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Contractibility of space of stability conditions on the projective plane via global dimension function

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We compute the global dimension function gldim on the principal component $\mathrm{Stab}^{\dagger}(\mathbb{P}^2)$ of the space of Bridgeland stability conditions on $\mathbb{P}^2$. It admits 2 as the minimum value and the preimage $\mathrm{gldim}^{-1}(2)$ is contained in the closure $\overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)}$ of the subspace consisting of geometric stability conditions. We show that $\mathrm{gldim}^{-1}[2,x)$ contracts to $\mathrm{gldim}^{-1}(2)$ for any real number $x \geq 2$ and that $\mathrm{gldim}^{-1}(2)$ is contractible.

1. Introduction

1.1. Stability conditions

The notion of stability conditions on triangulated categories was introduced by Bridgeland [8], with motivation coming from string theory and mirror symmetry. Let $\mathcal{D}$ be a triangulated category and $K_{\text{num}}(\mathcal{D})$ be its numerical Grothendieck group. A stability condition $\sigma = (Z, \mathcal{P})$ consists of a central charge $Z \in \text{Hom}(K(\mathcal{D}), \mathbb{C})$ and a slicing $\mathcal{P}$, which is an $\mathbb{R}$-collection of t-structures on $\mathcal{D}$. In this paper, we denote $\text{Stab}(\mathcal{D})$ as the stability manifold of stability conditions with support property with respect to $K_{\text{num}}(\mathcal{D})$. By the seminal result in [8], when $K_{\text{num}}(\mathcal{D})$ is of finite rank, the space $\text{Stab}(\mathcal{D})$ is a complex manifold with local coordinate given by the central charge. The original conjecture [9, Conjecture 1.2] in the K3 surface case is that $\text{Stab}(\mathcal{D})$ has a connected component $\text{Stab}^{\dagger}(\mathcal{D})$ which is simply-connected and preserved by the autoequivalence group of $\mathcal{D}$. A more ambitious conjecture expects that the stability manifold $\text{Stab}(\mathcal{D})$ is contractible in general. The contractibility is confirmed in a couple of examples at least for the principal component of the space, namely:

- The smooth curves case in [8, 22, 24].
- The K3 surfaces with Picard rank one in [2, 9].
- The local $\mathbb{P}^1$ in [16]; the local $\mathbb{P}^2$ in [3].
- The projective plane $\mathbb{P}^2$ in [17].
- The Abelian surfaces in [9] and Abelian threefolds with Picard rank one in [4].
- The finite type (connected) component Stab_0 in [28], where the heart of any stability conditions in Stab_0 is a length category with finite many torsion pairs. The key examples are (Calabi–Yau) ADE Dynkin quiver case and new classes of examples are studied in [1].
- The Calabi–Yau-3 affine type A case in [25].
- The acyclic triangular quiver case in [10].
- The wild Kronecker quiver case in [11].

The proofs in each case are quite different.

1.2. Global dimension functions

Recently, Ikeda and the fourth-named author [14, 26] introduce the global dimension function gldim on $\mathrm{Stab}(\mathcal{D})$, namely:

$$\mathrm{gldim}\colon \mathrm{Stab}(\mathcal{D}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}, \tag{1.1}$$

which is given by

$$\mathrm{gldim}\,\sigma = \mathrm{gldim}\,\mathcal{P} \coloneqq \sup\{\phi_2 - \phi_1 \mid \mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0\}. \tag{1.2}$$

Such a function is continuous and invariant under the natural left action by $\mathrm{Aut}(\mathcal{D})$ and the right action of $\mathbb{C}$, and thus descends to a continuous function

$$\mathrm{gldim}\colon \mathrm{Aut}(\mathcal{D})\backslash \mathrm{Stab}(\mathcal{D})/\mathbb{C} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}. \tag{1.3}$$

The philosophy in [26] is as follows:

(i) The infimum of gldim on $\mathrm{Stab}(\mathcal{D})$ (or the principal component of it) should be considered as the global dimension $\mathrm{gd}\,\mathcal{D}$ of the category $\mathcal{D}$.

(ii) If the subspace $\mathrm{gldim}^{-1}(\mathrm{gd}\,\mathcal{D})$ is non-empty, then it is contractible. Moreover, the preimage $\mathrm{gldim}^{-1}([\mathrm{gd}\,\mathcal{D}, x))$ contracts to $\mathrm{gldim}^{-1}(\mathrm{gd}\,\mathcal{D})$ for any real number $\mathrm{gd}\,\mathcal{D} < x$.

(iii) If $\mathrm{gldim}^{-1}(\mathrm{gd}\,\mathcal{D})$ is empty, then the preimage $\mathrm{gldim}^{-1}(\mathrm{gd}\,\mathcal{D}, x)$ contracts to $\mathrm{gldim}^{-1}(\mathrm{gd}\,\mathcal{D}, y)$ for any real number $\mathrm{gd}\,\mathcal{D} < y < x$.

Note that for a Calabi–Yau category, the global dimension function is constant. If the global dimension function gldim is not constant, it sheds some lights on why $\mathrm{Stab}(\mathcal{D})$ should be contractible.

The theme in [14] is to q-deform stability conditions. More precisely, given a Calabi–Yau-∞ category $\mathcal{D}_\infty$ (e.g. bounded derived category of $\mathbb{P}^2$), the corresponding Calabi–Yau-N category $\mathcal{D}_N$ (e.g. local $\mathbb{P}^2$ for $\mathbb{P}^2$ and $N = 3$) can be obtained by Calabi–Yau-$\mathbb{X}$ completing $\mathcal{D}_\infty$ to $\mathcal{D}_\mathbb{X}$ and specializing $\mathbb{X}$ to be N, in other words, taking the orbit category $\mathcal{D}_N = \mathcal{D}_\mathbb{X} /\!\!/ [\mathbb{X} - N]$. Under this procedure, a stability condition σ on $\mathcal{D}_\infty$ such that

$$\mathrm{gldim}\,\sigma \leq N - 1 \tag{1.4}$$

induces a stability condition on $\mathcal{D}_N$ via q-stability conditions on $\mathcal{D}_\mathbb{X}$. We will discuss such inducing in Section 7 for the example from $\mathbb{P}^2$ to local $\mathbb{P}^2$ where $N = 3$.

1.3. The projective plane case

In this paper, we study the case of the projective plane $\mathbb{P}^2$ for the above conjectures/philosophy. The main result is a computation of the global dimension function for the principal component $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ (i.e. the connected component which contains geometric stability conditions, where a stability condition $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$ is called geometric if all skyscraper sheaves are σ-stable of the same phase). Details are in Propositions 3.4 and 5.1. Based on the computation of gldim, we prove the following theorem.

Theorem 1.1 (Corollary 5.10 and Theorem 6.1). *Consider the function*

$$\mathrm{gldim}\colon \mathrm{Stab}^\dagger(\mathbb{P}^2) \to \mathbb{R}_{\geq 0}$$

on the principal component $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ *of the space of stability conditions on the bounded derived category* $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}\,\mathbb{P}^2)$ *of coherent sheaves on* $\mathbb{P}^2$. *Then*

- $\mathrm{gd}\,\mathcal{D} = 2$ *and* $\mathrm{gldim}\,\mathrm{Stab}^\dagger(\mathbb{P}^2) = [2, \infty)$,
- *the subspace* $\mathrm{gldim}^{-1}[2, x)$ *contracts to* $\mathrm{gldim}^{-1}(2)$, *for any* $x \geq 2$,
- *the subspace* $\mathrm{gldim}^{-1}(2)$ *is contractible and is contained in* $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$, *where* $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ *consists of geometric stability conditions.*

The contractibility of $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ is already proved by the second-named author [17]. The new approach here shows how this stability manifold contracts along the values of the global dimension function.

1.4. Topological Fukaya case

In the parallel work [27], we use the same philosophy to study the contractibility of the space of stability conditions on the topological Fukaya category of a graded marked surface. We prove a slightly weaker version of the corresponding Theorem 1.1, that gldim induces the contractible flow except for certain possible critical values.

We hope that these works will shed lights on how this philosophy would apply to other cases.

2. Preliminaries

2.1. The category

In this paper, we let $\mathbb{P}^2$ be the projective plane over the complex number field. We write

$$\mathcal{D}_\infty(\mathbb{P}^2) := \mathcal{D}^b(\mathbb{P}^2) = \mathcal{D}^b(\operatorname{Coh}\mathbb{P}^2) \tag{2.1}$$

for the bounded derived category of coherent sheaves on $\mathbb{P}^2$. Due to the well-known result by Beĭlinson [5], we have the equivalent description $\mathcal{D}_\infty(\mathbb{P}^2) \cong \mathcal{D}^b(\mathbf{k}Q/R)$, where (Q, R) is the quiver

$$1 \xrightarrow{x_1,\, y_1,\, z_1} 2 \xrightarrow{x_2,\, y_2,\, z_2} 3$$

with commutative relations

$$a_1 b_2 = b_1 a_2, \quad a, b \in \{x, y, z\}.$$

The Serre functor on $\mathcal{D}_\infty(\mathbb{P}^2)$ is given by (see [6] or [13])

$$\mathbb{S} = \mathbb{S}_{\mathbb{P}^2} := (-) \otimes \omega_{\mathbb{P}^2}[2] = (-) \otimes \mathcal{O}_{\mathbb{P}^2}(-3)[2].$$

An object $E \in \mathcal{D}_\infty(\mathbb{P}^2)$ is called exceptional if $\operatorname{Hom}(E, E[i]) = 0$ for $i \neq 0$ and $\operatorname{Hom}(E, E) = \mathbb{C}$. The right and left mutations of an object F with respect to an exceptional object E are defined by

$$\mathsf{R}_E(F) := \operatorname{Cone}\left(F \xrightarrow{\text{ev}} E \otimes \operatorname{Hom}(F, E)^*\right)[-1], \tag{2.2}$$

$$\mathsf{L}_E(F) := \operatorname{Cone}\left(E \otimes \operatorname{Hom}(E, F) \xrightarrow{\text{ev}} F\right). \tag{2.3}$$

2.2. An affine plane

Let $\mathcal{D} = \mathcal{D}_\infty(\mathbb{P}^2)$. Let H be the hyperplane divisor of $\mathbb{P}^2$. For $E \in \mathcal{D}$, we identity the Chern character $\operatorname{ch}(E)$ with the triple of numbers

$$\tilde{\mathrm{v}}(E) = (\operatorname{ch}_0(E), \operatorname{ch}_1(E).H, \operatorname{ch}_2(E)).$$

When we say the *point* E (or the *point* $\tilde{\mathrm{v}}(E)$), we mean the point in the real projective space $\mathbb{P}(\mathbb{R}^3)$ with homogeneous coordinate

$[\mathrm{ch}_0(E), \mathrm{ch}_1(E).H, \mathrm{ch}_2(E)]$. We call the locus $\mathrm{ch}_0 = 0$ as the line at infinity and its complement as the affine $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Moreover, we always assume that the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$-axis is horizontal and the $\frac{\mathrm{ch}_2}{\mathrm{ch}_0}$-axis is vertical. If $\mathrm{ch}_0(E) \neq 0$, the *reduced character* of E corresponds to the point

$$\mathrm{v}(E) := (1, s(E), q(E)), \quad \text{with } s(E) := \frac{\mathrm{ch}_1(E).H}{\mathrm{ch}_0(E)}, \quad q(E) := \frac{\mathrm{ch}_2(E)}{\mathrm{ch}_0(E)}, \tag{2.4}$$

in the affine $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. In particular $\mathrm{v}(E) = \mathrm{v}(E[n])$, i.e. E and its any shift $E[n]$ will be the same point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane.

The $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane provides a playground for studying both geometric and algebraic stability conditions in the following part of the paper.

2.3. Stability conditions

A *stability condition* $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ called the *central charge* and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ called the *slicing* satisfying certain conditions. We refer to [8] and the lecture notes [23, Definition 5.8] for the details. Nonzero objects in $\mathcal{P}(\phi)$ are called *semistable of phase* ϕ and simple objects in $\mathcal{P}(\phi)$ are called *stable of phase* ϕ. For semistable object $E \in \mathcal{P}(\phi)$, denote by $\phi_\sigma(E) = \phi$ its *phase*.

Let $\mathcal{D} = \mathcal{D}_\infty(\mathbb{P}^2)$ and

$$\mathrm{Stab}(\mathbb{P}^2) := \mathrm{Stab}(\mathcal{D}_\infty(\mathbb{P}^2))$$

be the space of stability conditions on $\mathcal{D}_\infty(\mathbb{P}^2)$.

A stability condition $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$ is called *geometric* if all skyscraper sheaves are σ-stable of the same phase. We denote the set of all geometric stability conditions by $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$.

Let us briefly recall the construction of geometric stability conditions. There is a fractal curve C_{LP}, the so called Le Potier curve, and a region $\mathrm{Geo}_{\mathrm{LP}}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, as in Definition 4.5. For each $(1, s, q) \in \mathrm{Geo}_{\mathrm{LP}}$, one can associate a geometric stability condition $\sigma_{s,q} = (Z_{s,q}, \mathcal{P}_{s,q})$ as follows. The central charge $Z_{s,q}$ is given by

$$\begin{aligned} Z_{s,q}(E) := {} & (-\mathrm{ch}_2(E) + q \cdot \mathrm{ch}_0(E)) \\ & + i(\mathrm{ch}_1(E).H - s \cdot \mathrm{ch}_0(E)), \quad \text{for } E \in \mathcal{D}. \end{aligned} \tag{2.5}$$

Denote H-slope of coherent sheaves by $\frac{\mathrm{ch}_1(-).H}{\mathrm{ch}_0(-)}$. We make a convention that H-slope of a torsion sheaf is $+\infty$. The heart $\mathcal{P}_{s,q}((0,1])$ is the tilting

$$\mathrm{Coh}_{\#s} := \langle \mathrm{Coh}_{\leq s}[1], \mathrm{Coh}_{>s} \rangle,$$

where $\mathrm{Coh}_{\leq s}$ (resp. $\mathrm{Coh}_{>s}$) is the subcategory of $\mathrm{Coh}(\mathbb{P}^2)$ generated by H-slope semistable sheaves of slope $\leq s$ (resp. $> s$) by extension. The slicing for $\phi \in (0,1]$ is defined by

$$\mathcal{P}_{s,q}(\phi) = \{E \in \mathrm{Coh}_{\#s} \mid E \text{ is } \sigma_{s,q} \text{ -semistable of phase } \phi\} \cup \{0\}.$$

For general $\phi \in \mathbb{R}$, we have $\mathcal{P}_{s,q}(\phi + 1) = \mathcal{P}_{s,q}(\phi)[1]$.

The $\widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ acts freely on $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ ([17, Definition 1.4, Corollary 1.15]) with quotient

$$\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)/\widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cong \mathrm{Geo}_{\mathrm{LP}}\,.$$

We refer to Section 4 for the definition of algebraic stability conditions $\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2)$. Let $\mathrm{Stab}^{\dagger}(\mathbb{P}^2)$ be the connected component in $\mathrm{Stab}(\mathbb{P}^2)$ which contains the geometric stability conditions. It is still a *conjecture* that $\mathrm{Stab}(\mathbb{P}^2) = \mathrm{Stab}^{\dagger}(\mathbb{P}^2)$. The second-named author [17] shows that

$$\mathrm{Stab}^{\dagger}(\mathbb{P}^2) = \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \bigcup \mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2) \tag{2.6}$$

and it is contractible. In the following sections, we will compute the global dimension function gldim on $\mathrm{Stab}^{\dagger}(\mathbb{P}^2)$ and show that the contraction is along the value of gldim.

3. Geometric stability conditions in the parabolic region

Let $\mathcal{D} = \mathcal{D}_\infty(\mathbb{P}^2)$. For $a \in \mathbb{R}$, denote by Δ_a the parabola in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane:

$$\Delta_a := \Big\{(1,s,q) \in \{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}\text{-plane} \mid \frac{1}{2}s^2 - q = a\Big\}.$$

Similarly we have the notation $\Delta_{<a}$ or $\Delta_{\geq a}$. We study geometric stability conditions in the parabolic region $\Delta_{<0}$. Denote by L_{PE} the line passing through the two points P and E in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Recall a lemma due to Bayer [19, Lemma 3].

Lemma 3.1. *Let P and Q be two points in the region $\Delta_{<0}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Let F be a σ_P-stable object in* Coh_P *with* $\mathrm{ch}(F) \neq (0,0,1)$. *Let C and D be the intersection points*

$$\{C, D\} := L_{PF} \cap \Delta_0,$$

of the line L_{PF} and the parabola Δ_0. Denote the σ_Q-HN semistable factors of F by F_i. Then for each factor, the phase $\phi_Q(F_i)$ lies in between $\phi_Q(C)$ and $\phi_Q(D)$.

Proof. The case that $\mathrm{ch}_0(F) \neq 0$ is proved in [19, Lemma 3].

So we assume that $\mathrm{ch}(F) = (0, \mathrm{ch}_1(F), \mathrm{ch}_2(F))$ with $\mathrm{ch}_1(F).H > 0$. Now the point F is in the ∞-line outside the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. But the line L_{PF} still makes sense: it is the line passing through the point P with slope $\frac{\mathrm{ch}_2(F)}{\mathrm{ch}_1(F).H}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, see [20, Corollary 2.8]. So we still have the notation l^+_{PF}, which is the ray starting at the point P on the line L_{PF} with $s \geq s(P)$. Note that L_{QF} is parallel to L_{PF}. Then the proof follows by Li-Zhao's original argument. □

For a point $P = (1, s, q)$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, we say that we move it along the parabola to the left by a number b if we move it along the unique parabola of the form Δ_a passing through P (so $a = \frac{1}{2}s^2 - q$), and the result point is still on the same parabola Δ_a with $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$-coordinate $s - b$. Let K be the canonical divisor of $\mathbb{P}^2$, and $\omega_{\mathbb{P}^2}$ be the dualizing sheaf. Let $\sigma = \sigma_{s,q}$ with $(1, s, q) \in \mathrm{Geo}_{\mathrm{LP}}$. We identify σ with the point $(1, s, q)$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Then $\sigma(-3) := \sigma \otimes \omega_{\mathbb{P}^2}$ is the point of moving σ along the parabola to the left by $-H.K = 3$. Similarly for $F \in \mathcal{D}$ as a point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, if $\mathrm{ch}_0(F) \neq 0$, then $F(-3) := F \otimes \omega_{\mathbb{P}^2}$ is the point of moving the point F along the parabola to the left by 3.

Let A, B, $\tilde{A}$, $\tilde{B}$ be the corresponding intersection points

$$\{A, B\} := L_{F\sigma} \cap \Delta_0, \quad \{\tilde{A}, \tilde{B}\} := L_{F(-3)\sigma(-3)} \cap \Delta_0,$$

with $s(B) > s(A)$ and $s(\tilde{B}) > s(\tilde{A})$. We have the following observation.

Lemma 3.2.

$$s(B) - s(A) = s(\tilde{B}) - s(\tilde{A}). \tag{3.1}$$

Proof. This is an elementary calculation. □

We prove a lemma, which is the key calculation for proving $\mathrm{gldim}\,\sigma_{s,q} = 2$ in the region $\Delta_{<0}$.

Lemma 3.3. *Let $\sigma_{s,q}$ be a geometric stability condition in the region $\Delta_{<0}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Denote $\sigma_{s,q}$ by σ. Let F, G be two σ stable objects in a same heart satisfying: $0 < \phi_\sigma(F) < \phi_\sigma(G) \leq 1$, $\mathrm{ch}_0(F) \neq 0$ and $s < s(F)$. Then* $\mathrm{Hom}(F, G[2]) = 0$.

Proof. Let $P \coloneqq L_{F\sigma} \cap L_{F(-3)\sigma(-3)}$. We have two cases.
Case A. P is in the region $\Delta_{\geq 0}$. Then by Lemma 3.2, we must have $s(\tilde{B}) \leq s(A)$ (i.e. left above of Figure 1) instead of $s(\tilde{B}) > s(A)$ (i.e. left below or right below of Figure 1). So $l^+_{\sigma F}$ is above or equal to $l^+_{\sigma\tilde{A}}$ and $l^+_{\sigma\tilde{B}}$.

By [21, Lemma A.3], $F(-3)$ is $\sigma(-3)$-stable. By Lemma 3.1, the σ-HN factor $F(-3)_i$ of $F(-3)$ lies between $\phi_\sigma(\tilde{A})$ and $\phi_\sigma(\tilde{B})$. By [19, Lemma 2], $\phi_\sigma(F(-3)_i) \leq \phi_\sigma(F)$. So $\phi^+_\sigma(F(-3)) \leq \phi_\sigma(F) < \phi_\sigma(G)$ and $\mathrm{Hom}(G, F(-3)) = 0$. By Serre duality, we have

$$\mathrm{Hom}(F, G[2]) \cong (\mathrm{Hom}(G[2], \mathbb{S}(F)))^* = (\mathrm{Hom}(G, F(-3)))^* = 0.$$

Case B. P is in the region $\Delta_{<0}$. So both F and $F(-3)$ are σ_P-stable with

$$\phi_P(F) > \phi_P(F(-3)). \tag{3.2}$$

Let $Q \coloneqq L_{G\sigma} \cap L_{F(-3)\sigma(-3)}$. We have three subcases.
Case B.(i) Q is in the region $\Delta_{>0}$. Then Q is to the right of $\tilde{B}$ since $\tilde{B}$ is on the Δ_0. Now l^+_{QG} is above $l^+_{\sigma\tilde{A}}$ and $l^+_{\sigma\tilde{B}}$. We must have $l^+_{\sigma G}$ is above $l^+_{\sigma\tilde{A}}$ and $l^+_{\sigma\tilde{B}}$. By Lemma 3.1 again, we have $\mathrm{Hom}(G, F(-3)) = 0$. By Serre duality, we have $\mathrm{Hom}(F, G[2]) = 0$.
Case B.(ii) Q is in the region $\Delta_{<0}$. We illustrate the picture in right above of Figure 1. Since G is σ-stable, it is also σ_Q-stable. Since $F(-3)$ is $\sigma(-3)$-stable, it is also σ_Q-stable. We then compare their phases at Q and have

$$\phi_Q(G) = \phi_\sigma(G) > \phi_\sigma(F) = \phi_P(F) > \phi_P(F(-3)) = \phi_Q(F(-3)),$$

where each equality is because of colinear condition, and the first inequality is given by the assumption of the Lemma and the second inequality is given by (3.2). So $\mathrm{Hom}(G, F(-3)) = 0$. By Serre duality, we have $\mathrm{Hom}(F, G[2]) = 0$.
Case B.(iii) Q is on the parabola Δ_0. Since F is σ-stable, we may perturb σ a little bit and reduce to the previous cases. □

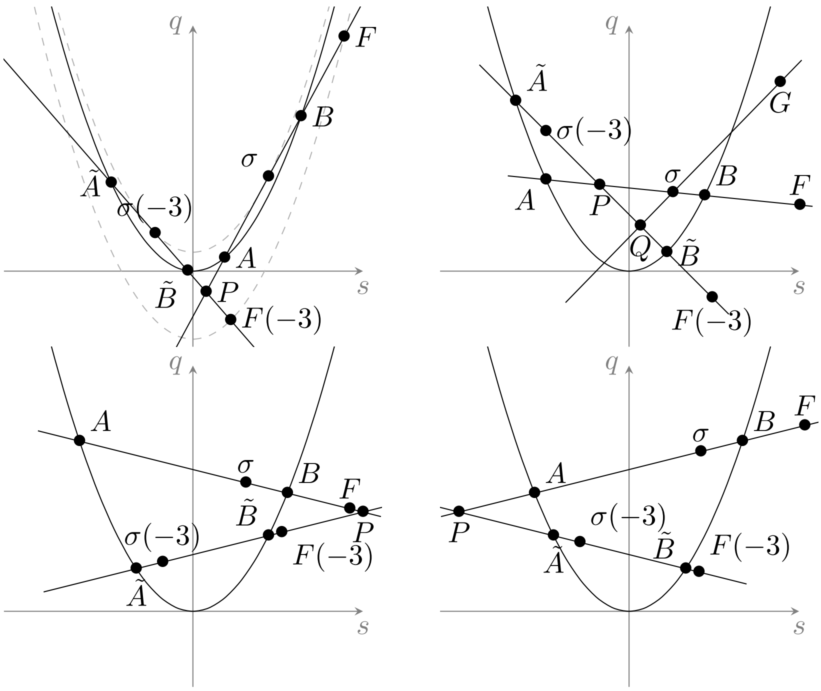


Figure 1: Relative positions of $L_{F\sigma}$ and $L_{F(-3)\sigma(-3)}$: $P \in \Delta_{\geq 0}$ (left above); $P \in \Delta_{<0}$ (right above). The below pictures are impossible by Lemma 3.2.

Proposition 3.4. *Let $\sigma_{s,q}$ be in the region $\Delta_{<0}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Then*

$$\operatorname{gldim} \sigma_{s,q} = 2. \tag{3.3}$$

Proof. Denote $\sigma_{s,q}$ by σ. Let F and G be two σ-semistable objects such that

$$\mathrm{Hom}(F, G[2]) \neq 0.$$

Then by Serre duality,

$$\mathrm{Hom}(F, G[2]) \cong (\mathrm{Hom}(G, F(-3)))^* \neq 0. \tag{3.4}$$

The object $F(-3)$ may not be σ-semistable. We consider its σ-HN factors. Thus by [8, Lemma 3.4] we have $\phi_\sigma(F) \leq \phi_\sigma(G[2]) \leq \phi^+_\sigma(F(-3)) + 2$. So

$$0 \leq \phi_\sigma(G[2]) - \phi_\sigma(F) \leq \phi^+_\sigma(F(-3)) - \phi_\sigma(F) + 2. \tag{3.5}$$

We need to show that

$$\phi_\sigma(G[2]) - \phi_\sigma(F) \leq 2. \tag{3.6}$$

The idea is to give an estimate of $\phi^+_\sigma(F(-3)) - \phi_\sigma(F)$. We could assume that $F \in \mathrm{Coh}_{\#s}$, i.e. $\phi_\sigma(F) \in (0,1]$. We could also assume that F is σ-stable since we can take its Jordan-Hölder factors. So by [21, Lemma A.3], $F(-3)$ is $\sigma(-3)$-stable.

We have the following three cases according to the Chern characters of F.

Case 1. Assume $\mathrm{ch}_0(F) = 0$, $\mathrm{ch}_1(F) = 0$ and $\mathrm{ch}_2(F) > 0$. Then F is supported at point(s) and $\phi_\sigma(F(-3)) = \phi_\sigma(F)$. So (3.6) holds. On the other hand, for any closed point $x \in \mathbb{P}^2$, we have $\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x[2]) \neq 0$ and

$$\phi_\sigma(\mathcal{O}_x[2]) - \phi_\sigma(\mathcal{O}_x) = 2. \tag{3.7}$$

Case 2. Assume that $\mathrm{ch}_0(F) \neq 0$. We have the following three subcases.

(i) σ is to the left of F. This is precisely Lemma 3.3.

(ii) If σ is to the right of F, by applying a shifted derived dual functor, we reduce to case (i).

(iii) If the H-slope of F is s, by local finiteness of walls, we could replace σ by σ' in a small open neighbourhood of σ so that F is σ'-stable. So we reduce to case (i) or (ii).

Case 3. Assume $\mathrm{ch}_0(F) = 0$ and $\mathrm{ch}_1(F).H > 0$. Now we have

$$\mathrm{ch}(F(-3)) = (0, \mathrm{ch}_1(F), \mathrm{ch}_2(F) + \mathrm{ch}_1(F).K).$$

The line $L_{F\sigma}$ is the line passing through σ of the slope $\frac{\mathrm{ch}_2(F)}{\mathrm{ch}_1(F).H}$. Similarly, the line $L_{F(-3)\sigma(-3)}$ is the line passing through $\sigma(-3)$ of the slope $\frac{\mathrm{ch}_2(F)}{\mathrm{ch}_1(F).H} + H.K$ by [21, Lemma A.3]. By Lemma 3.1, the phase of $\phi_\sigma((F(-3))_i)$ lies between $\phi_\sigma(\tilde{A})$ and $\phi_\sigma(\tilde{B})$. We have similar analysis as the **Case 2** and still have (3.6).

Therefore for $\sigma \in \Delta_{<0}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, we have $\mathrm{gldim}(\sigma) = 2$. Moreover, the value 2 can be obtained by (3.7). This finishes the proof. □

4. Algebraic stability conditions

4.1. Reviews

We first recall the construction of algebraic stability conditions with respect to exceptional triples from [17].

Definition 4.1. We call an ordered set $\mathcal{E} = \{E_1, E_2, E_3\}$ *exceptional triple* on $\mathcal{D}^b(\mathbb{P}^2)$ if $\mathcal{E}$ is a full strong exceptional collection of coherent sheaves on $\mathcal{D}^b(\mathbb{P}^2)$.

There is a one-to-one correspondence between the dyadic integers $\frac{p}{2^m}$ and exceptional bundles $E(\frac{p}{2^m})$:

$$\frac{p}{2^m} \Longleftrightarrow E(\frac{p}{2^m}), \text{ for } p \in \mathbb{Z} \text{ and } m \in \mathbb{Z}_{\geq 0}.$$

The exceptional triples have been classified by Gorodentsev and Rudakov [12]. The exceptional triples are labeled by the following three cases,

$$\left\{\frac{p-1}{2^m}, \frac{p}{2^m}, \frac{p+1}{2^m}\right\}, \quad \left\{\frac{p}{2^m}, \frac{p+1}{2^m}, \frac{p-1}{2^m}+3\right\}, \quad \left\{\frac{p+1}{2^m}-3, \frac{p-1}{2^m}, \frac{p}{2^m}\right\},$$

for $p \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$. Note that the last two cases are mutations of the first case.

Proposition 4.2 ([22, Section 3]). *Let $\mathcal{E}$ be an exceptional triple on $\mathcal{D}^b(\mathbb{P}^2)$. For any positive real numbers m_1, m_2, m_3 and real numbers ϕ_1, ϕ_2, ϕ_3 such that:*

$$\phi_1 < \phi_2 < \phi_3, \text{ and } \phi_1 + 1 < \phi_3,$$

there is a unique stability condition $\sigma = (Z, \mathcal{P})$ such that

1°. each E_j is stable with phase ϕ_j;

2°. $Z(E_j) = m_j e^{i\pi\phi_j}$.

Definition 4.3. For an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$ on $\mathcal{D}^b(\mathbb{P}^2)$, we write $\Theta_{\mathcal{E}}$ as the space of all stability conditions in Proposition 4.2, which is parametrized by

$$S := \{(m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}^3 \,|\, \phi_1 < \phi_2 < \phi_3,\ \phi_1 + 1 < \phi_3\}.$$

We make the following notations for some subsets of $\Theta_{\mathcal{E}}$.

$$\begin{aligned}
\Theta_{\mathcal{E}}(A) &:= \{\sigma \in \Theta_{\mathcal{E}} | \sigma \in A\}, \text{ where } A \text{ is a subset of } S;\\
\Theta_{\mathcal{E}}^{\mathrm{Pure}} &:= \{\sigma \in \Theta_{\mathcal{E}} | \phi_2 - \phi_1 \geq 1 \text{ and } \phi_3 - \phi_2 \geq 1\};\\
\Theta_{\mathcal{E},E_3}^{\mathrm{left}} &:= \{\sigma \in \Theta_{\mathcal{E}} | \phi_2 - \phi_1 < 1 \text{ and } E_3(3) \text{ is not } \sigma\text{-stable}\};\\
\Theta_{\mathcal{E},E_1}^{\mathrm{right}} &:= \{\sigma \in \Theta_{\mathcal{E}} | \phi_3 - \phi_2 < 1 \text{ and } E_1(-3) \text{ is not } \sigma\text{-stable}\};\\
\Theta_{\mathcal{E}}^{\mathrm{Geo}} &:= \Theta_{\mathcal{E}} \cap \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2);\\
\Theta_{\mathcal{E},E_3}^{-} &:= \Theta_{\mathcal{E}}(\phi_2 - \phi_1 < 1) \setminus \Theta_{\mathcal{E}}^{\mathrm{Geo}};\\
\Theta_{\mathcal{E},E_1}^{+} &:= \Theta_{\mathcal{E}}(\phi_3 - \phi_2 < 1) \setminus \Theta_{\mathcal{E}}^{\mathrm{Geo}}.
\end{aligned}$$

We denote

$$\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2) := \bigcup_{\mathcal{E} \text{ exceptional triples}} \Theta_{\mathcal{E}}$$

and call the elements of it as the *algebraic* stability conditions.

Lemma 4.4 ([17, Lemma 2.4]). *Let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional triple, and σ be a stability condition in $\Theta_{\mathcal{E}}^{\mathrm{Pure}}$. The only σ-stable objects are $E_i[n]$ for $i = 1, 2, 3$ and $n \in \mathbb{Z}$.*

4.2. Five points associated to an exceptional bundle

For an object $A \in \mathcal{D}$ with $\mathrm{ch}_0(A) \neq 0$, by abusing of notations, we write A for $\mathrm{v}(A) = (1, s(A), q(A))$ in (2.4) as the associated point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, and call it the *point* A in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Moreover, by the Riemann–Roch formula, we have

$$\chi(A, A) = \mathrm{ch}_0^2(A)(1 - s(A)^2 + 2q(A)). \tag{4.1}$$

In particular, for an exceptional bundle E, we have $\mathrm{ch}_0(E) \neq 0$, $\chi(E, E) = 1$, and

$$\frac{1}{2}s(E)^2 - q(E) = \frac{1}{2} - \frac{1}{2\mathrm{ch}_0^2(E)}. \tag{4.2}$$

So for each exceptional bundle E as a point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, the point E is in the region $\Delta_{[0, \frac{1}{2})}$.

In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, for each exceptional bundle E, we define the following two pairs of parallel lines:

$$\{\chi(E,-)=0\}, \quad \{\chi(E,-)=\frac{\mathrm{ch}_0(-)}{\mathrm{ch}_0(E)}\}; \tag{4.3}$$

$$\{\chi(-,E)=0\}, \quad \{\chi(-,E)=\frac{\mathrm{ch}_0(-)}{\mathrm{ch}_0(E)}\}. \tag{4.4}$$

We now give a geometric description of above lines. By the Riemann–Roch formula, one can check that the line $\{\chi(E,-)=\frac{\mathrm{ch}_0(-)}{\mathrm{ch}_0(E)}\}$ is the line $L_{E(-3)E}$ passing through the points $E(-3)$ and E. Similarly, the line $\{\chi(-,E)=\frac{\mathrm{ch}_0(-)}{\mathrm{ch}_0(E)}\}$ is the line $L_{E(E(3))}$ passing through the points E and $E(3)$.

The line $\{\chi(E,-)=0\}$ is the line passing through points E_1 and E_2 for any choice of exceptional triple $\{E_1,E_2,E\}$ ending with E. It is clearly that this line is independent of the choice of E_1 and E_2. Similarly, the line $\{\chi(-,E)=0\}$ is the line passing through points E_2 and E_3 for any choice of exceptional triple $\{E,E_2,E_3\}$ starting with E. This line is independent of the choice of E_2 and E_3.

For each exceptional bundle E, we define five points in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane as intersection points of the following lines or curves,

$$\begin{aligned}
E^l &:= L_{E(E(3))} \cap \{\chi(E,-)=0\},\\
E^r &:= L_{E(-3)E} \cap \{\chi(-,E)=0\},\\
E^+ &:= \{\chi(E,-)=0\} \cap \{\chi(-,E)=0\},\\
e^l &:= \Delta_{\frac{1}{2}} \cap \{\chi(E,-)=0\} \text{ as the first intersection point staring from } E^+,\\
e^r &:= \Delta_{\frac{1}{2}} \cap \{\chi(-,E)=0\} \text{ as the first intersection point staring from } E^+.
\end{aligned}$$

We now give a geometric description of above points. One can also refer to Figure 2 and Figure 3. By the Riemann–Roch formula, (4.2) and (2.4), we have

$$s(E^+)=s(E), \quad q(E^+)=q(E)-\frac{1}{(\mathrm{ch}_0(E))^2}. \tag{4.5}$$

So E^+ is the point of moving E downward of length $\frac{1}{(\mathrm{ch}_0(E))^2}$. By (4.2) and (4.5), we have

$$\frac{1}{2}s(E^+)^2-q(E^+)=\frac{1}{2}+\frac{1}{2\mathrm{ch}_0^2(E)}. \tag{4.6}$$

So the point E^+ is in the region $\Delta_{(\frac{1}{2},1]}$.

We observe that the point E^l stands for $\mathrm{v}(\mathsf{L}_E(E(3)))$, i.e. the reduced character of $\mathsf{L}_E(E(3))$. This is because by the definition (2.3), the point $\mathsf{L}_E(E(3))$ is on the line $L_{EE(3)}$. Also, the object $\mathsf{L}_E(E(3)) \in E^\perp = \langle E_1, E_2\rangle$ has a resolution (5.2) (by taking $E_3 = E$). Thus the point $\mathsf{L}_E(E(3))$ is on the line $\{\chi(E,-) = 0\}$.

By the Riemann–Roch formula, we have

$$\chi(E, E(3)) = 1 + 9\mathrm{ch}_0^2(E), \quad \chi(E(3), E) = 1, \quad \chi(E(3), E(3)) = 1.$$

Since $[\mathsf{L}_E(E(3))] = [E(3)] - \chi(E, E(3))[E]$ in $K_{\mathrm{num}}(\mathbb{P}^2)$, we have

$$\chi(\mathsf{L}_E(E(3)), \mathsf{L}_E(E(3))) = 1 - \chi(E(3), E)\chi(E, E(3)) = -9\mathrm{ch}_0^2(E) < 0.$$

Then by (4.1), the point E^l is in the region $\Delta_{>\frac{1}{2}}$. In particular, E^l is in the line segment $\overline{E^+e^l}$. Similarly, E^r stands for the reduced character of $\mathsf{R}_E(E(-3))$. It is in the region $\Delta_{>\frac{1}{2}}$ and in the line segment $\overline{E^+e^r}$. One can check that both of points E^l and E^r are in the parabola $\frac{1}{2}s^2 - q = \frac{1}{2} + \frac{1}{18\mathrm{ch}_0^4(E)}$.

Definition 4.5. ([17, Definition 1.4]) The Le Potier curve C_{LP} is a fractal curve defined in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane as

$$\mathrm{C}_{\mathrm{LP}} \coloneqq \bigsqcup_{\{E = E(\frac{p}{2^m})\,|\, p\in\mathbb{Z},\, m\in\mathbb{Z}_{\geq 0}\}} \left(\overline{E^+e^l} \cup \overline{E^+e^r}\right) \bigsqcup \{\text{Cantor pieces of } \Delta_{\frac{1}{2}}\}.$$

The region $\mathrm{Geo}_{\mathrm{LP}}$ is defined as $\mathrm{Geo}_{\mathrm{LP}} \coloneqq \big\{(1,s,q) \in \{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}\text{-plane} \,|\, (1,s,q) \text{ is above the curve } \mathrm{C}_{\mathrm{LP}} \text{ and is not on line segment } \overline{EE^+} \text{ for any exceptional bundle } E\big\}$.

4.3. Special regions associated to an exceptional triple

Definition 4.6. For an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$, the region $\mathrm{MZ}^{\mathrm{c}}_{\mathcal{E}}$ is defined as the open region in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane bounded by the line segments $\overline{E_1E_1^r}$, $\overline{E_1^rE_2}$, $\overline{E_2E_3^l}$, $\overline{E_3^lE_3}$ and $\overline{E_3E_1}$ (see Figure 2). The region $\mathrm{MZ}_{\mathcal{E}}$ is defined as the open region in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane bounded by line segments $\overline{E_1E_1^+}$, $\overline{E_1^+E_2}$, $\overline{E_2E_3^+}$, $\overline{E_3^+E_3}$ and $\overline{E_3E_1}$ (see Figure 3).

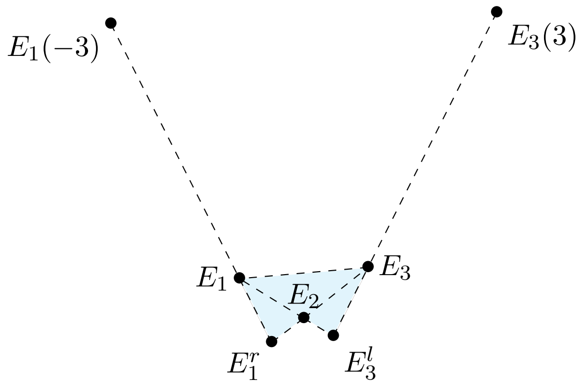


Figure 2: The region of $\mathrm{MZ}^c_{\mathcal{E}}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane.

We have $\mathrm{MZ}_{\mathcal{E}} \subset \mathrm{Geo}_{\mathrm{LP}}$ and ([17, Proposition 2.5])

$$\Theta^{\mathrm{Geo}}_{\mathcal{E}} = \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \{\sigma_{s,q} \in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \mid (1,s,q) \in \mathrm{MZ}_{\mathcal{E}}\}.$$

Remark 4.7. Let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional triple. Note that the region $\mathrm{MZ}^c_{\mathcal{E}}$ is a subregion of $\mathrm{MZ}_{\mathcal{E}}$.

1°. Since E_2 is in the region $\Delta_{[0,\frac{1}{2})}$ by (4.2) and E_1^+, E_3^+ are in the region $\Delta_{>\frac{1}{2}}$ by (4.6), we have

$$e_1^r = \Delta_{\frac{1}{2}} \cap \overline{E_1^+E_2}, \qquad e_3^l = \Delta_{\frac{1}{2}} \cap \overline{E_2E_3^+}.$$

2°. The line $L_{E_3(E_3(3))}$ is given as $\{\chi(-,E_3) = \frac{\mathrm{ch}_0(-)}{\mathrm{ch}_0(E_3)}\}$. For every stable vector bundle A with slope between the slopes of E_3 and $E_3(3)$, we have $\chi(A,E_3) \leq 0$. The line segment $\overline{E_3E_3(3)}$ is contained in $\mathrm{Geo}_{\mathrm{LP}}$.

3°. The point E_3^l is on the line segment $\overline{e_3^lE_3^+}$. In particular, The reduced character of any exceptional bundles with slope smaller than that of E_3 is to the left of e_3^l.

4°. By [17, Corollary 1.19], the exceptional object $E_3(3)$ is stable with respect to $\sigma_{s,q}$ for any $(1,s,q)$ in $\mathrm{MZ}^c_{\mathcal{E}}$, and is destabilized by E_3 on the line segment $\overline{E_3E_3^l}$. In particular, the region $\mathrm{MZ}^c_{\mathcal{E}}$ is a subregion of $\mathrm{MZ}_{\mathcal{E}}$ by removing the region that either $E_3(3)$ or $E_1(-3)$ is not stable. In particular, we can identify the region $\mathrm{MZ}^c_{\mathcal{E}}$ as the following algebraic stability conditions.

Lemma 4.8. *Let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional triple, then*

$$\begin{aligned}&\Theta_{\mathcal{E}} \setminus (\Theta^{\text{right}}_{\mathcal{E},E_1} \cup \Theta^{\text{left}}_{\mathcal{E},E_3} \cup \Theta^{\text{Pure}}_{\mathcal{E}}) \\ &\quad = \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \left\{\sigma_{s,q} \in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \mid (1,s,q) \in \mathrm{MZ}^c_{\mathcal{E}}\right\}.\end{aligned}$$

Proof. By the previous Remark 4.7.4, the proof is the same as that for [18, Lemma 1.29]. □

Definition 4.9. For an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$ on $\mathcal{D}^b(\mathbb{P}^2)$, we define $\mathrm{MZ}^l_{E_3}$ and $\mathrm{MZ}^r_{E_1}$ as subregions of $\mathrm{MZ}_{\mathcal{E}}$ as follows:

$$\mathrm{MZ}^l_{E_3} := \left\{(1,s,q) \in \mathrm{MZ}_{\mathcal{E}} \mid s < s(E_3), (1,s,q) \text{ is not above the line segment } \overline{E_3E^l_3}\right\},$$
$$\mathrm{MZ}^r_{E_1} := \left\{(1,s,q) \in \mathrm{MZ}_{\mathcal{E}} \mid s > s(E_1), (1,s,q) \text{ is not above the line segment } \overline{E_1E^r_1}\right\}.$$

Lemma 4.10 (Definition of Θ^{left}_E and Θ^{right}_E). *For any two exceptional triples $\mathcal{E}$ and $\mathcal{E}'$ on $\mathcal{D}^b(\mathbb{P}^2)$ ending with the same $E_3 = E'_3 = E$, we have $\Theta^{\text{left}}_{\mathcal{E},E_3} = \Theta^{\text{left}}_{\mathcal{E}',E'_3}$. We denote* this subspace *by Θ^{left}_E. In a similar way, we define the subspace $\Theta^{\text{right}}_E = \Theta^{\text{right}}_{E_1} := \Theta^{\text{right}}_{\mathcal{E},E_1}$ for any exceptional triple $\mathcal{E}$ starting with $E_1 = E$. Moreover, we have*

$$\Theta^{\text{left}}_{E_3} = \Theta^-_{E_3} \bigsqcup \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \left\{\sigma_{s,q} \in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \mid (1,s,q) \in \mathrm{MZ}^l_{E_3}\right\}, \tag{4.7}$$
$$\Theta^{\text{right}}_{E_1} = \Theta^+_{E_1} \bigsqcup \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cdot \left\{\sigma_{s,q} \in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \mid (1,s,q) \in \mathrm{MZ}^r_{E_1}\right\}. \tag{4.8}$$

Proof. By Remark 4.7.4, Lemma 4.8 and [17, Proposition and Definition 3.1], we have the equation (4.7), where $\Theta^-_{E_3} = \Theta^-_{\mathcal{E},E_3}$ is independent of the choice of E_1 and E_2. Note that by Remark 4.7.3, the boundary segment $\overline{E_3E^l_3}$ of $\mathrm{MZ}^l_{E_3}$ is also independent of the choice of E_1 and E_2 in the exceptional triple. The subspace Θ^{left}_E is well-defined. Similarly, we have the equation (4.8). □

Remark 4.11. We illustrate the regions in Figure 3. Then we could state Remark 4.7.4 in a precise way, namely for an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$,

$$\mathrm{MZ}_{\mathcal{E}} = \mathrm{MZ}^r_{E_1} \bigsqcup \mathrm{MZ}^c_{\mathcal{E}} \bigsqcup \mathrm{MZ}^l_{E_3}. \tag{4.9}$$

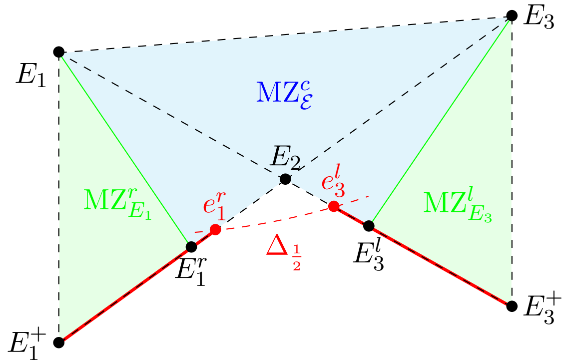


Figure 3: The regions of $\mathrm{MZ}_{\mathcal{E}}$, $\mathrm{MZ}^c_{\mathcal{E}}$, $\mathrm{MZ}^l_{E_3}$ and $\mathrm{MZ}^r_{E_1}$ with relation (4.9) in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. The line segments $\overline{E_1^+ e_1^r}$ and $\overline{E_3^+ e_3^l}$ give parts of the Le Potier curve $\mathrm{C_{LP}}$, and $\mathrm{MZ}_{\mathcal{E}} \subset \mathrm{Geo_{LP}}$.

5. Calculation of global dimension functions

The main result of this section is to compute the global dimension function on the algebraic stability conditions.

Proposition 5.1. *Let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional triple on $\mathcal{D}^b(\mathbb{P}^2)$ and $\Theta_{\mathcal{E}}$ be the algebraic stability conditions with respect to $\mathcal{E}$. The value of the global dimension function is*

$$\mathrm{gldim}(\sigma) = \begin{cases} 2, & \textit{when } \sigma \in \Theta_{\mathcal{E}} \setminus \left(\Theta_{E_1}^{\mathrm{right}} \cup \Theta_{E_3}^{\mathrm{left}} \cup \Theta_{\mathcal{E}}^{\mathrm{Pure}}\right); \\ \phi(\mathsf{R}_{E_1}(\mathbb{S}E_1)) - \phi_1, & \textit{when } \sigma \in \Theta_{E_1}^{\mathrm{right}}; \\ \phi_3 - \phi(\mathsf{L}_{E_3}(\mathbb{S}^{-1}E_3)), & \textit{when } \sigma \in \Theta_{E_3}^{\mathrm{left}}; \\ \phi_3 - \phi_1, & \textit{when } \sigma \in \Theta_{\mathcal{E}}^{\mathrm{Pure}}. \end{cases}$$

Recall that R and L are the right and left mutations in Section 2.1. The rest of the section is devoted to the proof of the proposition above.

5.1. The locus with minimum global dimension

The other three cases are much more subtle, we first discuss the case when $\sigma \in \Theta_{\mathcal{E}} \setminus \left(\Theta_{E_1}^{\mathrm{right}} \cup \Theta_{E_3}^{\mathrm{left}} \cup \Theta_{\mathcal{E}}^{\mathrm{Pure}}\right)$.

Proposition 5.2. *Let* σ *be a stability condition in* $\Theta_{\mathcal{E}} \setminus \left(\Theta^{\text{right}}_{E_1} \cup \Theta^{\text{left}}_{E_3} \cup \Theta^{\text{Pure}}_{\mathcal{E}}\right)$, *then* $\text{gldim}(\sigma) = 2$.

The non-trivial part is the '$\leq$' part. As for a brief idea of the proof, we will view σ both as a stability condition in the region $\text{MZ}^c_{\mathcal{E}}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$-plane, and as a quiver stability condition. We will show that we only need to concern about $\text{Hom}(F, G[2]) \neq 0$ for two σ stable objects F and G in a same heart with $\phi_\sigma(F) < \phi_\sigma(G)$. The line segments in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$-plane where F and G are σ stable (i.e. $W_{F\sigma}$ and $W_{G\sigma}$ below) are 'long' enough so that the line segment where $F(-3)$ is stable with respect to $\sigma(-3)$(i.e. $W_{F\sigma}(-3)$ below) intersects with previous two line segments $W_{F\sigma}$ and $W_{G\sigma}$. Then by the argument as that for stability conditions $\sigma_{s,q}$ above the parabola we show that $\text{Hom}(F, G[2]) = 0$ and get a contradiction. Details of the proof is given as follows.

Proof for Proposition 5.2. By Lemma 4.8, skyscraper sheaves are all stable with respect to σ. For any closed point $x \in \mathbb{P}^2$, since $\text{Hom}(\mathcal{O}_x, \mathcal{O}_x[2]) = \mathbb{C}$, we have $\text{gldim}(\sigma) \geq 2$.

By Lemma 4.8, up to a $\widetilde{\text{GL}^+(2, \mathbb{R})}$-action, we can view σ as a stability $\sigma_{s,q}$ condition in the region $\text{MZ}^c_{\mathcal{E}}$ in the $\{1, \frac{\text{ch}_1}{\text{ch}_0}, \frac{\text{ch}_2}{\text{ch}_0}\}$-plane. On the other hand, up to a suitable $\mathbb{C}$-action on σ, we may let the heart contain $E_1[2]$, $E_2[1]$ and E_3. Denote this stability condition and its heart by $\tilde{\sigma}$ and $\tilde{\mathcal{A}}$ respectively.

Step 1: We reduce the equation in the proposition to the statement that for all stable objects F and G in $\tilde{\mathcal{A}}$ with $\phi(F) < \phi(G)$, one must have $\text{Hom}(F, G[2]) = 0$.

As $\{E_1[2], E_2[1], E_3\}$ is an Ext-exceptional collection ([22, Definition 3.10]), an object in the heart is always of the form

$$E_1^{\oplus a_1} \to E_2^{\oplus a_2} \to E_3^{\oplus a_3}$$

for some non-negative integers a_i's.

For any generators $E_i[3-i]$ in $\tilde{\mathcal{A}}$, we always have

$$\text{Hom}(E_i[3-i], E_j[3-j][m]) = 0$$

for every $m \geq 3$. Therefore, for any objects F and G in $\tilde{\mathcal{A}}$, we have

$$\text{Hom}(F, G[m]) = 0$$

for every $m \geq 3$. To prove the '$\leq$' part, we only need to show that for any σ-stable F and G with $\phi(F) < \phi(G)$ in the heart $\tilde{\mathcal{A}}$, we have $\text{Hom}(F, G[2]) = 0$.

Step 2: We show that the phases of F and G are both in $[\phi(E_3(3)), \phi(E_1(-3)[2])]$.

Suppose there are σ-stable F and G with $\phi(F) < \phi(G)$ in the heart $\tilde{\mathcal{A}}$, such that $\mathrm{Hom}(F, G[2]) \neq 0$. Note that $\mathrm{Hom}(E_i[3-i], E_j[3-j][2]) \neq 0$ if and only if $i = 1$ and $j = 3$, we must have

$$\mathrm{Hom}(F, E_3[2]) \neq 0 \text{ and } \mathrm{Hom}(E_1[2], G[2]) \neq 0.$$

By Serre duality, we have

$$\mathrm{Hom}(E_3(3), F) \neq 0 \text{ and } \mathrm{Hom}(G, E_1(-3)[2]) \neq 0.$$

By [17, Corollary 1.19], both objects $E_3(3)$ and $E_1(-3)[2]$ are $\sigma_{s,q}$-stable (hence $\tilde{\sigma}$-stable). Both objects are in the heart $\tilde{\mathcal{A}}$. Therefore, their phases satisfy the inequality:

$$\phi(E_3(3)) \leq \phi(F) < \phi(G) \leq \phi(E_1(-3)[2]). \tag{5.1}$$

Step 3: We show that the walls $W_{F\sigma}$ and $W_{G\sigma}$ are 'long' enough so that the wall $W_{F\sigma}(-3)$ intersects the walls $W_{F\sigma}$ and $W_{G\sigma}$. We compare their slopes and get the contradiction.

Here the *wall* $W_{F\sigma} := \{(1, s, q) \in \{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}\text{-plane}|$ the line segment along the line $L_{F\sigma}$ that is above the Le Potier curve $\mathrm{C_{LP}}\}$ and the wall

$$W_{F\sigma}(-3) := \{(1, s-3, q-3s+\frac{9}{2}) \mid (1, s, q) \in W_{F\sigma}\}.$$

By Bertram's nested wall theorem, [18, Corollary 1.24], the object F is stable along the wall $W_{F\sigma}$. Let $F_a = (1, s(F_a), q(F_a))$ and $F_b = (1, s(F_b), q(F_b))$ be the two edges of the wall $W_{F\sigma}$ as that in the Figure 4. We denote similar notations for G as that for F.

By the relation of phases as (5.1), counter-clockwisely, one has the line segment $\overline{\sigma_{s,q}(E_3(3))}$, $\overline{\sigma_{s,q}F_b}$, $\overline{\sigma_{s,q}G_b}$ and $\overline{\sigma_{s,q}(E_1(-3))}$. In particular, either the wall $W_{F\sigma}$ is a vertical wall (parallel to the $\frac{\mathrm{ch}_2}{\mathrm{ch}_0}$-axis) or $|s(F_b) - s| > 3$. Same statement holds for $W_{G\sigma}$. In every case, the segment

$$\overline{\sigma_{s,q}(-3)F_b(-3)} = \overline{(1, s-3, q-3s+\frac{9}{2})(1, s(F_b)-3, q(F_b)-3s(F_b)+\frac{9}{2})}$$

intersects both segments $\overline{\sigma_{s,q}F_b}$ and $\overline{\sigma_{s,q}G_b}$ at P and Q respectively. The object $F(-3)$ is stable at both P and Q. By comparing the slopes, we have

$$\phi_Q(F(-3)) = \phi_P(F(-3)) < \phi_P(F) = \phi_{s,q}(F) < \phi_{s,q}(G) = \phi_Q(G).$$

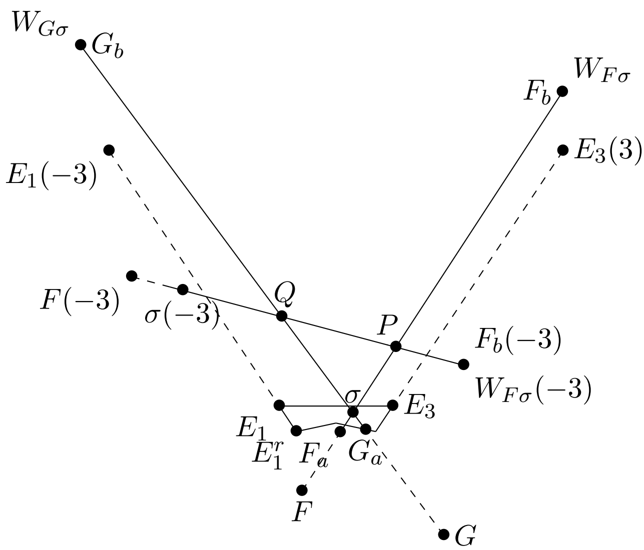


Figure 4: Compare the slopes of the wall $W_{G\sigma}$ and the wall $W_{F\sigma}(-3)$.

By Serre duality,

$$\mathrm{Hom}(F, G[2]) \cong (\mathrm{Hom}(G, F(-3)))^* = 0.$$

We get the contradiction. □

5.2. The global dimension on the leg locus

We discuss the case that $\sigma \in \Theta_{E_3}^{\mathrm{left}}$. We first recall the following basic properties for an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$. Denote by $\mathrm{rk}(E) = \mathrm{ch}_0(E)$ and $\hom(E, F) = \dim \mathrm{Hom}(E, F)$.

Lemma 5.3. *For an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$, the ranks and homs of these exceptional objects satisfy the following equations.*

$$(\mathrm{rk}E_1)^2 + (\mathrm{rk}E_2)^2 + (\mathrm{rk}E_3)^2 = 3\mathrm{rk}E_1\mathrm{rk}E_2\mathrm{rk}E_3 \quad \textit{(Markov equation)},$$
$$\hom(E_1, E_2) = 3\mathrm{rk}E_3, \quad \hom(E_2, E_3) = 3\mathrm{rk}E_1,$$
$$\hom(E_1, E_3) = 9\mathrm{rk}E_1\mathrm{rk}E_3 - 3\mathrm{rk}E_2.$$

The object $\mathsf{L}_{E_3}(E_3(3))[-1]$ admits a resolution:

$$0 \to E_1^{\oplus \hom(E_1, E_3)} \to E_2^{\oplus r} \to \mathsf{L}_{E_3}(E_3(3))[-1] \to 0, \tag{5.2}$$

where $r = \hom(E_1, E_3)\hom(E_1, E_2) - \hom(E_2, E_3)$.

Proof. The equations of rank and hom are well-known in [12]. As for the last statement, we consider the resolution of $E_3(3)$. Note that $\mathcal{D}_\infty(\mathbb{P}^2)$ has the semiorthorgonal decomposition $\langle E_1, E_2, E_3\rangle$, so an object A admits a unique filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 = A$$

such that $\mathrm{Cone}(F_i \to F_{i+1}) \in \langle E_{3-i}\rangle$ for $i = 0, 1, 2$. The term $\mathrm{Cone}(F_0 \to F_1)$ is given by $\bigoplus_i E_3[i] \otimes \mathrm{Hom}(E_3[i], A)$, while the term $\mathrm{Cone}(F_2 \to F_3)$ is given by $\bigoplus_i E_1[i] \otimes \mathrm{Hom}(A, E_1[i])^*$.

When $A = E_3(3)$, we have $\mathrm{Cone}(F_0 \to F_1) = E_3 \otimes \mathrm{Hom}(E_3, E_3(3)) = E_3^{\oplus 9(\mathrm{rk}E_3)^2+1}$ and $\mathrm{Cone}(F_2 \to F_3) = E_1^{\oplus \hom(E_1,E_3)}[2]$. The factor $\mathrm{Cone}(F_1 \to F_2)$ can only be $E_2^{\oplus r}[1]$. By the equations of rank and hom in the lemma, the rank

$$\begin{aligned} r &= \frac{9(\mathrm{rk}E_3)^3 + 9\mathrm{rk}E_3(\mathrm{rk}E_1)^2}{\mathrm{rk}E_2} - 3\mathrm{rk}E_1 \\ &= \hom(E_1, E_3)\hom(E_1, E_2) - \hom(E_2, E_3). \end{aligned}$$

Note that $\mathsf{L}_{E_3}(E_3(3))[-1]$ is the kernel of the map $E_3 \otimes \mathrm{Hom}(E_3, E_3(3)) \xrightarrow{\mathrm{ev}} E_3(3)$, the resolution sequence is clear. □

Lemma 5.4. *Let σ be a stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$, suppose an object $F = \mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})$ is stable with respect to σ, then F is stable everywhere in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$.*

Proof. For any stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$, by a suitable $\mathbb{C}$-action, we may assume that the heart contains $E_1[2]$, $E_2[1]$ and $E_3[n]$ for some $n \leq 0$. As $\{E_1[2], E_2[1], E_3[n]\}$ is an Ext-exceptional collection, an object in the heart is always of the form

$$E_1^{\oplus a_1} \to E_2^{\oplus a_2} \to E_3^{\oplus a_3}.$$

The object $F[1]$ can only be destabilized by some subobjects $F' = \mathrm{Cone}(E_1^{\oplus a'} \to E_2^{\oplus b'})[1]$ in the heart generated by $\{E_1[2], E_2[1], E_3[n]\}$ with larger phase, which means $\frac{a'}{b'} > \frac{a}{b}$. Note that this is independent of the choice of σ in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$, the object F is stable everywhere in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$. □

Now we are ready to compute the example achieving the value of the global dimension function in the region of $\Theta_{E_3}^{\mathrm{left}}$.

Lemma 5.5. *Let σ be a stability condition in $\Theta^{\mathrm{left}}_{E_3}$, then $\mathsf{L}_{E_3}E_3(3)$ is σ-stable and it has a non-zero morphism to $E_3[2]$. In particular, we have* $\mathrm{gldim}(\sigma) \geq \phi_3 - \phi(\mathsf{L}_{E_3}E_3(3)) + 2$.

Proof. By [18, Corollary 3.2], the object

$$\mathsf{L}_{E_3}E_3(3) = \mathrm{Cone}(E_3 \otimes \mathrm{Hom}(E_3, E_3(3)) \xrightarrow{\mathrm{ev}} E_3(3))$$

is $\sigma_{s,q}$-stable for $(1, s, q)$ which is slightly above the line segment $\overline{E_3E_3(3)}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. The object $\mathsf{L}_{E_3}E_3(3)$ is stable along the line segment $\overline{(1, s, q)E^l_3}$. As this segment intersects $\mathrm{MZ}^c_{\mathcal{E}}$, by Lemma 4.8, the object $\mathsf{L}_{E_3}E_3(3)$ is stable with respect to some stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$. By Lemma 5.3 and 5.4, the object $\mathsf{L}_{E_3}E_3(3)$ is σ-stable for every $\sigma \in \Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$.

By applying $\mathrm{Hom}(-, E_3[2])$ on the distinguished triangle

$$E_3 \otimes \mathrm{Hom}(E_3, E_3(3)) \xrightarrow{\mathrm{ev}} E_3(3) \to \mathsf{L}_{E_3}E_3(3) \xrightarrow{+},$$

we have $\mathrm{Hom}(\mathsf{L}_{E_3}E_3(3), E_3[2]) \cong \mathrm{Hom}(E_3(3), E_3[2]) = \mathbb{C}$. □

As for the '$\leq$' direction, we first treat with the easier case that the stable objects can be classified.

Proposition 5.6. *Let σ be a stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1 < \phi_3 - 1)$, then up to a homological shift, a σ-stable object is either*

- *E_3 or*
- $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})$

induced by a stable quiver representation $\mathbb{C}^{\oplus a} \xRightarrow{\mathrm{hom}(E_1,E_2)\ arrows} \mathbb{C}^{\oplus b}$. Moreover, we have $\mathrm{gldim}(\sigma) = \phi_3 - \phi(\mathsf{L}_{E_3}E_3(3)) + 2$.

Proof. By a suitable $\mathbb{C}$-action, we may assume that the heart contains $E_1[2]$, $E_2[1]$ and $E_3[n]$ for some $n \leq -1$. As $\{E_1[2], E_2[1], E_3[n]\}$ is an Ext-exceptional collection, an object in the heart is $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})[1] \bigoplus E_3^{\oplus c}[n]$. An object $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})[1]$ is σ-stable if and only if for any non-zero proper subobject $\mathrm{Cone}(E_1^{\oplus a_1} \to E_2^{\oplus b_1})[1]$ we have $\frac{a_1}{b_1} < \frac{a}{b}$. The first part of the statement is clear.

As for the second part of the statement, by Lemma 5.5, we only need to show the '$\leq$' side, note that for any two stable objects F and F' in the form of $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})$, we always have $\mathrm{Hom}(F, F'[m]) = 0$ for $m \geq 2$.

By the classification of stable objects, we only need to consider potential non-zero morphisms from $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})$ to $E_3[m]$ for $m \geq 1$. When $\phi(\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})) < \phi(\mathsf{L}_{E_3}E_3(3)[-1])$, by Lemma 5.3, we have

$$\frac{b}{a} > \hom(E_1, E_2) - \frac{\hom(E_2, E_3)}{\hom(E_1, E_3)}. \tag{5.3}$$

Let ϕ_3' be $\phi(\mathsf{L}_{E_3}E_3(3))$, which is greater than $\phi_1 + 1$. We consider the stability condition σ' in $\Theta_{\mathcal{E}}$ given by $(m_1, m_2, m_3, \phi_1, \phi_2, \phi_3')$. By Lemma 5.4, $\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})$ is σ'-stable and $E(3)$ is σ'-semistable with phase $\phi_3' = \phi'(E_3) = \phi'(\mathsf{L}_{E_3}E_3(3))$. By (5.3), we have $\phi'(\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b})) < \phi'(\mathsf{L}_{E_3}E_3(3))) - 1 = \phi'(E(3)) - 1$. Therefore, for any $m \geq 1$, by Serre duality, we have

$$\begin{aligned} &\mathrm{Hom}(\mathrm{Cone}(E_1^{\oplus a} \to E_2^{\oplus b}), E_3[m]) \\ &\quad \cong (\mathrm{Hom}(\mathrm{Cone}(E_3(3)[m-2], E_1^{\oplus a} \to E_2^{\oplus b})))^* = 0. \end{aligned}$$

As a summary, the global dimension at σ is $\phi_3 - \phi(\mathsf{L}_{E_3}E_3(3)) + 2$, and is achieved via the morphism between $\mathsf{L}_{E_3}E_3(3)$ and $E_3[2]$. □

We finally treat with region $\Theta_{\mathcal{E}}(\phi_2 < \phi_3 - 1 < \phi_1 + 1) \cap \Theta_{E_3}^{\mathrm{left}}$, where the stable objects are more complicated. In this case, the potential stable characters are away from the kernel of central charge of every σ in $\Theta_{\mathcal{E}}(\phi_2 < \phi_3 - 1 < \phi_1 + 1)$. We will think both the stable characters and (kernels of central charges of) stability conditions in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. This will allow us to show the vanishing of certain morphisms by comparing slopes.

We first prove a nested wall result for the algebraic stability conditions. Denote $\Theta_{\mathcal{E}}^{+}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2) := \{\sigma \in \Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2) |$ the kernel of central charge of σ is spanned by $(1, s, q)$ for some $s > s(E_1)\}$.

Lemma 5.7. *Let σ be a stability condition in $\Theta_{\mathcal{E}}^{+}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2)$ and G be a σ-stable object. Then G is σ'-stable for every σ' in $\Theta_{\mathcal{E}}^{+}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2)$ with kernel of central charge on the line through G and σ.*

Proof. In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, the kernel of the central charge of σ is in the region bounded by rays through E_1E_3, E_1E_2 as shown in the Figure 5 (Area I).

By a suitable $\mathbb{C}$-action, we may assume that the heart contains $\{E_1[2], E_2[1], E_3\}$. Denote this heart by $\tilde{\mathcal{A}}$, then an object in $\tilde{\mathcal{A}}$ is of the

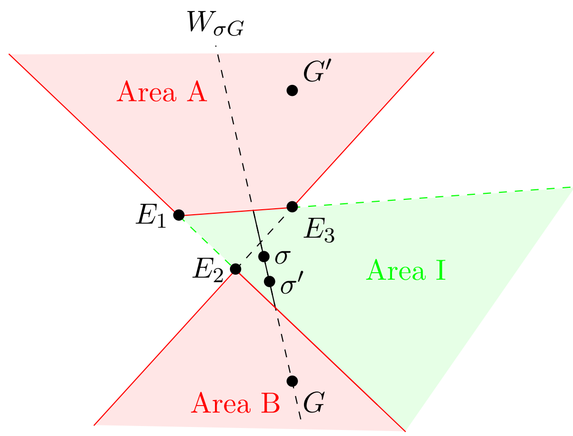


Figure 5: Stability conditions through $W_{\sigma G}$.

form $E_1^{\oplus a_1} \to E_2^{\oplus a_2} \to E_3^{\oplus a_3}$. In particular, the reduced character of a stable object is in the closed region (Area A $\cup$ Area B in Figure 5) bounded by the rays through E_1E_2, E_2E_3 and line segment through E_1E_3.

The phase of G is determined by the slope of line through σ and $\mathrm{v}(G)$. As for another object G', its phase $\phi(G') < \phi(G)$ if and only if the line through σ and $\mathrm{v}(G')$ rotates counter-clockwisely to the line through σ and $\mathrm{v}(G)$ without passing though the line through σ and $E_1[2]$.

For every non-zero proper subobject G' of G in $\tilde{\mathcal{A}}$, since G is stable, G' has smaller phase than that of G. In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, that is equivalent to the following description for $\mathrm{v}(G')$:

The reduced character of G' is either to the right of the line through G and σ when it is in Area A, or it is to the left of the line through G and σ when it is in Area B.

Note that for every stability condition σ' in $\Theta^+_{\mathcal{E}}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2)$ with kernel of central charge on the line through G and σ, the line through G' and σ' rotates counter-clockewisely to the line through σ, σ' and G. The object G is σ'-stable. □

Proposition 5.8. *Let σ be a stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_3 - 1 < \phi_1 + 1) \cap \Theta^{\mathrm{left}}_{E_3}$. Then* $\mathrm{gldim}(\sigma) = \phi(E_3) - \phi(\mathsf{L}_{E_3}E_3(3)) + 2$.

Proof. By Lemma 5.5, we only need to show the '$\leq$' part.

By a suitable $\mathbb{C}$-action, we may assume that the heart contains $\{E_1[2], E_2[1], E_3\}$. Denote this heart by $\tilde{\mathcal{A}}$, we have the same description for objects in $\tilde{\mathcal{A}}$ as that in Lemma 5.7.

Step 1: We reduce the claim in the proposition to the following statement: for all stable objects F and G in $\tilde{\mathcal{A}}$ with $\mathrm{Hom}(F, G[2]) \neq 0$, the difference of their phases $\phi(G) - \phi(F) \leq \phi(E_3) - \phi(\mathsf{L}_{E_3}E_3(3))$.

In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, the kernel of the central charge of σ is in the region bounded by rays through E_1E_3, E_1E_2 and line segment $\overline{E_3E_3^l}$ as shown in the Figure 6 (Area I $\cup$ Area II). Recall that the point E_3^l and $\mathsf{L}_{E_3}(E_3(3))$ are the same point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane.

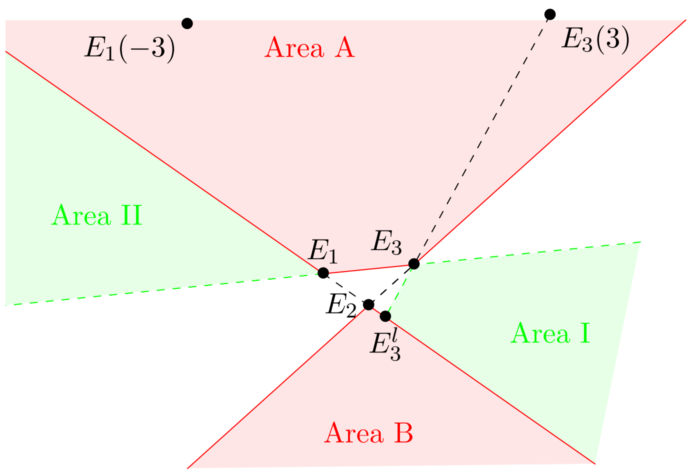


Figure 6: Stable characters are in Area A $\cup$ Area B. The kernels of the central charges are in Area I $\cup$ Area II.

As for any generators in $\{E_1[2], E_2[1], E_3\}$, we have $\mathrm{Hom}(-, -[m]) = 0$ for any $m \geq 3$. For any objects F and G in the heart, we have $\mathrm{Hom}(F, G[m]) = 0$ for any $m \geq 3$. To prove the '$\leq$' part of the statement, we only need consider $\mathrm{Hom}(F, G[2]) \neq 0$ for stable objects F, G in the heart with $\phi(F) < \phi(G)$.

Suppose there are σ-stable objects F and G with

$$\phi(G) - \phi(F) > \phi(E_3) - \phi(\mathsf{L}_{E_3}E_3(3)) \tag{5.4}$$

in the heart $\tilde{\mathcal{A}}$, such that $\mathrm{Hom}(F, G[2]) \neq 0$. By the same argument as that in Proposition 5.2, we must have

$$\mathrm{Hom}(E_3(3), F) \neq 0 \text{ and } \mathrm{Hom}(G, E_1(-3)[2]) \neq 0. \tag{5.5}$$

Step 2: We show that $\phi(G) > \phi(E_3)$.

Suppose $\phi(F) < \phi(\mathsf{L}_{E_3}E_3(3))$, then $\phi(F) < \phi(E_3)$. Therefore, the object F is of the form $\mathrm{Cone}(E_1^{\oplus a_F} \to E_2^{\oplus b_F})[1]$. By Lemma 5.4, F is stable everywhere in $\Theta(\phi_1 + 1 > \phi_2)$. In particular, it is stable with every stability condition σ' on the line segment $\overline{E_3E_3^l}$, where $E_3(3)$ is σ'-semistable. Since $\mathrm{Hom}(E_3(3), F) \neq 0$, we have $\phi'(F) \geq \phi'(E_3(3)) = \phi'(\mathsf{L}_{E_3}E_3(3))$. Therefore, we have

$$\frac{b_F}{a_F} \leq \hom(E_1, E_2) - \frac{\hom(E_2, E_3)}{\hom(E_1, E_3)}, \quad \phi'(F) \geq \phi'(\mathsf{L}_{E_3}E_3(3)),$$

which contradicts the assumption that $\phi(F) < \phi(\mathsf{L}_{E_3}E_3(3))$.

By (5.4), we must have

$$\phi(G) > \phi(E_3). \tag{5.6}$$

Step 3: We show that the kernel of the central charge of σ is in Area I and is below the line through $\overline{E_1(-3)[2]}$ and E_3, i.e. the open region bounded by line segments $\overline{RE_3}$, $\overline{E_3E_3^l}$ and $\overline{E_3^lR}$, with $R \coloneqq L_{E_1(-3)E_3} \cap L_{E_1E_2}$ as in Figure 7.

Let the central charge of $E_i[3-i]$ be z_i for $i = 1, 2, 3$. Let the object G be of the form $E_1^{\oplus n_1} \to E_2^{\oplus n_2} \to E_3^{\oplus n_3}$.

By Lemma 5.4 and a same argument as that in Lemma 5.5, we know that the object $\mathsf{L}_{E_3}(E_1(-3)[2])$ is of the form $\mathrm{Cone}(E_1^{\oplus r_1} \to E_2^{\oplus r_2})$ in the heart $\tilde{\mathcal{A}}$, and it is stable with respect to every stability condition in $\Theta_{\mathcal{E}}(\phi_2 < \phi_1 + 1)$. By (5.6), we have $\mathrm{Hom}(G, E_3) = 0$. By (5.5), we have $\mathrm{Hom}(G, \mathsf{L}_{E_3}(E_1(-3)[2])) \neq 0$. Therefore, we have

$$\phi(\mathsf{L}_{E_3}(E_1(-3)[2])) > \phi(G) > \phi(E_3). \tag{5.7}$$

Therefore, the kernel of the central charge of σ is in Area I and is below the line $L_{E_1(-3)E_3}$ as in Figure 7.

Step 4: We show that the wall $W_{G\sigma}$ intersects the wall $W_{G\sigma}(-3)$. We denote the intersection point by $P \coloneqq W_{G\sigma} \cap W_{G\sigma}(-3)$.

Consider the line $L_{G\sigma}$ through the reduced character of G and the kernel of the central charge of σ, which is in Area I. In particular, the stability condition

$$\sigma \in \Theta_{\mathcal{E}}^{+}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2).$$

By (5.7), the line $L_{G\sigma}$ intersects the line segment $\overline{E_1(-3)E_3}$. Therefore, the line $L_{G\sigma}$ intersects the region $\mathrm{MZ}_{\mathcal{E}}^{c}$.

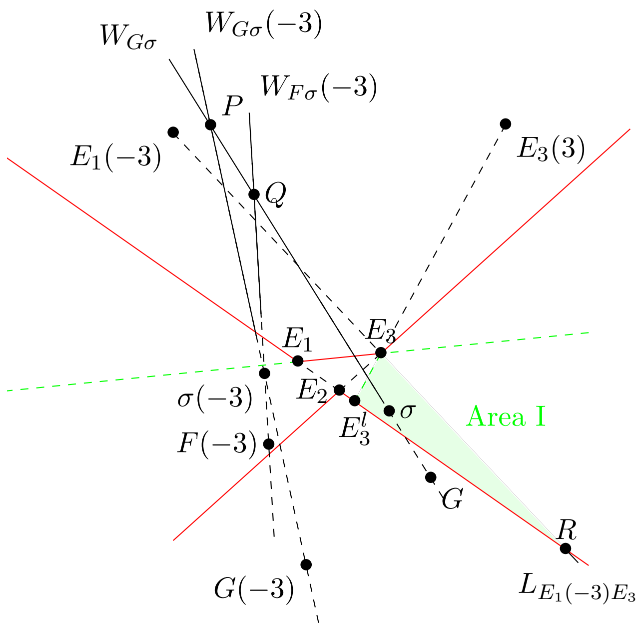


Figure 7: Comparing the phases of G and $F(-3)$.

By Lemma 5.7, the object G is stable with respect to every stability condition in σ in $\Theta^+_{\mathcal{E}}(\phi_2 < \phi_1 + 1, \phi_3 < \phi_1 + 2)$ with kernel on $L_{G\sigma}$. Note that there exists a point $(1, s_0, q_0)$ in $\mathrm{MZ}^c_{\mathcal{E}} \cap L_{G\sigma}$, by Lemma 4.8, the object G is σ_{s_0,q_0}-stable.

Recall the wall $W_{G\sigma} \coloneqq \{(1, s, q) \in \{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}\text{-plane}|$ the line segment along the line $L_{G\sigma}$ that is above the Le Potier curve $\mathrm{C_{LP}}\}$. By the Bertram's nest wall theorem [18, Corollary 1.24], the object G is $\sigma_{s,q}$-stable for every $(1, s, q)$ on $W_{G\sigma}$. Note that $W_{G\sigma}$ intersects the line segment $\overline{E_1E_3}$, but does not intersects the line segment $\overline{E_1(-3)E_1}$ or $\overline{E_3E_3(3)}$, both of which are above the Le Potier curve $\mathrm{C_{LP}}$. Therefore the horizontal length of $W_{G\sigma}$ is greater than 3 when $W_{G\sigma}$ is not the vertical wall. Let

$$W_{G\sigma}(-3) \coloneqq \{(1, s-3, q-3s+\frac{9}{2}) \mid (1, s, q) \in W_{G\sigma}\},$$

then $G(-3)$ is $\sigma_{s,q}$-stable for every $(1, s, q)$ on $W_{G\sigma}(-3)$. The wall $W_{G\sigma}(-3)$ intersects the wall $W_{G\sigma}$ at some point P and

$$\phi_P(G(-3)) < \phi_P(G). \tag{5.8}$$

As for the only exceptional case that $W_{G\sigma}$ is the vertical wall, we can view that the point P is at $(0,0,1)$. This will not affect the statement in the next step.

Step 5: When $s(F) > s(E_3)$, we show that the wall $W_{F\sigma}(-3)$ intersects the wall $W_{G\sigma}$. We denote the intersection point by $Q \coloneqq W_{G\sigma} \cap W_{F\sigma}(-3)$.

By (5.7), we have the same bounds for F

$$\phi(\mathsf{L}_{E_3}(E_1(-3)[2])) > \phi(G) > \phi(F) > \phi(E_3). \tag{5.9}$$

The horizontal length of $W_{F\sigma}(-3)$ is greater than 3 when it is not vertical. Note that the slope of $W_{F\sigma}(-3)$ is less than that of $W_{G\sigma}(-3)$, the segment $W_{F\sigma}(-3)$ intersects $W_{G\sigma}$ at Q on the line segment $\overline{P\sigma}$. The fact that $\phi_\sigma(G) > \phi_\sigma(F)$ implies $\phi_{\sigma(-3)}(G(-3)) > \phi_{\sigma(-3)}(F(-3))$. Both $F(-3)$ and G are σ_Q stable. We then compare their phases at Q by using (5.8) as follows:

$$\begin{aligned}\phi_Q(G) = \phi_P(G) > \phi_P(G(-3)) &= \phi_{\sigma(-3)}(G(-3)) \\ &> \phi_{\sigma(-3)}(F(-3)) = \phi_Q(F(-3)).\end{aligned}$$

So $\mathrm{Hom}(G, F(-3)) = 0$. By Serre duality, we have $\mathrm{Hom}(F, G[2]) = 0$.

Step 6: When $s(F) \leq s(E_3)$, we reduce this case to Proposition 5.2 .

Note that F is of the form $E_1^{\oplus a_1} \to E_2^{\oplus a_2} \to E_3^{\oplus a_3}$, we have $\mathrm{Hom}(E_3, F) \neq 0$ when $a_3 \neq 0$. The object F is either of the form $\mathrm{Cone}(E_1^{\oplus a_1} \to E_2^{\oplus a_2})[1]$ or E_3. Let $(1, s_0, q_0)$ be a point in $\mathrm{MZ}^c_{\mathcal{E}} \cap L_{G\sigma}$. By Lemma 5.4, in any case, F is σ_{s_0,q_0}-stable. By Lemma 5.7 and Lemma 4.8, the object G is also σ_{s_0,q_0}-stable and has phase

$$\phi_{s_0,q_0}(G) > \phi_{s_0,q_0}(F).$$

By Proposition 5.2, we have $\mathrm{Hom}(F, G[2]) = 0$.

As a summary, we have shown that $\mathrm{Hom}(F, G[2]) = 0$ when $\phi(F) < \phi(\mathsf{L}_{E_3} E_3(3))$ or $\phi(G) > \phi(E_3)$. In particular, we have $\mathrm{gldim}(\sigma) = \phi(E_3) - \phi(\mathsf{L}_{E_3} E_3(3)) + 2$. □

Proof for Proposition 5.1. When $\sigma \in \Theta_{\mathcal{E}} \setminus \left(\Theta^{\mathrm{right}}_{E_1} \cup \Theta^{\mathrm{left}}_{E_3} \cup \Theta^{\mathrm{Pure}}_{\mathcal{E}}\right)$, the global dimension is computed in Proposition 5.2.

When $\sigma \in \Theta^{\mathrm{left}}_{E_3}$, the global dimension is computed in Propositions 5.6 and 5.8.

When $\sigma \in \Theta^{\mathrm{right}}_{E_1}$, we take the derived dual stability condition $\sigma^\vee \in \Theta^{\mathrm{left}}_{\mathcal{E}^\vee, E_1^\vee}$, where $\mathcal{E}^\vee$ is the dual exceptional triple $\{E_3^\vee, E_2^\vee, E_1^\vee\}$. We reduce to the previous case and have

$$\begin{aligned}\operatorname{gldim}(\sigma) &= \operatorname{gldim}(\sigma^\vee) = \phi^\vee(E_1^\vee) - \phi^\vee(\mathsf{L}_{E_1^\vee}(\mathbb{S}^{-1}E_1^\vee)) \\ &= -\phi(E_1) - \phi^\vee\left((\mathsf{R}_{E_1}(\mathbb{S}E_1))^\vee\right) = \phi(\mathsf{R}_{E_1}(\mathbb{S}E_1)) - \phi_1.\end{aligned}$$

When $\sigma \in \Theta_{\mathcal{E}}^{\mathrm{Pure}}$, by Lemma 4.4, the only stable objects are $E_i[m]$ for $E_i \in \mathcal{E}$ and $m \in \mathbb{Z}$. As $\mathcal{E}$ is a strong exceptional collection, we have $\operatorname{Hom}(E_i, E_j[m]) \neq 0$ if and only $j \geq i$ and $m = 0$. So the result is clear. □

Remark 5.9. Following the notations in Remark 4.11, for any exceptional bundle E, we associate two regions MZ_E^l and MZ_E^r, which consist of geometric stability conditions. Moreover, if $\sigma \in \mathrm{MZ}_E^l \setminus \overline{E^l E}$ or $\sigma \in \mathrm{MZ}_E^r \setminus \overline{EE^r}$, we have $2 < \operatorname{gldim}(\sigma) < 3$.

Corollary 5.10. *The global dimension function*

$$\operatorname{gldim}\colon \operatorname{Stab}^\dagger(\mathbb{P}^2) \to \mathbb{R}_{\geq 0}$$

has minimum value 2 *and* $\underline{\operatorname{gldim}\operatorname{Stab}^\dagger \mathbb{P}^2} = [2, \infty)$*. Moreover, the subspace* $\operatorname{gldim}^{-1}(2)$ *is contained in* $\overline{\operatorname{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)}$*, and is contractible.*

Proof. The image of gldim follows from Proposition 3.4, Proposition 5.1 and the description of $\operatorname{Stab}^\dagger \mathbb{P}^2$ (2.6). The contractibility of $\operatorname{gldim}^{-1}(2)$ is clear. □

6. Contractibility via global dimension

We denote by $\operatorname{gldim}^{-1}(I)$ by the space of all stability conditions in the component $\operatorname{Stab}^\dagger(\mathbb{P}^2)$ with global dimension in I for an interval $I \subset \mathbb{R}$. Based on Proposition 5.1 and the cell-decomposition description for $\operatorname{Stab}^\dagger(\mathbb{P}^2)$, our main result shows that the connected component $\operatorname{Stab}^\dagger(\mathbb{P}^2)$ is contractible via the global dimension:

Theorem 6.1. *For any $x > 2$, the space* $\operatorname{gldim}^{-1}([2, x))$ *contracts to* $\operatorname{gldim}^{-1}(2)$*.*

Proof. By Proposition 5.1, Remark 4.7 and [17, Corollary 3.5, Theorem 3.9], the space of preimage $\operatorname{gldim}^{-1}([2, x))$ has a cell decomposition as

$$\operatorname{gldim}^{-1}(2) \bigcup \left(\bigsqcup_E \left(\Theta_E^{\mathrm{left}}(x) \bigsqcup \Theta_E^{\mathrm{right}}(x) \right) \bigsqcup \left(\bigsqcup_{\mathcal{E}} \Theta_{\mathcal{E}}^{\mathrm{Pure}}(x) \right) \right),$$

where E runs all exceptional bundles, and $\mathcal{E}$ runs all exceptional triples, and the notation $\Theta^{\dagger}_{*}(x)$ stands for $\Theta^{\dagger}_{*} \cap \mathrm{gldim}^{-1}([2,x))$.

By Proposition 5.1, we have $\Theta^{\mathrm{Pure}}_{\mathcal{E}}(x) = \Theta^{\mathrm{Pure}}_{\mathcal{E}}(\phi_3 - \phi_1 < x)$. Each $\Theta^{\mathrm{Pure}}_{\mathcal{E}}(x)$ has an open neighborhood, say, $\Theta_{\mathcal{E}}(\phi_3 - \phi_2 > \frac{1}{2}, \phi_2 - \phi_1 > \frac{1}{2}, \phi_3 - \phi_1 < x)$, in $\Theta_{\mathcal{E}}(x)$ which does not intersect any other $\Theta^{\mathrm{Pure}}_{\mathcal{E}'}(x)$. As $\mathrm{Stab}^{\dagger}(\mathbb{P}^2)$ admits a metric, we may then choose open neighborhoods of $\Theta^{\mathrm{Pure}}_{\mathcal{E}}(x)$'s which do not intersect with each other. By the cell decomposition, the space $\mathrm{gldim}^{-1}([2,x))$ contracts to its subspace

$$A(x) := \mathrm{gldim}^{-1}(2) \bigcup \left(\bigsqcup_{E \text{ exceptional bundles}} \left(\Theta^{\mathrm{left}}_{E}(x) \bigsqcup \Theta^{\mathrm{right}}_{E}(x) \right) \right).$$

For each exceptional object E, let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional collection such that $E_3 = E$. By Proposition 5.1 and Lemma 5.3, we have

$$\Theta^{\mathrm{left}}_{E}(x) \cong \Bigg\{ (m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}^3 | \phi_1 < \phi_2 < \phi_1 + 1,$$
$$\phi_3 > \phi_2 + 1 + \frac{1}{\pi} \arctan\left(\frac{\sin((\phi_1 + 1 - \phi_2)\pi)}{\cos((\phi_1 + 1 - \phi_2)\pi) + \frac{m_2}{m_1} h} \right) > \phi_3 - x + 2 \Bigg\},$$

where $h = \mathrm{hom}(E_1, E_2) - \frac{\mathrm{hom}(E_2,E_3)}{\mathrm{hom}(E_1,E_3)}$. Therefore, the space $\Theta^{\mathrm{left}}_{E}(x)$ contracts to $\Theta^{\mathrm{left}}_{E}(x) \cap \mathrm{gldim}^{-1}(2)$.

By Remark 4.7 and [17, Lemma 3.7], each $\Theta^{\mathrm{left}}_{E}(x)$ has an open neighborhood in $A(x)$, which does not intersect any other $\Theta^{\mathrm{left}}_{E'}(x)$ or $\Theta^{\mathrm{right}}_{E'}(x)$. Same argument works for all $\Theta^{\mathrm{right}}_{E}(x)$, we may therefore contract all $\Theta^{\mathrm{left}}_{E}(x)$ and $\Theta^{\mathrm{right}}_{E}(x)$ in $A(x)$ simultaneously to $\mathrm{gldim}^{-1}(2)$, which is a contractible space. □

7. Inducing stability conditions from projective plane to the local projective plane

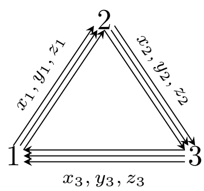
Let Y be the total space of the canonical bundle of $\mathbb{P}^2$, and $i : \mathbb{P}^2 \hookrightarrow Y$ be the inclusion of the zero-section. We write $\mathcal{D}^b_{\mathbb{P}^2}(Y)$ for the subcategory of $\mathcal{D}^b(Y)$ of complexes with bounded cohomology, such that all of its cohomology sheaves are supported on the zero-section. The space of Bridgeland stability conditions on $\mathcal{D}^b_{\mathbb{P}^2}(Y)$ has been studied by Bayer and Macrì [3]. In this section, we prove that the stability conditions in $\mathrm{gldim}^{-1}(2) \subset \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$

can be used to induce stability conditions on $\mathcal{D}^b_{\mathbb{P}^2}(Y)$ by Ikeda-Qiu's inducing theorem, via q-stability conditions on Calabi–Yau-$\mathbb{X}$ categories.

Following the notion in [14], we have the Calabi–Yau-$\mathbb{X}$ version of $\mathcal{D}_\infty(\mathbb{P}^2)$

$$\mathcal{D}_{\mathbb{X}}(\mathbb{P}^2) \coloneqq \mathcal{D}^b_{c,\mathbb{C}^*}(Y). \tag{7.1}$$

By [14, Proposition 3.14], we have $\mathcal{D}_{\mathbb{X}}(\mathbb{P}^2) \cong \mathcal{D}_{\mathrm{fd}}(\Gamma_{\mathbb{X}}(\widetilde{Q}_{\mathrm{gr}}, W_{\mathrm{gr}}))$ with $\mathbb{Z} \oplus \mathbb{Z}[\mathbb{X}]$ graded quiver $\widetilde{Q}_{\mathrm{gr}}$ as follows and potential $W_{\mathrm{gr}} = \sum_{i=1}^3 (x_i y_i z_i - x_i z_i y_i)$,



where $\deg x_3, y_3, z_3 = 3 - \mathbb{X}$ and gradings of other arrows are zero. Here $\Gamma_{\mathbb{X}}(\widetilde{Q}_{\mathrm{gr}}, W_{\mathrm{gr}})$ is the Calabi–Yau-$\mathbb{X}$ Ginzburg dg algebra [14, 15]. Note that there is a canonical fully faithful embedding

$$\mathcal{D}_\infty(\mathbb{P}^2) \to \mathcal{D}_{\mathbb{X}}(\mathbb{P}^2)$$

whose image is an $\mathbb{X}$-baric heart of $\mathcal{D}_{\mathbb{X}}(\mathbb{P}^2)$ in the sense of [14, Definition 2.17].

Finally, we have the 3-reduction of $\mathcal{D}_{\mathbb{X}}(\mathbb{P}^2)$ (see [14, Example 3.16])

$$\mathcal{D}_3(\mathbb{P}^2) \coloneqq \mathcal{D}_{\mathbb{X}}(\mathbb{P}^2) /\!\!/ [\mathbb{X} - 3] \cong \mathcal{D}^b_{\mathbb{P}^2}(Y) \tag{7.2}$$

which is equivalent to the derived category of coherent sheaves on the local $\mathbb{P}^2$.

Lemma 7.1. *Consider the composition of functors in the inducing process*

$$\begin{aligned}\Phi : \mathcal{D}_\infty(\mathbb{P}^2) &\xrightarrow{\sim} \mathcal{D}^b(\mathbf{k}Q/R) \to \mathcal{D}_{\mathrm{fd}}(\Gamma_{\mathbb{X}}(\widetilde{Q}_{\mathrm{gr}}, W_{\mathrm{gr}})) \to \\ &\mathcal{D}_{\mathrm{fd}}(\Gamma_{\mathbb{X}}(\widetilde{Q}_{\mathrm{gr}}, W_{\mathrm{gr}})) /\!\!/ [\mathbb{X} - 3] \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}-J(\widetilde{Q}, W)) \xrightarrow{\sim} \mathcal{D}^b_{\mathbb{P}^2}(Y).\end{aligned}$$

Then $\Phi = i_* : \mathcal{D}_\infty(\mathbb{P}^2) \to \mathcal{D}^b_{\mathbb{P}^2}(Y)$.

Proof. Let $E_i = \mathcal{O}_{\mathbb{P}^2}(i)$. Then the first equivalence $\mathcal{D}_\infty(\mathbb{P}^2) \cong \mathcal{D}^b(\mathbf{k}Q/R)$ in Φ is given by

$$\mathrm{Hom}^\bullet(\bigoplus_{i=0}^{2} E_i, -) : \mathcal{D}_\infty(\mathbb{P}^2) \to \mathcal{D}^b(\mathbf{k}Q/R),$$

and the last equivalence $\mathcal{D}^b_{\mathbb{P}^2}(Y) \cong \mathcal{D}^b(\mathrm{mod}-J(\widetilde{Q}, W))$ in Φ is given by

$$\mathrm{Hom}^\bullet(\bigoplus_{i=0}^{2} \pi^* E_i, -) : \mathcal{D}^b_{\mathbb{P}^2}(Y) \to \mathcal{D}^b(\mathrm{mod}-J(\widetilde{Q}, W)),$$

where $\pi : Y \to \mathbb{P}^2$ is the projection [7]. The lemma then follows from

$$\mathrm{Hom}^\bullet(\pi^*\mathcal{E}, i_*\mathcal{F}) = \mathrm{Hom}^\bullet(\mathcal{E}, \pi_* i_*\mathcal{F}) = \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}).$$

□

Now we recall the inducing construction of stability conditions from the projective plane to the local projective plane, through the 'q-stability conditions' introduced by Ikeda and Qiu [14].

Construction 7.2. Let $\sigma_\infty = (Z_\infty, \mathcal{P}_\infty)$ be a stability condition in $\mathrm{gldim}^{-1}(2) \subset \overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)}$.

- By [14, Theorem. 2.25], there is an induced q-stability conditions (σ, s) in $\mathrm{QStab}\,\mathcal{D}_{\mathbb{X}}(\mathbb{P}^2)$ with parameter $s = 3$, as constructed in [14, Cons. 2.18].
- By [14, Theorem. 2.16], (σ, s) projects to a stability condition σ_3 in the principal (connected) component $\mathrm{Stab}^\dagger\,\mathcal{D}_3(\mathbb{P}^2)$.

Denote by

$$\iota_3\colon\ \mathrm{gldim}^{-1}(2) \to \mathrm{Stab}^\dagger\,\mathcal{D}_3(\mathbb{P}^2)$$

the map of the above inducing process.

Proposition 7.3. *The inducing map ι_3 is injective. Moreover, it factors through the isomorphism between the spaces of geometric stability conditions on $\mathbb{P}^2$ and local $\mathbb{P}^2$:*

$$\iota_3\colon\ \mathrm{gldim}^{-1}(2) \hookrightarrow \overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)} \xrightarrow{\sim} \overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}^b_{\mathbb{P}^2}(Y))} \hookrightarrow \mathrm{Stab}^\dagger\,\mathcal{D}_3(\mathbb{P}^2).$$

Proof. A stability condition σ on $\mathcal{D}^b_{\mathbb{P}^2}(Y)$ is called *geometric* if all skyscraper sheaves $i_*\mathcal{O}_x$ of closed points $x \in \mathbb{P}^2$ are σ-stable of the same phase. By Lemma 7.1, the inducing map ι_3 maps geometric stability conditions on $\mathbb{P}^2$ with global dimension 2 to geometric stability conditions on local $\mathbb{P}^2$.

Let $\sigma = (Z, P) \in \mathrm{gldim}^{-1}(2)$ and $\iota_3(\sigma) = (\widetilde{Z}, \widetilde{P}) \in \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}^b_{\mathbb{P}^2}(Y))$. By Lemma 7.1, we have $Z = \widetilde{Z} \circ [i_*]$, where $[i_*] : K_0(\mathcal{D}_\infty(\mathbb{P}^2)) \xrightarrow{\sim} K_0(\mathcal{D}^b_{\mathbb{P}^2}(Y))$. By [3, Theorem 2.5] and [17, Proposition 1.12], any geometric stability condition on $\mathbb{P}^2$ or local $\mathbb{P}^2$ is uniquely determined by its central charge. Moreover, the open set $U \subset \mathrm{Hom}(K_0(\mathcal{D}_\infty(\mathbb{P}^2)), \mathbb{C})$ consists of central charges of geometric stability conditions on $\mathbb{P}^2$ and the open set $\widetilde{U} \subset \mathrm{Hom}(K_0(\mathcal{D}^b_{\mathbb{P}^2}(Y)), \mathbb{C})$ of central charges of geometric stability conditions on local $\mathbb{P}^2$ coincide via the isomorphism $[i_*]$. This proves the proposition. □

Finally, we remark that the whole connected component of stability conditions in $\mathrm{Stab}^\dagger \mathcal{D}_3(\mathbb{P}^2)$ can be obtained by inducing from stability conditions on $\mathcal{D}_\infty(\mathbb{P}^2)$ and autoequivalences, since the translates of $\overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}^b_{\mathbb{P}^2}(Y))}$ under the group of autoequivalences cover the whole connected component $\mathrm{Stab}^\dagger \mathcal{D}_3(\mathbb{P}^2)$ [3, Theorem 1].

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