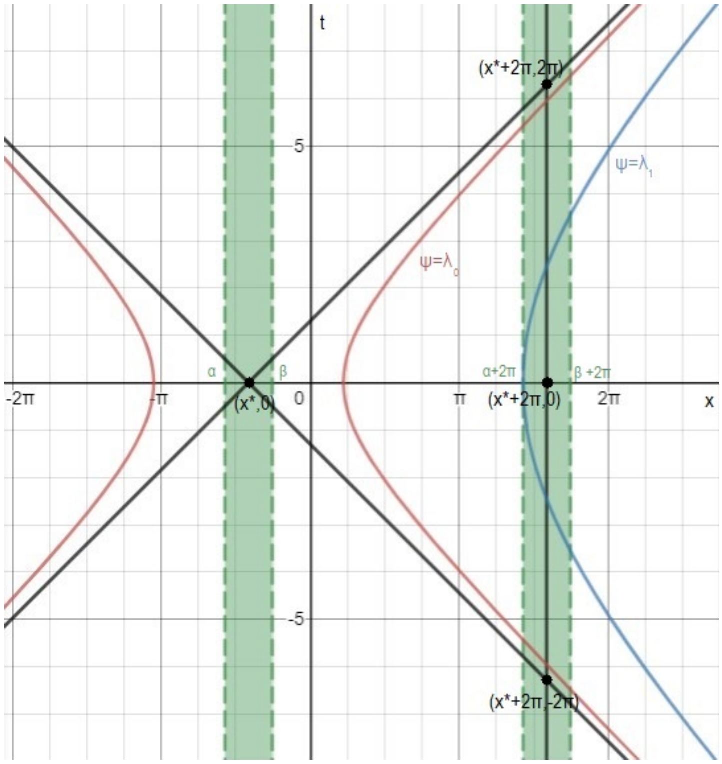
Theorem 11. *Let K be an open set of $\mathbb{R}^n$ and P a differential operator of ordre m such that*

1) *The principal symbol of P is real and whose coefficients are independent of t and assumed to be in $\mathcal{C}^1(\mathbb{T})$,*
2) *The coefficients of lower order terms of P are analytic from $I\subset\mathbb{R}$ into $L^\infty(\mathbb{T})$,*

Let $(x_0,t_0)\in\mathbb{T}\times\mathbb{R}$ and ψ a smooth function such that $\nabla\psi(x_0,t_0)\neq 0$. Assume that the level surface $\{\psi(x,t)=\psi(x_0,t_0)\}$ is strongly pseudo-convex in (x_0,t_0) with respect to P on Γ. Then there exists an open neighbourhood V of (x_0,t_0) such that if u is solution of $P(x,\mathcal{D})u=0$ in K and $u=0$ in $\{\psi>\psi(x_0,t_0)\}$, then $u=0$ in V.

Clearly P verifies the first assumption of the theorem. The second assumption is also verified due to Proposition 5 and Theorem 7.

We will show now that the level surfaces of the function ψ are strongly pseudo-convex on Γ. We give the following figure.



Proposition 11. *Let* $\lambda_1 = \psi(\alpha + 2\pi, 0) = (\alpha + 2\pi - x^*)^2$*. The level surfaces* $\{\psi = \lambda\}$ *for* $\lambda \in]0, \lambda_1]$ *are strongly pseudo-convex with respect to* P *on* Γ*.*

Proof. We will show that the sets

$$\{(x_0, t_0, \xi, 0) \in \Gamma \;/\; p(x_0, t_0, \xi, 0) = \{p, \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \xi \neq 0\},$$

and

$$\begin{aligned}\{(x_0, t_0, \xi, 0) \in \Gamma \;/\; & p(x_0, t_0, \xi + i\gamma\psi'_x, i\gamma\psi'_t) \\ & = \{p(x, t, \xi + i\gamma\psi'_x, \tau + i\gamma\psi'_t), \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \gamma > 0\}\end{aligned}$$

are empty. For the first set, we already notice that the case $p(x_0, t_0, \xi, 0) = \xi^3 = 0$ give necessarily $\xi = 0$, which is impossible. For the second one, we have $\gamma > 0$, then $p(x_0, t_0, \xi + i\gamma\psi'_x, i\gamma\psi'_t) = (\xi + 2i\gamma(x - x^*))^2(\xi + 2i\gamma(t + x - x^*))$ is equal to 0 when $\xi = 0$ and $x = x^*$, or when $\xi = 0$ and $t + x - x^* =$

0. The first case is impossible since $x = x^*$ means that $\psi = -t^2 < 0$, but we want that the surfaces $\{\psi = \lambda\}$ are stongly pseudo-convex for $\lambda \in]0, \lambda_1]$. For the same reason, the second one is also impossible since $t + x - x^* = 0$ correspond for all $t \in \mathbb{R}$ to $\{\psi = 0\}$. □

The assumptions of Theorem 11 are satisfied. So we can use it to show the following result.

Theorem 12. *Let $\mathcal{T}$ be the triangle of vertices $\{(x^*, 0); (x^* + 2\pi, 2\pi); (x^* + 2\pi, -2\pi)\}$, then the function v is identically zero on $\mathcal{T}$.*

The proof is given in Annex.

Remark 5. *The constant λ_1 is chosen so that $\big(\{\psi \geq \lambda_1\} \cap \mathcal{T}\big) \subset \big(\{a > 0\} \times \mathbb{R}\big)$. So combinning this with the last line of equation (69), we have clearly that $v = 0$ on $\{\psi \geq \lambda_1\} \cap \mathcal{T}$.*

The proof of Proposition 9 is easy now.

Proof. We conclude from the preceding result that v is equal to zero everywhere since $v = 0$ on $]x^*, x^* + 2\pi] \times \{t = 0\}$, and then, the function v_0 in (69) is equal to 0. Thus we obtain $v(x,t) = u_\infty(x,t) - c^* = 0$ for all $(x,t) \in \mathbb{T} \times \mathbb{R}$. □

Remark 6. *Note that [20, Theorem 28.3.4] is not applicable here since the level sufaces of ψ are not strongly pseudo-convex with respect to P in the sense of the theorem. Indeed, by a simple calculus we can prove that the assumption*

$$\mathfrak{Re}\,\{\overline{p}, \{p, \psi\}\}\,(x_0, t_0, \xi, \tau) > 0,$$

in the definition of pseudo-convexity in [20] is not verified on the set

$$\left\{(\xi,\tau) \neq (0,0) \in \mathbb{R}^2 \;/\; p(x_0, t_0, \xi, \tau) = \left\langle p'_{(\xi,\tau)}(x_0, t_0, \xi, \tau), \psi'_{(x,t)}(x_0, t_0)\right\rangle = 0\right\},$$

since the choice $\xi = 0$ and $\tau \neq 0$ vanish the two quantities $\left\langle p'_{(\xi,\tau)}(x_0, t_0, \xi, \tau), \psi'_{(x,t)}(x_0, t_0)\right\rangle$, $p(x_0, t_0, \xi, \tau)$ and vanish also the term $\mathfrak{Re}\,\{\overline{p}, \{p, \psi\}\}$.

We give finally a proof of Theorem 1.

Proof. Let $u = \sum_{k\in\mathbb{Z}} u^k(t)e^{ikx}$ be the solution of the nonlinear problem. Combining Theorem 7 and Proposition 9 we can write for all $T > 0$

$$\lim_{n\to+\infty} \sup_{s\in[-T,T]} \|u(t_n + s) - c^*\|_{H^1(\mathbb{T})} = 0.$$

Thus

$$\lim_{n\to+\infty} \sup_{s\in[-T,T]} \left(\left(u^0(t_n + s) - c^*\right)^2 + \sum_{k\in\mathbb{Z}^*} (1 + k^2) \left(u^k(t_n + s)\right)^2 \right)^{\frac{1}{2}} = 0.$$

We already know that $u^0(t_n + s)$ is constant in time, in particular $u^0(t_n + s) = u^0(0)$, and since the term $\sum_{k\in\mathbb{Z}^*} (1 + k^2) \left(u^k(t_n + s)\right)^2$ goes to 0, we deduce that $c^* = u^0(0)$. We finish by proving that $\lim_{t\to+\infty} u(t, \cdot) = c^*$. Suppose that there exists $(s_n)_n$ a sequence of times which goes to $+\infty$ and $\|u(s_n) - c^*\|_{H^1(\mathbb{T})} \geq \epsilon$, for some $\epsilon > 0$. From Theorem 7, there exists a subsequence $(s_{\phi(n)})_n$ such that $\lim_{n\to+\infty} \left\|u(s_{\phi(n)}) - c^*\right\|_{H^1(\mathbb{T})} = 0$, which is absurd. □

5. A remark on the BBM equation

We consider this BBM equation

$$\text{(70)} \quad \begin{cases} \partial_t v + \partial_x v - \partial_{xxt} v - \partial_x(a(x + ct)\partial_x)v + v\partial_x v = 0, & x \in \mathbb{T},\ t > 0, \\ v(\cdot, 0) = v_0, & x \in \mathbb{T}, \end{cases}$$

where $c > 0$ and $a \geq 0$ is assumed to be a bounded function in $\mathcal{C}^\infty(\mathbb{T})$ such that $\{a > 0\} \neq \emptyset$.

We can prove the result of stabilization by following the same approach in the preceding paragraphs. After a change of variables we can write the equation under the form

$$\partial_t v - c\partial_{xxx} v - \partial_{xxt} v + (c + 1)\partial_x v - \partial_x(a(x)\partial_x)v + v\partial_x v = 0.$$

We prove that if v is the solution of the linear equation then

$$\frac{d}{dt} \left(\frac{1}{2} \|v(t)\|^2_{H^1} \right) = -2 \int_{\mathbb{T}} a(x) \left|\partial_x v(t, x)\right|^2 dx \leq 0.$$

Except that there are different constants that appear in the equation compared to what we did previously, the results are essentially the same, and we give the following theorem.

Theorem 13. *For all $v_0 \in H^1(\mathbb{T})$, there exists a unique solution $v = v(t,x)$ of* (70) *global in time such that*

$$\lim_{t\to+\infty} v(t,\cdot) = \frac{1}{2\pi}\int_{\mathbb{T}} v_0(x)dx, \quad \text{in } H^1(\mathbb{T}).$$

6. Annex

Proof. (of Lemma 2) Note that if $[\Psi(-ih\partial_x), a]\, u^k = \sum_{n\in\mathbb{Z}} \gamma_n^k e^{inx}$, then

$$\gamma_n^k = \Psi(hn)\sum_{j\in\mathbb{Z}} a_{n-j}u_j^k - \sum_{j\in\mathbb{Z}} a_{n-j}\Psi(hj)u_j^k = \sum_{j\in\mathbb{Z}} a_{n-j}\left(\Psi(hn)-\Psi(hj)\right)u_j^k.$$

We want to prove that

(71)

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left(1+n^2\right)|\gamma_n|^2 &= \sum_{n\in\mathbb{Z}}\left(1+n^2\right)\left(\sum_{j\in\mathbb{Z}} a_{n-j}\,|\Psi(hn)-\Psi(hj)|\,|z_j|\right)^2 \\ &\leq c\sum_{n\in\mathbb{Z}}|z_n|^2 .\end{aligned}$$

The constant δ is the one used in (18).

We write $\mathbb{Z}^2 = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5$, where

$$\Gamma_1 = \left\{(n,j)\in\mathbb{Z}^2\ ;\ n\leq 0 \text{ and } |hj-1|\leq 2\delta\right\},$$

$$\Gamma_2 = \left\{(n,j)\in\mathbb{Z}^2\ ;\ 0\leq hn\leq 1-3\delta \text{ and } |hj-1|\leq 2\delta\right\},$$

$$\Gamma_3 = \left\{(n,j)\in\mathbb{Z}^2\ ;\ hn\geq 1+3\delta \text{ and } |hj-1|\leq 2\delta\right\},$$

$$\Gamma_4 = \left\{(n,j)\in\mathbb{Z}^2\ ;\ |hn-1|\leq 3\delta\right\},$$

and

$$\Gamma_5 = \left\{(n,j)\in\mathbb{Z}^2\ ;\ |hn-1|\geq 3\delta \text{ and } |hj-1|\geq 2\delta\right\}.$$

In Γ_1, we have $\Psi(hn)=0$ and $|n-j|=|n|+j$. We know also that for all σ we have $|a_n|\leq \dfrac{1}{(1+|n|)^{\sigma}}$. Thus

$$\begin{aligned}
\sqrt{1+n^2}|\gamma_n| &\leq c\sum_{j\in\mathbb{Z}}\underbrace{\frac{\sqrt{1+n^2}}{1+|n|+j}}_{\leq c}\frac{1}{(1+|n-j|)^{\sigma-1}}\underbrace{|\Psi(hj)|}_{\leq 1}|z_j| \\
&\leq c\left(\sum_{j\in\mathbb{Z}}\frac{1}{(1+|n-j|)^{\sigma-1}}\right)^{\frac{1}{2}}\left(\sum_{j\in\mathbb{Z}}\frac{1}{(1+|n-j|)^{\sigma-1}}|z_j|^2\right)^{\frac{1}{2}} \\
&\leq c\left(\sum_{j\in\mathbb{Z}}\frac{1}{(1+|n-j|)^{\sigma-1}}|z_j|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\sum_{n\in\mathbb{Z}}\left(1+n^2\right)|\gamma_n|^2\leq c\sum_{j\in\mathbb{Z}}|z_j|^2\sum_{n\in\mathbb{Z}}\frac{1}{(1+|n-j|)^{\sigma-1}}.$$

The term $\displaystyle\sum_{n\in\mathbb{Z}}\frac{1}{(1+|n-j|)^{\sigma-1}}$ converges and it does not depend on j. We deduce the estimate (71) on Γ_1.

In Γ_2, $\Psi(hn)=0$, and we have

$$h+|hn-hj|\geq h+|hn-1|-|hj-1|\geq h+3\delta-2\delta\geq\delta.$$

It follows that $\dfrac{1}{1+|n-j|}\leq ch$, thus

$$\sqrt{1+n^2}|\gamma_n|\leq c\sum_{j\in\mathbb{Z}}\frac{h\sqrt{1+n^2}}{(1+|n-j|)^{\sigma-1}}\left|z_j\right|.$$

Since in Γ_2 we have $hn\leq 1-3\delta$, the term $h\sqrt{1+n^2}$ is bounded independently of n. We obtain with the same Hölder inequality as in Γ_1 the estimate (71).

In Γ_3, $\Psi(hn)=0$ and we have also

$$h+|hn-hj|\geq h+hn-hj\geq h+1+3\delta-(2\delta+1)\geq\delta.$$

So

$$\frac{1}{1+|n-j|} \lesssim h. \tag{72}$$

Furthermore

$$n \lesssim |n-j| + |j| \lesssim |n-j| + \frac{1}{h}. \tag{73}$$

By multiplying (72) and (73) we get

$$\frac{n}{1+|n-j|} \lesssim h|n-j| + 1.$$

Hence we estimate

$$\sqrt{1+n^2}|\gamma_n| \lesssim \sum_{j\in\mathbb{Z}} \frac{h|n-j|+1}{(1+|n-j|)^{\sigma-1}} |z_j| \lesssim \sum_{j\in\mathbb{Z}} \frac{1}{(1+|n-j|)^{\sigma-2}} |z_j|,$$

and we use again Hölder inequality as in Γ_1 to obtain (71).

In Γ_4, we use the mean value theorem to write

$$|\Psi(hn) - \Psi(hj)| \leq ch|n-j|.$$

Then

$$\sqrt{1+n^2}|\gamma_n| \leq c\sum_{j\in\mathbb{Z}} \frac{|n-j|}{(1+|n-j|)^{\sigma}} |z_j| \leq c\sum_{j\in\mathbb{Z}} \frac{1}{(1+|n-j|)^{\sigma-1}} |z_j|.$$

As in Γ_2, we obtain (71).

In Γ_5, we have $\Psi(hn) = \Psi(hj) = 0$. The result follows directly. □

Proof. (of Theorem 10) We suppose (65). Let $y(t) = S(t)y_0$ be a solution of (64) and $E(t) = \frac{1}{2}\|y(t)\|_H^2$.

We have for every $t \geq 0$

$$\left(y'(t), y(t)\right)_H + (A_1 y(t), y(t))_H = -(BB^* y(t), y(t))_H.$$

Since A_1 is skew-adjoint, we obtain

$$E'(t) = -\|B^* y(t)\|_H^2 \leq 0. \tag{74}$$

Thus E is nonincreasing. Now let $T = kT_0$, where k is to be chosen further, by integrating (74) between 0 and T we get

$$\frac{1}{2}\|y(T)\|_H^2 - \frac{1}{2}\|y(0)\|_H^2 = -\int_0^T \|B^* y(s)\|_H^2\, ds.$$

Then

$$\int_0^T \|B^* y(s)\|_H^2\, ds \le E(0). \tag{75}$$

This implies that there exists $p \in \{0, ..., k-1\}$ such that

$$\int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H^2\, ds \le \frac{1}{k}E(0). \tag{76}$$

Let v be the solution of

$$\begin{cases} v'(t) + A_1 v(t) = 0 \\ v(pT_0) = y(pT_0). \end{cases} \tag{77}$$

The function $w = y - v$ satisfy $w'(t) + A_1 w(t) = -BB^* y(t),$ and $w(pT_0) = 0$. Thus

$$w(t) = -\int_{pT_0}^{t} V(t-s)BB^* y(s) ds,$$

where $\{V(t)\}_{t\ge 0}$ is the semigroup generated by A_1, which is a semigroup of contractions by the Hille-Yosida Theorem.

We set $M = \|B\|_{\mathcal{L}(H)}$. For $t \in [pT_0, (p+1)T_0]$ we have

$$\begin{aligned} \|w(t)\|_H &\le M \int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H\, ds \\ &\le M\sqrt{T_0}\left(\int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H^2\, ds\right)^{\frac{1}{2}} \le M\sqrt{\frac{T_0 E(0)}{k}}. \end{aligned} \tag{78}$$

It follows

$$\int_{pT_0}^{(p+1)T_0} \|B^* w(s)\|_H^2\, ds \le M^2 \int_{pT_0}^{(p+1)T_0} \|w(s)\|_H^2\, ds \le M^4 T_0^2 \frac{E(0)}{k}. \tag{79}$$

As we have

$$\|B^* v(t)\|_H^2 = \|B^* y(t) - B^* w(t)\|_H^2 \le 2\left(\|B^* y(t)\|_H^2 + \|B^* w(t)\|_H^2\right),$$

combinning (76) and (79) we get

$$\int_{pT_0}^{(p+1)T_0} \|B^*v(s)\|_H^2\, ds \le 2(1+M^4T_0^2)\frac{E(0)}{k}. \tag{80}$$

Since v verifies (77), we have from (65) and (80)

$$2E(pT_0) = \|v(pT_0)\|_H^2 \le 2c(1+M^4T_0^2)\frac{E(0)}{k}.$$

Since E is nonincreasing, then $E(kT_0) \le c(1+M^4T_0^2)\dfrac{E(0)}{k}$. So, for $k > 4c(1+M^4T_0^2)$ and $T = kT_0$ we obtain $\|y(T)\|_H \le \dfrac{1}{2}\|y_0\|_H$, and it is classical that this implies (66).

Reciprocally, suppose that we have (66). Since E decreases towards 0, we can find $T_0 > 0$ such that every solution y of (64) verifies

$$\int_0^{T_0} \|B^*y(s)\|_H^2\, ds = E(0) - E(T_0) > \frac{1}{2}E(0) = \frac{1}{4}\|y_0\|_H^2. \tag{81}$$

Now let φ be a solution of (63) and y a solution of (64) with the initial data $\varphi(0) = y(0)$. The function $z = y - \varphi$ verifies $z'(t) + A_1 z + BB^*z = -BB^*\varphi$. Then

$$z(t) = -\int_0^t S(t-s)BB^*\varphi(s)ds.$$

For all $t \in [0, T_0]$ we have

$$\|z(t)\|_H^2 \le c\int_0^{T_0} \|B^*\varphi(s)\|_H^2\, ds \tag{82}$$

Since $\|B^*y(s)\|_H \le \|B^*\varphi(s)\|_H + \|B^*z(s)\|_H$, using (82) and the fact that B^* is bounded

$$\int_0^{T_0} \|B^*y(s)\|_H^2\, ds \le c\int_0^{T_0} \|B^*\varphi(s)\|_H^2\, ds.$$

From (81), we get (65). □

Proof. (of Proposition 8) We recall that for $u \in \dot{H}^1(\mathbb{T})$, $G(u) = -(1-\partial_{xx})^{-1}(u\partial_x u)$. We know that $G(u) \in \dot{H}^1(\mathbb{T})$ since

$$\begin{aligned}(G(u),1)_{L^2} &= -\left((1-\partial_{xx})^{-1}(u\partial_x u),1\right)_{L^2}\\ &= -\frac{1}{2}\left((1-\partial_x)^{-1}\partial_x\left(u^2\right),1\right)_{L^2} = 0.\end{aligned}$$

We give the following definition of a holomorphic function on a Banach space. See [19] for more details.

Definition 3. *Let X and Y be complex Banach spaces and let $\mathcal{D}$ be an open subset of X. A function $G : \mathcal{D} \longrightarrow Y$ is holomorphic if for each $u \in \mathcal{D}$, there exists a continuous complex-linear mapping $DG(u) : X \longrightarrow Y$ such that*

$$\lim_{h\longrightarrow 0}\frac{\|G(u+h)-G(u)-DG(u)h\|_Y}{\|h\|_X} = 0. \tag{83}$$

Let u and h be two functions in $\dot{H}^1(\mathbb{T})$. We have

$$\begin{aligned}G(u+h) &= -(1-\partial_{xx})^{-1}((u+h)\partial_x(u+h))\\ &= G(u) - (1-\partial_{xx})^{-1}(u\partial_x h + h\partial_x u) + G(h)\end{aligned}$$

The map $DG(u) : h \longrightarrow -(1-\partial_{xx})^{-1}(u\partial_x h + h\partial_x u)$ is linear and continuous, indeed

$$\begin{aligned}\|DG(u)h\|_{H^1(\mathbb{T})} &\le \|DG(u)h\|_{H^2(\mathbb{T})}\\ &\le \big\|u\big\|_{L^\infty(\mathbb{T})}\|h\|_{H^1(\mathbb{T})} + \|h\|_{L^\infty(\mathbb{T})}\|\partial_x u\|_{L^2(\mathbb{T})}\\ &\le 2\big\|u\big\|_{H^1(\mathbb{T})}\|h\|_{H^1(\mathbb{T})}.\end{aligned} \tag{84}$$

Furthermore

$$\|G(u+h)-G(u)-DG(u)h\|_{H^1} = \|G(h)\|_{H^1} \le \|h\|_{L^\infty}\|\partial_x h\|_{L^2} \le \|h\|_{H^1}^2.$$

We have then (83). Now let u_∞ be the solution of the nonlinear problem with the initial data $u_\infty(0)$, and $h \in \dot{H}^1(\mathbb{T})$ such that $\|h\|_{H^1(\mathbb{T})} \le 1$. We know that $u_\infty \in L^\infty\left(\mathbb{R}, \dot{H}^1(\mathbb{T})\right)$ from Proposition 5. Estimation (84) shows that $\{DG(u_\infty(t))h \;/\; t \in \mathbb{R}, \|h\|_{\dot{H}^1(\mathbb{T})} \le 1\}$ is a bounded set in $\dot{H}^2(\mathbb{T})$.

Thanks to the compact injection $\dot{H}^2(\mathbb{T}) \hookrightarrow \dot{H}^1(\mathbb{T})$, we deduce the assumption 6 of Theorem 8. □

Proof. (of Theorem 12) Let

$$\lambda_0 = \inf\left\{\lambda \in \left]0,(2\pi)^2\right[\ / \ v(x,t)=0,\ \forall (x,t)\in\{\psi>\lambda\}\cap\mathcal{T}\right\}.$$

It is sufficient to prove that $\lambda_0 = 0$. Assume that $\lambda_0 > 0$.

We know that $v = 0$ on $\{\psi \geq \lambda_1\}\cap\mathcal{T}$ from Remark 5. By applying Theorem 11 to each point $(x,t)\in\{\psi=\lambda_0\}\cap\mathcal{T} =: \mathcal{T}_0$, there exists an open neighborhood $B(x,t)$ of (x,t) such that $v = 0$ on $B(x,t)$.

So, if $\mathcal{T}_0 \subset \bigcup_{(x_i,t_i)\in\mathcal{T}_0} B(x_i,t_i)$, there exists $B(x_1,t_1),...,B(x_p,t_p)$ by a compactness argument such that $\mathcal{T}_0 \subset \bigcup_{i=1}^{p} B(x_i,t_i)$.

We will show that there exists $\lambda \in [0,\lambda_0[$ such that $\{\psi=\lambda\}\cap\mathcal{T} \subset \bigcup_{i=1}^{p} B(x_i,t_i)$.

Suppose that for all $n\in\mathbb{N}^*$, $\left\{\psi = \lambda_0 - \frac{1}{n}\right\}\cap\mathcal{T} \not\subset \bigcup_{i=1}^{p} B(x_i,t_i)$.

Let $(y_n,s_n)\in\left\{\psi=\lambda_0-\frac{1}{n}\right\}\cap\mathcal{T}$. We know that for all $n\in\mathbb{N}^*$ $(y_n,s_n)\in \operatorname{supp}(v)$. Since $\left\{\psi=\lambda_0-\frac{1}{n}\right\}\cap\mathcal{T}$ is a compact set, there exists $(y_{\varphi(n)},s_{\varphi(n)})$ a subsequence of (y_n,s_n) such that $y_{\varphi(n)}\to y$ and $s_{\varphi(n)}\to s$. This give us that $\psi(y,s)=\lambda_0$. Hence, there exists $i\in\{1,..,p\}$ such that $(y,s)\in B(x_i,t_i)$. So for n large enough, we have $(y_{\varphi(n)},s_{\varphi(n)})\in B(x_i,t_i)$, and then, $(y_{\varphi(n)},s_{\varphi(n)})\notin \operatorname{supp}(v)$, which reach a contradiction. We deduce that $\lambda_0=0$ and then $v=0$ on $\mathcal{T}$. □

Acknowledgements

We thank the referee for his comments which helped us to precise some parts of this work and to improve the article. The first author would like to thank his thesis co-supervisor Luc Robbiano for his support throughout this work and for his patience.

References

[1] K. Ammari and E. Crépeau, *Well-posedness and stabilization of the Benjamin-Bona-Mahony equation on star-shaped networks*, Systems Control Lett. **127** (2019) 39–43.

[2] L. Baudouin, E. Crépeau, and J. Valein, *Two approaches for the stabilization of nonlinear KdV equation with boundary time-delay feedback*, IEEE Trans. Automat. Control 64 **no. 4** (2019) 1403–1414.

[3] C. Besse, P. Noble, and D. Sanchez, *Discrete transparent boundary conditions for the mixed KDV-BBM equation*, J. Comput. Phys. **345 35Q53 (76B15)** (2017) 484–509.

[4] J. Bona, X. Carvajal, M. Panthee, and M. Scialom, *Higher-order Hamiltonian model for unidirectional water waves*, J Nonlinear Sci **28** (2018) 543–577.

[5] J. Bona and H. Chen, *Comparison of model equations for small-amplitude long waves*, Nonlinear Analysis **38** (1999) 625–647.

[6] J. Bona, M. Chen, and J.-C. Saut, *Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory*, J. Nonlinear Sci. **12** (2002) 283–318.

[7] N. Burq, *Mesures semi classiques et mesures de défauts*, Séminaire Bourbaki, 49 ème année **826** (1997).

[8] ———, *Semi-classical estimates for the resolvant in non trapping geometries*, International Mathematics Research Notices **5** (2002) 221–241.

[9] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la controlabilité exacte des ondes*, Les Comptes rendus de l'Academie des sciences **325** (1997) 749–752.

[10] X. Carvajal, M. Panthee, and M. Scialom, *Comparison between model equations for long waves and blow-up phenomena*, J. Math. Anal. Appl **442 (1)** (2016) 273–290.

[11] E. Cerpa and E. Crépeau, *On the controllability of the improved Boussinesq equation*, SIAM J. Control Optim **56, no. 4** (2018) 3035–3049.

[12] D. Denys and E. Pelinovsky, *Numerical simulation of a solitonic gas in KdV and KdV-BBM equations*, Phys. Lett. A **378, no. 42** (2014) 3102–3110.

[13] P. Gérard, *Mesures semi-classiques et ondes de Bloch*, Séminaire Équations aux dérivées partielles (Polytechnique) **16** (1991) 1–19.

[14] P. Gérard and E. Leichtnam, *Microlocal defect measures*, Comm. Partial Differential Equations **16** (1991) 1761–1794.

[15] ———, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Mathematical Journal **71, no. 2** (1993) 559–607.

[16] J. Hale and G. Raugel, *Regularity, determining modes and Galerkin methods*, Journal de Mathématiques Pures et Appliquées **82** (2003) 1075–1136.

[17] A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Portugaliae Mathematica **Vol. 46 Fasc. 3** (1989).

[18] P. Harinck, A. Plagne, and C. Sabbah, *Chaos en mécanique quantique*, Les éditions de l'école polytechnique (2014).

[19] L. A. Harris, *Fixed Point Theorems for Infinite Dimensional Holomorphic Functions*, J. Korean Math. Soc. **41, No. 1** (2004) 175–192.

[20] L. Hörmander, The Analysis of Linear PD Operators. IV, Fourier Integral Operators, Springer (1985).

[21] F. Huang, *Caracteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. of Diff. Eqs. **1 (1)** (1985) 43–56.

[22] R. Joly and C. Laurent, *Stabilization for the semilinear wave equation with geometric control condition*, Analysis & PDE, Mathematical Sciences Publishers **6 (5)** (2013) 1089–1119.

[23] C. Laurent, L. Rosier, and Z. B-Y, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain.*, Comm. Partial Differential Equations **35, no. 4** (2010) 707–744.

[24] G. Lebeau, *Equation des ondes amorties.*, Séminaire Équations aux dérivées partielles (Polytechnique) **exp. no 15** (1994) 1–14.

[25] S. Mancas and A. Ronald, *Elliptic solutions and solitary waves of a higher order KdV-BBM long wave equation*, J. Math. Anal. Appl. **452, no. 2** (2017) 1168–1181.

[26] L. Miller, *Refraction of high-frequency waves density by sharp interfaces and semi classical measures at the boundary*, Journal de Mathematiques Pures et Appliquées **79, no. 3** (2000) 227–269.

[27] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer (1983).

[28] L. Robbiano and C. Zuily, *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients*, Inventiones mathematicae, Springer-Verlag **131** (1998) 493–539.

[29] ———, *The Kato Smoothing Effect for Schrödinger Equations with Unbounded Potentials in Exterior Domains*, International Mathematics Research Notices (2009).

[30] L. Rosier, *On the Benjamin-Bona-Mahony equation with a localized damping*, J. Math. Study **49, no. 2** (2016) 195–204.

[31] D. Tataru, *Unique continuation for solutions to PDE's; between Hörmander's Theorem and Holmgren's Theorem*, Communications in Partial Differential Equations **20:5-6** (1995) 855–884.

[32] M. Zworski, Semiclassical analysis, Vol. 138 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI (2012).

Laboratoire de Mathématiques de Versailles
Université Paris-Saclay, UVSQ, CNRS
78000, Versailles, France
Laboratoire de Mathématiques:
Modélisation Déterministe et Aléatoire
Université de Sousse, 4011, Hammam Sousse, Tunisie
E-mail address: mohamed-melek.jellouli@ens.uvsq.fr

Laboratoire de Mathématiques:
Modélisation Déterministe et Aléatoire
Université de Sousse, 4011, Hammam Sousse, Tunisie
E-mail address: moez.khenissi@essths.u-sousse.tn

Received October 14, 2020
Accepted October 19, 2021