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Representations of surface groups with universally finite mapping class group orbit

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Let $\Sigma_{g,n}$ be the orientable genus g surface with n punctures, where $2-2g-n<0$. Let

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_m(\mathbb{C})$$

be a representation. Suppose that for each finite covering map $f : \Sigma_{g',n'} \to \Sigma_{g,n}$, the orbit of (the isomorphism class of) $f^*(\rho)$ under the mapping class group $MCG(\Sigma_{g',n'})$ of $\Sigma_{g',n'}$ is finite. Then we show that ρ has finite image. The result is motivated by the Grothendieck-Katz p-curvature conjecture, and gives a reformulation of the p-curvature conjecture in terms of isomonodromy.

1. Introduction

1.1. The main result

The goal of this paper is to prove a result on mapping class group actions on character varieties, motivated by questions from algebraic and arithmetic geometry.

Our main result may be stated purely topologically. Let Σ be an orientable surface (possibly with finitely many punctures and boundary components) with $\chi(\Sigma)<0$. Note that the mapping class group $\mathrm{MCG}(\Sigma)$ of Σ has a natural outer action on $\pi_1(\Sigma)$, and hence acts on the set of isomorphism classes of complex representations of $\pi_1(\Sigma)$.

Theorem 1.1.1. *Let*

$$\rho : \pi_1(\Sigma) \to GL_m(\mathbb{C})$$

be a representation. Suppose that for each finite covering map

$$f : \Sigma' \to \Sigma,$$

the orbit of (the isomorphism class of) $f^*(\rho)$ under the mapping class group $MCG(\Sigma')$ is finite. Then ρ has finite image.

We will give a proof of Theorem 1.1.1 in Section 3.2.

Remark 1.1.2. *Note that if $\rho : \pi_1(\Sigma) \to GL_m(\mathbb{C})$ has finite image, then its orbit under the mapping class group is finite. In general, the converse is not true; see e.g. [2, Proposition 1.2] or Example 3.3.1 of this paper. See also [2, Theorem 1.1] for a result related to our Theorem 1.1.1, where the mapping class group is replaced by $Aut(\pi_1(\Sigma))$.*

See also [1] for stronger results in the case of representations into $SL_2(\mathbb{C})$.

As a corollary of our main theorem, we have the following purely group-theoretic statement:

Corollary 1.1.3. *Let*

$$\rho : \pi_1(\Sigma) \to GL_m(\mathbb{C})$$

be a representation. Suppose that for each finite index subgroup $G \subset \pi_1(\Sigma)$, the orbit of (the isomorphism class) of $\rho|_G$ under $Out(G)$ is finite. Then ρ has finite image.

Remark 1.1.4. *Note that the analogue of Corollary 1.1.3 is not true for general groups. For example, let $n > 2$ and let*

$$\rho_{std} : SL_n(\mathbb{Z}) \to GL_n(\mathbb{C})$$

be the standard representation. For any $G \subset SL_n(\mathbb{Z})$ of finite index, the orbit of $\rho_{std}|_G$ under $Out(G)$ is a singleton, by e.g. Margulis super-rigidity — but of course ρ_{std} has infinite image.

Remark 1.1.5. *To clarify ideas, we'll explain the special case where the representation ρ has rank $m = 1$, i.e. is given by a map*

$$\rho\colon \pi_1(\Sigma) \to \mathbb{C}^*.$$

Such a ρ must factor through the abelianization $H_1(\Sigma)$ of $\pi_1(\Sigma)$. Choosing a basis for $H_1(\Sigma)$, we see that the set of such ρ is in bijection with

$$\mathrm{Hom}(H_1(\Sigma), (\mathbb{C}^*)) \cong (\mathbb{C}^*)^{2g} \cong (\mathbb{C}/\mathbb{Z})^{2g}$$

(the second isomorphism being given by a suitably normalized logarithm).

The mapping class group acts through its quotient $\mathrm{Sp}_{2g}(\mathbb{Z})$ *on* $(\mathbb{C}/\mathbb{Z})^{2g}$ *in the obvious way. In order that* ρ *be MCG-finite, the corresponding point of* $(\mathbb{C}/\mathbb{Z})^{2g}$ *must have finite orbit under the action of* $\mathrm{Sp}_{2g}(\mathbb{Z})$*. One verifies that the only such points are torsion points, i.e. elements of*

$$(\mathbb{Q}/\mathbb{Z})^{2g}.$$

Hence, if ρ *is MCG-finite, then it has finite image.*

Theorem 1.1.1 and Corollary 1.1.3 are in fact equivalent in the case that Σ is a closed surface, as in this case $MCG(\Sigma)$ has finite index in $\mathrm{Out}(\pi_1(\Sigma))$ by the Dehn-Nielsen-Baer Theorem. The case of surfaces with punctures (and possibly boundary components) also admits a purely group-theoretic reformulation, but we omit it here.

For the rest of the introduction, we explain the motivation for this theorem, arising from the p-curvature conjecture, and its implications for isomonodromic deformations of flat vector bundles on algebraic curves.

1.2. The algebro-geometric setting

Let C be smooth proper algebraic curve over the field of complex numbers, and let $D \subset C$ be a finite set. The Riemann-Hilbert correspondence is an equivalence of categories between the category of algebraic flat vector bundles with regular singularities at infinity on $C \setminus D$, (that is, flat vector bundles on $C \setminus D$ which extend to objects of the category

$$\mathrm{MIC}(C(\log D))$$

of vector bundles with flat holomorphic connection

$$(\mathscr{E}, \nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_C(\log D))$$

on C) and the category $\mathrm{LocSys}(C \setminus D)$ of complex local systems on $C \setminus D$. If we choose a base-point $x \in C$, then monodromy gives an equivalence of both categories above with the category $\mathrm{Rep}_{\mathbb{C}}(\pi_1(C \setminus D, x))$ of representations

$$\rho : \pi_1(C \setminus D, x) \to \mathrm{GL}_n(\mathbb{C})$$

of the topological fundamental group of C. Let $\rho_{\mathscr{E},\nabla}$ be the representation associated to a flat vector bundle $(\mathscr{E}, \nabla)$.

Consider the relative situation, where we have a family $\pi : \mathcal{C} \to S$ of smooth proper curves over a smooth base S, which we take to be a scheme over $\mathbb{C}$. Locally for the complex topology, we can choose a section $x : S \to \mathcal{C}$, and (the isomorphism class of) the fundamental group of the fiber is locally constant on S. If we are given a base-point $s_0 \in S$, and a vector bundle with connection

$$(\mathscr{E}_0, \nabla_0 : \mathscr{E}_0 \to \mathscr{E}_0 \otimes \Omega^1_C)$$

on the fiber $C_0 = \mathcal{C}_{s_0}$, there is a unique (up to canonical isomorphism) analytic deformation

$$(\mathscr{E}, \nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{\mathcal{C}/S})$$

of $(E_0^{\text{an}}, \nabla_0^{\text{an}})$ to a relative flat vector bundle on $\pi^{-1}(U)$, where $U \subset S$ is any contractible analytic open set containing s_0, such that (the isomorphism class of) the corresponding representation ρ of the fundamental group is constant. Explicitly, as U is contractible, $\pi^{-1}(U)$ is naturally homotopy equivalent to C_0, so the composition

$$\pi_1(\pi^{-1}(U)) \xrightarrow{\sim} \pi_1(C_0) \xrightarrow{\rho_{\mathscr{E},\nabla_0}} GL(\mathscr{E}_{x(s_0)})$$

yields a local system on $\pi^{-1}(U)$, hence an (analytic) flat vector bundle. We call this the *isomonodromic deformation* of $(\mathscr{E}_0, \nabla_0)$. Such isomonodromic deformations are sometimes referred to as flat sections to the non-abelian Gauss-Manin connection.

Typically, if S is not simply connected, the isomonodromic deformation does not extend to all of $\mathcal{C}/S$, even after étale base change. If it does, we say that $(\mathscr{E}, \nabla)$ admits an algebraic isomonodromic deformation.

Definition 1.2.1. *Let $(\mathscr{E}, \nabla)$ be a flat vector bundle on a smooth proper curve C of genus $g > 1$. Let $\mathscr{C}_g \to \mathscr{M}_g$ be the universal curve over the Deligne-Mumford moduli stack of genus g curves. We say (following [3]) that $(\mathscr{E}, \nabla)$ admits a universal algebraic isomonodromic deformation if there exists an étale $U \to \mathscr{M}_g$ containing $[C]$ in its image such that $(\mathscr{E}, \nabla)$ admits an isomonodromic deformation to $U \times_{\mathscr{M}_g} \mathscr{C}_g$.*

By e.g. [3, Theorem A], $(\mathscr{E}, \nabla)$ admits a universal algebraic isomonodromic deformation if and only if the orbit of $\rho_{\mathscr{E},\nabla}$ under the mapping class group of C is finite. (See Section 2.4 of [3] for an extension of these notions to the case of non-proper curves.)

Thus, using the Riemann existence theorem, Theorem 1.1.1 for surfaces without boundary admits a purely algebro-geometric statement:

Theorem 1.2.2. *Let C be a curve over $\mathbb{C}$ with $\chi(C) < 0$, and let $(\mathscr{E}, \nabla)$ be a flat vector bundle on C. Suppose that for all finite étale maps of curves $f : C' \to C$, $f^*(\mathscr{E}, \nabla)$ admits a universal algebraic isomonodromic deformation. Then $(\mathscr{E}, \nabla)$ has finite monodromy.*

1.3. The arithmetic setting and the p-curvature conjecture

The authors became interested in isomonodromic deformations by way of the Grothendieck-Katz p-curvature conjecture [6]. The strategy is based on an idea of Kisin, and is closely related to work of Papaioannou [9], Shankar [14], and Patel-Shankar-Whang [11]. Given a vector bundle with connection $(\mathscr{E}, \nabla)$ on a curve C over an arbitrary field, we say that $(\mathscr{E}, \nabla)$ *admits a full set of algebraic sections* if there exist some curve C' and finite map $C' \to C$ such that the pullback of $(\mathscr{E}, \nabla)$ to C' is spanned as an $\mathscr{O}_{C'}$-module by flat global sections.

Let K be a finitely-generated field of characteristic zero, and take C and $(\mathscr{E}, \nabla)$ as above. We can spread this picture out to some integral domain $R \subset K$ with $\mathrm{Frac}(R) = K$, and reduce modulo any maximal ideal $\mathfrak{m}$ of R. Let $(C_{\mathfrak{m}}, \mathscr{E}_{\mathfrak{m}}, \nabla_{\mathfrak{m}})$ denote the base change of the spreading-out of $(C, \mathscr{E}, \nabla)$ to $R/\mathfrak{m}$.

Conjecture 1.3.1 (The p-curvature conjecture, Grothendieck--Katz). *In order that $(\mathscr{E}, \nabla)$ admit a full set of algebraic sections, it is necessary and sufficient that $(\mathscr{E}_{\mathfrak{m}}, \nabla_{\mathfrak{m}})$ admit a full set of algebraic sections for all $\mathfrak{m}$ in a non-empty open subset of* $\mathrm{Spec}(R)$.

Note that the hypothesis is independent of the chosen spreading-out, and that necessity above is clear. See [6] for a discussion of Conjecture 1.3.1, and a proof in the case $(\mathscr{E}, \nabla)$ arises from the de Rham cohomology of a family of varieties over C, endowed with the Gauss-Manin connection.

The authors' main motivation for this paper is the observation that the hypothesis of the p-curvature conjecture (namely that $(\mathscr{E}_{\mathfrak{m}}, \nabla_{\mathfrak{m}})$ admit a full set of algebraic sections for all $\mathfrak{m}$ in a non-empty open subset of $\mathrm{Spec}(R)$) is stable under pullback. In particular, Theorem 1.2.2 suggests the following reformulation of the p-curvature conjecture, in terms of the so-called non-abelian Gauss-Manin connection (i.e. isomonodromic deformation). Let $C, K, \mathscr{E}, \nabla$ be as above. Choose an embedding $K \hookrightarrow \mathbb{C}$.

Conjecture 1.3.2. *If $(\mathscr{E}_{\mathfrak{m}}, \nabla_{\mathfrak{m}})$ admit a full set of algebraic sections for all $\mathfrak{m}$ in a non-empty open subset of* $\mathrm{Spec}(R)$*, then the flat vector bundle $(\mathscr{E}, \nabla)_{\mathbb{C}}$ on $C_{\mathbb{C}}$ admits a universal algebraic isomonodromic deformation.*

Conjectures 1.3.1 and 1.3.2 are equivalent by Theorem 1.2.2, and the well-known fact that the p-curvature conjecture may be reduced to the case of smooth proper curves of genus at least 2.

Remark 1.3.3. *Let $\mathscr{X}/\mathscr{O}_{K,S}$ be a smooth proper curve over the ring of S-integers of a number field K, and let $(\mathscr{E}, \nabla)$ be an arithmetic $\mathscr{D}_{\mathscr{X}/\mathscr{O}_{K,S}}$-module on $\mathscr{X}$ (this condition is a priori much stronger than the hypotheses of the p-curvature conjecture). Then one can use the main result of [14] to see that the analogue of Conjecture 1.3.2 for such $(\mathscr{E}, \nabla)$ implies finiteness of monodromy.*

Remark 1.3.4. *To connect our work to the p-curvature conjecture, one would like to know something about the behavior of p-curvature under isomonodromic deformation. Unfortunately, it seems very difficult to say anything concrete here. For instance, one might like to say that the condition of vanishing p-curvature is preserved under isomonodromic deformation. But it's not clear how to make sense of this statement, since isomonodromic deformations don't in general exist integrally.*

1.4. Plan of the proof of Theorem 1.1.1

The argument is a proof by induction on the dimension of the representation. Roughly speaking, if there is some $\gamma \in \pi_1(C)$ such that $\rho(\gamma)$ is not of finite order, we pass to a finite cover, make γ a simple closed curve, and cut along γ. After cutting, we show that the representation ρ becomes reducible, so we can reduce the problem to a lower-dimensional case (on the cut surface). If there is no such γ, we conclude by Lemma 3.1.3.

1.5. Questions

Our proof of Theorem 1.1.1 is geometric; Corollary 1.1.3 suggests that one might look for a purely group-theoretic proof. More generally, one might ask for an intrinsic characterization of those groups for which the analogue of Corollary 1.1.3 holds true.

Definition 1.5.1 (Locally Extended Residually Finite (LERF)). *A group G is said to be Locally Extended Residually Finite (LERF) if for every finitely-generated subgroup $H \subset G$, H is closed in the profinite topology on G.*

Question 1.5.2. *Suppose G is finitely-generated and LERF. Let*

$$\rho : G \to GL_n(\mathbb{C})$$

be a representation such that for each finite-index subgroup $H \subset G$, the orbit of the isomorphism class of $\rho|_H$ under $Out(H)$ is finite. Does ρ necessarily have finite image?

Note that Scott shows that surface groups are LERF [13]; this fact is crucially used in the proof of Theorem 1.1.1.

The next question was suggested to us by Junho Peter Whang; it asks whether, when the genus is large compared to the rank, we can eliminate the finite covers Σ' from the statement of Theorem 1.1.1. A positive answer for SL_2 is given by [1].

Question 1.5.3. *Suppose m is a positive integer. Is the following statement true for all Σ of sufficiently large genus: For any representation*

$$\rho : \pi_1(\Sigma) \to GL_m(\mathbb{C}),$$

if the orbit of (the isomorphism class of) $f^*(\rho)$ under the mapping class group $MCG(\Sigma)$ is finite, then ρ has finite image?

Finally, we propose two variants on our main theorem; we suspect both are true, and could be proven by similar methods, but we have not verified either.

Question 1.5.4. *Does the statement of Theorem 1.1.1 remain true, if Σ' is instead allowed to range over all branched covers of Σ of degree 2?*

Question 1.5.5. *(Junho Peter Whang) Suppose*

$$\rho : \pi_1(\Sigma) \to GL_m(\mathbb{C})$$

is an absolutely irreducible representation, such that for each finite covering map

$$f : \Sigma' \to \Sigma,$$

the orbit of (the isomorphism class of) $f^*(\rho)$ under the mapping class group $MCG(\Sigma')$ has compact closure in the character variety classifying conjugacy classes of maps $\pi_1(\Sigma') \to GL_m(\mathbf{C})$. Must ρ be unitarizable? (See [15] for the definition and properties of character variety.)

2. MCG-finiteness

2.1. Definitions

Let Σ be an orientable surface, possibly with boundary/punctures; let $p \in \Sigma$ be a point. Let $\mathrm{MCG}(\Sigma)$ be the mapping class group of Σ. (See [5, Section 2.1] for a discussion of mapping class groups. In particular, recall that an element of $MCG(\Sigma)$ must fix $\partial\Sigma$ point-wise, but may permute punctures.)

The natural map

$$MCG(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma, p))$$

induces an action of $MCG(\Sigma)$ on the set of isomorphism classes of representations of $\pi_1(\Sigma, p)$, as we now explain.

We say that two representations

$$\rho_1, \rho_2 \colon \pi_1(\Sigma, p) \to GL_r(\mathbb{C})$$

are *isomorphic* if there is some $g \in GL_r(\mathbb{C})$ such that $\rho_2 = g\rho_1 g^{-1}$.

Any mapping class in $MCG(\Sigma)$ has a representative $T \colon \Sigma \to \Sigma$ that fixes the basepoint p. Then T gives an automorphism $\pi_1(T)$ of $\pi_1(\Sigma)$; we denote by $T^*\rho$ the representation of $\pi_1(\Sigma)$ obtained by precomposing ρ with $\pi_1(T)$. If T' is another representative of the same mapping class, also fixing p, then T' and T agree up to an inner automorphism of $\pi_1(T)$, so $(T')^*\rho$ is isomorphic (i.e. conjugate) to $T^*\rho$.

Definition 2.1.1. *Say a representation*

$$\rho : \pi_1(\Sigma, p) \to GL_r(\mathbb{C})$$

is MCG-finite *if the orbit of its isomorphism class under the action of* $MCG(\Sigma)$ *is finite.*

Say ρ *is* universally MCG-finite *if, for any finite covering map* $\Sigma' \to \Sigma$*, its pullback to* Σ' *is MCG-finite.*

Remark 2.1.2. *One way to produce MCG-finite representations is as follows. Let* Σ *be a closed orientable surface,* $p \in \Sigma$ *a point, and consider the*

Birman exact sequence

$$1 \to \pi_1(\Sigma, p) \to MCG(\Sigma \setminus p) \to MCG(\Sigma) \to 1.$$

Then if $\rho : MCG(\Sigma \setminus p) \to GL_m(\mathbb{C})$ is any representation, $\rho|_{\pi_1(\Sigma,p)}$ is MCG-finite, by the normality of $\pi_1(\Sigma, p)$ in $MCG(\Sigma \setminus p)$. See Example 3.3.1 for an example of a representation ρ of $MCG(\Sigma \setminus p)$ such that $\rho|_{\pi_1(\Sigma,p)}$ has infinite image.

Our goal in this paper is to show that *universally* MCG-finite representations have finite image.

In order to prove this result, we will need a refined notion of universal MCG-finiteness for surfaces obtained by cutting a given surface along special collections of simple closed curves.

Definition 2.1.3. *Let Σ be an orientable surface, possibly with boundary/punctures. Let $\gamma_1, \cdots, \gamma_r$ be disjoint simple closed curves on Σ. We say that $\{\gamma_1, \cdots, \gamma_r\}$* are not jointly separating *if $\Sigma \setminus \bigcup_i \gamma_i$ is connected.*

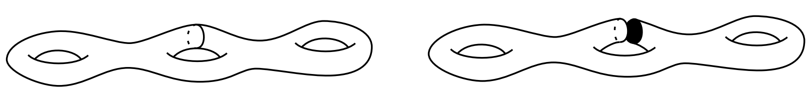

Figure 1: The surfaces Σ and Σ_{cut}.

Suppose Σ is an orientable surface and $\{\gamma_1, \cdots, \gamma_r\}$ is a collection of disjoint curves in Σ. Then we define $\Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r)$ to be the surface with with boundary obtained by removing an ϵ-neighborhood of $\bigcup_i \gamma_i$ from Σ.

Definition 2.1.4. *Let Σ be an orientable surface, possibly with boundary/punctures. Let $(\gamma_1, \cdots, \gamma_r)$ be a collection of simple closed curves in Σ which are not jointly separating, with $r \geq 0$. Then a representation*

$$\rho : \pi_1(\Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r)) \to GL_m(\mathbb{C})$$

is universally MCG-finite relative to Σ *if for each finite covering space $f : \Sigma' \to \Sigma$, and each connected component X of $\Sigma'_{\text{cut}}(f^{-1}(\gamma_1), \cdots, f^{-1}(\gamma_r))$, the representation $\rho|_{\pi_1(X)}$ is MCG-finite.*

We will in fact prove a version of Theorem 1.1.1 in the relative setting:

Theorem 2.1.5. *Let Σ be an orientable surface, possibly with boundary/punctures. Let $(\gamma_1, \cdots, \gamma_r)$ be a collection of simple closed curves in Σ which are not jointly separating. Suppose that $\chi(\Sigma_{\mathrm{cut}}(\gamma_1, \cdots, \gamma_r)) < 0$ and let*

$$\rho : \pi_1(\Sigma_{\mathrm{cut}}(\gamma_1, \cdots, \gamma_r)) \to GL_m(\mathbb{C})$$

be a representation which is universally MCG-finite relative to Σ. Then ρ^{ss} has finite image.

Here ρ^{ss} is the semi-simplification of ρ.

Theorem 1.1.1 will follow from Theorem 2.1.5 by Lemma 2.2.6; the modified statement of Theorem 2.1.5 is more amenable to geometric constructions, e.g. to cutting the surface in question.

It is not clear to us whether semi-simplification is necessary in Theorem 2.1.5.

Question 2.1.6. *Let Σ be an orientable surface, possibly with boundary/punctures. Let $(\gamma_1, \cdots, \gamma_r)$ be a collection of simple closed curves in Σ which are not jointly separating. Suppose that $\chi(\Sigma_{\mathrm{cut}}(\gamma_1, \cdots, \gamma_r)) < 0$ and let*

$$\rho : \pi_1(\Sigma_{\mathrm{cut}}(\gamma_1, \cdots, \gamma_r)) \to GL_m(\mathbb{C})$$

be a representation which is universally MCG-finite relative to Σ. Does ρ necessarily have finite image?

2.2. Basic properties of MCG-finiteness

Lemma 2.2.1. *The following hold:*

- *The semi-simplification of an MCG-finite representation is MCG-finite.*
- *Any sub-quotient of the semi-simplification of an MCG-finite representation is MCG-finite.*
- *Let Σ be an orientable surface, possibly with boundary/punctures. Let $(\gamma_1, \cdots, \gamma_r)$ be a collection of simple closed curves in Σ which are not jointly separating. Then any sub-quotient of the semi-simplification of a representation of $\pi_1(\Sigma_{\mathrm{cut}}(\gamma_1, \cdots, \gamma_r))$ which is universally MCG-finite relative to Σ is universally MCG-finite relative to Σ.*

Proof. The first statement follows from the fact that for representations V, W, we have $V^{ss} \simeq W^{ss}$ if $V \simeq W$; thus, if the mapping class group orbit of V is finite, then the same is true of V^{ss}.

To see the second statement, take V an MCG-finite representation and W a sub-quotient of V^{ss}. Then for any $T \in MCG(\Sigma)$, we know that T^*W is a sub-quotient of T^*V^{ss}. The set of sub-quotients of $\{T^*V^{ss}\}_{m\in\mathbb{Z}}$ is finite by MCG-finiteness; hence there are only finitely many possibilities for T^*W, as desired.

The third claim is proved in exactly the same way. □

Next we will see that the property of universal MCG-finiteness is preserved when we cut Σ along a non-separating simple curve γ.

Lemma 2.2.2. *Let Σ be an orientable surface, possibly with boundary, and let $(\gamma_1, \cdots, \gamma_r, \cdots, \gamma_s)$ be a collection of disjoint simple closed curves on Σ which are not jointly separating. Suppose $\rho : \pi_1(\Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r)) \to \mathrm{GL}(V)$ is universally MCG-finite relative to Σ (or, if $r = 0$, that ρ is universally MCG-finite). Then the restriction $\rho|_{\Sigma_{\text{cut}}(\gamma_1,\cdots,\gamma_s)}$ is universally MCG-finite relative to Σ.*

Proof. This is immediate from the definitions. Indeed, let $f : \Sigma' \to \Sigma$ be any finite covering map, and let X be a component of $\Sigma'_{\text{cut}}(f^{-1}(\gamma_1), \cdots, f^{-1}(\gamma_s))$. Then any mapping class on X extends to a mapping class on Σ' (recall that mapping classes on surface with boundary must fix the boundary, by definition). The result follows. □

Next we'll work out a concrete consequence of MCG-finiteness. Suppose ρ is MCG-finite, and let γ be a simple closed curve on Σ, not passing through the base-point p. Consider the Dehn twist T_γ. MCG-finiteness implies that for some $m > 0$, there is an isomorphism

$$(T_\gamma^m)^*\rho \cong \rho. \tag{1}$$

The explicit geometric construction of the Dehn twist T_γ gives a bona fide automorphism (not just an outer automorphism) of $\pi_1(\Sigma, p)$. Hence, there is a preferred choice of isomorphism between the underlying vector spaces of $(T_\gamma^m)^*\rho$ and ρ; we'll use this isomorphism without comment in what follows.

The data of the isomorphism (1) is the same as that of a linear map $g : V \xrightarrow{\sim} V$, intertwining the actions of $\pi_1(\Sigma, p)$ via $(T_\gamma^m)^*\rho$ and ρ — we call g an "intertwining operator." Thus we have shown:

Lemma 2.2.3. *Suppose ρ is MCG-finite and γ is a simple closed curve on C. Then there exist a positive integer m and an automorphism g of V such that, for every $\delta \in \pi_1(\Sigma, p)$, we have*

$$\rho(T_\gamma^m \delta) = g\rho(\delta)g^{-1}.$$

Note that if ρ is simple, the intertwining operator g above is unique up to scaling, by Schur's lemma.

Definition 2.2.4. *Let (Σ, p) be an orientable surface, possibly with boundary/punctures. We denote by $H_1(\Sigma) = H_1(\Sigma, \mathbb{Z})$ the homology of Σ with* integral *coefficients.*

Let $\overline{H_1(\Sigma)}$ denote the quotient of $H_1(\Sigma)$ by the span of classes of boundary components and loops around punctures.

We say that $\gamma \in \pi_1(\Sigma, p)$ is nontrivial modulo boundary *(in $H_1(\Sigma)$) if the class of γ in $\overline{H_1(\Sigma)}$ is non-trivial.*

Remark 2.2.5. *Note that $\overline{H_1(\Sigma)}$ is naturally identified with the first homology of the compact orientable surface without boundary obtained by filling in the punctures of Σ and gluing caps onto the boundary components of Σ. In particular, $\overline{H_1(\Sigma)}$ has a well-defined intersection product $\langle -, - \rangle$; we abuse notation and denote the induced product on $H_1(\Sigma)$ (which is in general no longer non-degenerate) via $\langle -, - \rangle$ as well.*

We now show that the property of having finite image is preserved under extensions of universally MCG-finite representations.

Lemma 2.2.6. *Suppose $\chi(\Sigma) < 0$, and $\rho : \pi_1(\Sigma) \to \mathrm{GL}(V)$ fits in an exact sequence of representations*

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0,$$

where ρ_1 and ρ_2 have finite image, and suppose ρ itself is universally MCG-finite. Then ρ has finite image.

Proof. Let V_1 be the vector space underlying ρ_1 and $V_2 = V/V_1$ the vector space underlying ρ_2. By passing to a finite cover of Σ, we may assume ρ_1 and ρ_2 are in fact trivial.

The representation $\rho : \pi_1(\Sigma, p) \to GL(V)$ factors through the group

$$\mathrm{Aut}_{V_1, V_2}(V)$$

of automorphisms g of V fixing V_1 and acting trivially on both V_1 and V_2.

Given $f \in \mathrm{Hom}(V_2, V_1)$, let $\bar{f} : V \to V$ be the composition

$$V \twoheadrightarrow V_2 \xrightarrow{f} V_1 \hookrightarrow V.$$

It is a standard fact from linear algebra that the map $f \mapsto \bar{f} + \mathrm{Id}$ is an isomorphism

$$\mathrm{Hom}(V_2, V_1) \cong \mathrm{Aut}_{\mathrm{V_1,V_2}}(V).$$

Hence the representation ρ has abelian image, so it factors through a homomorphism

$$\sigma_\Sigma : H_1(\Sigma) \to \mathrm{Hom}(V_2, V_1).$$

The same remains true if we pull back to any finite cover $\Sigma' \to \Sigma$, and the resulting diagram

$$\begin{array}{ccc} \sigma_{\Sigma'} : H_1(\Sigma') & \longrightarrow & \mathrm{Hom}(V_2, V_1) \\ \downarrow & & \downarrow = \\ \sigma_\Sigma : H_1(\Sigma) & \longrightarrow & \mathrm{Hom}(V_2, V_1) \end{array}$$

commutes.

It suffices to show that σ_Σ is identically zero. Assume for a contradiction that $\sigma_\Sigma \neq 0$. Then it is possible to produce a cover Σ' and two classes $\gamma_1, \gamma_2 \in H_1(\Sigma')$ such that

- $\sigma_{\Sigma'}(\gamma_1) \neq 0$
- $\sigma_{\Sigma'}(\gamma_2) = 0$
- γ_1 and γ_2 have intersection number $\langle \gamma_1, \gamma_2 \rangle = i \neq 0$.
- γ_1 is represented by a simple closed loop.

To see this, we first choose some nontrivial cover Σ' of Σ so that $\mathrm{genus}(\Sigma') > \mathrm{genus}(\Sigma)$ (using that $\chi(\Sigma) < 0$), and let γ_2 be a class in $H_1(\Sigma')$, nontrivial modulo boundary, and in the kernel of the $H_1(\Sigma') \to H_1(\Sigma)$.

Then, we choose a simple closed loop γ_1 in Σ' whose class in $H_1(\Sigma')$ satisfies: $\sigma_{\Sigma'}(\gamma_1) \neq 0$ and $\langle \gamma_1, \gamma_2 \rangle \neq 0$. This is possible as the set of γ failing one of these conditions is a union of two proper linear subspaces of $H_1(\Sigma')$, and thus does not contain every primitive element in $H_1(\Sigma')$; but any primitive element is represented by a simple closed loop by e.g. [5, Proposition 6.2].

Now apply Lemma 2.2.3 to the Dehn twist T_{γ_1}. We find some m such that

$$(T_{\gamma_1}^m)^*\rho \cong \rho$$

and hence

$$(T_{\gamma_1}^m)^*\sigma_{\Sigma'} = \sigma_{\Sigma'}.$$

On the other hand, we have

$$\sigma_{\Sigma'}(\gamma_2) = 0$$

and

$$(T_{\gamma_1}^m)^*\sigma_{\Sigma'}(\gamma_2) = \sigma_{\Sigma'}(\gamma_2\gamma_1^{im}) = \sigma_{\Sigma'}(\gamma_1^{im}) \neq 0,$$

so the two representations $(T_{\gamma_1}^m)^*\rho$ and ρ cannot be isomorphic. This is our desired contradiction. □

Remark 2.2.7. *We do not know if an analogue of Lemma 2.2.6 holds in the relative setting.*

3. Proof of Theorems 1.1.1 and 2.1.5

Before proceeding to the proof of Theorem 1.1.1, we record a few useful lemmas.

3.1. Some useful lemmas

First, we prove some variants on a lemma that appears in [16] (Proposition 2.5).

Recall that a matrix is called *quasi-unipotent* if some power of it is unipotent.

Lemma 3.1.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finitely-generated group on a complex vector space. Suppose that for every $g \in G$, the transformation $\rho(s)$ is quasi-unipotent. Then the semisimplification ρ^{ss} has finite image.*

Proof. First, suppose ρ is simple, so $\rho(G)$ spans the algebra $\mathrm{End}(V)$ as a $\mathbb{C}$-vector space, by Burnside's Theorem [4, Theorem 27.4]. Let $g_1, \ldots, g_{n^2}$

be elements in G such that $\{\rho(g_i)\}$ forming a basis for $\mathrm{End}(V)$, and let $e_1, \ldots, e_{n^2}$ be the dual basis under the trace pairing. For $g \in G$, we have

$$\rho(g) = \sum_i \mathrm{Tr}(\rho(g_i)^{-1}\rho(g))e_i.$$

By the proof of [16, Lemma 2.4], $\mathrm{Tr}(g)$ takes on only finitely many values as g ranges over all elements of G. We conclude that $\rho(G)$ is finite.

The general result follows immediately from the simple case. □

Lemma 3.1.2. *Let G be a finitely-generated group, and $H \subseteq G$ a subgroup of finite index. Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation such that $(\rho|_H)^{ss}$ has finite image. Then ρ^{ss} also has finite image.*

Proof. We know that $\rho(g)$ is quasi-unipotent for $g \in H$. Now let $g \in G$ be arbitrary; there exists some $n > 0$ such that $g^n \in H$. Since $\rho(g)^n$ is quasi-unipotent, $\rho(g)$ itself is as well. We conclude by Lemma 3.1.1 that ρ^{ss} has finite image. □

Lemma 3.1.3. *Let (Σ, p) be a pointed orientable surface (possibly with punctures or boundary), and let $\rho : \pi_1(\Sigma, p) \to \mathrm{GL}(V)$ be a simple representation of $\pi_1(\Sigma, p)$ on a complex vector space. (Here simple means that V has no proper nontrivial ρ-stable subspace W.)*

Assume that Σ has genus at least 1. Suppose that, for every $\gamma \in \pi_1(\Sigma, p)$ nontrivial modulo boundary in $H_1(\Sigma)$ (Def. 2.2.4), $\rho(\gamma)$ is quasi-unipotent. Then ρ has finite image.

Proof. Let $G = \pi_1(\Sigma, p)$.

Since ρ is simple, $\rho(G)$ spans the algebra $\mathrm{End}(V)$ as a $\mathbb{C}$-vector space. Let $g_1, \ldots, g_{n^2}$ be elements in G such that $\{\rho(g_i)\}$ forming a basis for $\mathrm{End}(V)$, and let $e_1, \ldots, e_{n^2}$ be the dual basis under the trace pairing. For $g \in G$, we have

$$\rho(g) = \sum_i \mathrm{Tr}(\rho(g_i)^{-1}\rho(g))e_i.$$

Let $\Gamma \subset G$ be the subset consisting of those elements $\gamma \in G$ such that $g_i^{-1}\gamma$ is non-trivial mod boundary for all i.

By the proof of [16, Lemma 2.4], $\mathrm{Tr}(\gamma)$ takes on only finitely many values as γ ranges over all elements of $\pi_1(\Sigma, p)$ nontrivial modulo boundary in H_1. Thus there is a finite subset $G_0 \subseteq \mathrm{End}\, V$ such that, if γ is contained in Γ, then in fact $\rho(\gamma) \in G_0$.

Now we claim that any γ can be written as a product $\gamma = \gamma_1\gamma_2$, with in $\gamma_1, \gamma_2 \in \Gamma$. Let $\bar{g}_j$ be the image of g_j in $H_1(C)$. It is enough to show that any $h \in H_1(\Sigma)$ can be written as $h = h_1 + h_2$, where $h_j - \bar{g}_i$ has non-zero image in $\overline{H_1(\Sigma)}$ for all i, j. But now we may choose h_1 to be any element of $H_1(\Sigma)$ whose image in $\overline{H_1(\Sigma)}$ is not equal to that of $\bar{g}_i$ or $h - \bar{g}_i$ for any i, and set $h_2 = h - h_1$.

We conclude that $\rho(G) \subseteq G_0G_0$, so $\rho(G)$ is finite. □

Corollary 3.1.4. *Let (Σ, p) be a pointed orientable surface (possibly with punctures or boundary), and let $\rho : \pi_1(\Sigma, p) \to \mathrm{GL}(V)$ be any representation of $\pi_1(\Sigma, p)$ on a complex vector space.*

Assume that Σ has genus at least 1. Suppose that, for every $\gamma \in \pi_1(\Sigma, p)$ nontrivial modulo boundary in $H_1(\Sigma)$, $\rho(\gamma)$ is quasi-unipotent. Then the semi-simplification ρ^{ss} of ρ has finite image.

Proof. Immediate from Lemma 3.1.3. □

Not every class $\gamma \in \pi_1(\Sigma, p)$ can be represented by a simple curve (i.e. a curve with no self-intersection). But any γ becomes a simple curve after pullback to a cover.

The result we need is a simple application of a theorem of Scott [13, Theorem 3.3]. Scott's theorem allows one to find covers such that a given curve lifts to a simple closed curve; we observe that one may do so while keeping it disjoint from a collection of other curves. (A detailed exposition of Scott's proof may be found in [12].)

Lemma 3.1.5. *Let (Σ, p) be a pointed orientable surface with $\chi(\Sigma) < 0$, and $\gamma_1, \ldots, \gamma_r$ simple closed curves on Σ, not passing through p and not jointly separating. Suppose $\gamma \in \pi_1(\Sigma, p)$ is represented by a closed curve that is disjoint from $\gamma_1, \ldots, \gamma_r$.*

Then there exists a finite cover $f\colon (\Sigma', p') \to (\Sigma, p)$ such that the subgroup $\pi_1(\Sigma', p') \subseteq \pi_1(\Sigma, p)$ contains γ, and the class $\gamma \in \pi_1(\Sigma', p')$ is represented by a simple *closed curve, disjoint from the curves $f^{-1}(\gamma_i)$.*

Proof. Scott's theorem ([13, Theorem 3.3]) shows that there is some $f\colon (\Sigma', p') \to (\Sigma, p)$ such that $\gamma \in \pi_1(\Sigma', p')$ is represented by a simple closed curve in Σ'. We need to show that γ may be taken to avoid the curves $f^{-1}(\gamma_i)$.

We can put a hyperbolic metric on Σ' in such a way that each γ_i is a geodesic. The universal cover $\tilde{\Sigma}$ of Σ' is a convex region in the hyperbolic plane $\mathbf{H}$, bounded by lines. (If Σ – hence Σ' – has no boundary components, then the universal cover is all of $\mathbf{H}$. See [12, end of §5].)

The curves γ_i lift to lines in $\tilde{\Sigma} \subseteq \mathbf{H}$; let L be the union of these lines, and let X be the connected component of p in $\tilde{\Sigma} - L$. Again, X is a convex region in $\mathbf{H}$, bounded by a union of lines.

The fundamental group of Σ' acts on the universal cover $\tilde{\Sigma}$ by deck transformations, which extend to translations of $\mathbf{H}$. Let g be the translation of $\mathbf{H}$ corresponding to γ. Since γ is represented by a curve in Σ that avoids the curves γ_i, we find that $g^n p' \in X$ for all $n \in \mathbf{Z}$. The translation g acts on $\partial\mathbf{H}$ with two fixed points $g^{\pm\infty}p'$. By a limiting argument, both these fixed points lie in ∂X, so the geodesic ℓ between them lies in X.

Now ℓ descends to a curve isotopic to γ in Σ'; this is again a simple closed curve by [5, Proposition 1.6], and it avoids the curves $f^{-1}(\gamma_i)$ by construction. □

Finally, we observe:

Lemma 3.1.6. *Let V be a complex vector space, and let $g \in \mathrm{GL}(V)$ be an element not equal to a scalar matrix. Then there exists a nontrivial proper subspace $W \subset V$ such that, if $h \in \mathrm{GL}(V)$ commutes with g, then h stabilizes W.*

Proof. Take W to be any nontrivial eigenspace for any eigenvalue of g. □

3.2. Proof of the main theorem

We may now proceed with the proof of our main result. First we prove the result for relatively universally MCG-finite representations.

Proof of Theorem 2.1.5. For the reader's convenience, we briefly recall the theorem we are trying to prove. Namely, let Σ be an orientable surface, possibly with boundary/punctures, and let $\gamma_1, \ldots, \gamma_r$ be disjoint simple closed curves on Σ, not jointly separating. Let $p \in \Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r)$ be a point. Suppose $\pi_1(\Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r), p)$ is nonabelian, and we have a representation

$$\rho : \pi_1(\Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r), p) \to GL(V)$$

that is universally MCG-finite relative to Σ. Then we wish to show that the semi-simplification ρ^{ss} of ρ has finite image.

Write $\Sigma_{\text{cut}} = \Sigma_{\text{cut}}(\gamma_1, \cdots, \gamma_r)$. We will proceed by induction on the rank of ρ.

By Lemma 3.1.2, there is no harm in passing to a finite cover of Σ. Hence we may assume that Σ_{cut} is of genus at least 2.

Let $S \subseteq \pi_1(\Sigma_{\text{cut}}, p)$ be the set of elements that are nontrivial modulo boundary in $H^1(\Sigma_{\text{cut}})$. We will show that for $\gamma \in S$, $\rho(\gamma)$ is quasi-unipotent.

Fix $\gamma \in S$. Pulling back to a further finite cover Σ' of Σ, we may assume by Lemma 3.1.5 that γ is isotopic to a simple closed curve γ_{r+1}, such that $\gamma_1, \dots, \gamma_{r+1}$ are not jointly separating in Σ (as γ is non-trivial mod boundary in Σ_{cut}).

By Lemma 2.2.3, there exist a positive integer m and an automorphism g of V such that, for every $\delta \in \pi_1(\Sigma_{\text{cut}}, p)$, we have

$$\rho(T^m_{\gamma_{r+1}}\delta) = g\rho(\delta)g^{-1}.$$

First, assume g is a scalar matrix. Then we have

$$\rho(T^m_{\gamma_{r+1}}\delta) = \rho(\delta)$$

for all δ. Taking δ a curve in Σ_{cut} which meets γ_{r+1} exactly once, and is transverse to γ_{r+1} at that point (possible because γ_{r+1} is non-separating), we have

$$\rho(\delta)\rho(\gamma_{r+1})^m = \rho(\delta\gamma^m_{r+1}) = \rho(T^m_{\gamma_{r+1}}\delta) = \rho(\delta)$$

and we find that

$$\rho(\gamma_{r+1})^m = 1,$$

as desired. In particular, if $\dim(V) = 1$, g is always a scalar matrix, and so we may conclude the base case of the induction.

Now, assume g is not a scalar matrix. Let Σ'_{cut} be the surface obtained from Σ_{cut} by cutting along γ_{r+1}, with base-point p not on one of the boundary components coming from γ_{r+1}. Since γ_{r+1} is a non-separating closed curve, and Σ_{cut} has genus at least 2, Σ'_{cut} is again a surface with nonabelian π_1, and our local system on Σ_{cut} pulls back to a local system on Σ'_{cut}.

The representation

$$\rho|_{\pi_1(\Sigma'_{\text{cut}}, p)} : \pi_1(\Sigma'_{\text{cut}}, p) \to \mathrm{GL}(V)$$

factors through the centralizer of g, as $\pi_1(\Sigma'_{\text{cut}}, p)$ is generated by a loop isotopic to γ_{r+1} and by loops in Σ not intersecting γ_{r+1}. By Lemma 3.1.6, there is thus a non-trivial proper subspace $W \subset V$ which is stable under this representation.

By Lemma 2.2.2, $\rho|_{\pi_1(\Sigma'_{\text{cut}}, p)}$ is universally MCG-finite relative to Σ'. Hence by Lemma 2.2.1, W^{ss} and $(V/W)^{\text{ss}}$ are universally MCG-finite relative to Σ' and thus by the inductive hypothesis, the representations on W^{ss} and $(V/W)^{\mathrm{ss}}$ have finite image. Thus $\rho(\gamma_{r+1})$ is quasi-unipotent.

Now we conclude the result by Corollary 3.1.4. □

We now deduce the main result of the paper.

Proof of Theorem 1.1.1. By Theorem 2.1.5, ρ^{ss} has finite image. Now we may conclude by Lemma 2.2.6, by induction on the number of components in the composition series for ρ. □

3.3. An example: the Parshin representation

In the course of proving Theorem 1.1.1, we also prove the following intermediate result.

Suppose

$$\rho : \pi_1(\Sigma, p) \to GL(V)$$

is MCG-finite, and γ is some simple closed non-separating curve in Σ. Let Σ_{cut} be the surface obtained by cutting Σ along γ. Take m such that

$$(T_\gamma^m)^*\rho \cong \rho,$$

and let

$$g\colon V \to V$$

be an intertwining operator such that

$$\rho(T_\gamma^m \delta) = g\rho(\delta)g^{-1}.$$

Then for any δ disjoint from γ, $\rho(\delta)$ must commute with g. In particular, if g is not a scalar matrix, then the restriction of ρ to Σ_{cut} is reducible.

As an example, we'll see what this looks like for ρ the Parshin representation. The idea goes back to Kodaira and Parshin [10, Appendix]; an explicit construction with the properties we describe is given in [8, Section 7].

Example 3.3.1. *Let $\Sigma = C$ be a complex algebraic curve of genus at least 2, and let p be a point of C. There are finitely many isomorphism classes of degree-3 covers Y_p^i of C, branched at p and nowhere else, and having Galois group S_3. There is an algebraic family $\pi\colon Y \to C$ whose fiber over any point $p \in C(\mathbf{C})$ is the disjoint union of the curves Y_p^i. Let $Y \to C' \to C$ be the Stein factorization of π. The cohomology of this family $R^1\pi_*(\mathbf{C}_Y)$*

gives a local system on C; the corresponding representation ρ is exactly the monodromy representation of $\pi_1(C)$ on the cohomology of a fiber.

First, note that ρ is MCG-finite, as by construction it extends to a representation of $MCG(C \setminus \{p\})$ (see Remark 2.1.2).

Next, note that ρ is virtually reducible: on a finite-index subgroup of $\pi_1(C)$ (namely, $\pi_1(C')$), it splits up as a direct sum of the $H^1(Y_p^i)$. Each $H^1(Y_p^i)$ is virtually irreducible; this follows from the big monodromy result [8, Lemma 4.3]. In particular, ρ has infinite image; it is thus not universally MCG-finite.

Now fix a simple closed curve γ on C, not passing through p, and take m sufficiently divisible. Then γ^m lifts to a disjoint union of simple closed curves on each cover Y_p^i. Thus T_γ^m acts unipotently, as a product of commuting Dehn twists.

The corresponding g is also unipotent: it's given by the action of a Dehn twist about γ, as an element of MCG(C).

Let V_i be the subspace of H^1 dual to the subspace of H_1 spanned by the components of the lift of γ^m to Y_p^i. Then $\sum_i V_i$ (the direct sum of the V_i) is a g-stable subspace of $\rho|_{\pi_1(\Sigma_{\mathrm{cut}})}$.

In particular, we see that $\rho|_{\pi_1(\Sigma_{\mathrm{cut}})}$ is reducible, as expected.

3.4. An example from topological quantum field theory

Unlike universally MCG-finite representations, MCG-finite representations can be quite interesting.

Example 3.4.1. *In [7, Thm. 1.1], Koberda and Santharoubane construct representations ρ of $\pi_1(\Sigma)$ with the following properties:*

- *ρ has infinite image.*
- *$\rho(\gamma)$ has finite order, for any simple closed curve γ.*
- *ρ is MCG-fixed ([7, §1.2]).*

This example is particularly relevant to the strategy outlined in the introduction for proving the p-curvature conjecture; it shows that it does not suffice to study the monodromy of simple closed loops in a surface.

3.5. An example: a unipotent, MCG-finite representation

We conclude with an example philosophically relevant to Question 2.1.6. Namely, for every surface Σ with $\chi(\Sigma) < 0$, we observe that there exist non-trivial unipotent representations of the fundamental group of Σ which are stable under the action of the mapping class group of Σ.

Example 3.5.1. *Let Σ be a surface with $\chi(\Sigma) < 0$, and let $p \in \Sigma$ be a point, so that $\pi_1(\Sigma, p)$ is non-abelian. Let $\mathbb{Q}[\pi_1(\Sigma, p)]$ be the group algebra of Σ and $\mathscr{I} \subset \mathbb{Q}[\pi_1(\Sigma, p)]$ the augmentation ideal. Then we claim that for any $n > 1$, the representation of $\pi_1(\Sigma, p)$ on $V_n := \mathbb{Q}[\pi_1(\Sigma, p)]/\mathscr{I}^n$ induced by the action of $\pi_1(\Sigma)$ on itself by conjugation is non-trivial, unipotent, and fixed by the action of the mapping class group.*

Indeed, direct computation shows that these representations are non-trivial; they are unipotent as for each i, the action of $\pi_1(\Sigma)$ on $\mathscr{I}^i/\mathscr{I}^{i+1}$ is trivial. Finally, these representations are MCG-finite (indeed, they are fixed by $MCG(\Sigma)$) by Remark 2.1.2, as each V_n is naturally a representation of $MCG(\Sigma \setminus p)$.

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