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On Kakeya maps with regularity assumptions

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In $\mathbb{R}^n$, we parametrize Kakeya sets using Kakeya maps. A Kakeya map is defined to be a map

$$\phi : B^{n-1}(0,1) \times [0,1] \to \mathbb{R}^n, \qquad (v,t) \mapsto (c(v) + tv, t),$$

where $c : B^{n-1}(0,1) \to \mathbb{R}^{n-1}$. The associated Kakeya set is defined to be $K := \mathrm{Im}(\phi)$.

We show that the Kakeya set K has positive measure if either one of the following conditions is true.
(1) c is continuous and $c|_{S^{n-2}} \in C^\alpha(S^{n-2})$ for some $\alpha > \frac{(n-2)n}{(n-1)^2}$,
(2) c is continuous and $c|_{S^{n-2}} \in W^{1,p}(S^{n-2})$ for some $p > n-2$.

1. Introduction

The Kakeya set conjecture says if a set $K \subset \mathbb{R}^n$ contains a unit line segment in each direction (such a set is called a Kakeya set), then K has Hausdorff dimension n. The $n = 2$ case is solved by Davies [1], so we shall restrict ourselves to $n \geq 3$. Although we cannot solve the full conjecture, we can prove some positive results by assuming some regularity on the Kakeya set.

We start by defining the Kakeya map. Notation-wise all the balls are closed balls. For example, by $B^{n-1}(0,1)$ we mean the closed unit ball in $\mathbb{R}^{n-1}$.

Definition 1 (Kakeya map). Given a direction map $c : B^{n-1}(0,1) \to \mathbb{R}^{n-1}$, we define the associated Kakeya map to be the map

$$\phi : B^{n-1}(0,1) \times [0,1] \to \mathbb{R}^n, \qquad (v,t) \mapsto (c(v) + tv, t). \tag{1}$$

We define the associated Kakeya set to be $K := \mathrm{Im}(\phi)$.

Remark. By construction, for any $v \in B^{n-1}(0,1)$, K contains a line segment in direction $(v,1)$. Actually, this line segment in direction $(v,1)$ has one of its endpoint at $(c(v),0)$. This is the reason that we call c the direction map. Sometimes, it is good to write $\phi=\phi_c$ to highlight the dependence on c, but we just omit the subscript and write as ϕ since c is always priorly fixed and there is no ambiguity.

Let us talk about the regularity assumptions that we will impose on ϕ. First of all, we assume c is a continuous map. Second, we assume $c|_{S^{n-2}}$ (which is the restriction of c to S^{n-2}) lies in some function spaces of high regularity, for example, $C^\alpha(S^{n-2}), W^{1,p}(S^{n-2})$.

We state our main results.

Theorem 1. *If $c: B^{n-1}(0,1)\to\mathbb{R}^{n-1}$ is continuous and $c|_{S^{n-2}}$ is α-Hölder continuous for some $\alpha>\frac{(n-2)n}{(n-1)^2}$, then $\mathrm{Im}(\phi)$ has positive Lebesgue measure.*

Theorem 2. *If $c: B^{n-1}(0,1)\to\mathbb{R}^{n-1}$ is continuous and $c|_{S^{n-2}}$ lies in $W^{1,p}(S^{n-2})$ for some $p>n-2$, then $\mathrm{Im}(\phi)$ has positive Lebesgue measure.*

Remark. In a previous version of Theorem 2, the regularity assumption was $c\in H^s(B^{n-1}(0,1))$ for some $s>(n-1)/2$. However, the definition of fractional Sobolev space on bounded domains as well as on manifolds is quite tricky and is not the main purpose of this paper, so we switch to a less tricky space $W^{1,p}(S^{n-2})$, which is defined by pulling back to the Euclidean space. To define $W^{1,p}(S^{n-2})$, we first choose two charts $\{U_1,U_2\}$ to cover S^{n-2}. Let $\psi_i: U_i\to B^{n-2}(0,1)$ be diffeomorphisms. For f being a function on S^{n-2}, we define the norm:

$$\|f\|_{W^{1,p}(S^{n-2})} := \sum_{i=1}^{2}\|f\circ\psi_i^{-1}\|_{W^{1,p}(B^{n-2}(0,1))}. \tag{2}$$

It is not hard to check by the chain rule that for different choices of the charts, the norms defined as above are comparable.

The proofs will largely rely on the winding number from topology. In the rest of this section, we briefly discuss the winding number and its properties.

1.1. Winding number

We first set up some notation. Given a continuous function

$$f: S^n\to S^n,$$

we use $\deg f$ to denote the degree of f. There are many ways to define the degree of a function, which all turn out to be equivalent. In [2] Section 2.2, the degree of f is defined to be the integer d such that the induced homomorphism

$$f_* : H_n(S^n) \to H_n(S^n)$$

satisfies $f_*(\alpha) = d\alpha$ (noting that $H_n(S^n) \approx \mathbb{Z}$).

We also note that S^n and $\mathbb{R}^{n+1} \setminus \{0\}$ are homotopically equivalent, so $H_n(S^n) \approx H_n(\mathbb{R}^n \setminus \{0\})$ are isomorphic. Therefore, we can define the degree of

$$f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$$

using homology groups in the same way. In this case, we usually call it the winding number of f at 0, denoted by $\mathrm{wind}(f, 0)$. We can replace 0 by any other point x and define $\mathrm{wind}(f, x)$ as well.

We remark that we can also define the winding number in the following way. Suppose we are given a continuous function

$$g : S^{n-1} \to \mathbb{R}^n,$$

then for $z \in \mathbb{R}^n \setminus \mathrm{Im}(g)$ we define

$$g_z : S^{n-1} \to S^{n-1}, \quad z \mapsto \frac{g(x) - z}{|g(x) - z|}.$$

The winding number of g at z can be defined equivalently as

$$\mathrm{wind}(g, z) := \deg g_z.$$

Morally speaking, $\mathrm{wind}(g, z)$ is the number of times that the "hypersurface" $g(S^{n-1})$ wraps around z.

Next, we interpret the winding number of f from an analytic point of view. More precisely, we have the following lemma.

Lemma 1. *Assume $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is smooth. Consider another smooth function*

$$\widetilde{f} : B^{n+1}(0, 1) \to \mathbb{R}^{n+1}$$

satisfying $\widetilde{f}|_{S^n} = f$. Suppose $0 \in \mathbb{R}^{n+1}$ is a regular value of $\widetilde{f}$ in the sense that $D\widetilde{f}(x)$ is nonsingular for any $x \in \widetilde{f}^{-1}(0)$. Then we have

$$\mathrm{wind}(f, 0) = \sum_{x \in \widetilde{f}^{-1}(0)} \mathrm{sgn}(x). \tag{3}$$

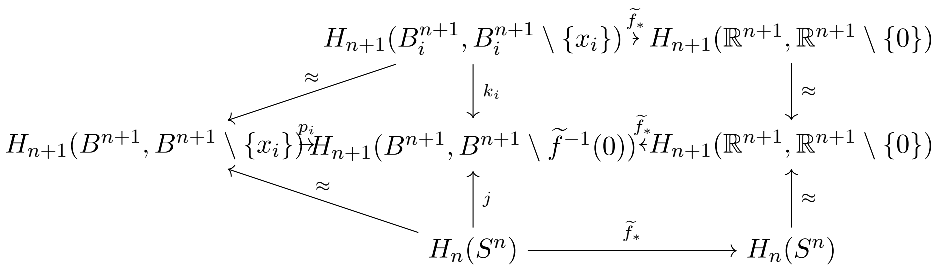

Figure 1.

Here, $\operatorname{sgn}(x) = 1$ *if* $\det(D\widetilde{f}(x)) > 0$ *and* $= -1$ *if* $\det(D\widetilde{f}(x)) < 0$.

One noticeable thing according to this lemma is that the right hand side of (3) only depends on the value of $\widetilde{f}$ on S^n. This lemma is fundamental from the point of view of algebraic topology, but we still provide the proof.

Proof. Write $\widetilde{f}^{-1}(0) = \{x_1, \ldots, x_m\}$. This is a finite set because 0 is a regular value of $\widetilde{f}$. Of course, it could also be an empty set. We can find a small number $\delta > 0$ so that $\{B^{n+1}(x_i, \delta)\}_{i=1}^m$ are disjoint and each $\widetilde{f}|_{B^{n+1}(x_i,\delta)}$ is a diffeomorphism onto some neighborhood of 0. For simplicity, we denote $B^{n+1}(0, 1)$ by B^{n+1} and denote $B^{n+1}(x_i, \delta)$ by B_i^{n+1}.

Note that $\widetilde{f}$ induces a commutative diagram as in Figure 1. This is essentially the same as the diagram in [2] page 136. We explain what those arrows mean. The arrows with $\approx$ mean that the relative homology groups are isomorphic. k_i and p_i are induced by inclusions. The top two groups are isomorphic to $\mathbb{Z}$, and the top homomorphism

$$H_n(B_i^{n+1}, B_i^{n+1} \setminus \{x_i\}) \xrightarrow{\widetilde{f}_*} H_n(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$$

becomes the multiplication by an integer called the *local degree* of $\widetilde{f}$ at x_i, written $\deg \widetilde{f}|_{x_i}$. Since $\widetilde{f}|_{B_i^{n+1}}$ is a diffeomorphism, we have $\deg \widetilde{f}|_{x_i} = \operatorname{sgn}(x_i)$. For the definition of homomorphism j, we first consider the inclusion

$$(B^{n+1}, S^n) \hookrightarrow (B^{n+1}, B^{n+1} \setminus \widetilde{f}^{-1}(0)),$$

which induces the homomorphism

$$H_{n+1}(B^{n+1}, S^n) \to H_{n+1}(B^{n+1}, B^{n+1} \setminus \widetilde{f}^{-1}(0)).$$

Since $H_{n+1}(B^{n+1}, S^n) \approx H_n(S^n)$, we denote the above map by

$$j : H_n(S^n) \to H_{n+1}(B^{n+1}, B^{n+1} \setminus \widetilde{f}^{-1}(0)).$$

In the bottom homomorphism

$$H_n(S^n) \xrightarrow{\widetilde{f}_*} H_n(S^n),$$

it is a multiplication by an integer and this integer is exactly $\deg \widetilde{f}|_{S^n}$ which also equals the winding number $\text{wind}(f, 0)$. Similar to [2] Proposition 2.30, we can show that

$$\text{wind}(f, 0) = \deg \widetilde{f}|_{S^n} = \sum_{i=1}^{m} \deg \widetilde{f}|_{x_i} = \sum_{i=1}^{m} \text{sgn}(x_i).$$

This finishes the proof. □

The way we connect the winding number with the Kakeya problem is through the following lemmas.

Lemma 2. *Given a continuous map $g : B^n(0,1) \to \mathbb{R}^n$, let $g|_{S^{n-1}}$ be the restriction of g to S^{n-1}. For any $z \notin g(S^{n-1})$, we have* $\text{wind}(g|_{S^{n-1}}, z) \neq 0$ *implies $z \in \text{Im}(g)$.*

Lemma 2 is a direct corollary of Lemma 1. The next lemma is known as the isoperimetric inequality.

Lemma 3. *Given a smooth map $g : S^{n-1} \to \mathbb{R}^n$, we have*

$$\left(\int_{\mathbb{R}^n \setminus \text{Im}(g)} |\text{wind}(g, z)|^{\frac{n}{n-1}} dz \right)^{\frac{n-1}{n}} \lesssim A(g). \tag{4}$$

Here, $A(g) = \int_{S^{n-1}} |\det(\sqrt{Dg^*(\omega) Dg(\omega)})| d\omega$ is the area of the self-intersecting "hypersurface" $\text{Im}(g)$. *Dg^* denotes the transpose of Dg. Locally, $|\det(\sqrt{Dg^*(\omega) Dg(\omega)})| d\omega$ is the volume form on* $\text{Im}(g)$.

Lemma 3 can be found in an equivalent form as equation (2.10) of [4]. We note that $z \mapsto \text{wind}(g, z)$ is constant on each connected component of $\mathbb{R}^n \setminus \text{Im}(g)$. So if we denote the volumes these components by $\{V_k\}_k$ and the values of $\mathrm{wind}(g, z)$ on these components by $\{n_k\}_k$, we see that (4) is equivalent to

$$(\sum_k |n_k|^{\frac{n}{n-1}} V_k)^{\frac{n-1}{n}} \lesssim A, \tag{5}$$

which is equation (2.10) of [4].

Now we briefly discuss the main idea of the paper. Given a continuous Kakeya map $\phi : B^{n-1}(0,1) \times [0,1] \to \mathbb{R}^n$, we want to show the Kakeya set $K = \mathrm{Im}(\phi)$ has positive Lebesgue measure. It will be helpful to consider two maps:

- $\phi_t(v) := \phi(v, t)$, the restriction of ϕ to the t-slice,
- $\gamma_t := \phi_t|_{S^{n-2}}$, the restriction of ϕ_t to the boundary sphere.

For $v_0 \notin \mathrm{Im}(\gamma_t)$, we denote by $\mathrm{wind}_t(v_0)$ the winding number of γ_t at point v_0, that is, the degree of the map

$$S^{n-2} \to S^{n-2},\ v \mapsto \frac{\gamma_t(v) - v_0}{|\gamma_t(v) - v_0|}.$$

If we can find some t_0 such that $\mathrm{wind}_{t_0}(v_0) \neq 0$ for some $v_0 \notin \mathrm{Im}(\gamma_{t_0})$, then by continuity, we have $v \notin \mathrm{Im}(\gamma_t)$ and $\mathrm{wind}_t(v) \neq 0$ for t close enough to t_0 and v close enough to v_0. Thus by Lemma 2, we know there is an open neighborhood of (t_0, v_0) contained in $\mathrm{Im}(\phi)$, and hence it has positive Lebesgue measure.

From the above discussion we see our main obstacle is the case that $|\mathrm{Im}(\phi)| = 0$ (so $\mathrm{wind}_t(v)$ is defined for almost every v) and the winding number is 0 where it is defined. We will show this cannot happen if we assume some regularity property on the Kakeya map.

Let us first consider an easy case: ϕ is Lipschitz. By the area formula, we have

$$\int_{B^{n-1}\times[0,1]} |\det(D\phi(x))| dx = \int_{\mathrm{Im}(\phi)} \#\{x : \phi(x) = y\} dy. \tag{6}$$

Using parameter $x = (v, t)$ and recalling the definition of ϕ in (1), one can calculate that

$$D_{v,t}\phi(v,t) = \begin{pmatrix} D_v c(v) + tI_{n-1} & 0 \\ v & 1 \end{pmatrix}.$$

Therefore, $\det(D\phi) = \det(D_{v,t}\phi(v,t))$ is a monic polynomial of degree $n-1$ in variable t. Therefore, $\det(D\phi)$ is nonzero almost everywhere. This implies (6) is nonzero, and hence $|\mathrm{Im}(\phi)| > 0$. In fact we shall see in Section 4 that using the winding numbers we can show $|\mathrm{Im}(\phi)|$ is bounded from below by some positive constant depending only on the Lipschitz constant of ϕ.

Let us come back to Theorem 1 and Theorem 2 where the regularity assumption of the Kakeya map is weaker than Lipschitz. The strategy is still proof by contradiction. We assume $\mathrm{Im}(\phi)$ has zero Lebesgue measure. Then we use smooth maps to approximate Hölder continuous or Sobolev-regular Kakeya maps. We will eventually derive a contradiction using an isoperimetric inequality and the following key estimate:

$$1 \lesssim \int_0^1 |\int_{\mathbb{R}^{n-1}} \mathrm{wind}_t(x)dx|dt. \tag{7}$$

One intuition for this to be true is that the inner integral is a polynomial in t with leading term t^{n-1}. This is very similar to an observation by Katz and Rogers in [3]. They showed that if c is a polynomial of degree d, then $|\mathrm{Im}(\phi)|$ is bounded from below by some constant $c(n,d) > 0$. The key observation there is, by the area formula (6) and Bezout's theorem, $|\mathrm{Im}(\phi)| \geq C_d \int |\det(D\phi)|dvdt$, and the latter integral is always at least $c(n)$ for some constant $c(n) > 0$. The difference in our approach is that instead of considering the integral of $|\det(D\phi)|$ in (t, v), we consider the integral of $\int \det(D\phi_t)dv$ for each fixed t. This is the signed volume on the t-slice and can be related to the winding number.

This paper is structured as follows. In Section 2 we will prove Theorem 1, and in Section 3 we will prove Theorem 2. Section 4 will be a discussion of two other problems related to the Kakeya maps with regularity assumptions.

Notation. We use $A \lesssim B$ to denote that $A \leq CB$ for constant C which depends only on the dimension n. $A \sim B$ will mean $A \lesssim B$ and $B \lesssim A$. We will use $A \lesssim_q B$ to denote $A \leq C_q B$ for some constant C_q depending on q (and n). The closed unit ball in $\mathbb{R}^k$ is denoted by $B^k(0,1)$, and the unit sphere in $\mathbb{R}^k$ is denoted by S^{k-1}.

2. Hölder continuous Kakeya map

Let $n \geq 3$. Suppose we have a Kakeya map

$$\phi : B^{n-1}(0,1) \times [0,1] \to \mathbb{R}^n, \qquad (v,t) \mapsto (c(v) + tv, t)$$

where $c : B^{n-1}(0,1) \to \mathbb{R}^{n-1}$ is continuous and C^α on S^{n-2}. We will prove the following theorem.

Theorem 3. *If $\alpha > \frac{(n-2)n}{(n-1)^2}$, then $\mathrm{Im}(\phi)$ has positive Lebesgue measure.*

We start with some definitions. Denote the restriction of c to S^{n-2} by $\underline{c}$. Let $\underline{c}_\epsilon$ be the ϵ-mollification of $\underline{c}$. To be precise, let $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ be a radial compactly supported smooth bump function in $\mathbb{R}^{n-1}$ adapted to $B^{n-1}(0,1)$, and let

$$\rho_\epsilon(y) := d_\epsilon \epsilon^{2-n} \rho(y/\epsilon),$$

where the normalization constant d_ϵ is set to be

$$d_\epsilon = \epsilon^{n-2} \Big(\int_{S^{n-2}} \rho((y_0 - y)/\epsilon) dy \Big)^{-1} \tag{8}$$

for any $y_0 \in S^{n-2}$. Note that the right hand side of (8) is independent of $y_0 \in S^{n-2}$ since ρ is a radial function. Also, we have $d_\epsilon \sim 1$.

We write $\underline{c}$ in components as $\underline{c} = (\underline{c}_1, \cdots, \underline{c}_{n-1})$. Finally we define $\underline{c}_\epsilon$ to be $(\underline{c}_1 * \rho_\epsilon, \cdots, \underline{c}_{n-1} * \rho_\epsilon)$, where $\underline{c}_i * \rho_\epsilon(x) := \int_{S^{n-2}} \underline{c}_i(y) \rho_\epsilon(x-y) dy$ for $x \in S^{n-2}$. So the convolution $\underline{c}_i * \rho_\epsilon(x)$ averages the value of $\underline{c}_i$ over an ϵ-neighborhood of x on S^{n-2}.

Define γ_t to be the map

$$S^{n-2} \to \mathbb{R}^{n-1}, \qquad v \mapsto \underline{c}(v) + tv,$$

and $\gamma_{t,\epsilon}$ to be the map

$$S^{n-2} \to \mathbb{R}^{n-1}, \qquad v \mapsto \underline{c}_\epsilon(v) + tv.$$

Then from the Hölder continuity of c we have $|\gamma_t(v) - \gamma_{t,\epsilon}(v)| \lesssim_c \epsilon^\alpha$, which implies:

$$\mathrm{wind}_{t,\epsilon}(x) = \mathrm{wind}_t(x) \quad \text{if} \quad \mathrm{dist}(x, \mathrm{Im}\gamma_t) \gtrsim_c \epsilon^\alpha. \tag{9}$$

Here $\mathrm{wind}_t(x) = \mathrm{wind}(\gamma_t, x)$ is the degree of the map

$$S^{n-2} \to S^{n-2}, \qquad y \to \frac{\gamma_t(y) - x}{|\gamma_t(y) - x|}, \tag{10}$$

and $\mathrm{wind}_{t,\epsilon}(x) = \mathrm{wind}(\gamma_{t,\epsilon}, x)$ is the degree of the map

$$S^{n-2} \to S^{n-2}, \qquad y \to \frac{\gamma_{t,\epsilon}(y) - x}{|\gamma_{t,\epsilon}(y) - x|}. \tag{11}$$

We will need the following identity from differential topology that relates the integral of winding numbers with the integral of the determinant of the differential.

Lemma 4. *Suppose* $f : S^{n-2} \to \mathbb{R}^{n-1}$ *is smooth. Let* $\widetilde{f} : B^{n-1}(0,1) \to \mathbb{R}^{n-1}$ *be a smooth map satisfying* $\widetilde{f}|_{S^{n-2}} = f$. *Then, we have*

$$\int_{\mathbb{R}^{n-1}} \mathrm{wind}(f,x)dx = \int_{B^{n-1}(0,1)} \det(D\widetilde{f}(y))dy.$$

Proof. We recall that we say $x \in \mathbb{R}^{n-1}$ is a regular value of f if for any $y \in f^{-1}(x)$ we have $\det(Df(y)) \neq 0$. In particular, x is a regular value if $f^{-1}(x)$ is an empty set.

By Lemma 1, if $x \in \mathbb{R}^{n-1} \setminus \mathrm{Im}(f)$ is a regular value of $\widetilde{f}$, then

$$\mathrm{wind}(f,x) = \sum_{y \in \widetilde{f}^{-1}(x)} \mathrm{sgn}(y),$$

where $\mathrm{sgn}(y)$ equals 1 if $\det(D\widetilde{f}(y)) > 0$, and -1 if $\det(D\widetilde{f}(y)) < 0$. We also note that by Sard's theorem almost every $x \in \mathbb{R}^{n-1}$ is a regular value of $\widetilde{f}$. So,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \mathrm{wind}(f,x)dx &= \int_{\mathbb{R}^{n-1}} \sum_{y \in f^{-1}(x)} \mathrm{sgn}(y)dx \\ &= \int_{B^{n-1}(0,1)} \det(D\widetilde{f}(y))dy.\end{aligned}$$

(Rigorously speaking, we should write the integration domain as $\mathbb{R}^{n-1} \setminus \mathrm{Im}(f)$. But since $\mathrm{Im}(f)$ has zero measure, we still write it as $\mathbb{R}^{n-1}$ without any ambiguity.) □

Remark. From the previous lemma, we see that the value of the integral

$$\int_{B^{n-1}(0,1)} \det(D\widetilde{f}(y))dy$$

only depends on the value of $\widetilde{f}$ on S^{n-2}. Therefore, for any smooth function $f : S^{n-2} \to \mathbb{R}^{n-1}$, it makes sense to define the integral

$$\int_{B^{n-1}(0,1)} \det(Df(y))dy.$$

Proposition 1. *Let* $\mathrm{wind}_{\epsilon,t}(x)$ *be defined as in* (11). *We have*

$$1 \lesssim \int_0^1 \Big| \int_{\mathbb{R}^{n-1}} \mathrm{wind}_{t,\epsilon}(x) dx \Big| dt. \tag{12}$$

The implicit constant is independent of ϵ.

Proof. Fix $\epsilon > 0$. By Lemma 4 and the remark above, we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \mathrm{wind}_{t,\epsilon}(x) dx &= \int_{B^{n-1}(0,1)} \det(D\gamma_{t,\epsilon}(v)) dv \\ &= \int_{B^{n-1}(0,1)} \gamma_{t,\epsilon}^*(dx_1 \wedge \cdots dx_{n-1}).\end{aligned}$$

Here $\gamma_{t,\epsilon}^*(dx_1 \wedge \cdots dx_{n-1})$ is the pullback of the differential form. Further by Stokes' theorem, we have

$$\begin{aligned}\int_{B^{n-1}(0,1)} \gamma_{t,\epsilon}^*(dx_1 \wedge \cdots dx_{n-1}) &= \int_{B^{n-1}(0,1)} d\big(\gamma_{t,\epsilon}^*(x_1 dx_2 \wedge \cdots dx_{n-1})\big) \\ &= \int_{S^{n-2}} \gamma_{t,\epsilon}^*(x_1 dx_2 \wedge \cdots dx_{n-1}).\end{aligned}$$

If we write $\underline{c}_\epsilon(v) = (\underline{c}_{1,\epsilon}(v), \ldots, \underline{c}_{n-1,\epsilon}(v))$, we have

$$\gamma_{t,\epsilon}(v) = (tv_1 + \underline{c}_{1,\epsilon}(v), \ldots, tv_{n-1} + \underline{c}_{n-1,\epsilon}(v)).$$

Therefore

$$\begin{aligned}&\int_{S^{n-2}} \gamma_{t,\epsilon}^*(x_1 dx_2 \wedge \cdots dx_{n-1}) \\ &\quad= \int_{S^{n-2}} (tv_1 + \underline{c}_{1,\epsilon}(v)) d(tv_2 + \underline{c}_{2,\epsilon}(v)) \wedge \cdots \wedge d(tv_{n-1} + \underline{c}_{n-1,\epsilon}(v)) \\ &\quad= t^{n-1} \int_{S^{n-2}} v_1 dv_2 \wedge \cdots \wedge dv_{n-1} + e(t) \\ &\quad= t^{n-1} + e(t),\end{aligned}$$

where $e(t)$ is a polynomial in t with degree at most $n-2$ and coefficients determined by c, ϵ. In summary, we have shown

$$\int_{\mathbb{R}^{n-1}} \mathrm{wind}_{t,\epsilon}(x) dx = t^{n-1} + e(t).$$

Next we claim that

$$\int_0^1 \big|t^{n-1} + e(t)\big| dt \gtrsim 1. \tag{13}$$

If the claim (13) is true, we may then conclude

$$1 \lesssim \int_0^1 \big|t^{n-1} + e(t)\big| dt = \int_0^1 \Big| \int_{\mathbb{R}^{n-1}} \mathrm{wind}_{t,\epsilon}(x) dx \Big| dt.$$

So it suffices to prove the claim (13). The proof can be found in [3], but we give the proof here for the sake of completeness. Since $t^{n-1} + e(t)$ is a monic polynomial of degree $n-1$, we write

$$t^{n-1} + e(t) = (t - r_1) \cdots (t - r_{n-1}),$$

where $r_1 \cdots r_{n-1}$ are complex numbers. We observe that for t in a subset of $[0, 1]$ with measure greater than $1/4$, we have the estimate

$$|t - r_j| \geq \frac{1}{4n} \gtrsim 1, \quad j = 1, \cdots n-1,$$

which means for t in a subset of $[0, 1]$ with measure greater than $1/4$ we have

$$|t^{n-1} + e(t)| = |t - r_1| \cdots |t - r_{n-1}| \gtrsim 1.$$

Therefore, (13) holds. □

Proof of Theorem 3. Let $\underline{\phi}$ be the restriction of ϕ to $S^{n-2} \times [0, 1]$. For the sake of contradiction we assume that $|\mathrm{Im}\underline{\phi}| = 0$ and $\mathrm{wind}_t(x) = 0$ where it is defined. Then by Proposition 1 and (9) we have

$$\int_0^1 \int_{\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}\gamma_t)} |\mathrm{wind}_{t,\epsilon}(x)| dx dt \gtrsim 1. \tag{14}$$

Here $\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}\gamma_t)$ denotes the $C\epsilon^\alpha$-neighborhood of $\mathrm{Im}\gamma_t$. C is a constant depending on the Hölder constant of c.

Next we show

$$|\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}\gamma_t)| \lesssim \epsilon^{-n+2}(\epsilon^{\alpha(n-1)}) = \epsilon^{(n-1)\alpha-n+2}, \tag{15}$$

and

$$A(\gamma_{t,\epsilon}) \lesssim \epsilon^{(\alpha-1)(n-2)}. \tag{16}$$

(For the definition of $A(\gamma_{t,\epsilon})$, see the line next to equation (4).)

Indeed to see (15), we choose a maximal ϵ-separated subset S of S^{n-2}, so $|S| \sim \epsilon^{2-n}$. We claim that the union of balls $\bigcup_{x_i\in S} B_{C_1\epsilon^\alpha}(\gamma_t(x_i))$ covers $\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}\gamma_t)$, when C_1 is large enough. In fact for any $y \in \mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}\gamma_t)$, there exists an $x \in S^{n-2}$, such that $|y - \gamma_t(x)| \lesssim \epsilon^\alpha$. Also by the choice of S, there exists an $x_i \in S$ such that $|x - x_i| \le \epsilon$. So by the Hölder continuity, we have

$$|y - \gamma_t(x_i)| \le |y - \gamma_t(x)| + |\gamma_t(x) - \gamma_t(x_i)| \le C_1\epsilon^\alpha,$$

which means $y \in \mathcal{N}_{C_1\epsilon^\alpha}(\gamma_t(x_i)) \subset \bigcup_{x_i\in S} B_{C_1\epsilon^\alpha}(\gamma_t(x_i))$ if C_1 is sufficiently large. So (15) follows.

To see (16), we only need to show $|\nabla\gamma_{t,\epsilon}| \lesssim \epsilon^{\alpha-1}$ (here ∇ is the gradient on S^{n-2}), which will imply

$$A(\gamma_{t,\epsilon}) \lesssim \int_{S^{n-2}} |\det(D\gamma_{t,\epsilon})| \lesssim \int_{S^{n-2}} |\nabla\gamma_{t,\epsilon}|^{n-2} \lesssim \epsilon^{(\alpha-1)(n-2)}.$$

Recall that ρ is the mollifier with $\rho_\epsilon(y) = d_\epsilon\epsilon^{2-n}\rho(y/\epsilon)$ and $d_\epsilon \sim 1$, so we have

$$\begin{aligned}
|\nabla\gamma_{t,\epsilon}(y)| &= d_\epsilon|\nabla \int_{S^{n-2}} \gamma_{t,\epsilon}(x)\epsilon^{2-n}\rho(\frac{y-x}{\epsilon})dx| \\
&\sim |\int_{S^{n-2}} \gamma_{t,\epsilon}(x)\epsilon^{2-n}\nabla(\rho(\frac{y-x}{\epsilon}))dx| \\
&= |\int_{S^{n-2}} (\gamma_{t,\epsilon}(x) - \gamma_{t,\epsilon}(y))\epsilon^{1-n}\nabla\rho(\frac{y-x}{\epsilon})dx| \\
&\lesssim \int_{\mathcal{N}_{C\epsilon}(x)\cap S^{n-2}} |\gamma_{t,\epsilon}(x) - \gamma_{t,\epsilon}(y)|\epsilon^{1-n}dy \\
&\lesssim \int_{\mathcal{N}_{C\epsilon}(x)\cap S^{n-2}} |x-y|^\alpha\epsilon^{1-n}dy \\
&\lesssim \epsilon^{\alpha-1}.
\end{aligned}$$

So we indeed have (16).

Lemma 3 states

$$\left(\int_{\mathbb{R}^{n-1}} |\mathrm{wind}_{t,\epsilon}(x)|^{\frac{n-1}{n-2}}dx\right)^{n-2} \lesssim A(\gamma_{t,\epsilon})^{n-1}.$$

Combining what we have so far with Hölder's inequality we obtain

$$\begin{aligned}
1 &\lesssim \int_0^1 \int_{\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}(\gamma_t))} |\mathrm{wind}_{t,\epsilon}(x)| dx dt \\
&\leq |\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}(\gamma_t))|^{1/(n-1)} \\
&\quad \times \left(\int_0^1 \int_{\mathcal{N}_{C\epsilon^\alpha}(\mathrm{Im}(\gamma_t))} |\mathrm{wind}_{t,\epsilon}(x)|^{(n-1)/(n-2)} dx dt \right)^{(n-2)/(n-1)} \\
&\lesssim \epsilon^{((n-1)\alpha-n+2)/(n-1)} \epsilon^{(\alpha-1)(n-2)} \\
&= \epsilon^{\alpha - \frac{n-2}{n-1} + (\alpha-1)(n-2)}.
\end{aligned}$$

This is a contradiction if $\alpha - \frac{n-2}{n-1} + (\alpha - 1)(n-2) > 0$, that is, $\alpha > \frac{(n-2)n}{(n-1)^2}$. □

3. Sobolev regular Kakeya map

We use the same notation as in Section 2 but instead of assuming $c|_{S^{n-2}}$ is Hölder continuous C^α, we assume c is continuous and $c|_{S^{n-2}}$ lies in the Sobolev space $W^{1,p}(S^{n-2})$ for some $p > n-2$. We write $p = n-2+\delta$ for some small $\delta > 0$.

To compare this regularity assumption with that in Theorem 1, by the Sobolev embedding, we know that $c|_{S^{n-2}}$ is δ'-Hölder continuous for some $\delta' = O(\delta)$. On the other hand when δ is small the space $W^{1,p}$ and C^α for $\alpha > \frac{(n-2)n}{(n-1)^2}$ are mutually non-inclusive.

We will prove the following theorem.

Theorem 4. *If c is continuous and $c|_{S^{n-2}} \in W^{1,n-2+\delta}(S^{n-2})$ for some $\delta > 0$ then $\mathrm{Im}(\phi)$ has positive Lebesgue measure.*

Proof. Since $c|_{S^{n-2}} \in W^{1,n-2+\delta}(S^{n-2})$, we also have $\gamma_t \in W^{1,n-2+\delta}(S^{n-2})$. We consider the mollified c_ϵ as we did in Section 2. Also recall the definitions of $\mathrm{wind}_t(x)$ and $\mathrm{wind}_{t,\epsilon}(x)$ in (10) and (11).

By Proposition 1, we have

$$\int_0^1 |\int_{\mathbb{R}^{n-1}} \mathrm{wind}_{t,\epsilon}(x) dx| dt \gtrsim 1. \tag{17}$$

Suppose for the sake of contradiction that $|\mathrm{Im}(\phi)| = 0$ and $\mathrm{wind}_t(x) = 0$ wherever it is defined. Since by the Sobolev embedding $c|_{S^{n-2}}$ is in some

Hölder space $C^{\delta'}$ for some $\delta' > 0$, the same reasoning as in (9) yields

$$\text{wind}_{t,\epsilon}(x) = \text{wind}_t(x) \quad \text{if} \quad \text{dist}(x, \text{Im}\gamma_t) \gtrsim_c \epsilon^{\delta'}. \tag{18}$$

So if we split

$$\begin{aligned}\int_0^1 \Big|\int_{\mathbb{R}^{n-1}} \text{wind}_{t,\epsilon}(x)dx\Big|dt &\le \int_0^1 \Big|\int_{\mathcal{N}_{C\epsilon^{\delta'}}(\text{Im}\gamma_t)} \text{wind}_{t,\epsilon}(x)dx\Big|dt \\ &\quad + \int_0^1 \Big|\int_{\mathbb{R}^{n-1}\setminus\mathcal{N}_{C\epsilon^{\delta'}}(\text{Im}\gamma_t)} \text{wind}_{t,\epsilon}(x)dx\Big|dt \\ &=: I_1 + I_2,\end{aligned}$$

then

$$I_2 = \int_0^1 \Big|\int_{\mathbb{R}^{n-1}\setminus\mathcal{N}_{C\epsilon^{\delta'}}(\text{Im}\gamma_t)} \text{wind}_{t,\epsilon}(x)dx\Big|dt = 0,$$

since we assumed $\text{wind}_t(x) = 0$ and we have (18).

In the rest of the proof we will show that $I_1 \to 0$ as $\epsilon \to 0$, which contradicts (17). We claim that

$$A(\gamma_{t,\epsilon}) := \int_{S^{n-2}} \Big|\det\left(\sqrt{(D_\omega(c * \rho_\epsilon + t\omega))^*(D_\omega(c * \rho_\epsilon + t\omega))}\right)\Big|d\omega \lesssim_c 1. \tag{19}$$

Here the notation is from Lemma 3. In the integral, we think of $c * \rho_\epsilon + t\omega$ as a function $S^{n-2} \to \mathbb{R}^{n-1}$.

To prove (19), we cover S^{n-2} by two coordinate charts $\{U_1, U_2\}$ with coordinate maps $\psi_i : U_i \to B^{n-2}(0,1)$. Note that the function $t\omega$ is smooth and $\|c * \rho_\epsilon\|_{W^{1,p}(S^{n-2})} \lesssim \|c\|_{W^{1,p}(S^{n-2})}$. So after change of variables and pulling back using ψ_i, the inequality (19) becomes $\int_{B^{n-2}(0,1)} |\det(\sqrt{(Df)^*(Df)})| \lesssim_c 1$, where f is a function on $B^{n-2}(0,1)$ satisfying $\|f\|_{W^{1,p}(B^{n-2}(0,1))} \lesssim 1 + \|c\|_{W^{1,p}(S^{n-2})}$. Expanding the integrand we see that

$$\begin{aligned}\int |\det(\sqrt{(Df)^*(Df)})| &= \int |\det((Df)^*(Df))|^{\frac{1}{2}} \\ &\lesssim \int |Df|^{n-2} \lesssim \|f\|^{n-2}_{W^{1,n-2}} \lesssim_c 1.\end{aligned}$$

So, we prove the claim (19).

Therefore applying the isoperimetic inequality (Lemma 3) gives us

$$
\begin{aligned}
|\int_{\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)} \mathrm{wind}_{t,\epsilon}(x)dx| &\lesssim |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)|^{1/(n-1)}\|\mathrm{wind}_{t,\epsilon}\|_{L^{(n-1)/(n-2)}} \\
&\lesssim |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)|^{1/(n-1)}A(\gamma_{t,\epsilon}) \lesssim_c |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)|^{1/(n-1)}.
\end{aligned}
$$

In the last inequality, we used (19).

Since

$$
\begin{aligned}
\int_0^1 |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)|^{1/(n-1)}dt &\lesssim \left(\int_0^1 |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\gamma_t)|dt\right)^{1/(n-1)} \\
&\leq |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\underline{\phi})|^{1/(n-1)},
\end{aligned}
$$

we conclude

$$
I_1 \lesssim_c |\mathcal{N}_{C\epsilon^{\delta'}}(\mathrm{Im}\underline{\phi})|^{1/(n-1)} \to 0
$$

as $\epsilon \to 0$ (because by assumption $|\mathrm{Im}\underline{\phi}| = 0$). Hence we finish the proof of Theorem 4. □

4. Other results related to the Kakeya problem

In this section we discuss two Kakeya-type problems which are under different settings from the previous sections, and may be of independent interest.

4.1. Tube-Kakeya set with Lipschitz spacing condition

In the previous sections, we studied the Kakeya set which is the union of line segments. In this subsection, we study the tube-version of the Kakeya set which is the union of δ-tubes.

Definition 2 (Tube-Kakeya set). For $0 < \delta < 1$, we choose V_δ to be a maximal δ-separated subset of $B^{n-1}(0,1)$. For a map $c : B^{n-1}(0,1) \to \mathbb{R}^{n-1}$, we consider the set of tubes $\{T^\delta_{c(v),v}\}_{v\in V_\delta}$, where $T^\delta_{x,v}$ is the δ-neighborhood of the segment $\{(x + tv, t) : t \in [0,1]\}$. We call the union of these tubes

$$
\bigcup_{v\in V_\delta} T^\delta_{c(v),v}
$$

the tube-Kakeya set.

It is conjectured that for any map c we have

$$|\bigcup_{v\in V_\delta} T^{\delta}_{c(v),v}| \gtrsim_\epsilon \delta^{\epsilon},$$

for $\epsilon > 0$. In this subsection, we will assume some regularity on the map c and prove the result. We define the Lipschitz constant of c by

$$\|c\|_{\text{Lip}} := \max_{v,v'\in B^{n-1}(0,1)} \frac{|c(v)-c(v')|}{|v-v'|}.$$

Our result is:

Proposition 2. *If* $\{T^{\delta}_{c(v),v}\}_{v\in V_\delta}$ *is a collection of tubes as in Definition 2, then*

$$|\bigcup_{v\in V_\delta} T^{\delta}_{c(v),v}| \gtrsim (\|c\|_{\text{Lip}}+1)^{-(n-1)^2}.$$

Proof. Consider the map

$$\phi : B^{n-1}(0,1)\times[0,1]\to\mathbb{R}^n, \qquad (v,t)\mapsto(c(v)+tv,t),$$

which is the Kakeya map corresponding to c.

Denote the Lipschitz constant $L=\|c\|_{\text{Lip}}$. Note that $B^{n-1}(0,1)\subset \bigcup_{v\in V_\delta} B(v,2\delta)$, and $\phi(B(v,2\delta)\times[0,1])\subset T^{100L\delta}_{c(v),v}$ (recall that $T^{100L\delta}_{c(v),v}$ is the $100L\delta$-neighborhood of the segment $\{(c(v)+tv,t) : t\in[0,1]\}$). Therefore,

$$\text{Im}(\phi)\subset \bigcup_{v\in V_\delta} T^{100L\delta}_{c(v),v}. \tag{20}$$

Choose a maximal δ-separated subset of $B^{n-1}(0,100L\delta)$, denoted by $\{x_1,\cdots,x_M\}$ with $M\sim L^{n-1}$. We see for any $v\in V_\delta$,

$$T^{100L\delta}_{c(v),v}\subset \bigcup_{1\le i\le M} T^{\delta}_{c(v)+x_i,v}.$$

Combined with (20), we have

$$\text{Im}(\phi)\subset \bigcup_{1\le i\le M}\bigcup_{v\in V_\delta} T^{\delta}_{c(v)+x_i,v}.$$

We also observe that $\bigcup_{v\in V_\delta} T^{\delta}_{c(v)+x_i,v} = \bigcup_{v\in V_\delta} T^{\delta}_{c(v),v} + (x_i,0)$, which implies

$$|\bigcup_{v\in V_\delta} T^{\delta}_{c(v)+x_i,v}| = |\bigcup_{v\in V_\delta} T^{\delta}_{c(v),v}|,$$

for any x_i. As a result, we have

$$|\mathrm{Im}(\phi)| \leq M|\bigcup_{v\in V_\delta} T^\delta_{c(v),v}| \lesssim \|c\|^{n-1}_{\mathrm{Lip}}|\bigcup_{v\in V_\delta} T^\delta_{c(v),v}|.$$

It remains to find a lower bound of $|\mathrm{Im}(\phi)|$. We note the following inequalities

$$\begin{aligned}
1 &\lesssim \int_0^1 \int_{\mathbb{R}^{n-1}} |\mathrm{wind}_t(x)| dx dt \\
&\lesssim (\int_0^1 \int_{\mathbb{R}^{n-1}} |\mathrm{wind}_t(x)|^{\frac{n-1}{n-2}} dx dt)^{\frac{n-2}{n-1}} |\mathrm{Im}(\phi)|^{\frac{1}{n-1}} \\
&\lesssim (\int_0^1 A(\phi_t|_{S^{n-2}})^{\frac{n-1}{n-2}} dt)^{\frac{n-2}{n-1}} |\mathrm{Im}(\phi)|^{\frac{1}{n-1}} \\
&\lesssim (\|c\|_{\mathrm{Lip}} + 1)^{n-2} |\mathrm{Im}(\phi)|^{\frac{1}{n-1}}
\end{aligned}$$

Here, the first inequality is by Proposition 1 , the second inequality is Hölder's inequality, the third inequality is by the isoperimetric inequality (Lemma 3), and the fourth inequality is by $A(\phi_t|_{S^{n-2}}) \lesssim \|\det\sqrt{D\phi_t^* D\phi_t}\|_\infty \lesssim \|D\phi_t\|^{n-2}_\infty \lesssim (\|c\|_{\mathrm{Lip}} + 1)^{n-2}$.

We obtain

$$|\bigcup_{v\in V_\delta} T^\delta_{c(v),v}| \gtrsim \|c\|^{-(n-1)}_{\mathrm{Lip}} |\mathrm{Im}(\phi)| \gtrsim (\|c\|_{\mathrm{Lip}} + 1)^{-(n-1)^2}.$$

□

Furthermore we may ask the estimate for the tube-Kakeya set under the α-Hölder regularity assumption. Motivated by Theorem 1, we wonder whether the following is true.

Question 1. *Does there exists $\alpha < 1$ so that*

$$|\bigcup_{v\in V_\delta} T^\delta_{c(v),v}| \gtrsim_{\epsilon, \|c\|_{C^\alpha}} \delta^\epsilon, \tag{21}$$

for any $\epsilon > 0$?

4.2. Line-Kakeya set

In this subsection we would like to use a slightly different notation for the Kakeya map. The direction set is parametrized by S^{n-1}, as opposed to Definition 1 where the direction set is parametrized by $B^{n-1}(0,1)$.

Definition 3. For any function $c : S^{n-1} \to \mathbb{R}^n$, we let ϕ_c be the map

$$\phi_c : S^{n-1} \times [0,1] \to \mathbb{R}^n, \qquad (v,t) \mapsto c(v) + tv.$$

We call ϕ_c a Kakeya map, and call $K_c := \mathrm{Im}(\phi_c) = \bigcup_{v\in S^{n-1}} c(v) + [0,1] \cdot v$ the associated Kakeya set.

We can also define the line-Kakeya set where line segments are replaced by infinite lines

$$\tilde{K}_c := \bigcup_{v\in S^{n-1}} c(v) + \mathbb{R}_{\geq 0} \cdot v.$$

Proposition 3. *If c is continuous and $\mathrm{Im}(c) \subset B(0,R)$, then $\tilde{K}_c \supset \mathbb{R}^n \backslash B(0,R)$.*

Proof. We prove it using the degree theory from topology. For any point $x \notin B(0,R)$, we define a map

$$f : S^{n-1} \to S^{n-1}$$

$$f(v) = \frac{x - c(v)}{|x - c(v)|}.$$

We see that f is not surjective (actually $\mathrm{Im}(f)$ is contained in a half sphere), and hence f has degree 0, which implies f has a fixed point (see for example Section 2.2 of [2]). Let v be a fixed point of f. Then

$$v = \frac{x - c(v)}{|x - c(v)|}$$

or equivalently,

$$x = c(v) + |c(v) - x| \cdot v \in \tilde{K}_c.$$

□

We could immediately obtain the following result for a segment-Kakeya set provided that c has small image.

Proposition 4. *If c is continuous and $\mathrm{diam}(\mathrm{Im}(c)) < \frac{1}{2}$. Then K_c has positive Lebesgue measure.*

Proof. By the assumption, we have $c(S^{n-1}) \subset B_{0.9}$, a ball of radius 0.9. By Proposition 3, we have $\tilde{K}_c \supset \mathbb{R}^n \setminus B_{0.9}$. Therefore $K_c \supset B_1 \setminus B_{0.9}$. □

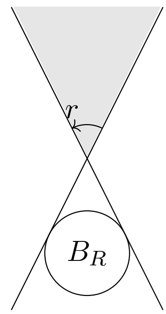

Figure 2.

We could also prove a local version of the theorem. To be precise, for each subset U of S^{n-1} and a continuous c as above, we define the line-Kakeya set with directions in U to be $\tilde{K}_c(U) = \bigcup_{v \in U} c(v) + \mathbb{R}_{\geq 0} \cdot v$.

Proposition 5. *If B_r is a small closed ball of radius r in S^{n-1}, then $\tilde{K}_c(B_r)$ contains an infinite cone with the cone angle $\gtrsim r$.*

Proof. Suppose $c(B_r)$ lies in $B_R(\subset \mathbb{R}^n)$, a ball of radius R. We can find another continuous map $\tilde{c}$ on the whole sphere S^{n-1}, such that $\tilde{c}(S^{n-1}) \subset B_R$ and $\tilde{c}|_{B_r} = c|_{B_r}$. Without loss of generality, we assume the center of $B_R(\subset \mathbb{R}^n)$ is 0 and the center of $B_r(\subset S^{n-1})$ is the north pole $(0, \cdots, 0, 1)$ of S^{n-1}.

Consider the cone $C = \{(\bar{x}, x_n) \in \mathbb{R}^n : x_n - \frac{R}{r} > \frac{|\bar{x}|}{r}\}$ which is the shaded region in Figure 2. By Proposition 3, $C \subset \tilde{K}_{\tilde{c}}$. Also note that for any $x \in C$ and $y \in B_R$, we have $\frac{x-y}{|x-y|} \in B_r \subset S^{n-1}$, so actually we have $C \subset \tilde{K}_{\tilde{c}}(B_r) = \tilde{K}_c(B_r)$. □

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