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Continuous time soliton resolution for two-bubble equivariant wave maps

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We consider the energy-critical wave maps equation $\mathbb{R}^{1+2} \to \mathbb{S}^2$ in the equivariant case. We prove that if a wave map decomposes, along a sequence of times, into a superposition of at most two rescaled harmonic maps (bubbles) and radiation, then such a decomposition holds for continuous time. We deduce, as a consequence of sequential soliton resolution results of Côte [5], and Jia and Kenig [25], that any topologically trivial equivariant wave map with energy less than four times the energy of the bubble asymptotically decomposes into (at most two) bubbles and radiation.

1. Introduction

1.1. Setting of the problem

This paper concerns wave maps from the Minkowski space $\mathbb{R}^{1+2}_{t,x}$ into the two-sphere $\mathbb{S}^2$, with k-equivariant symmetry. These are formal critical points of the Lagrangian action,

$$\mathscr{L}(\Psi) = \frac{1}{2} \iint_{\mathbb{R}^{1+2}_{t,x}} \left(-|\partial_t \Psi(t,x)|^2 + |\nabla \Psi(t,x)|^2 \right) \mathrm{d}x\, \mathrm{d}t, \tag{1.1}$$

restricted to the class of maps $\Psi : \mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2 \subset \mathbb{R}^3$ that take the form,

$$\Psi(t, r\mathrm{e}^{i\theta}) = (\sin\psi(t,r)\cos k\theta, \sin\psi(t,r)\sin k\theta, \cos\psi(t,r)) \in \mathbb{S}^2 \subset \mathbb{R}^3, \tag{1.2}$$

for some fixed $k \in \{1, 2, \dots\}$. Here ψ is the colatitude measured from the north pole of the sphere, the metric on $\mathbb{S}^2$ is given by $\mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi\, \mathrm{d}\omega^2$, and (r, θ) are polar coordinates on $\mathbb{R}^2$.

Wave maps are called nonlinear σ-models in the high energy physics literature, see for example [19, 32]. They are a canonical example of a geometric wave equation as they generalize the free scalar wave equation to the geometric setting of manifold-valued maps. The $2d$ case we consider is of particular interest, since the static solutions given by finite energy harmonic maps are amongst the simplest examples of topological solitons; other examples include kinks in scalar field equations, vortices in Ginzburg-Landau equations, magnetic monopoles, Skyrmions, and Yang-Mills instantons; see [32]. Wave maps under k-equivariant symmetry possess intriguing features from the point of view of nonlinear dynamics, for example, bubbling harmonic maps, multi-soliton solutions, etc., in the relatively simple setting of a geometrically natural scalar semilinear wave equation.

The Cauchy problem for k-equivariant wave maps is given by

$$\begin{aligned} &\partial_t^2\psi - \partial_r^2\psi - \frac{1}{r}\partial_r\psi + k^2\frac{\sin 2\psi}{2r^2} = 0, \\ &(\psi(T_0), \partial_t\psi(T_0)) = (\psi_0, \dot\psi_0), \quad T_0 \in \mathbb{R}. \end{aligned} \tag{1.3}$$

The conserved energy is

$$E(\boldsymbol\psi(t)) := 2\pi\int_0^\infty \frac{1}{2}\left((\partial_t\psi)^2 + (\partial_r\psi)^2 + k^2\frac{\sin^2\psi}{r^2}\right) r\,\mathrm{d}r, \tag{1.4}$$

where we have used bold font to denote the vector $\boldsymbol\psi(t) := (\psi(t), \partial_t\psi(t))$. We will write pairs of functions as $\boldsymbol\phi = (\phi, \dot\phi)$, noting that the notation $\dot\phi$ will not, in general, refer to a time derivative of ϕ but rather just to the second component of $\boldsymbol\phi$. With this notation (1.3) can be rephrased as the Hamiltonian system

$$\partial_t\boldsymbol\psi(t) = J\circ \mathrm{D}E(\boldsymbol\psi(t)), \qquad \boldsymbol\psi(T_0) = \boldsymbol\psi_0, \tag{1.5}$$

where

$$\begin{aligned} J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathrm{D}E(\boldsymbol\psi(t)) &= \begin{pmatrix} -\partial_r^2\psi(t) - r^{-1}\partial_r\psi(t) + \frac{1}{2}k^2 r^{-2}\sin(2\psi(t)) \\ \partial_t\psi(t) \end{pmatrix}. \end{aligned} \tag{1.6}$$

We remark that both (1.5) and (1.4) are invariant under the scaling

$$(\psi(t,r), \partial_t\psi(t,r)) \mapsto \big(\psi(t/\lambda, r/\lambda), \lambda^{-1}\partial_t\psi(t/\lambda, r/\lambda)\big), \quad \lambda > 0, \tag{1.7}$$

which makes this problem energy-critical.

For $\psi_0 : (0, \infty) \to \mathbb{R}$, we denote

$$E_p(\psi_0) = 2\pi \int_0^\infty \frac{1}{2}\Big((\partial_r \psi)^2 + k^2 \frac{\sin^2 \psi}{r^2}\Big)\, r\, \mathrm{d}r \tag{1.8}$$

the potential part of the energy (1.4). It is easy to check that any k-equivariant state ψ_0 of finite potential energy must satisfy $\lim_{r\to 0}\psi_0(r) = \ell\pi$ and $\lim_{r\to\infty}\psi_0(r) = m\pi$ for some $\ell, m \in \mathbb{Z}$, which splits the set of states of finite potential energy into disjoint classes, which we denote $\mathcal{H}_{\ell,m}$. These classes are related to the topological degree of the full map $\Psi(t) : \mathbb{R}^2 \to \mathbb{S}^2$: if $m - \ell$ is even and $\psi_0 \in \mathcal{H}_{\ell,m}$, then the corresponding map Ψ is topologically trivial, whereas for odd $m - \ell$ we obtain maps of degree k.

The sets $\mathcal{H}_{\ell,m}$ are affine spaces, parallel to the linear space $\mathcal{H} := \mathcal{H}_{0,0}$, which we equip with the norm

$$\|\psi_0\|_{\mathcal{H}}^2 := \int_0^\infty \Big((\partial_r \psi_0(r))^2 + k^2 \frac{\psi_0(r)^2}{r^2}\Big)\, r\, \mathrm{d}r. \tag{1.9}$$

We denote $L^2 := L^2(r\mathrm{d}r)$ and $\mathcal{E}_{\ell,m} := \mathcal{H}_{\ell,m} \times L^2$ the set of finite energy initial data corresponding to the class $\mathcal{H}_{l,m}$. It is natural to consider the Cauchy problem (1.3) within a fixed class $\mathcal{E}_{\ell,m}$. The set $\mathcal{E} := \mathcal{E}_{0,0}$ is a linear space, which comes with the norm

$$\begin{aligned} \|\boldsymbol{\psi}_0\|_{\mathcal{E}}^2 &:= \|\psi_0\|_{\mathcal{H}}^2 + \|\dot\psi_0\|_{L^2}^2 \\ &= \int_0^\infty \Big((\partial_r \psi_0(r))^2 + k^2 \frac{\psi_0(r)^2}{r^2}\Big)\, r\, \mathrm{d}r + \int_0^\infty \dot\psi_0(r)^2\, r\, \mathrm{d}r. \end{aligned} \tag{1.10}$$

The unique (up to scaling and sign change, and adding a multiple of π) k-equivariant harmonic map is given explicitly by

$$Q(r) := 2\arctan(r^k). \tag{1.11}$$

The function Q, and its rescaled versions $Q_\lambda(r) := Q(\lambda^{-1} r)$ for $\lambda > 0$, are minimizers of E_p within the class $\mathcal{H}_{0,1}$. On can compute that $E_p(Q_\lambda) = 4\pi k$. We denote $\boldsymbol{Q}_\lambda := (Q_\lambda, 0)$ the initial data yielding the stationary solution of (1.3), $\boldsymbol{\psi}(t) = \boldsymbol{Q}_\lambda$.

Linearizing (1.3) around $\psi = 0$ leads to the equation

$$\partial_t^2 \psi_{\mathrm{L}} + L_0 \psi_{\mathrm{L}} = 0, \qquad \text{where} \quad L_0 := -\partial_r^2 - r^{-1}\partial_r + k^2 r^{-2}. \tag{1.12}$$

We say a solution $\boldsymbol{\psi}$ of (1.3) *scatters* in the energy space in the positive time direction if there exists a solution $\boldsymbol{\psi}_{\mathrm{L}}(t) \in \mathcal{E}$ to (1.12) such that

$$\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}_{\mathrm{L}}(t)\|_{\mathcal{E}} \to 0 \quad \text{as } t \to \infty. \tag{1.13}$$

1.2. Sub-threshold theorems, bubbling and soliton resolution

The regularity theory for k-equivariant wave maps is well understood, see [2, 3, 40–42], and recent research has been focused on the nonlinear dynamics of solutions with large energy. A guiding principle is called the *soliton resolution conjecture*, which asserts that every finite energy k-equivariant wave map asymptotically decouples into a superposition of finitely many harmonic maps (bubbles) with dynamically separating scaling parameters plus a term capturing the linear radiation. There has been substantial progress towards this conjecture over the last twenty years.

Struwe's sequential characterization of singular wave maps [45] can be viewed as a first step in the direction of soliton resolution. He proved that any wave map that blows up in finite time converges locally along a well chosen sequence of times and a well chosen sequence of scales to a non-constant harmonic map. Struwe's bubbling theorem has an immediate consequence for the regularity theory: every wave map with energy less than that of the ground state harmonic map must be globally regular. Since only topologically trivial wave maps, and more specifically those in the class $\mathcal{E}_{\ell,\ell}$, can scatter, one is led to the following formulation of the threshold theorem proved in [6] using the Kenig-Merle road map [26, 27]: every wave map $\boldsymbol{\psi} \in \mathcal{E}_{\ell,\ell}$ with $E(\boldsymbol{\psi}) < 2E(\boldsymbol{Q})$ must scatter in both time directions. That the threshold is $2E(\boldsymbol{Q})$ rather than $E(\boldsymbol{Q})$ reflects the fact that any k-equivariant element of $\mathcal{E}_{\ell,\ell}$ that develops a bubble must use a least another quantum of energy $E(\boldsymbol{Q})$ to connect back to $\ell\pi$; see [31, 43] for generalizations of these results outside equivariant symmetry.

Using similar logic, a natural threshold in the class $\mathcal{E}_{0,1}$ is $E < 3E(\boldsymbol{Q})$ since this is the maximal energy level allowing for at most one bubble to form. It was proved in [6, 7], using ideas from Duyckaerts, Kenig, and Merle [12–15], that continuous soliton resolution does hold in this regime. For every global-in-forward-time 1-equivariant wave map $\boldsymbol{\psi} \in \mathcal{E}_{0,1}$ with $E(\boldsymbol{\psi}) < 3E(\boldsymbol{Q})$ one can find a *continuous* function $\lambda(t) \ll t$ and a finite energy linear wave $\boldsymbol{\psi}_{\mathrm{L}}^*$ so that $\boldsymbol{\psi}(t)$ satisfies,

$$\boldsymbol{\psi}(t) = \boldsymbol{Q}_{\lambda(t)} + \boldsymbol{\psi}_{\mathrm{L}}^*(t) + o_{\mathcal{E}}(1) \text{ as } t \to \infty. \tag{1.14}$$

For wave maps in the same class that blow up at a finite time T_+ an analogous decomposition holds with $\lambda(t) = o(T_+ - t)$; see [9] for a related theorem for general wave maps with energy slightly above the ground state. Côte [5] generalized this result to allow for an arbitrary number of bubbles, but at the cost of only establishing the decomposition along a well chosen *sequence of times*. Later, Jia and Kenig [25] generalized the result from [5] to $k = 2$-equivariant wave maps using some different techniques, and we note that a minor technical observation can be used to generalize such sequential decompositions to all equivariance classes; see Remark 1.2. Since we will use these results, we state them precisely. Before giving the theorem, we introduce the notation

$$(1.15)\qquad \begin{aligned} \|\boldsymbol{\psi}_0\|^2_{\mathcal{E}(r\le\rho)} &= \|(\psi_0, \dot\psi_0)\|^2_{\mathcal{E}(r\le\rho)} \\ &:= \int_0^\rho \Big(\dot\psi_0(r)^2 + (\partial_r\psi_0(r))^2 + k^2\frac{\psi_0(r)^2}{r^2}\Big)\, r\,\mathrm{d}r, \\ \|\boldsymbol{\psi}_0\|^2_{\mathcal{E}(r\ge\rho)} &= \|(\psi_0, \dot\psi_0)\|^2_{\mathcal{E}(r\ge\rho)} \\ &:= \int_\rho^\infty \Big(\dot\psi_0(r)^2 + (\partial_r\psi_0(r))^2 + k^2\frac{\psi_0(r)^2}{r^2}\Big)\, r\,\mathrm{d}r. \end{aligned}$$

Theorem 1.1 (Sequential soliton resolution). [5–7, 25] *Fix any $k \in \mathbb{N}$, $\ell, m \in \mathbb{Z}$ and let $\boldsymbol{\psi}(t) \in \mathcal{E}_{\ell,m}$ be a finite energy solution to (1.3) on its maximal forward interval of existence, $I_+ = [0, T_+)$. Then, there exists a sequence of times $t_n \to T_+$, an integer $N \in \{0, 1, 2, \dots\}$, and sequences of scales $0 < \lambda_{n,1} \ll \lambda_{n,2} \ll \cdots \ll \lambda_{n,N}$, and signs $\iota_{n,1}, \dots, \iota_{n,N} \in \{-1, 1\}^N$ with the following properties:*

(Finite time blow up) *Assume $T_+ < \infty$. Then there exists $p \in \mathbb{Z}$ and $\boldsymbol{\psi}_0^* \in \mathcal{E}_{p,m}$, called the radiation, such that $\boldsymbol{\phi}(t) := \boldsymbol{\psi}(t) - \boldsymbol{\psi}_0^*$ satisfies*

$$(1.16)\qquad \lim_{t\to T_+} \|\boldsymbol{\phi}(t)\|_{\mathcal{E}(r\ge T_0 - t)} = 0,$$

The scale $\lambda_{n,N}$ satisfies, $\lambda_{n,N} \ll T_+ - t_n$ and $\boldsymbol{\psi}(t_n)$ satisfies,

$$(1.17)\qquad \|\boldsymbol{\psi}(t_n) - \boldsymbol{\psi}_0^* - (m\pi, 0) - \sum_{j=1}^N \iota_{n,j}(\boldsymbol{Q}_{\lambda_{n,j}} - (\pi, 0))\|_{\mathcal{E}} \to 0 \quad \text{as } n \to \infty.$$

(Global solution) *Assume $T_+ = \infty$. There exists a solution $\boldsymbol{\psi}_{\mathrm{L}}^* : \mathbb{R} \to \mathcal{E}$ of (1.12) and an increasing function $A : [0, \infty) \to [0, \infty)$ such that*

$\lim_{t\to\infty} A(t) = \infty$, *and* $\boldsymbol{\phi}(t) := \boldsymbol{\psi}(t) - \boldsymbol{\psi}^*_{\mathrm{L}}(t)$ *satisfies*

$$\lim_{t\to\infty} \|\boldsymbol{\phi}(t) - (m\pi, 0)\|_{\mathcal{E}(r\geq t-A(t))} = 0, \quad \lim_{t\to\infty} \|\boldsymbol{\psi}^*_{\mathrm{L}}(t)\|_{\mathcal{E}(r\leq t-A(t))} = 0 \tag{1.18}$$

The scale $\lambda_{n,N}$ *satisfies,* $\lambda_{n,N} \ll t_n$ *and* $\boldsymbol{\psi}(t_n)$ *satisfies,*

$$\|\boldsymbol{\psi}(t_n) - \boldsymbol{\psi}^*_{\mathrm{L}}(t_n) - (m\pi, 0) - \sum_{j=1}^{N} \iota_{n,j}(\boldsymbol{Q}_{\lambda_{n,j}} - (\pi, 0))\|_{\mathcal{E}} \to 0 \quad \text{as } n\to\infty. \tag{1.19}$$

Remark 1.2. The proof of Theorem 1.1 was carried out in full detail in [5, 25] only in the cases $k = 1, 2$. The missing technical ingredient in their proofs is the observation that an $L^3_t L^6_x$-type Strichartz estimate (see (2.1) for the precise norm) holds for the linearization of k-equivariant wave maps equation for every $k \geq 1$ using the estimates proved by Planchon, Stalker, Tahvildar-Zadeh [36] for the radially symmetric wave equation with inverse square potential; see [23, Section 2.2] for a more detailed explanation. With this observation, the arguments in [5] generalize to any odd k and the arguments in [25] generalize to all k. For the statements in the theorem about the radiation terms $\boldsymbol{\psi}^*_0$ and $\boldsymbol{\psi}^*_{\mathrm{L}}$ see [5, Propositions 5.1 and 5.2].

In this paper we will address the question of how to refine such sequential decompositions to ones that hold continuously in time when the sequential decomposition has at most two bubbles. Recently, a remarkable preprint by Duyckaerts, Kenig, Martel, and Merle [10] completely resolved this question (i.e. for an arbitrary number of bubbles) for the equivariance class $k = 1$ and for the related $4d$ critical focusing NLW in the radial case; see also [11, 16–18] for a complete resolution of this question for the critical radial focusing NLW in odd dimensions. We emphasize that the proof given in this paper is distinct and independent of the arguments in [10, 11, 16–18]. In particular, we do not make use of "channels of energy" estimates.

1.3. Statement of the results

We prove continuous time soliton resolution for a class of initial data not covered in [5–7, 10, 25].

Theorem 1. *Fix* $k \in \{1, 2, 3, \dots\}$. *Let* $\boldsymbol{\psi}_0 \in \mathcal{E}$ *such that* $E(\boldsymbol{\psi}_0) < 4E(\boldsymbol{Q}) = 16k\pi$, *and let* $\boldsymbol{\psi} : [0, T_+) \to \mathcal{E}$ *be the corresponding solution of* (1.3). *One of the following alternatives holds:*

(i) *(Scattering)* $T_+ = \infty$ *and* $\boldsymbol{\psi}(t)$ *scatters as* $t \to \infty$,

(ii) *(One-bubble blow-up)* $T_+ < \infty$, *and there exist* $\lambda : [0, T_+) \to (0, \infty)$, $\boldsymbol{\psi}_0^* \in \mathcal{E}_{0,1}$ *and* $\iota \in \{-1, 1\}$ *such that* $\lambda(t) \ll T_+ - t$ *as* $t \to T_+$ *and*

$$\lim_{t \to T_+} \big\| \boldsymbol{\psi}(t) - \iota\big(\boldsymbol{Q}_{\lambda(t)} - \boldsymbol{\psi}_0^*\big) \big\|_{\mathcal{E}} = 0. \tag{1.20}$$

(iii) *(Two-bubble blow-up)* $T_+ < \infty$, *and there exist* $\lambda, \mu : [0, T_+) \to (0, \infty)$, $\boldsymbol{\psi}_0^* \in \mathcal{E}$ *and* $\iota \in \{-1, 1\}$ *such that* $\lambda(t) \ll \mu(t) \ll T_+ - t$ *as* $t \to T_+$ *and*

$$\lim_{t \to T_+} \big\| \boldsymbol{\psi}(t) - \iota\big(\boldsymbol{Q}_{\lambda(t)} - \boldsymbol{Q}_{\mu(t)} + \boldsymbol{\psi}_0^*\big) \big\|_{\mathcal{E}} = 0. \tag{1.21}$$

(iv) *(Global two-bubble)* $T_+ = \infty$, *and there exist* $\lambda, \mu : [0, \infty) \to (0, \infty)$, *a solution* $\boldsymbol{\psi}_{\mathrm{L}}^* : [0, \infty) \to \mathcal{E}$ *of* (1.12) *and* $\iota \in \{-1, 1\}$ *such that* $\lambda(t) \ll \mu(t) \ll t$ *as* $t \to \infty$ *and*

$$\lim_{t \to \infty} \big\| \boldsymbol{\psi}(t) - \iota\big(\boldsymbol{Q}_{\lambda(t)} - \boldsymbol{Q}_{\mu(t)} + \boldsymbol{\psi}_{\mathrm{L}}^*(t)\big) \big\|_{\mathcal{E}} = 0. \tag{1.22}$$

The theorem stated above will easily follow from the sequential soliton resolution of Côte [5], and Jia and Kenig [25], once we prove the following result, which is our main contribution.

Theorem 2. *Fix* $k \in \{1, 2, 3, \ldots\}$, $m \in \{0, 1, \ldots\}$, *and let* $\boldsymbol{\psi} : [0, T_+) \to \mathcal{E}_{0,m}$ *be a solution of* (1.3).

1. *(Blow-up case.) Assume* $T_+ < \infty$, *there exists* $\boldsymbol{\psi}_0^* \in \mathcal{E}_{0,m}$, *and a sequence* $t_n \to T_+$ *such that* $\lambda_n \ll \mu_n \ll T_+ - t_n$ *and*

$$\lim_{n \to \infty} \big\| \boldsymbol{\psi}(t_n) - \iota\big(\boldsymbol{Q}_{\lambda_n} - \boldsymbol{Q}_{\mu_n} + \boldsymbol{\psi}_0^*\big) \big\|_{\mathcal{E}} = 0. \tag{1.23}$$

Then there exist continuous functions $\lambda, \mu : [T_0, T_+) \to (0, \infty)$ *such that* $\lambda(t) \ll \mu(t) \ll T_+ - t$ *as* $t \to T_+$ *and*

$$\lim_{t \to T_+} \big\| \boldsymbol{\psi}(t) - \iota\big(\boldsymbol{Q}_{\lambda(t)} - \boldsymbol{Q}_{\mu(t)} + \boldsymbol{\psi}_0^*\big) \big\|_{\mathcal{E}} = 0. \tag{1.24}$$

2. *(Global case.) Assume* $T_+ = \infty$, *there exists* $\boldsymbol{\psi}_{\mathrm{L}}^* : [0, \infty) \to \mathcal{E}$ *a solution of* (1.12), *and a sequence* $t_n \to \infty$ *such that* $\lambda_n \ll \mu_n \ll t_n$ *and*

$$\lim_{n \to \infty} \big\| \boldsymbol{\psi}(t_n) - \iota\big(\boldsymbol{Q}_{\lambda_n} - \boldsymbol{Q}_{\mu_n} + \boldsymbol{\psi}_{\mathrm{L}}^*(t_n)\big) \big\|_{\mathcal{E}} = 0. \tag{1.25}$$

Then there exist continuous functions $\lambda, \mu : [T_0, \infty) \to (0, \infty)$ *such that* $\lambda(t) \ll \mu(t) \ll t$ *as* $t \to \infty$ *and*

$$\lim_{t\to\infty} \big\| \boldsymbol{\psi}(t) - \iota\big(\boldsymbol{Q}_{\lambda(t)} - \boldsymbol{Q}_{\mu(t)} + \boldsymbol{\psi}_{\mathrm{L}}^*(t)\big)\big\|_{\mathcal{E}} = 0. \tag{1.26}$$

1.4. Comments on the results

While the soliton resolution conjecture itself is a qualitative description of the dynamics, it is of central importance to understand which configurations of bubbles and radiation are actually realized by solutions in either the finite time singularity or global-in-time case. Finite-time blow up wave maps with one concentrating bubble were first constructed in a series of influential works by Krieger, Schlag, Tataru [30], Rodnianski, Sterbenz [38], and Raphaël, Rodnianski [37], with the latter work yielding a stable blow-up regime; see also the recent works [28, 29] for stability properties of the solutions from [30], as well as [24] for a classification of blowup solutions with a given radiation profile, and [34, 35] for constructions of various types of solutions with one bubble in infinite time. While these solutions are all constructed within the class $\mathcal{E}_{0,1}$, the examples that blow up in finite time can be smoothly truncated outside the light cone to yield solutions in $\mathcal{E}$ blowing up in finite time with one bubble, thus the scenario (ii) in Theorem 1 is realized.

The first examples of wave maps with two bubbles were constructed by the first author in [20] in equivariance classes $k \geq 2$. The solutions in [20] take the form

$$\boldsymbol{\psi}(t) = Q_{\lambda(t)} - Q_{\mu(t)} + o_{\mathcal{E}}(1) \text{ as } t \to \infty \tag{1.27}$$

with $\lambda(t) \to 0$ and $\mu(t) \to 1$ as $t \to \infty$. The radiation term $\boldsymbol{\psi}_{\mathrm{L}}^* = 0$ and thus the solution has threshold energy, i.e., $E(\boldsymbol{\psi}) = 2E(\boldsymbol{Q})$. No such example is known to exist for $k = 1$.

In [23] the authors classified the dynamics of every k-equivariant wave map with energy $E = 2E(\boldsymbol{Q}) = 8k\pi$ in *both time directions*, showing, for example, that every such wave map must scatter in at least one time direction. Rodriguez [39] proved an analogous result in the case $k = 1$ including a construction of a threshold wave map blowing up in finite time in one direction and scattering in the other. The collision analysis in these papers will play a key role in the proof of Theorem 2; see Section 1.5. Recently the authors proved that the 2-bubble solution constructed in [20] is unique and of regularity H^2 for classes $k \geq 4$; see [21, 22].

Côte [5] observed that a result analogous to Theorem 2 holds, both in the blow-up case and in the global case, if both bubbles have the same sign. In fact, Côte's result allows an arbitrary number of bubbles, all having the same sign. However, it is worth noting that existence of solutions developing more than one bubble *of the same sign* is unknown.

In the setting of Theorem 2, we know that, at least in the global case, the set of the initial data satisfying the assumptions is non-empty, as it contains the two-bubble solution constructed in [20]. Of course, we expect this set to be much bigger. Whether the set of initial data satisfying the assumptions of the blow-up case in Theorem 2 is non-empty, is unclear to us. Also, in the case $k = 1$, we do not know if there exist any solutions satisfying the assumptions of Theorem 2.

A natural question is whether our strategy could lead to a proof of soliton resolution for any number of bubbles. While we believe that studying threshold N-bubble solutions for $N \geq 3$ is an interesting topic in itself, currently it is unknown if this can lead to a proof of soliton resolution in the general case.

Remark 1.3. To be precise, the paper [20] provided a construction for the radial Yang-Mills equation, which is very similar to equivariant wave maps with $k = 2$.

1.5. Comments on the proofs

The following result follows from [6, Theorem 1.1], [23, Theorem 1.6] and [39, Theorem 1.6].

Theorem 1.4. *Let* $k \in \{1, 2, \ldots\}$.

1. *If* $\boldsymbol{\psi}$ *is a solution of* (1.3) *with initial data* $\boldsymbol{\psi}_0 \in \mathcal{E}$ *and energy* $E(\boldsymbol{\psi}) < 2E(\boldsymbol{Q}) = 8k\pi$*, then* ψ *scatters to a linear wave in both time directions.*
2. *If* $\boldsymbol{\psi}$ *is a solution of* (1.3) *with initial data* $\boldsymbol{\psi}_0 \in \mathcal{E}$ *and energy* $E(\boldsymbol{\psi}) = 2E(\boldsymbol{Q}) = 8k\pi$*, then* ψ *scatters to a linear wave in at least one time direction.*

Settling the threshold case $E(\boldsymbol{\psi}) = 8k\pi$ is essential for our proof of continuous time soliton resolution of two-bubble wave maps. Indeed, one immediate enemy, when one attempts to deduce continuous time soliton resolution from sequential soliton resolution, is the possibility of *elastic collisions*. If colliding solitons could recover their shape after a collision, then one could potentially encounter the following scenario: the solution approaches a multi-soliton configuration for a sequence of times, but in between infinitely many collisions take place, so that there is no soliton resolution in continuous time. The threshold case in Theorem 1.4 shows in particular that two bubbles cannot collide in an elastic manner.

Transforming the intuition described above into a proof might not be immediate, see for example the recent preprints [16–18]. In our case, however, we know that threshold two-bubbles not only collide in an inelastic way, but *scatter* in one time direction. As we demonstrate, this very strong information makes the proof of continuous time soliton resolution almost immediate.

We stress as well that our proofs crucially use the observation from the earlier works cited above that *the radiation part* of the solution is extracted for continuous time, and not only for a sequence. We further comment on this issue at the beginning of Section 3 below.

2. Preliminaries

2.1. Profile decomposition

Our proof is based on the nonlinear profile decomposition method due to Bahouri and Gérard [1], and Merle and Vega [33]. For the presentation of this theory in the setting of the equivariant wave maps for arbitrary k, see [23, Section 2.3]. Here, we only state the relevant results.

For a time interval I and $\psi : I \times (0, \infty) \to \mathbb{R}$, we define the *Strichartz norm*

$$\|\psi\|_{S(I)} := \left(\int_I \left(\int_0^\infty \frac{\psi(t,r)^6}{r^3} \, \mathrm{d}r \right)^{\frac{1}{2}} \right)^{\frac{1}{3}}. \tag{2.1}$$

If $I = \mathbb{R}$, we write S instead of $S(\mathbb{R})$. It is worth noting that a solution $\boldsymbol{\psi}$ of (1.3) scatters for positive times if and only if $\|\psi\|_{S([T_0,\infty))} < \infty$; see [36] for the relevant Strichartz estimates, [23, Section 2.2] for a review of the local Cauchy theory for (1.3) using the S norm, and e.g., Strauss's book [44] for the if and only if statement about scattering.

We also introduce the following notation for the scale change: if $\phi \in \mathcal{H}_{\ell,m}$, then $\phi_\lambda(r) := \phi(r/\lambda)$ for all $\lambda > 0$; if $\boldsymbol{\phi} = (\phi, \dot\phi) \in \mathcal{E}_{\ell,m}$, then $\boldsymbol{\phi}_\lambda(r) := (\phi(r/\lambda), \lambda^{-1}\dot\phi(r/\lambda))$.

Definition 2.1. We say that a bounded sequence $(\boldsymbol{\psi}_n) \subset \mathcal{E}$ has a *profile decomposition* with profiles $\boldsymbol{U}_0^j \in \mathcal{E}$ and displacements (λ_n^j, t_n^j) if the following conditions are satisfied:

1) if $j \neq j'$, then $\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|t_n^{j'} - t_n^j|}{\lambda_n^j} = \infty$,
2) if $\boldsymbol{w}_{n,0}^J$ is the *remainder term* defined by

$$\boldsymbol{\psi}_n = \sum_{j=1}^{J} \boldsymbol{U}_{\mathrm{L}}^j(-t_n^j/\lambda_n^j)_{\lambda_n^j} + \boldsymbol{w}_{n,0}^J, \tag{2.2}$$

then

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_{n,\mathrm{L}}^J\|_S = 0, \tag{2.3}$$

where $\boldsymbol{U}_{\mathrm{L}}^j : \mathbb{R} \to \mathcal{E}$ and $\boldsymbol{w}_{n,\mathrm{L}}^J : \mathbb{R} \to \mathcal{E}$ are the solutions of (1.12) such that $\boldsymbol{U}_{\mathrm{L}}^j(0) = \boldsymbol{U}_0$ and $\boldsymbol{w}_{n,\mathrm{L}}^J(0) = \boldsymbol{w}_{n,0}^J$.

Lemma 2.2 (Linear Profile Decomposition). *Every bounded sequence $(\boldsymbol{\psi}_n) \subset \mathcal{E}$ has a subsequence which has a profile decomposition.* □

Without loss of generality, upon taking subsequences and modifying the profiles, one can assume that for all j one of the following holds: $t_n^j = 0$ for all n, $\lim_{n\to\infty} t_n^j/\lambda_n^j = \infty$, $\lim_{n\to\infty} t_n^j/\lambda_n^j = -\infty$. The *nonlinear profile* $\boldsymbol{U}^j$ associated with the profile $\boldsymbol{U}_0^j$ is a solution of (1.3) defined by the condition

$$\lim_{n\to\infty} \|\boldsymbol{U}^j(-t_n^j/\lambda_n^j) - \boldsymbol{U}_{n,\mathrm{L}}^j(-t_n^j/\lambda_n^j)\|_{\mathcal{E}} = 0. \tag{2.4}$$

Lemma 2.3 (Nonlinear Profile Decomposition). *Let $\boldsymbol{\psi}_{n,0} \in \mathcal{E}$ be a bounded sequence with a profile decomposition, and let $\boldsymbol{U}^j$ be the associated nonlinear profiles, with the maximal forward time of existence $T_+(\boldsymbol{U}^j)$. The following "Pythagorean formula" holds:*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \Big| E(\boldsymbol{\psi}_{n,0}) - \sum_{j=1}^{J} E(\boldsymbol{U}^j) - \|\boldsymbol{w}_{n,0}^J\|_{\mathcal{E}}^2 \Big| = 0. \tag{2.5}$$

Furthermore, let $s_n \in (0,\infty)$ be any sequence such that for all j and n

$$\frac{s_n - t_n^j}{\lambda_n^j} < T_+(\boldsymbol{U}^j), \quad \limsup_{n\to\infty} \|U^j\|_{S(I_n^j)} < \infty, \quad \textit{where } I_n^j := \Big[-\frac{t_n^j}{\lambda_n^j}, \frac{s_n - t_n^j}{\lambda_n^j}\Big]. \tag{2.6}$$

Let $\boldsymbol{\psi}_n(t)$ denote the solution of (1.3) *with initial data $\boldsymbol{\psi}_n(0) = \boldsymbol{\psi}_{n,0}$. Then for n large enough $\boldsymbol{\psi}_n(t)$ exists on the interval $s \in [0, s_n]$ and satisfies,*

$$\limsup_{n\to\infty} \|\psi_n\|_{S([0,s_n])} < \infty. \tag{2.7}$$

Moreover, the following nonlinear profile decomposition holds for all $s \in [0, s_n]$,

$$\boldsymbol{\psi}_n(s) = \sum_{j=1}^{J} \boldsymbol{U}^j\Big(\frac{s - t_n^j}{\lambda_n^j}\Big)_{\lambda_n^j} + \boldsymbol{w}_{n,\mathrm{L}}^J(s) + \boldsymbol{g}_n^J(s) \tag{2.8}$$

with $\boldsymbol{w}_{n,\mathrm{L}}^J(t)$ as in Definition 2.1 and

$$\lim_{J\to\infty} \limsup_{n\to\infty} \big(\|g_n^J\|_{S([0,s_n])} + \|\boldsymbol{g}_n^J\|_{L^\infty([0,s_n];\mathcal{E})}\big) = 0. \tag{2.9}$$

The analogous statement holds for sequences $s_n \in (-\infty, 0)$. □

2.2. Bubbles and two-bubbles

In this section, we state a few useful facts about states $\boldsymbol{\psi}_0 \in \mathcal{E}$ which are close to a two-bubble.

First, we recall the following variational characterization of Q in $\mathcal{H}_{0,1}$ from [4], which amounts to the coercivity of the energy functional near Q.

Lemma 2.4. [4, Proposition 2.3] *For any $\epsilon > 0$ there exists $\delta > 0$ such that if $\psi_0 \in \mathcal{H}_{0,1}$ and $E_p(\psi_0) \le 4k\pi + \delta$, then there exists $\lambda > 0$ such that $\|\psi_0 - Q_\lambda\|_{\mathcal{H}} \le \epsilon$.* □

Next, we consider two-bubble configurations.

Definition 2.5. Given a map $\boldsymbol{\psi}_0 \in \mathcal{E}$, we define its proximity $\mathbf{d}_+(\boldsymbol{\psi}_0)$ to a positive pure 2-bubble and its proximity $\mathbf{d}_-(\boldsymbol{\psi}_0)$ to a negative pure 2-bubble by

$$\mathbf{d}_\pm(\boldsymbol{\psi}_0) := \inf_{\lambda,\mu>0} \big(\|\boldsymbol{\psi}_0 \mp (\boldsymbol{Q}_\lambda - \boldsymbol{Q}_\mu)\|_{\mathcal{E}}^2 + (\lambda/\mu)^k\big). \tag{2.10}$$

We also set

$$\mathbf{d}(\boldsymbol{\psi}_0) := \min(\mathbf{d}_+(\boldsymbol{\psi}_0), \mathbf{d}_-(\boldsymbol{\psi}_0)). \tag{2.11}$$

Lemma 2.6. *For any $\epsilon > 0$ there exists $\delta > 0$ such that if $\boldsymbol{\psi}_0 \in \mathcal{E}$, $E(\boldsymbol{\psi}_0) \leq 8k\pi + \delta$ and $\|\boldsymbol{\psi}_0\|_{\mathcal{E}} \geq \delta^{-1}$, then $\mathbf{d}(\boldsymbol{\psi}_0) \leq \epsilon$.*

Proof. The result follows from the proof of Lemma 2.13 in [23], with minor modifications left to the Reader. □

Remark 2.7. The hypothesis of Theorem 2 yields a sequence of times $t_n \to T_+$ for which $\mathbf{d}(\boldsymbol{\psi}(t_n)) \to 0$ as $n \to \infty$. We will show, using Lemma 2.6, that if the hypothesis of Theorem 2 hold then we must have $\mathbf{d}(\boldsymbol{\psi}(t)) \to 0$ as $t \to T_+$. Given this, the existence of continuous (in fact C^1) functions $\lambda(t), \mu(t)$ as in the statement of Theorem 2 is standard; see for example [23, Lemma 3.1].

3. Proofs of the theorems

Our proof of Theorem 2 uses the observation from [5–7] that the sequential decomposition provides one profile which is independent of the time sequence: $\boldsymbol{\psi}_0^*$ in the blow-up case and $\boldsymbol{\psi}_{\mathrm{L}}^*$ in the global case. In the paper [5], only the cases $k = 1$ and $k = 2$ were considered, but the proofs are valid for general k, using the Strichartz estimates from [36]; see also Section 5.1 of [6] and Section 3.2 of [7] for related arguments.

In the global case, we will also need the following fact.

Lemma 3.1. *Let t_n be an increasing sequence such that $\lim_{n\to\infty} t_n = \infty$, and let ρ_n be a sequence such that $\lim_{n\to\infty}(t_n - \rho_n) = \infty$.*

1. *If $\boldsymbol{\phi}_{\mathrm{L}} : \mathbb{R} \to \mathcal{E}$ is a solution of (1.12), then $\lim_{n\to\infty} \|\boldsymbol{\phi}_{\mathrm{L}}(t_n)\|_{\mathcal{E}(r\leq\rho_n)} = 0$.*
2. *Let $\boldsymbol{\phi}_{0,n} \in \mathcal{E}$ be a bounded sequence of initial data such that*

$$\lim_{n\to\infty} \|\boldsymbol{\phi}_{0,n}\|_{\mathcal{E}(r\geq\rho_n)} = 0. \tag{3.1}$$

   Let $\boldsymbol{\phi}_{n,\mathrm{L}}$ be the solution of (1.12) corresponding to the initial data $\boldsymbol{\phi}_{n,\mathrm{L}}(t_n) = \boldsymbol{\phi}_{0,n}$. Then $\boldsymbol{\phi}_{n,\mathrm{L}}(0) \rightharpoonup 0$ as $n \to \infty$.

Proof. The first claim is a well-known property of the linear wave equation, see for example [8, Proposition 4].

Regarding the second claim, it is sufficient to prove that every subsequence of $\boldsymbol{\phi}_{n,\mathrm{L}}(0)$ has a subsequence weakly converging to 0. By weak compactness, we only need to show that $\boldsymbol{\phi}_{n,\mathrm{L}}(0) \rightharpoonup \boldsymbol{\phi}_0$ implies $\boldsymbol{\phi}_0 = 0$.

Let $\boldsymbol{\phi}_{n,\mathrm{L}}(0) = \boldsymbol{\phi}_0 + \widetilde{\boldsymbol{\phi}}_{0,n}$, and let $\widetilde{\boldsymbol{\phi}}_{n,\mathrm{L}}$ be the solution of (1.12) corresponding to the initial data $\widetilde{\boldsymbol{\phi}}_{n,\mathrm{L}}(0) = \widetilde{\boldsymbol{\phi}}_{0,n}$. Let $\boldsymbol{\phi}_{\mathrm{L}}$ be the solution of (1.12) for the initial data $\boldsymbol{\phi}_{\mathrm{L}}(0) = \boldsymbol{\phi}_0$, so that $\boldsymbol{\phi}_{n,\mathrm{L}}(t) = \boldsymbol{\phi}_{\mathrm{L}}(t) + \widetilde{\boldsymbol{\phi}}_{n,\mathrm{L}}(t)$ for all $t \geq 0$. It is easy to see that (1.12) defines a unitary group in $\mathcal{E}$, hence

$$
\begin{aligned}
(3.2) \qquad \langle \boldsymbol{\phi}_{\mathrm{L}}(t_n), \boldsymbol{\phi}_{0,n} - \boldsymbol{\phi}_{\mathrm{L}}(t_n)\rangle_{\mathcal{E}} &= \langle \boldsymbol{\phi}_{\mathrm{L}}(t_n), \widetilde{\boldsymbol{\phi}}_{n,\mathrm{L}}(t_n)\rangle_{\mathcal{E}} \\
&= \langle \boldsymbol{\phi}_0, \widetilde{\boldsymbol{\phi}}_{0,n}\rangle_{\mathcal{E}} \to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{E}$.

By the first part of the lemma, we have $\|\boldsymbol{\phi}_{\mathrm{L}}(t_n)\|_{\mathcal{E}(r\leq\rho_n)} \to 0$, which yields

$$
(3.3) \qquad \langle \boldsymbol{\phi}_{\mathrm{L}}(t_n), \boldsymbol{\phi}_{0,n}\rangle_{\mathcal{E}} \to 0,
$$

and we obtain $\|\boldsymbol{\phi}_0\|_{\mathcal{E}}^2 = \langle \boldsymbol{\phi}_{\mathrm{L}}(t_n), \boldsymbol{\phi}_{\mathrm{L}}(t_n)\rangle_{\mathcal{E}} \to 0$. □

Proof of Theorem 2. In the proof, we use several times the following fact. If $\boldsymbol{\phi}_n, \boldsymbol{\psi}_n$ are sequences such that $E(\boldsymbol{\phi}_n), E(\boldsymbol{\psi}_n)$ are bounded, and $\rho_n \in (0,\infty)$ is a sequence such that

$$
(3.4) \qquad \lim_{n\to\infty} \|\boldsymbol{\phi}_n\|_{\mathcal{E}(r\leq\rho_n)} = 0, \qquad \lim_{n\to\infty} \|\boldsymbol{\psi}_n\|_{\mathcal{E}(r\geq\rho_n)} = 0,
$$

then

$$
(3.5) \qquad \lim_{n\to\infty} \big(E(\boldsymbol{\phi}_n + \boldsymbol{\psi}_n) - E(\boldsymbol{\phi}_n) - E(\boldsymbol{\psi}_n)\big) = 0.
$$

The blow-up case. We first prove that it can be assumed without loss of generality that $m = 0$, so that $\boldsymbol{\psi}_0^* \in \mathcal{E}$.

To see this, consider the solution $\widetilde{\boldsymbol{\psi}}$ of (1.3) corresponding to the initial data $\widetilde{\boldsymbol{\psi}}(T_0) = \chi\boldsymbol{\psi}(T_0)$, where χ is a smooth cut-off function such that $\chi(r) = 1$ if $r \leq \frac{1}{2}$ and $\chi(r) = 0$ if $r \geq 1$, and $T_+ - \frac{1}{8} < T_0 < T_+$. By finite speed of propagation, $\widetilde{\boldsymbol{\psi}}(t,r) = \boldsymbol{\psi}(t,r)$ for all $t \in [T_0, T_+)$ and $r \leq 3/8$ (since an equivariant wave map can only blow up at $r = 0$, it is clear that $\widetilde{\psi}$ does not blow up until time T_+). Let $\widetilde{\boldsymbol{\psi}}_0^* \in \mathcal{E}$ be given by Theorem 1.1, so that $\widetilde{\boldsymbol{\phi}}(t) := \widetilde{\boldsymbol{\psi}}(t) - \widetilde{\boldsymbol{\psi}}_0^*$ satisfies (1.16). It follows that $\widetilde{\boldsymbol{\psi}}_0^*(r) = \boldsymbol{\psi}_0^*(r)$ if $r \leq \frac{3}{8}$, implying $\widetilde{\boldsymbol{\phi}}(t,r) = \boldsymbol{\phi}(t,r)$ if $r \leq \frac{3}{8}$. Thus, for any sequence $s_n \to T_+$, $\mathbf{d}_\pm(\boldsymbol{\phi}(s_n)) \to 0$ if and only if $\mathbf{d}_\pm(\widetilde{\boldsymbol{\phi}}(s_n)) \to 0$. We deduce that (sequential or continuous-time) soliton resolution holds for $\boldsymbol{\psi}$ if and only if it holds for $\widetilde{\boldsymbol{\psi}}$.

We thus assume $m = 0$. Let $\boldsymbol{\psi}_0^* \in \mathcal{E}$ be given by Theorem 1.1. We decompose

$$\boldsymbol{\psi}(t) = \boldsymbol{\phi}(t) + \boldsymbol{\psi}_0^*. \tag{3.6}$$

We have $\lim_{n\to\infty} E(\boldsymbol{\phi}(t_n)) = 8k\pi$, thus $E(\boldsymbol{\psi}_0^*) = E(\boldsymbol{\psi}) - 8k\pi$. Since

$$\lim_{t\to T_+} \|\boldsymbol{\phi}(t)\|_{\mathcal{E}(r\geq T_+-t)} = 0, \qquad \lim_{t\to T_+} \|\boldsymbol{\psi}_0^*\|_{\mathcal{E}(r\leq T_+-t)} = 0, \tag{3.7}$$

we obtain $\lim_{t\to T_+}\big(E(\boldsymbol{\phi}(t) + \boldsymbol{\psi}_0^*) - E(\boldsymbol{\phi}(t)) - E(\boldsymbol{\psi}_0^*)\big) = 0$, in other words

$$\lim_{t\to T_+} E(\boldsymbol{\phi}(t)) = 8k\pi. \tag{3.8}$$

Suppose there exists a sequence $\tau_n \to T_+$ such that $\sup_n \|\boldsymbol{\phi}(\tau_n)\|_{\mathcal{E}} < \infty$. Upon extracting a subsequence, we can assume without loss of generality that the sequence $\boldsymbol{\phi}(\tau_n)$ has a profile decomposition. For $j \in \{1, 2, \ldots\}$, let $\boldsymbol{U}^j$ be the nonlinear profiles, with the corresponding parameters λ_n^j, t_n^j. Let $\boldsymbol{U}^0$ be the solution of (1.3) with the initial data $\boldsymbol{U}^0(0) = \boldsymbol{\psi}_0^*$, $t_n^0 = 0$, $\lambda_n^0 = 1$. Since $\boldsymbol{\phi}(\tau_n) \rightharpoonup 0$ as $n \to \infty$, see Theorem 1.1, the sequence $\boldsymbol{\psi}(\tau_n)$ has a profile decomposition with profiles $\boldsymbol{U}^j$, $j \in \{0, 1, 2, \ldots\}$, and parameters λ_n^j, t_n^j.

Thanks to the Pythagorean formula (2.5), either there is just one nonzero profile of energy $8k\pi$ and $\boldsymbol{w}_{n,L}^J = \boldsymbol{w}_{n,L}^1 \to 0$ in $\mathcal{E}$ for all $J \geq 1$, or all the profiles scatter in both time directions.

Case 1. All the profiles $\boldsymbol{U}^j$ scatter in both time directions. Fix any $0 < T < T_+(\boldsymbol{U}^0)$. The assumptions of Lemma 2.3 are satisfied with $s_n = T$, which implies that $\boldsymbol{\psi}$ exists on the time interval $[\tau_n, \tau_n + T]$ for all n large enough. This is in contradiction with the fact that $\boldsymbol{\phi}$ blows up at $t = T_+$.

Case 2. There is just one profile $\boldsymbol{U}^1$, and $\boldsymbol{w}_{n,\mathrm{L}}^J = 0$ for $J \geq 1$. By taking a subsequence and adjusting $\boldsymbol{U}^1$, we can assume that $\lim_{n\to\infty} t_n^1/\lambda_n^1 \in \{-\infty, \infty\}$, or $t_n^1 = 0$ for all n.

Case 2.1. We either have $\lim_{n\to\infty} t_n^1/\lambda_n^1 = -\infty$, or $t_n^1 = 0$ for all n and $\boldsymbol{U}^1$ scatters in the forward time direction. In this situation, the same argument as in Case 1 yields a contradiction.

Case 2.2. We either have $\lim_{n\to\infty} t_n^1/\lambda_n^1 = \infty$, or $t_n^1 = 0$ for all n and $\boldsymbol{U}^1$ scatters in the backward time direction.

Let $\boldsymbol{\psi}_n(t) := \boldsymbol{\psi}(\tau_n + t)$. The assumptions of Lemma 2.3 are satisfied with $s_n = -\infty$. For t_m fixed and n large enough so that $t_m < \tau_n$, we obtain

$$\boldsymbol{\psi}(t_m) = \boldsymbol{\psi}_n(-(\tau_n - t_m)) = \boldsymbol{U}^1\Big(\frac{-(\tau_n - t_m) - t_n^1}{\lambda_n^1}\Big) + \boldsymbol{\psi}_0^* + \boldsymbol{h}_n, \tag{3.9}$$

with $\lim_{n\to\infty} \|\boldsymbol{h}_n\|_{\mathcal{E}} = 0$. Since $\boldsymbol{U}^1$ scatters in the backward time direction, there exists $\epsilon > 0$ such that $\mathbf{d}(\boldsymbol{U}^1(t)) \geq 2\epsilon$ for all $t \leq 0$. Taking $n \to \infty$ in (3.9), we obtain

$$\mathbf{d}(\boldsymbol{\psi}(t_m) - \boldsymbol{\psi}_0^*) \geq \epsilon. \tag{3.10}$$

This is true for all m, with ϵ independent of m, in contradiction with the assumptions of Theorem 2.

Thus, we have proved that $\lim_{t\to T_+} \|\boldsymbol{\phi}(t)\|_{\mathcal{E}} = \infty$. By Lemma 2.6, $\boldsymbol{\phi}(t)$ converges to a two-bubble in continuous time; see Remark 2.7.

The global case. The proof is completely analogous. We decompose

$$\boldsymbol{\psi}(t) = \boldsymbol{\phi}(t) + \boldsymbol{\psi}_{\mathrm{L}}^*(t). \tag{3.11}$$

and we claim that

$$\lim_{t\to\infty} E(\boldsymbol{\phi}(t)) = 8k\pi. \tag{3.12}$$

To see this, note that the limits

$$\begin{aligned} \lim_{n\to\infty} E(\boldsymbol{\phi}(t_n)) &= 8k\pi, \\ \lim_{n\to\infty} \|\boldsymbol{\phi}(t_n)\|_{\mathcal{E}(r\geq t_n - A(t_n))} &= 0, \\ \lim_{t\to\infty} \|\boldsymbol{\psi}_{\mathrm{L}}^*(t)\|_{\mathcal{E}(r\leq t-A(t))} &= 0 \end{aligned} \tag{3.13}$$

imply that

$$\lim_{n\to\infty} E(\boldsymbol{\psi}_{\mathrm{L}}^*(t_n)) = E(\boldsymbol{\psi}) - 8k\pi.$$

Next, let $\boldsymbol{\psi}^*(t) \in \mathcal{E}$ denote the solution to (1.3) that scatters to $\boldsymbol{\psi}_{\mathrm{L}}^*(t)$, i.e.,

$$\|\boldsymbol{\psi}^*(t) - \boldsymbol{\psi}_{\mathrm{L}}^*(t)\|_{\mathcal{E}} \to 0 \text{ as } t \to \infty, \tag{3.14}$$

which, together with the previous displayed equation implies that $E(\boldsymbol{\psi}^*) = E(\boldsymbol{\psi}) - 8k\pi$. Now define $\widetilde{\boldsymbol{\phi}}(t)$ by $\boldsymbol{\psi}(t) = \widetilde{\boldsymbol{\phi}}(t) + \boldsymbol{\psi}^*(t)$. Since $\lim_{t\to\infty} \|\boldsymbol{\psi}^*(t)\|_{\mathcal{E}(r\leq t-A(t))} = 0$ and $\lim_{t\to\infty} \|\widetilde{\boldsymbol{\phi}}(t)\|_{\mathcal{E}(r\geq t-A(t))} = 0$, we obtain $\lim_{t\to\infty} \big(E(\widetilde{\boldsymbol{\phi}}(t) + \boldsymbol{\psi}^*(t)) - E(\widetilde{\boldsymbol{\phi}}(t)) - E(\boldsymbol{\psi}^*)\big) = 0$, or in other words

$$\lim_{t\to\infty} E(\boldsymbol{\phi}(t)) = \lim_{t\to\infty} E(\widetilde{\boldsymbol{\phi}}(t)) = E(\boldsymbol{\psi}) - E(\boldsymbol{\psi}^*) = 8k\pi, \tag{3.15}$$

proving (3.12).

Suppose there exists a sequence $\tau_n \to \infty$ such that $\sup_n \|\boldsymbol{\phi}(\tau_n)\|_{\mathcal{E}} < \infty$. Upon extracting a subsequence, there exists a profile decomposition of the sequence $\boldsymbol{\phi}(\tau_n)$. For $j \in \{1, 2, \ldots\}$, let $\boldsymbol{U}^j$ be the nonlinear profiles, with the corresponding parameters λ_n^j, t_n^j. Set $\boldsymbol{U}_{\mathrm{L}}^0 := \boldsymbol{\psi}_{\mathrm{L}}^*$, $t_n^0 = \tau_n$, $\lambda_n^0 = 1$. In order to check that $\boldsymbol{U}_{\mathrm{L}}^0$ is a profile, we need to verify that $\boldsymbol{V}_{n,\mathrm{L}}(0) \rightharpoonup 0$, where $\boldsymbol{V}_{n,\mathrm{L}}$ is the solution of (1.12) corresponding to the initial data $\boldsymbol{V}_{n,\mathrm{L}}(\tau_n) = \boldsymbol{\phi}(\tau_n)$. By Theorem 1.1, $\|\boldsymbol{\phi}(\tau_n)\|_{\mathcal{E}(r \geq \tau_n - A(\tau_n))} \to 0$. Thus Lemma 3.1, applied with $\rho_n := \tau_n - A(\tau_n)$, implies the claim.

Let $\boldsymbol{U}^0$ be the corresponding nonlinear profile, so that the sequence $\boldsymbol{\psi}(\tau_n)$ has a profile decomposition with nonlinear profiles $\boldsymbol{U}^j$, $j \in \{0, 1, 2, \ldots\}$, and parameters λ_n^j, t_n^j.

Either there is just one profile of energy $8k\pi$Received: 08 Mar 2021 Accepted: 1 Jun 2021 and $\boldsymbol{w}_{n,\mathrm{L}}^J = \boldsymbol{w}_{n,\mathrm{L}}^1 \to 0$ in $\mathcal{E}$ for all $J \geq 1$, or all the profiles scatter in both time directions.

Case 1. All the nonlinear profiles $\boldsymbol{U}^j$ scatter in both time directions. Since the nonlinear profile $\boldsymbol{U}^0$ scatters as $t \to \infty$, Lemma 2.3 yields a contradiction with the fact that $\boldsymbol{\phi}$ does not scatter as $t \to \infty$.

Case 2. There is just one profile $\boldsymbol{U}^1$, and $\boldsymbol{w}_{n,\mathrm{L}}^J = 0$ for $J \geq 1$. By taking a subsequence and adjusting $\boldsymbol{U}^1$, we can assume that $\lim_{n\to\infty} t_n^1/\lambda_n^1 \in \{-\infty, \infty\}$, or $t_n^1 = 0$ for all n.

Case 2.1. We either have $\lim_{n\to\infty} t_n^1/\lambda_n^1 = -\infty$, or $t_n^1 = 0$ for all n and $\boldsymbol{U}^1$ scatters in the forward time direction. In this situation, the same argument as in Case 1 yields a contradiction.

Case 2.2. We either have $\lim_{n\to\infty} t_n^1/\lambda_n^1 = \infty$, or $t_n^1 = 0$ for all n and $\boldsymbol{U}^1$ scatters in the backward time direction.

Let $\boldsymbol{\psi}_n(t) := \boldsymbol{\psi}(\tau_n + t)$. The assumptions of Lemma 2.3 are satisfied with $s_n = -\infty$. For t_m fixed and n large enough so that $t_m < \tau_n$, we obtain

$$\boldsymbol{\psi}(t_m) = \boldsymbol{\psi}_n(-(\tau_n - t_m)) = \boldsymbol{U}^1\Big(\frac{-(\tau_n - t_m) - t_n^1}{\lambda_n^1}\Big) + \boldsymbol{\psi}_{\mathrm{L}}^*(t_m) + \boldsymbol{h}_n, \tag{3.16}$$

with $\lim_{n\to\infty} \|\boldsymbol{h}_n\|_{\mathcal{E}} = 0$. Since $\boldsymbol{U}^1$ scatters in the backward time direction, there exists $\epsilon > 0$ such that $\mathbf{d}(\boldsymbol{U}^1(t)) \geq 2\epsilon$ for all $t \leq 0$. Taking $n \to \infty$ in (3.16), we obtain

$$\mathbf{d}(\boldsymbol{\psi}(t_m) - \boldsymbol{\psi}_{\mathrm{L}}^*(t_m)) \geq \epsilon. \tag{3.17}$$

This is true for all m, with ϵ independent of m, in contradiction with the assumptions of Theorem 2.

Thus, we have proved that $\lim_{t\to T_+} \|\boldsymbol{\phi}(t)\|_{\mathcal{E}} = \infty$. By Lemma 2.6, $\boldsymbol{\phi}(t)$ converges to a two-bubble in continuous time. $\square$

Proof of Theorem 1. The energy constraint implies that we have $N \leq 2$ in Theorem 1.1. If $N \neq 2$, then [5, Corollary 1.4] yields the result (we note that proof of [5, Corollary 1.4] in the cases $N = 0, 1$ immediately generalizes to $k \geq 2$). If $N = 2$, then the assumptions of our Theorem 2 are satisfied. □

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The importance of the collision problem for the continuous time soliton resolution was explained to us by Frank Merle.

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