Math. Res. Lett.
Volume 30, Number 1, 33–49, 2023

Higher Ext-groups in the triple product case

Li Cai and Yangyu Fan

In this short note, we compute higher extension groups for all irreducible representations and deduce the multiplicity formula for finite length representations in triple product case.

1. Introduction

Let F be a p-adic field. Let L/F be a cubic étale extension and D/F be a quaternion algebra. Let $G = \mathrm{Res}_{L/F} D^\times$ and $H = D^\times$. Note that the intersection of the center Z_G of G with H is $Z_H = \mathbb{G}_m$. Denote by $\mathrm{Rep}(F^\times \backslash G(F))$ the category of smooth $F^\times \backslash G(F)$-representations.

In this short note, we prove that the higher Ext groups vanish

$$\mathrm{Ext}^i_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = 0, \quad i > 0.$$

for any generic $\pi \in \mathrm{Rep}(F^\times \backslash G(F))$. Combining with results from the local trace formula approach, we obtain a multiplicity formula for any irreducible $\pi \in \mathrm{Rep}(F^\times \backslash G(F))$.

To state the result, we introduce more notations. For any irreducible $\pi \in \mathrm{Rep}(F^\times \backslash G(F))$, consider its geometric multiplicity

$$m_{\mathrm{geo}}(\pi) = \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \int_{F^\times \backslash T(F)} c_\pi(t) D^H(t) dt$$

where

- the support $\mathcal{T} = \mathcal{T}(G,H)$ is a set of tori in H. If D is split, $\mathcal{T}$ consists of $F^\times$ and the nonsplit maximal tori in H. If D is non-split, $\mathcal{T}$ consists of the nonsplit maximal tori in H.
- $W(H,T) = N_H(T)/Z_H(T)$.
- c_π is the regularized character on the semi-simple locus of $G(F)$ (See [12, Definition 2.5]).
- $D^H(t) = |\det(1 - \mathrm{ad}(t))|_{\mathfrak{h}/\mathfrak{h}_t}|$ is the Weyl discriminant on the semi-simple locus of $H(F)$.

- the measure dt is normalized so that the volume $\mathrm{Vol}(F^\times\backslash T(F), dt) = 1$.

Via the local trace formula approach, Wan proves the following multiplicity formula for tempered representations:

Theorem 1.1 (Wan [11], C.2 and C.3). *For any irreducible tempered $\pi \in \mathrm{Rep}(F^\times\backslash G(F))$,*

$$m(\pi) := \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(\pi, \mathbb{C}) = m_{\mathrm{geo}}(\pi).$$

Remark 1.2. In fact, the above multiplicity formula holds for generic representations combining the result of Prasad [8, 9] (See [11, Remark C. 2.3]).

In general, it is conjectured in [10, Conjecture 7.1] (for the Gan-Gross-Prasad models) and [12, Conjecture 7.6] (for all spherical pairs) that the multiplicity formula should hold for all $\pi \in \mathrm{Rep}(F^\times\backslash G(F))$ of finite length with the multiplicity $m(\pi)$ replaced by the Euler-Poincaré number

$$\mathrm{EP}_{F^\times\backslash H(F)}(\pi, \mathbb{C}) := \sum_{i\geq 0} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{F^\times\backslash H(F)}(\pi, \mathbb{C}).$$

Philosophically, such multiplicity formula can be viewed as a kind of Riemann-Roch theorem (See [10, Remark 7.2]).

Example 1.3. Assume D is split and $L = E \oplus F$ where E/F is a quadratic field extension. Let $\pi = \pi_1 \boxtimes \pi_2 \in \mathrm{Rep}(F^\times\backslash G(F))$ with

- $\pi_1 = I_{B(E)}^{\mathrm{GL}_2(E)} \chi_1 \boxtimes \chi_2 \in \mathrm{Rep}(F^\times\backslash \mathrm{GL}_2(E))$ being the normalized parabolic induction for characters $\chi_1, \chi_2 : E^\times \to \mathbb{C}^\times$ such that $\chi_1\chi_2|_{F^\times} = 1$;
- $\pi_2 = \mathbb{C} \in \mathrm{Rep}(F^\times\backslash \mathrm{GL}_2(F))$ being the trivial representation.

Then

$$\mathrm{Hom}_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = \mathrm{Hom}_{F^\times\backslash \mathrm{GL}_2(F)}(\pi_1, \mathbb{C}).$$

Set $\chi' := \chi_1\overline{\chi_2}$ where $-$ denotes the Galois conjugation with respect to E/F. It is known that (see e.g. [6, Theorem 5.2]) $m(\pi) \leq 1$ and the equality holds if and only if $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$ or $\chi' = 1$.

On the other hand, by a property of the regularized characters of parabolic inductions [12, Proposition 2.7]

$$m_{\mathrm{geo}}(\pi) = \frac{1}{2}\int_{F^\times\backslash E^\times} c_\pi(t) D^H(t) dt = \frac{1}{2}\int_{F^\times\backslash E^\times} (\chi'(t) + \chi'(\bar{t})) dt.$$

In particular, $m_{\mathrm{geo}}(\pi) \leq 1$ and the equality holds if and only if $\chi' = 1$.

Therefore, in the case $\chi' \neq 1$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$,

$$m(\pi) \neq m_{\text{geo}}(\pi).$$

This is compatible with Theorem 1.1 since π_2 (hence π) is neither generic nor tempered.

The following is the main result of this paper.

Theorem 1.4. *For any irreducible $\pi \in \text{Rep}(F^\times \backslash G(F))$, generic if D is split,*

$$\text{Ext}^i_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = 0, \quad i \geq 1.$$

Moreover, for any $\pi \in \text{Rep}(F^\times \backslash G(F))$ of finite length, the multiplicity formula holds

$$\text{EP}_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = m_{\text{geo}}(\pi).$$

Remark 1.5. Previously,

- The vanishing result is known for the Whittaker model: if G is a connected quasi-split reductive group over F, $B = TN$ a Borel subgroup of G, ψ a generic character on $N(F)$, then for any irreducible representation $\pi \in \text{Rep}(G(F))$ and any $i \geq 1$, by [10, Proposition 2.8]

  $$\text{Ext}^i_{N(F)}(\pi, \psi^{-1}) \cong \text{Ext}^i_{G(F)}\left(i_{N(F)}^{G(F)}\psi, \pi^\vee\right) \cong \text{Ext}^i_1(\pi_{N,\psi}, \mathbb{C}) = 0.$$

  For the Gan-Gan-Prasad models of general linear groups, the vanishing result is due to Chan-Savin [5].

- The only known cases of [12, Conjecture 7.6] are the group case [10, Proposition 2.1(4)], the Whittaker models [12, Section 8.1], and the Gan-Gross-Prasad models for general linear groups ([10, Theorem 4.2] and note that in this case the support $\mathcal{T} = \mathcal{T}(G, H)$ of m_{geo} is $\{1\}$).

We explain the proof of Theorem 1.4. In fact, it is known that $\text{Ext}^i_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = 0$ for any π when $i \geq 2$ (see [10, Proposition 2.9], also Proposition 2.1(1) below) and $\text{Ext}^1_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = 0$ if π is supercuspidal (see [3, Theorem 5.4.1], also Proposition 2.1(1) below). The proof of the vanishing result is reduced to showing that $\text{Ext}^1_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = 0$ for non-supercuspidal π. For this, we apply standard tools: the geometric lemma of Bernstein-Zelevinsky and the Schneider-Stuhler duality ([7, Theorem 2], see also Theorem 2.2 below). Once the vanishing results are available, the multiplicity formula for finite length representations is deduced from the tempered version (Theorem 1.1) by noting both sides are additive and constant in an unramified twisting family.

In fact, we compute $\mathrm{Ext}^i_{F^\times\backslash H(F)}(\pi,\mathbb{C})$ for all irreducible $\pi\in\mathrm{Rep}(F^\times\backslash G(F))$ and all i. The results for $L=E\oplus F$ together with the Schneider-Stuhler duality implies the following complete classification of irreducible $\mathrm{GL}_2(F)$-subrepresentations of irreducible $\mathrm{GL}_2(E)$-representations, which is analogous to [7, Proposition 9.1] for the case $L=F\oplus F\oplus F$ (in fact, there is an extra central character condition in [7, Proposition 9.1]: $\pi=\pi_1\boxtimes\pi_2\boxtimes\pi_3$ with π_3 having trivial central character. This condition is dropped here).

Proposition 1.6. *Assume $L=E\oplus F$ and D is split. For $\pi=\pi_1\boxtimes\pi_2\in\mathrm{Rep}(F^\times\backslash G(F))$ irreducible, $\pi_2^\vee$ is a $\mathrm{GL}_2(F)$-subrepresentation of π_1 if and only if*

- $\pi_1=\xi\circ\det$ *and* $\pi_2=\xi^{-1}|_{F^\times}\circ\det$*; or*
- $\pi_2^\vee$ *is supercuspidal, and appears as a quotient of* $\pi_1|_{\mathrm{GL}_2(F)}$*; or*
- $\pi_2=\mathrm{St}_F\otimes\xi$, $\pi_1=I_{B(E)}^{\mathrm{GL}_2(E)}\chi_1\boxtimes\chi_2$ *such that* $\xi\chi_1|_{F^\times}=\xi\chi_2|_{F^\times}=1$ *and* $\xi_E\chi'\neq 1$*. Here* $\xi_E=\xi\circ N_{E/F}$*,* $\chi'=\chi_1\overline{\chi_2}$ *and* St_F *is the Steinberg representation for* $\mathrm{GL}_2(F)$*.*

2. The proof

For any reductive group Γ over F and any center character $\omega:\ Z_\Gamma(F)\to\mathbb{C}^\times$, let $\mathrm{Rep}(\Gamma(F),\omega)$ denote the full subcategory of $\mathrm{Rep}(\Gamma(F))$ consisting of objects on which $Z_\Gamma(F)$ acts by ω. For any objects $\pi,\pi'\in\mathrm{Rep}(\Gamma(F),\omega)\subset\mathrm{Rep}(\Gamma(F))$, let $\mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi,\pi')$ (resp. $\mathrm{Ext}^i_{\Gamma(F)}(\pi,\pi')$) be the i-th extension group in the category $\mathrm{Rep}(\Gamma(F),\omega)$ (resp. $\mathrm{Rep}(\Gamma(F))$). Let $\Gamma'(F)$ be a closed subgroup of the p-adic group $\Gamma(F)$ with modulus character $\delta_{\Gamma'}$. Note that when Γ' is reductive, $\delta_{\Gamma'}$ is trivial. Denote by $I_{\Gamma'(F)}^{\Gamma(F)}$ (resp. $i_{\Gamma'(F)}^{\Gamma(F)}$) the normalized (resp. compact) induction. Note that for any $\sigma\in\mathrm{Rep}(\Gamma'(F))$, $(i_{\Gamma'(F)}^{\Gamma(F)}\sigma)^\vee=I_{\Gamma'(F)}^{\Gamma(F)}\sigma^\vee$ where $-^\vee$ stands for smooth dual in the proper category. Moreover if $\Gamma'=MN$ is a parabolic subgroup with Levi factor M, denote by J_N the normalized Jacquet functor for Γ'.

We record some basic properties of Ext-groups which are frequently used in the following.

Proposition 2.1. *We have the following results for the Ext-groups.*

1) *For any irreducible* $\pi \in \mathrm{Rep}(\Gamma(F))$*, smooth* $\pi' \in \mathrm{Rep}(\Gamma(F))$ *and any* $i >$ split rank of Z_Γ

$$\mathrm{Ext}^i_{\Gamma(F)}(\pi, \pi') \cong \mathrm{Ext}^i_{\Gamma(F)}(\pi', \pi) = 0.$$

If π *is supercuspidal with central character* ω*, then* π *is both projective and injective in* $\mathrm{Rep}(\Gamma(F), \omega)$*. In particular, for any* $i \geq 1$

$$\mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi, \pi') \cong \mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi', \pi) = 0.$$

2) *For any* $\pi \in \mathrm{Rep}(\Gamma(F), \omega)$*,* $\sigma \in \mathrm{Rep}(\Gamma(F), \omega^{-1})$ *and any* $i \geq 0$

$$\mathrm{Ext}^i_{Z_\Gamma(F)\backslash\Gamma(F)}(\pi \otimes \sigma, \mathbb{C}) \cong \mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi, \sigma^\vee) \cong \mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega^{-1})}(\sigma, \pi^\vee).$$

3) *The restriction functor* $\mathrm{Rep}(\Gamma(F)) \to \mathrm{Rep}(\Gamma'(F))$ *sends projective objects to projective objects, the normalized induction functor* $I^{\Gamma(F)}_{\Gamma'(F)}$: $\mathrm{Rep}(\Gamma'(F)) \to \mathrm{Rep}(\Gamma(F))$ *sends injective objects to injective objects. Moreover, for any* $\sigma \in \mathrm{Rep}(\Gamma'(F))$*,* $\pi \in \mathrm{Rep}(\Gamma(F))$ *and any* $i \geq 0$

$$\mathrm{Ext}^i_{\Gamma(F)}\left(\pi, I^{\Gamma(F)}_{\Gamma'(F)}\sigma\right) \cong \mathrm{Ext}^i_{\Gamma'(F)}\left(\pi, \sigma \otimes \delta^{1/2}_{\Gamma'}\right).$$

When $\Gamma' = MN$ *is a parabolic subgroup with Levi factor* M*, the normalized Jacquet functor* $J_N :$ $\mathrm{Rep}(\Gamma(F)) \to \mathrm{Rep}(M(F))$ *sends projective objects to projective objects. For any* $\sigma \in \mathrm{Rep}(M(F))$ *and any* $i \geq 0$

$$\mathrm{Ext}^i_{\Gamma(F)}\left(\pi, I^{\Gamma(F)}_{\Gamma'(F)}\sigma\right) \cong \mathrm{Ext}^i_{M(F)}(J_N(\pi), \sigma).$$

Proof. The property of supercuspidal representations can be found in [3, Theorem 5.4.1]. Other results are summarized in [10, Section 2]. □

For any $\pi \in \mathrm{Rep}(\Gamma(F), \omega)$ irreducible, let $d(\pi)$ (resp. $d'(\pi)$) be the split rank of Z_M (resp. $Z_M \cap [\Gamma, \Gamma]$), where M is any Levi subgroup carrying the cuspidal support of π. The key ingredient for computing higher Ext groups is the following Schneider-Stuhler duality theorem.

Theorem 2.2 (Theorem 1,2 in [7]). *Let* $\pi \in \mathrm{Rep}(\Gamma(F), \omega)$ *be any irreducible representation and* $D(\pi)$ *be the Aubert-Zelevinsky involution of* π*. Then*

- *for any* $\pi' \in \mathrm{Rep}(\Gamma(F))$*,* $\mathrm{Ext}^i_{\Gamma(F)}(\pi, \pi') = 0$ *for* $i > d(\pi)$ *and for* $0 \leq i \leq d(\pi)$*, there is a non-degenerate pairing*

$$\mathrm{Ext}^i_{\Gamma(F)}(\pi, \pi') \times \mathrm{Ext}^{d(\pi)-i}_{\Gamma(F)}(\pi', D(\pi)) \to \mathrm{Ext}^{d(\pi)}_{\Gamma(F)}(\pi, D(\pi)) \cong \mathbb{C},$$

- *for any* $\pi' \in \mathrm{Rep}(\Gamma(F),\omega)$, $\mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi,\pi') = 0$ *for* $i > d'(\pi)$ *and for* $0 \le i \le d'(\pi)$, *there is a non-degenerate pairing*

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi,\pi') \times \mathrm{Ext}^{d'(\pi)-i}_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi', D(\pi))\\ \to \mathrm{Ext}^{d'(\pi)}_{\mathrm{Rep}(\Gamma(F),\omega)}(\pi, D(\pi)) \cong \mathbb{C}.\end{aligned}$$

Remark 2.3. In general, the non-degeneracy means that if $\pi' = \varinjlim_n \pi'_n$ is the inductive limit of finite generated $\Gamma(F)$-submodules π'_n, then

$$\begin{aligned}\mathrm{Ext}^i_{\Gamma(F)}(\pi,\pi') &= \varinjlim_n \mathrm{Ext}^i_{\Gamma(F)}(\pi,\pi'_n),\\ \mathrm{Ext}^i_{\Gamma(F)}(\pi', D(\pi)) &= \varprojlim_n \mathrm{Ext}^i_{\Gamma(F)}(\pi'_n, D(\pi))\end{aligned}$$

is the inductive limit (resp. projective limit) of finite dimensional $\mathbb{C}$-spaces and the pairing is direct limit of perfect pairings on these finite dimensional spaces.

In the triple product case, the result of Aizenbud-Sayag [1] guarantees all the Ext-groups below are finite dimensional and the non-degeneracy has the usual meaning.

In the following, we shall concentrate on the triple product case.

The case D non-split is straightforward.

Proposition 2.4. *Assume* D *is non-split. Then for any* $\pi \in \mathrm{Rep}(F^\times\backslash G(F))$ *irreducible,*

$$\mathrm{Ext}^i_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0, \quad i > 0.$$

Proof. Note that $Z_H\backslash H$ is anisotropic. Then by Proposition 2.1 (1), $\mathbb{C}$ is injective and the statement follows. □

To deal with the D split case, we need to consider the following Waldspurger toric case.

Lemma 2.5. *Assume* D *is split. Let* $K \subset D$ *be an étale quadratic* F*-algebra and embed* $F^\times$ *into* $D^\times(F) \times K^\times$ *diagonally. Then for any generic irreducible* $\pi \in \mathrm{Rep}(F^\times\backslash(D^\times(F)\times K^\times))$,

$$\mathrm{Ext}^i_{F^\times\backslash K^\times}(\pi,\mathbb{C}) = 0, \quad i \ge 1.$$

Proof. Assume $\pi = \sigma \boxtimes \xi$ with $\sigma \in \mathrm{Rep}(D^\times(F))$ and $\xi \in \mathrm{Rep}(K^\times)$. If K/F is a field extension, then $\mathrm{Ext}^i_{F^\times\backslash K^\times}(\sigma\boxtimes\xi,\mathbb{C}) = 0$ for $i \ge 1$ by Proposition 2.1 (1).

If $K = F \oplus F$, $\mathrm{Ext}^i_{F^\times\backslash K^\times}(\sigma \boxtimes \xi, \mathbb{C}) = 0$ for $i \geq 2$ by Proposition 2.1 (1). For $i = 1$, one can apply Theorem 2.2 to $\mathbb{C} \in \mathrm{Rep}(F^\times\backslash K^\times)$, where $d(\mathbb{C}) = 1$ and $D(\mathbb{C}) = \mathbb{C}$. One has

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash K^\times}(\sigma \boxtimes \xi, \mathbb{C}) &= \dim_{\mathbb{C}} \mathrm{Hom}_{F^\times\backslash K^\times}(\mathbb{C}, \sigma \boxtimes \xi)\\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{K^\times}(\xi^{-1}, \sigma).\end{aligned}$$

Identify $K^\times$ with the split torus $T(F)$ and let $\mathcal{K}(\sigma)$ be the Kirillov model of σ. Write $\xi = \xi_1 \boxtimes \xi_2$. Then by [2, Proposition 4.7.2],

$$\begin{aligned}&\mathrm{Hom}_{T(F)}(\xi^{-1}, \mathcal{K}(\sigma)|_T)\\ &\quad = \{\phi \in \mathcal{K}(\sigma) \mid \phi(ax) = \xi_1^{-1}(a)\phi(x) \text{ for any } a \in F^\times\} = 0.\end{aligned}$$

□

Assume D is split. By Proposition 2.1(1), $\mathrm{Ext}^i_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = 0$ for $i \geq 2$. In the following, we compute $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C})$ case by case. Let $T \subset B \subset \mathrm{GL}_2$ be the diagonal torus and the group of upper triangular matrices respectively, St_F be the Steinberg representation for $\mathrm{GL}_2(F)$ and $|\cdot|_F$ be the norm character on $F^\times$.

Proposition 2.6. *Assume D is split and L/F is a cubic field extension. Then for $\pi \in \mathrm{Rep}(F^\times\backslash G(F))$ irreducible, $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = 0$.*

Proof. By Proposition 2.1 (1,3)(the modulus character of H is trivial), when π is supercuspidal,

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = \mathrm{Ext}^1_{F^\times\backslash G(F)}(\pi, I^{G(F)}_{H(F)}\mathbb{C}) = 0.$$

If π is one-dimensional, then by Theorem 2.2 for $\mathbb{C} \in \mathrm{Rep}(F^\times\backslash H(F))$, where $D(\mathbb{C}) = \mathrm{St}_F$ and $d(\mathbb{C}) = \mathbb{C}$,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(\mathrm{St}_F, \pi) = 0.$$

Note that when π is a special series, then $\pi \hookrightarrow I^{\mathrm{GL}_2(L)}_{B(L)}\chi_1 \boxtimes \chi_2$ with $\chi_1\chi_2^{-1} = |\cdot|_L$. Thus it suffices to show that for $\pi = I^{\mathrm{GL}_2(L)}_{B(L)}\chi \in \mathrm{Rep}(F^\times\backslash G(F))$ where $\chi = \chi_1 \boxtimes \chi_2$ with $\chi_1\chi_2^{-1} \neq |\cdot|_L^{-1}$,

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = 0.$$

By the geometric lemma (see [9, Section 6.1]), there exists an exact sequence of $F^\times\backslash H(F)$-representations

$$0 \to i^{H(F)}_{F^\times}\mathbb{C} \to \pi \to I^{H(F)}_{B(F)}\chi\delta_B \to 0$$

where δ_B is the modulus character of $B(F)$. The short exact sequence induces a long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{F^\times\backslash H(F)}(I_{B(F)}^{H(F)}\chi\delta_B,\mathbb{C}) \to \mathrm{Hom}_{F^\times\backslash H(F)}(\pi,\mathbb{C})\\ &\to \mathrm{Hom}_{F^\times\backslash H(F)}(i_{F^\times}^{H(F)}\mathbb{C},\mathbb{C}) \to \mathrm{Ext}^1_{F^\times\backslash H(F)}(I_{B(F)}^{H(F)}\chi\delta_B,\mathbb{C})\\ &\to \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) \to \mathrm{Ext}^1_{F^\times\backslash H(F)}(i_{F^\times}^{H(F)}\mathbb{C},\mathbb{C}) \to 0.\end{aligned}$$

By Proposition 2.1(2,3),

$$\mathrm{Hom}_{F^\times\backslash H(F)}(i_{F^\times}^{H(F)}\mathbb{C},\mathbb{C}) = \mathbb{C}, \quad \mathrm{Ext}^1_{F^\times\backslash H(F)}(i_{F^\times}^{H(F)}\mathbb{C},\mathbb{C}) = 0.$$

Note that

- if π is reducible, $I_{B(F)}^{H(F)}\chi\delta_B$ is irreducible;
- if π is irreducible, we may assume $I_{B(F)}^{H(F)}\chi\delta_B$ is irreducible by replacing χ by χ^w, the twisting of χ by $w = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, if necessary.

Thus

$$\mathrm{Hom}_{F^\times\backslash H(F)}(I_{B(F)}^{H(F)}\chi\delta_B,\mathbb{C}) = 0.$$

Moreover by Theorem 2.2 for $\mathbb{C} \in \mathrm{Rep}(F^\times\backslash H(F))$, where $D(\mathbb{C}) = \mathrm{St}_F$ and $d(\mathbb{C}) = 1$,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(I_{B(F)}^{H(F)}\chi\delta_B,\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{F^\times\backslash H(F)}(\mathrm{St}_F, I_{B(F)}^{H(F)}\chi\delta_B) = 0.$$

Consequently, $\mathrm{Hom}_{H(F)}(\pi,\mathbb{C}) = \mathbb{C}$ and $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$. ☐

Proposition 2.7. *Assume D is split and $L = E \oplus F$. For $\pi = \pi_1 \boxtimes \pi_2 \in \mathrm{Rep}(F^\times\backslash G(F))$ irreducible, $\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) \le 1$ with the equality holds if and only if*

- *$\pi_2 = \xi \circ \det$, $\pi_1 = I_{B(E)}^{\mathrm{GL}_2(E)}\chi_1 \boxtimes \chi_2$ such that $\chi_1|_{F^\times}\xi = \chi_2|_{F^\times}\xi = 1$ and $\xi_E\chi' \ne 1$ (See Proposition 1.6 for the notations ξ_E and χ'); or*
- *$\pi_1 = \xi \circ \det$ and $\pi_2 = \mathrm{St}_F \otimes \xi^{-1}|_{F^\times}$.*

Proof. Denote by ω^{-1} the central character of π_2. Then by Proposition 2.1 (2)

$$\mathrm{Ext}^i_{F^\times\backslash H(F)}(\pi,\mathbb{C}) \cong \mathrm{Ext}^i_{\mathrm{Rep}(H(F),\omega)}(\pi_1,\pi_2^\vee)$$

and if π_1 or π_2 is supercuspidal, $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$.

When π_1 is one-dimensional, by Theorem 2.2 for $\mathbb{C} \in \mathrm{Rep}(F^\times \backslash H(F))$, where $D(\mathbb{C}) = \mathrm{St}_F$ and $d(\mathbb{C}) = 1$,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(\mathrm{St}_F, \pi) \leq 1$$

with the equality holds if and only if $\pi_1 = \xi \circ \det$ and $\pi_2 = \mathrm{St}_F \otimes \xi^{-1}|_{F^\times}$.

Note that when π_1 is a special series, $\pi \hookrightarrow I^{\mathrm{GL}_2(E)}_{B(E)} \chi_1 \boxtimes \chi_2$ with $\chi_1\chi_2^{-1} = |\cdot|_E$. Thus it suffices to consider $\pi_1 = I^{\mathrm{GL}_2(E)}_{B(E)} \chi$ for $\chi = \chi_1 \boxtimes \chi_2$ where $\chi_1\chi_2^{-1} \neq |\cdot|_E^{-1}$. By the geometric lemma (See [9, Section 4.1]), there is an exact sequence of $H(F)$-representations

$$0 \to i^{H(F)}_{E^\times} \chi' \to \pi_1 \to I^{H(F)}_{B(F)} \chi \delta_B^{1/2} \to 0.$$

By Proposition 2.1(2,3) and Lemma 2.5,

$$\mathrm{Hom}_{H(F)}(i^{\mathrm{GL}_2(F)}_{E^\times} \chi', \pi_2^\vee) \cong \mathrm{Hom}_{E^\times}(\pi_2 \otimes \chi', \mathbb{C});$$

$$\mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(i^{\mathrm{GL}_2(F)}_{E^\times} \chi', \pi_2^\vee) \cong \mathrm{Ext}^1_{F^\times \backslash E^\times}(\pi_2 \otimes \chi', \mathbb{C}) = 0;$$

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{Rep}(H(F),\omega)}(I^{\mathrm{GL}_2(F)}_{B(F)} \chi \delta_B^{1/2}, \pi_2^\vee) &\cong \mathrm{Ext}^i_{\mathrm{Rep}(T(F),\omega^{-1})}(J_N(\pi_2), \delta_B^{-1/2}\chi^{-1}) \\ &\cong \mathrm{Ext}^i_{F^\times \backslash T(F)}(J_N(\pi_2) \otimes \delta_B^{1/2}\chi, \mathbb{C}).\end{aligned}$$

Then by Theorem 2.2 for $\mathbb{C} \in \mathrm{Rep}(F^\times \backslash T(F))$, where $D(\mathbb{C}) = \mathbb{C}$ and $d(\mathbb{C}) = 1$,

$$\begin{aligned}&\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(I^{\mathrm{GL}_2(F)}_{B(F)} \chi \delta_B^{1/2}, \pi_2^\vee) \\ &\quad = \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(I^{\mathrm{GL}_2(F)}_{B(F)} \chi \delta_B^{1/2}, \pi_2^\vee) \leq 1\end{aligned}$$

with the equality holds if and only if $\delta_B^{-1/2}\chi^{-1}|_{T(F)}$ is a Jordan factor of $J_N(\pi_2)$. By the long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{H(F)}(I^{H(F)}_{B(F)} \chi \delta_B^{1/2}, \pi_2^\vee) \to \mathrm{Hom}_{H(F)}(\pi_1, \pi_2^\vee) \\ &\to \mathrm{Hom}_{H(F)}(i^{H(F)}_{E^\times} \chi', \pi_2^\vee) \to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(I^{H(F)}_{B(F)} \chi \delta_B^{1/2}, \pi_2^\vee) \\ &\to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1, \pi_2^\vee) \to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(i^{H(F)}_{E^\times} \chi', \pi_2^\vee) \to 0,\end{aligned}$$

one has that

(i) $\mathrm{EP}_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{E^\times}(\pi_2 \otimes \chi', \mathbb{C}) \leq 1$

(ii) unless $\delta_B^{-1/2}\chi^{-1}|_{T(F)}$ is a Jordan factor of $J_N(\pi_2)$, $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$,

(iii) if $\mathrm{Hom}_{E^\times}(\pi_2\otimes\chi',\mathbb{C})=0$,

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(I_{B(F)}^{H(F)}\chi\delta_B^{1/2},\pi_2^\vee).$$

If $\delta_B^{-1/2}\chi^{-1}|_{T(F)}$ is a Jordan factor of $J_N(\pi_2)$ and $\pi_1 = I_{B(E)}^{\mathrm{GL}_2(E)}\chi$ sits in the exact sequence

$$0\to \mathrm{St}_E\otimes\mu\to\pi_1\to\mu\to 0,$$

one has $\pi_2 = I_{B(F)}^{\mathrm{GL}_2(F)}\chi^{-1}\delta_B^{-1/2}$ is irreducible. Then by Saito-Tunnell, $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C})=1$ and hence

$$\dim_{\mathbb{C}}\mathrm{Hom}_{H(F)}(\mathrm{St}_E\otimes\mu,\pi_2^\vee) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\mathrm{St}_E\otimes\mu,\pi_2^\vee)+1.$$

Note that

$$\dim_{\mathbb{C}}\mathrm{Hom}_{H(F)}(\mathrm{St}_E\otimes\mu,\pi_2^\vee) = \dim_{\mathbb{C}}\mathrm{Hom}_{F^\times\backslash H(F)}(\mathrm{St}_E\otimes\mu\otimes\pi_2,\mathbb{C})\le 1.$$

one deduce $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\mathrm{St}_E\otimes\mu\otimes\pi_2,\mathbb{C})=0$ and hence $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$ from the induced long exact sequence.

Assume $\pi_1 = I_{B(E)}^{\mathrm{GL}_2(E)}\chi$ is irreducible. If $\chi_1|_{F^\times}\neq\chi_2|_{F^\times}$, then up to replacing χ by χ^w, one can make $\delta_B^{-1/2}\chi^{-1}|_{T(F)}$ different from the Jordan Holder factors of $J_N(\pi_2)$. Then $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})=0$ by (ii).

Now assume moreover $\xi^{-1}=\chi_1|_{F^\times}=\chi_2|_{F^\times}$. Then $\delta_B^{-1/2}\chi^{-1}|_{T(F)}$ is a Jordan factor of $J_N(\pi_2)$ if and only if $\pi_2=\xi\circ\det$ or $\mathrm{St}_F\otimes\xi$. In the case $\pi_2=\mathrm{St}_F\otimes\xi$,

- if $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C})=1$, then

  $$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})=0$$

  since $\dim_{\mathbb{C}}\mathrm{Hom}_{H(F)}(\pi,\mathbb{C})\le 1$;
- if $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C})=\dim_{\mathbb{C}}\mathrm{Hom}_{E^\times}(\pi_2\otimes\chi',\mathbb{C})=0$, then by Theorem 2.2 for $\mathrm{St}_F\in\mathrm{Rep}(F^\times\backslash H(F))$, where $D(\mathrm{St}_F)=\mathbb{C}$ and $d(\mathrm{St}_F)=1$,

  $$\begin{aligned}\dim_{\mathbb{C}}\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) &= \dim_{\mathbb{C}}\mathrm{Ext}^1_{F^\times\backslash H(F)}(I_{B(F)}^{H(F)}\chi\delta_B^{1/2}\otimes\xi,\mathrm{St}_F)\\ &= \dim_{\mathbb{C}}\mathrm{Hom}_{F^\times\backslash H(F)}(\mathbb{C},I_{B(F)}^{\mathrm{GL}_2(F)}\delta_B^{1/2})=0.\end{aligned}$$

In the case $\pi_2=\xi\circ\det$, we shall give the criterion for $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})=\mathbb{C}$ via studying $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C})$. By the above (i), $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{E^\times}(\pi_2\otimes\chi',\mathbb{C}) \le 1$, where the equality holds if and only if $\xi_E\chi' = 1$.

If $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 1$, as the second term in the long exact sequence has multiplicity ≤ 1, we must have $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$.

If $\mathrm{EP}_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$, then as the last term in the last exact sequence vanishes, we must have

$$\begin{aligned}&\dim_{\mathbb{C}} \mathrm{Hom}_{E^\times}(\pi_2\otimes\chi',\mathbb{C}) = 0,\\ &\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(I_{B(F)}^{H(F)}\chi\delta_B^{1/2},\pi_2^\vee).\end{aligned}$$

It is then easy to see that $\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 1$ if and only if $\chi|_{F^\times}\xi = 1$ and $\xi_E\chi' \neq 1$. ☐

One can deduce Proposition 1.6 immediately from Proposition 2.7.

Proof of Proposition 1.6. Denote the central character of π_2 by ω^{-1}. By Theorem 2.2,

- when π_2 is supercuspidal, by Proposition 2.1 (1),

$$\mathrm{Hom}_{H(F)}(\pi_2^\vee,\pi_1) \cong \mathrm{Hom}_{H(F)}(\pi_1,\pi_2^\vee);$$

- when π_2 is non-supercuspidal, $d(\pi_2) = 1$ and

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(\pi_2^\vee,\pi_1) &= \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1,D(\pi_2^\vee))\\ &= \dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi_1\otimes D(\pi_2),\mathbb{C}).\end{aligned}$$

Since $D(\xi\circ\det) = \mathrm{St}_F\otimes\xi$ and $D(\mathrm{St}_F\otimes\xi) = \xi\circ\det$, the statement follows from Proposition 2.7 immediately. ☐

Proposition 2.8. *Assume D is split and $L = F\oplus F\oplus F$. For $\pi = \pi_1\boxtimes\pi_2\boxtimes\pi_3 \in \mathrm{Rep}(F^\times\backslash G(F))$ with $\pi_i \in \mathrm{Rep}(\mathrm{GL}_2(F))$ irreducible, $\dim_{\mathbb{C}} \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) \le 1$ with the equality holds if and only if up to reordering and twisting,*

- *$\pi_1 = \mathrm{St}_F$ and $\pi_2 = \pi_3 = \mathbb{C}$, or*
- *$\pi_1 = \pi_2^\vee$ are principal series and $\pi_3 = \mathbb{C}$.*

Proof. Denote the central character of π_3 by ω^{-1}. Then by Proposition 2.1 (i) (ii)

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) \cong \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1\otimes\pi_2,\pi_3^\vee)$$

and if π_3 is supercuspidal, $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C}) = 0$. In the following, we assume none of π_i is supercuspidal.

If π_3 is one dimensional, which we may assume to be $\mathbb{C}$ by twisting, then

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})\cong \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega_1)}(\pi_1,\pi_2^\vee)$$

where ω_1 is the central character of π_1. Then by Theorem 2.2 for $\pi_2^\vee$,

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})\le 1$$

with the equality holds if and only if up to reordering and twisting,

- $\pi_1=\mathrm{St}_F$ and $\pi_2=\mathbb{C}$,
- $\pi_1=\pi_2^\vee$ are principal series.

If π_1, π_2 are special series, which we assume to be St_F by twisting, and π_3 is generic, then by [8, Lemma 5.4], there exists an exact sequence of $\mathrm{GL}_2(F)$-representations

$$0\to \mathrm{St}_F\to i^{\mathrm{GL}_2(F)}_{T(F)}\mathbb{C}\to \pi_1\otimes\pi_2\to 0,$$

which induces the long exact sequence

$$\begin{aligned}0&\to \mathrm{Hom}_{F^\times\backslash H(F)}(\pi_1\otimes\pi_2,\pi_3^\vee)\to \mathrm{Hom}_{F^\times\backslash H(F)}(i^{H(F)}_{T(F)}\mathbb{C},\pi_3^\vee)\\ &\to \mathrm{Hom}_{F^\times\backslash H(F)}(\mathrm{St}_F,\pi_3^\vee)\to \mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi_1\otimes\pi_2,\pi_3^\vee)\\ &\to \mathrm{Ext}^1_{F^\times\backslash H(F)}(i^{H(F)}_{T(F)}\mathbb{C},\pi_3^\vee)\to \mathrm{Ext}^1_{F^\times\backslash H(F))}(\mathrm{St}_F,\pi_3^\vee)\to 0.\end{aligned}$$

By Lemma 2.5,

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(i^{H(F)}_{T(F)}\mathbb{C},\pi_3^\vee)=0,\quad \mathrm{Hom}_{F^\times\backslash H(F)}(i^{H(F)}_{T(F)}\mathbb{C},\pi_3^\vee)=\mathbb{C}.$$

Thus

- when $\pi_3=\mathrm{St}_F$, $\mathrm{Hom}_{H(F)}(\pi,\mathbb{C})=0$ by [8, Theorem 1.2] and hence $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi,\mathbb{C})=0$,
- when $\pi_3\neq\mathrm{St}_F$, $\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi|_H,\mathbb{C})=\mathrm{Hom}_{F^\times\backslash H(F)}(\mathrm{St}_F,\pi_3^\vee)=0$.

If $\pi_1=I^{\mathrm{GL}_2(F)}_{B(F)}\xi_1$, $\pi_2=I^{\mathrm{GL}_2(F)}_{B(F)}\xi_2$ and π_3 is generic. Then by the geometric lemma (see [8, Section 5]), there is an exact sequence of $\mathrm{GL}_2(F)$-representations

$$0\to i^{\mathrm{GL}_2(F)}_{T(F)}(\xi_1\xi_2^w)\to\pi_1\otimes\pi_2\to I^{\mathrm{GL}_2(F)}_{B(F)}(\xi_1\xi_2\delta^{1/2})\to 0,$$

which leads to a long exact sequence

$$
\begin{aligned}
0 &\to \mathrm{Hom}_{H(F)}(I_{B(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2\delta_B^{1/2}), \pi_3^\vee) \to \mathrm{Hom}_{H(F)}(\pi_1 \otimes \pi_2, \pi_3^\vee) \\
&\to \mathrm{Hom}_{H(F)}(i_{T(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2^w), \pi_3^\vee) \to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(I_{B(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2\delta_B^{1/2}), \pi_3^\vee) \\
&\to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1 \otimes \pi_2, \pi_3^\vee) \to \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(i_{T(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2^w), \pi_3^\vee) \to 0.
\end{aligned}
$$

By Proposition 2.1 (2)(3),

$$\mathrm{Ext}^i_{\mathrm{Rep}(H(F),\omega)}(i_{T(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2^w), \pi_3^\vee) = \mathrm{Ext}^i_{F^\times\backslash T(F)}(\pi_3 \otimes \xi_1\xi_2^w, \mathbb{C}).$$

Moreover by Theorem 2.2 for $\pi_3^\vee \in \mathrm{Rep}(F^\times\backslash H(F), \omega)$, where $d(\pi_3^\vee) = 1$,

$$
\begin{aligned}
&\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(I_{B(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2\delta_B^{1/2}), \pi_3^\vee) \\
&\quad = \dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(I_{B(F)}^{\mathrm{GL}_2(F)}(\xi_1\xi_2\delta_B^{1/2}), \pi_3^\vee).
\end{aligned}
$$

Then by Lemma 2.5, one has

$$
\begin{aligned}
\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1 \otimes \pi_2, \pi_3^\vee) &= \dim_{\mathbb{C}} \mathrm{Hom}_{F^\times\backslash H(F)}(\pi, \mathbb{C}) \\
&\quad - \dim_{\mathbb{C}} \mathrm{Hom}_{F^\times\backslash T(F)}(\pi_3 \otimes \xi_1\xi_2^w, \mathbb{C}) \\
&= \dim_{\mathbb{C}} \mathrm{Hom}_{F^\times\backslash H(F)}(\pi, \mathbb{C}) - 1 \geq 0.
\end{aligned}
$$

Since $\dim_{\mathbb{C}} \mathrm{Hom}_{H(F)}(\pi|_H, \mathbb{C}) \leq 1$, one has

$$\mathrm{Ext}^1_{F^\times\backslash H(F)}(\pi, \mathbb{C}) = \mathrm{Ext}^1_{\mathrm{Rep}(H(F),\omega)}(\pi_1 \otimes \pi_2, \pi_3^\vee) = 0.$$

Interchanging the roles of π_i, the statement follows. □

Immediately, we deduce the following corollary, which slightly generalizes [7, Proposition 9.1] (by dropping the central character condition).

Corollary 2.9. *Let $\pi = \boxtimes_{i=1}^3 \pi_i \in \mathrm{Rep}(F^\times\backslash G(F))$ be an irreducible smooth representation. Then $\pi_3^\vee$ is a $\mathrm{GL}_2(F)$-subrepresentation of $\pi_1 \otimes \pi_2$ if and only if*

- *$\pi_3^\vee$ is supercuspidal, and appears as a quotient of $\pi_1 \otimes \pi_2$;*
- *$\pi_3^\vee = \mathrm{St}_F \otimes \xi$ and $\pi_1 = \pi_2^\vee \otimes \xi$ is a principal series;*
- *π_1 or π_2 is one-dimensional and $\pi_3^\vee \cong \pi_1 \otimes \pi_2$.*

Proof. Denote the central character of π_3 by ω^{-1}. By Theorem 2.2 and Proposition 2.1 (2),

- when π_3 is supercuspidal,

$$\mathrm{Hom}_{H(F)}(\pi_3^\vee, \pi_1 \otimes \pi_2) = \mathrm{Hom}_{H(F)}(\pi_1 \otimes \pi_2, \pi_2^\vee),$$

- when π_3 is non-supercuspidal,

$$\mathrm{Hom}_{H(F)}(\pi_3^\vee, \pi_1 \otimes \pi_2) = \mathrm{Ext}^1_{F^\times \backslash H(F)}(\pi_1 \otimes \pi_2 \otimes D(\pi_3), \mathbb{C}).$$

Then the statement follows immediately from Proposition 2.8. □

Proof of Theorem 1.4. The vanishing of higher Ext-groups for generic representations is proved in Propositions 2.6, 2.7, 2.8. In particular, for $\pi \in \mathrm{Rep}(F^\times \backslash G(F))$ tempered,

$$m(\pi) = \mathrm{EP}_{F^\times \backslash H(F)}(\pi, \mathbb{C}).$$

Hence, by the multiplicity formula for tempered representations (Theorem 1.1),

$$\mathrm{EP}_{F^\times \backslash H(F)}(\pi, \mathbb{C}) = m(\pi) = m_{\mathrm{geo}}(\pi).$$

For any Levi subgroup M of G, denote by $\mathrm{Tmp}(F^\times \backslash M(F))$ the set of tempered representations on $F^\times \backslash M(F)$ and $\widehat{F^\times \backslash M(F)}^{\mathrm{un}}$ the set of unramified characters on $F^\times \backslash M(F)$. Consider the subcategory $\mathrm{Rep}^0(F^\times \backslash G(F))$ of $\mathrm{Rep}(F^\times \backslash G(F))$ consisting of $I_P^G(\sigma \otimes \chi)$ where $P = MN$ is a parabolic subgroup of G, $\sigma \in \mathrm{Tmp}(F^\times \backslash M(F))$ and $\chi \in \widehat{F^\times \backslash M(F)}^{\mathrm{un}}$. We have the following two lemmas.

Lemma 2.10. *The Grothendieck group of the abelian sub-category of finite length representations in* $\mathrm{Rep}(F^\times \backslash G(F))$ *can be generated by* $\mathrm{Rep}^0(F^\times \backslash G(F))$.

Proof. For any character $\omega : F^\times \to \mathbb{C}^\times$, set $\mathrm{Rep}^0(D^\times(F), \omega) := \mathrm{Rep}(D^\times(F), \omega) \cap \mathrm{Rep}^0(D^\times(F))$. Then by the classification of irreducible $D^\times(F)$-representations, for any irreducible $\pi \in \mathrm{Rep}(D^\times(F), \omega)$, there exists $\pi_0 \in \mathrm{Rep}^0(D^\times(F), \omega)$ such that the semisimplification of π_0 is the direct sum of π and another irreducible representation in $\mathrm{Rep}^0(D^\times, \omega)$. The lemma follows from this fact. □

Lemma 2.11. *Fix* $\sigma \in \mathrm{Tmp}(F^\times \backslash M(F))$ *for some Levi* M *of* G *and consider the unramified twisting family* $\pi_\chi = I_P^G \sigma \otimes \chi \in \mathrm{Rep}^0(F^\times \backslash G(F))$ *with*

$\chi \in \widehat{F^\times \backslash M(F)}^{\mathrm{un}}$. *For any $T \in \mathcal{T}$, the function*

$$\chi \mapsto c_{\pi_\chi}|_T$$

is constant. In particular, the geometric multiplicity is constant for an unramified twisting family in the sense that the function

$$\chi \mapsto m_{\mathrm{geo}}(\pi_\chi)$$

is constant.

Proof. We have the following fact on the regularized character of a parabolic induction [12, Proposition 2.7]. Let $\pi = I_P^G \sigma$ be a finite length $G(F)$-representation induced from $P = MN$. Then for any semi-simple $x \in G(F)$

$$D^G(x)^{1/2} c_\pi(x) = \sum_{y \in \mathfrak{X}_M(x)} D^M(y)^{1/2} c_\sigma(y)$$

where $\mathfrak{X}_M(x)$ is the set of representatives for $M(F)$-conjugacy classes of elements in $M(F)$ that are $G(F)$-conjugated to x.

Let $T \in \mathcal{T}$. If $\mathfrak{X}_M(t)$ is empty for any $t \in T(F)$, $c_{\pi_\chi}|_T = 0$ for any χ. Assume there exists $t \in T(F)$ such that $\mathfrak{X}_M(t)$ is nonempty, or equivalently, there is an embedding of T into M. This happens exactly when

- $M = G$;
- $L = E \oplus F$, $T = E^\times$ and $M = A_E \times D^\times$ where A_E is the diagonal torus of $\mathrm{GL}_{2,E}$. In this case,

$$\mathfrak{X}_M(t) = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}, t \right), \quad \left(\begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix}, t \right) \right\}.$$

To show the constancy of $c_{\pi_\chi}|_T$ for any unramified character χ on $M(F)$, it is enough to prove that

$$F^\times \backslash T(F) \subset \left(F^\times \backslash M\right)^0 = \bigcap_{\mu \in \mathrm{Rat}(F^\times \backslash M)} \ker |\mu|$$

where $\mathrm{Rat}\,(F^\times \backslash M)$ is the group of rational characters on M. As $F^\times \backslash T$ is anisotropic, $\mathrm{Rat}(F^\times \backslash T) = 0$ so that for any $\mu \in \mathrm{Rat}\,(F^\times \backslash M)$, $\mu(t) = 1$ for any $t \in T$. □

Now, as the both sides of the multiplicity formula is additive, by (1), we only need to consider representations in $\mathrm{Rep}^0(F^\times \backslash G(F))$, that is, unramified twists of tempered representations. By (2), the geometric multiplicity is constant for an unramified twisting family. Meanwhile, it is known that the Euler-Poincaré number is constant for an unramified twisting family (see [1, Theorem E(4)] or [4, Proposition 3.18] for a more general situation). Therefore, the multiplicity formula for tempered representations implies the formula for any irreducible representation. □

Remark 2.12. In fact, the constancy of the geometric multiplicity holds in general. It can be proved similarly as the above special case Lemma 2.11 (Note that in general, any torus T in the support $\mathcal{T}(G, H)$ satisfies $T/Z_{G,H}$ is anisotropic. See [12, Definition 4.3].)

Acknowledgements

The debt this work owes to [10] and [7] is clear. We thank Professor Ye Tian for his consistent encouragement. We want to thank Professor Kei-Yuen Chan for helpful discussions. L. Cai is partially supported by NSFC grant No.11971254.

References

[1] A. Aizenbud and E. Sayag, *Homological multiplicities in representation theory of p-adic groups.* (English summary) Math. Z. 294 (2020), no. 1-2, 451–469.

[2] D. Bump, *Automorphic forms and representations* Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997. xiv+574 pp. `ISBN 0-521-55098-X`.

[3] W.A. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, Draft 1995, available at `https://secure.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`

[4] L. Cai and Y. Fan, *Families of Canonical Local Periods on Spherical Varieties*, `arXiv:2107.05921`.

[5] K. Chan, G. Savin, *A vanishing Ext-branching theorem for* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$. Duke Math. J. 170 (2021), no. 10, 2237–2261.

[6] N. Matringe, *Distinguished generic representations of* $\mathrm{GL}(n)$ *over p-adic fields*, Int. Math. Res. Not. IMRN (2011), no. 1, 74–95.

[7] M. Nori and D. Prasad, *On a duality theorem of Schneider-Stuhler*, J. Reine Angew. Math. 2020 (2020), no. 762, 261–280.

[8] D. Prasad, *Trilinear forms for representations of* GL(2) *and local* ϵ-*factors.* Compositio Math. 75 (1990), no. 1, 1–46.

[9] D. Prasad, *Invariant forms for representations of* GL_2 *over a local field.* Amer. J. Math. 114 (1992), no. 6, 1317–1363.

[10] D. Prasad, *Ext-analogues of branching laws.* (English summary) Proceedings of the International Congress of Mathematicians Rio de Janeiro 2018. Vol. II. Invited lectures, 1367–1392, World Sci. Publ., Hackensack, NJ, 2018.

[11] C. Wan, *A local trace formula and the multiplicity one theorem for the Ginzburg-Rallis model*, Ph.D. thesis, 2017.

[12] C. Wan, *On multiplicity formula for spherical varieties*, accepted by Journal of the European Mathematical Society.

Academy for Multidisciplinary Studies
Beijing National Center for Applied Mathematics
Capital Normal University
Beijing 100048, China
E-mail address: caili@cnu.edu.cn
E-mail address: b452@cnu.edu.cn

Received March 1, 2022
Accepted August 4, 2022