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Linear subspaces of minimal codimension in hypersurfaces

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Let $\mathbf{k}$ be a perfect field and let $X \subset \mathbb{P}^N$ be a hypersurface of degree d defined over $\mathbf{k}$ and containing a linear subspace L defined over $\overline{\mathbf{k}}$ with $\operatorname{codim}_{\mathbb{P}^N} L = r$. We show that X contains a linear subspace L_0 defined over $\mathbf{k}$ with $\operatorname{codim}_{\mathbb{P}^N} L \le dr$. We conjecture that the intersection of all linear subspaces (over $\overline{\mathbf{k}}$) of minimal codimension r contained in X, has codimension bounded above only in terms of r and d. We prove this when either $d \le 3$ or $r \le 2$.

1. Introduction

Let $f(x_1, \dots, x_n)$ be a homogeneous polynomial of degree $d \ge 2$ over a field $\mathbf{k}$. Recall that the *slice rank* $\operatorname{srk}_{\mathbf{k}}(f)$ of f is the minimal number r such that there exists a decomposition

$$f = l_1 f_1 + \ldots + l_r f_r,$$

where l_i are linear forms defined over $\mathbf{k}$.

The above notion is a symmetric version of the notion of slice rank for tensors introduced by Tao and Sawin (see [10]) in connection with the bounds on the maximal size of subsets in $\mathbb{F}_3^n$ not containing any lines, obtained in [4], [6]. It is also related to the notion of Schmidt rank (or strength) which plays a role in the Ananyan-Hochster proof of Stillman's conjecture [2].

The slice rank $\operatorname{srk}_{\mathbf{k}}(f)$ has a simple geometric meaning: it is the minimal codimension in $\mathbb{P}^{n-1}$ of a linear $\mathbf{k}$-subspace $P \subset \mathbb{P}^{n-1}$ contained in the projective hypersurface $f = 0$. Note that if $\operatorname{srk}_{\overline{\mathbf{k}}}(f) < n/2$ then this hypersurface is necessarily singular.

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It is clear that $\mathrm{srk}_{\mathbf{k}}(f) \geq \mathrm{srk}_{\overline{\mathbf{k}}}(f)$ and it is easy to find examples when $\mathrm{srk}_{\mathbf{k}}(f) > \mathrm{srk}_{\overline{\mathbf{k}}}(f)$.

One can ask for an upper estimate for $\mathrm{srk}_{\mathbf{k}}(f)$ in terms of $\mathrm{srk}_{\overline{\mathbf{k}}}(f)$. The first result of this paper is precisely such an estimate, which we obtain by adapting to homogeneous polynomials the theory of G-rank of tensors from the work of Derksen [5]. The main point is that one can associate to such a polynomial f a (real-valued) invariant $r^G_{\mathbf{k}}(f)$, called *the G-rank*, which does not change when we pass from $\mathbf{k}$ to $\overline{\mathbf{k}}$ (for perfect $\mathbf{k}$), and which can be bounded in terms of the slice rank. Note that Jiang showed in [7] that $r^G_{\mathbf{k}}(f)$ is equal to Derksen's G-rank of the corresponding symmetric tensor.

A bit more generally, for a collection $f_1, \ldots, f_s$ of homogeneous polynomials of the same degree d, we set

$$\mathrm{srk}_{\mathbf{k}}(f_1, \ldots, f_s) = \inf_{(c_1,\ldots,c_s)\neq(0,\ldots,0)} \mathrm{srk}_{\mathbf{k}}(c_1 f_1 + \ldots + c_s f_s),$$

and we define a real number $r^G_{\mathbf{k}}(f_1, \ldots, f_s)$ (see Section 2).

Theorem A. *Assume that the field $\mathbf{k}$ is perfect. Then for homogeneous polynomials $f_1, \ldots, f_s$ over $\mathbf{k}$ of degree $d \geq 2$, one has*

$$\mathrm{srk}_{\mathbf{k}}(f_1, \ldots, f_s) \leq r^G_{\mathbf{k}}(f_1, \ldots, f_s) = r^G_{\overline{\mathbf{k}}}(f_1, \ldots, f_s) \leq ds \cdot \mathrm{srk}_{\overline{\mathbf{k}}}(f_1, \ldots, f_s).$$

In particular, for a single polynomial we get inequality

$$\mathrm{srk}_{\mathbf{k}}(f) \leq d \cdot \mathrm{srk}_{\overline{\mathbf{k}}}(f).$$

The inequality for a single f is sharp for every degree d: if $E/\mathbf{k}$ is a Galois extension of degree d then the norm $E \to \mathbf{k}$ is a polynomial of degree d that has slice rank d over $\mathbf{k}$ and slice rank 1 over E.

Note that although $r^G_{\mathbf{k}}$ coincides with Derksen's G-rank of the corresponding symmetric tensor (as proved in [7]), the bounds connecting $r^G_{\mathbf{k}}$ with the slice rank are sharper in the symmetric case: the bounds obtained by Derksen for arbitrary tensors would only give the inequality $\mathrm{srk}_{\mathbf{k}}(f) \leq \frac{d^2}{2} \cdot \mathrm{srk}_{\overline{\mathbf{k}}}(f)$.

Our second goal in this paper is to understand the inequality of Theorem A more constructively. Geometrically, the statement is that if a hypersurface $X \subset \mathbb{P}^N$ of degree d, defined over $\mathbf{k}$, contains a linear subspace L of codimension r in $\mathbb{P}^N$, defined over $\overline{\mathbf{k}}$, then X contains a linear subspace L_0 of codimension $\leq dr$ in $\mathbb{P}^N$, defined over $\mathbf{k}$. One can ask for an explicit construction of L_0 from L and its Galois conjugates. The simplest answer would be that one can just take L_0 to be the intersection of all the Galois conjugates of L. We conjecture that the following stronger geometric statement holds.

Conjecture B. *Let $X = (f = 0) \subset \mathbb{P}^N$ be a hypersurface of degree d (over any ground field), and let r be the minimal natural number such that X contains a linear subspace L with $\mathrm{codim}_{\mathbb{P}^N} L = r$. Set*

$$L_f := \cap_{L \subset X, \mathrm{codim}_{\mathbb{P}^n} L = r} L.$$

Then there exists a function $c(r, d)$ such that $\mathrm{codim} L_f \leq c(r, d)$.

It is an easy exercise to check that the conjecture holds if $d = 2$ or $r = 1$ with $c(r, 2) = 2r$ and $c(1, d) = d$. We prove the following cases of the conjecture: when $d = 3$ (and r is arbitrary) and when $r = 2$ (and d is arbitrary).

Theorem C. *(i) Conjecture B holds for cubic hypersurfaces with*

$$c(r, 3) = c(r) := \frac{1}{2}\Big(\frac{(r+1)^2}{4} + r + 3\Big) \cdot \Big(\frac{(r+1)^2}{4} + r\Big).$$

(ii) Conjecture B holds for $r = 2$ with

$$c(2, d) = d^2 + 1.$$

More precisely, for a polynomial f of slice rank 2 and degree d, either $\mathrm{codim} L_f \leq d^2 - 1$ or f is a pullback from a space of dimension $\leq d^2 + 1$.

One can ask how far are the estimates of Theorem C from being optimal. In the case $d = 3$ and $r = 2$ we show that Conjecture B holds with $c(2, 3) = 6$ by giving a partial classification of cubic hypersurfaces of rank 2 (see Theorem 3.7). Consider the cubic $f(x_i, y_{jk})$, where $i = 1, \ldots, n$, $1 \leq j < k \leq n$ (so the number of variables is $n(n+1)/2$), given by

$$f = \sum_{i<j} x_i x_j y_{ij}.$$

One can check that the rank of f is equal to $n - 1$. Furthermore, for every $i < j$, f is in the ideal $(y_{ij}, x_k \mid k \neq i, k \neq j)$, so we get $\mathrm{codim} L_f = n(n+1)/2$, which depends quadratically on the rank. So it seems that the optimal bound $c(r, 3)$ is at least quadratic in r.

On the other hand, let us consider a polynomial f in n groups of variables

$$(x_1(1),\dots,x_m(1)),\ \dots,(x_1(n),\dots,x_m(n))$$

given by

$$f=\sum_{i=1}^{m}x_1(1)\dots\widehat{x_i(1)}\dots x_m(1)\cdot x_i(2)\dots x_i(n).$$

Then $\deg(f)=m+n-2$ and it is easy to see that f has rank 2 and $\mathrm{codim} L_f=mn$. This shows that the optimal bound $c(2,d)$ grows quadratically in d.

Theorem C(i) implies that if L is a linear subspace of minimal codimension r in $\mathbb{P}^N$, defined over a Galois extension of $\mathbf{k}$, contained in a cubic hypersurface X (defined over $\mathbf{k}$), then by taking intersection of all Galois conjugates of L we get a linear subspace L_0 of codimension $\le c(r)$, contained in X and defined over $\mathbf{k}$.

Since we don't know the validity of Conjecture B for general r and d, we give a more complicated construction of a linear subspace $L_0\subset X$, defined over $\mathbf{k}$, starting from the Galois conjugates of $L\subset X$ defined over a Galois extension of $\mathbf{k}$. For this we introduce the following recursive definition, where for linear subspaces $L_1,\dots,L_s\subset\mathbb{P}^N$ we denote by $\langle L_1,\dots,L_s\rangle\subset\mathbb{P}^N$ their linear span.

Definition 1.1. For a collection $\mathcal{L}=\{L_1,\dots,L_s\}$ of linear subspaces of $\mathbb{P}^N$, we define a new collection of linear subspaces of $\mathbb{P}^N$ as follows. Let $L=\langle L_1,\dots,L_s\rangle$. For each minimal subset $J\subset[1,s]$ such that $\langle L_j\mid j\in J\rangle=L$, we set $L_J:=\cap_{j\in J}L_j$, and we denote by $\mathcal{L}^{(1)}$ the collection of all such subspaces L_J. We denote by $\mathcal{L}^{(i)}$, $i\ge 1$, the collections of linear subspaces obtained by iterating this construction.

Theorem D. *Let $X\subset\mathbb{P}^N$ be a hypersurface of degree $d\ge 2$ and let $\mathcal{L}=(L_1,\dots,L_s)$ be a collection of linear subspaces contained in X, such that $\mathrm{codim}_{\mathbb{P}^N}L_i\le r$, where $r\ge 2$. Then for the linear subspace*

$$L_0:=\langle L\mid L\in\mathcal{L}^{(d-1)}\rangle,$$

we have $L_0\subset X$ and

$$\mathrm{codim}_{\mathbb{P}^N}L_0\le r^{2^{d-1}}.$$

Applying the construction of Theorem D to the collection of all Galois conjugates of a linear subspace of codimension $r\ge 2$ in $\mathbb{P}^N$, defined over some Galois extension of $\mathbf{k}$, contained in a hypersurface X (defined over $\mathbf{k}$), we get an algorithm for producing a linear subspace of codimension $\leq r^{2^{d-1}}$ in $\mathbb{P}^N$, contained in X and defined over $\mathbf{k}$.

One can ask whether the second inequality in Theorem A can also be explained constructively. In other words, starting with an s-dimensional subspace F of homogeneous polynomials of the same degree d, defined over a perfect field $\mathbf{k}$, such that there exists a nonzero $f \in F_{\overline{\mathbf{k}}}$ and a subspace of linear forms of dimension r over $\overline{\mathbf{k}}$ such that $f \in (L)$, we want to produce an element $f_0 \in F \setminus 0$ and a subspace of linear forms L_0, both defined over $\mathbf{k}$, such that $f_0 \in (L_0)$ and dimension of L_0 is $\leq c(sr)$. In Remark 4.1 we show how to do this using the algorithm of Theorem D for a single polynomial.

Our study is partially motivated by the desire to understand the related notion of the *Schmidt rank* (also known as *strength*) of a homogeneous polynomial (see [1], [3] and references therein), defined as the minimal number r such that f admits a decomposition $f = g_1 h_1 + \ldots + g_r h_r$, with $\deg(g_i)$ and $\deg(h_i)$ smaller than $\deg(f)$. Similarly to Theorem A one can try to estimate the Schmidt rank of a polynomial over a non-closed field in terms of its Schmidt rank over an algebraic closure. In [8], we show how to do this for quartic polynomials.

2. *G*-rank for homogeneous polynomials

Throughout this section we assume that the ground field $\mathbf{k}$ is perfect.

2.1. Definition of the *G*-rank and the relation to the slice rank

Below we introduce an analog of G-rank for symmetric tensors, or equivalently, for homogeneous polynomials, $r^G_{\mathbf{k}}(f)$ (where $G = \mathrm{GL}_n$). We show that it enjoys similar properties to Derksen's G-rank of a non-symmetric tensor studied in [5], in particular, it does not change under algebraic extensions of perfect fields. We also introduce the notion $r^G_{\mathbf{k}}(f_1, \ldots, f_s)$ of a G-rank for a collection of polynomials of the same degree.

Let V be an n-dimensional space over $\mathbf{k}$. We consider the group $G = \mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbf{k})$ acting naturally on the space $S^d V$, and the induced action on $\bigwedge^s (S^d V)$.

We consider points of G and of $\bigwedge^s S^d V$ with values in the ring of formal power series $\mathbf{k}[\![t]\!]$. For a $\mathbf{k}$-vector space W and a vector $w \in W[\![t]\!]$, we denote by $\mathrm{val}_t(w)$ the minimal $m \geq 0$ such that $w \in t^m W[\![t]\!]$.

For $f \in S^dV \subset S^dV[\![t]\!]$ and $g(t) \in G(\mathbf{k}[\![t]\!])$ such that $\mathrm{val}_t(g(t) \cdot f) > 0$, we set

$$\mu(g(t), f) = d \cdot \frac{\mathrm{val}_t(\det(g(t)))}{\mathrm{val}_t(g(t) \cdot f)}.$$

The factor d in front is a matter of convention: it makes the factor d disappear in some of the statements below.

Definition 2.1. (i) For nonzero $f \in S^dV$ we define its G-rank by

$$r^G_{\mathbf{k}}(f) = \inf_{g(t)} \mu(g(t), f),$$

where we take the infimum over all $g(t) \in G(\mathbf{k}[\![t]\!])$ such that $\mathrm{val}_t(g(t) \cdot f) > 0$.
(ii) More generally, for linearly independent $f_1, \ldots, f_s \in S^dV$, we define the G-rank by

$$r^G_{\mathbf{k}}(f_1, \ldots, f_s) = \inf_{g(t)} \mu(g(t), f_1, \ldots, f_s),$$

where

$$\mu(g(t), f_1, \ldots, f_s) = ds \cdot \frac{\mathrm{val}_t(\det(g(t)))}{\mathrm{val}_t(g(t) \cdot f_1 \wedge \ldots \wedge f_s)},$$

and the infinum is taken over $g(t) \in G(\mathbf{k}[\![t]\!])$ such that $\mathrm{val}_t(g(t) \cdot f_1 \wedge \ldots \wedge f_s) > 0$.

The formula

$$\mathrm{val}_t(g(t) \cdot f^m) = m \cdot \mathrm{val}_t(g(t) \cdot f)$$

immediately implies the following property.

Lemma 2.2. *For any $f \in S^dV$ and any $m \geq 1$ one has*

$$r^G_{\mathbf{k}}(f^m) = r^G_{\mathbf{k}}(f).$$

Here is the main result connecting the G-rank with the slice rank and also with the Waring rank.

Theorem 2.3. *Assume the base field $\mathbf{k}$ is perfect.*

(i) For a homogeneous polynomial f of degree d over $\mathbf{k}$ one has

$$\mathrm{srk}_{\mathbf{k}}(f) \le r^G_{\mathbf{k}}(f) \le d \cdot \mathrm{srk}_{\mathbf{k}}(f).$$

For a collection $f_1, \dots f_s$ of homogeneous polynomials of degree d over $\mathbf{k}$ one has

$$\mathrm{srk}_{\mathbf{k}}(f_1, \dots, f_s) \le r^G_{\mathbf{k}}(f_1, \dots, f_s) \le ds \cdot \mathrm{srk}_{\mathbf{k}}(f_1, \dots, f_s).$$

(ii) Suppose $m_1, \dots, m_r$ are divisors of d, and $f_1, \dots, f_r$ are homogeneous polynomials of degrees $\deg(f_i) = d/m_i$. Then

$$r^G_{\mathbf{k}}(f_1^{m_1} + \dots + f_r^{m_r}) \le r^G_{\mathbf{k}}(f_1) + \dots + r^G_{\mathbf{k}}(f_r).$$

In particular,

$$r^G_{\mathbf{k}}(f) \le w_{\mathbf{k}}(f),$$

where $w_{\mathbf{k}}(f)$ is the Waring rank of f, i.e., the minimal number r such that

$$f = l_1^d + \dots + l_r^d,$$

where l_i are linear forms defined over $\mathbf{k}$.

The proof will be given in Sec. 2.3 after some preparations. The argument is very close to the one in [5].

2.1.1. Relation to the GIT stability. Let W be a finite dimensional algebraic representation of $G = \mathrm{GL}(V)$ over $\mathbf{k}$. Recall that a point $w \in W$ is called *G-semistable* if the orbit closure $\overline{G \cdot v}$ does not contain 0. Recall that Kempf's $\mathbf{k}$-rational version of the Hilbert-Mumford criterion (see [9]) states (assuming $\mathbf{k}$ is perfect) that if w is not G-semistable then there exists a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ *defined over* $\mathbf{k}$ such that $\lim_{t \to 0} \lambda(t) \cdot w = 0$. Here λ has form $g \cdot \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n}) \cdot g^{-1}$ for some $g \in G(\mathbf{k})$ and $\lambda_i \in \mathbb{Z}$.

In the following statement we relate the G-rank to G-semistability, using an auxiliary element $u \in V^n$ which has (usual) rank n, viewed as an $n \times n$ matrix (recall that $\dim V = n$).

Proposition 2.4. *For integers $p \ge 0$ and $q > 0$, let us consider the G-representation*

$$W = (\bigwedge^s S^d V)^{\otimes p} \otimes \det{}^{-dsq} \oplus V^n.$$

Let $u \in V^n$ be a fixed element of rank n. Then we have $r^G_{\mathbf{k}}(f_1, \dots, f_s) \ge \frac{p}{q}$ if and only if $w = ((f_1 \wedge \dots \wedge f_s)^{\otimes p} \otimes 1, u)$ is G-semistable.

Proof. By Hilbert-Mumford-Kempf's criterion, if w is not G-semistable then there exists a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ over $\mathbf{k}$, such that $\lim_{t\to 0} \lambda(t) \cdot w = (0,0)$. In particular, we have $\lim_{t\to 0} \lambda(t) \cdot u = 0$, so $\lambda(t) \in G(\mathbf{k}[t])$, and

$$\begin{aligned}&\mathrm{val}_t(\lambda(t) \cdot (f_1 \wedge \ldots \wedge f_s)^{\otimes p} \otimes 1)\\ &\qquad = p \cdot \mathrm{val}_t(\lambda(t) \cdot f_1 \wedge \ldots \wedge f_s) - dsq \cdot \mathrm{val}_t \det(\lambda(t)) > 0,\end{aligned}$$

which implies that $\mathrm{val}_t(\lambda(t) \cdot f_1 \wedge \ldots \wedge f_s) > 0$ and

$$\mu(\lambda(t), f_1, \ldots, f_s) < \frac{p}{q}.$$

Hence, $r^G(f_1, \ldots, f_s) < \frac{p}{q}$.

Conversely, assume there exists $g(t) \in G(\mathbf{k}[\![t]\!])$ such that $\mathrm{val}_t(g(t) \cdot f_1 \wedge \ldots \wedge f_s) > 0$ and $\mu(g(t), f_1, \ldots, f_s) < \frac{p}{q}$, i.e.,

$$\mathrm{val}_t(g(t) \cdot (f_1 \wedge \ldots \wedge f_s)^{\otimes p} \otimes 1) > 0.$$

Truncating $g(t)$ at high enough order in t, we can assume that $g(t) \in G(\mathbf{k}[t])$. Then the fact that $\lim_{t\to 0} g(t) \cdot w = (0, g(0) \cdot u)$ implies that $(0, g(0) \cdot u)$ lies in the closure of the G-orbit of w. Since 0 lies in the closure of the G-orbit of $g(0) \cdot u$ (we can just use the 1-parameter subgroup $t \cdot \mathrm{id}_V$ in G to see this), we see that $(0,0)$ lies in the closure of $G \cdot w$, so w is not G-semistable. □

As a consequence of Proposition 2.4, in the definition of $r^G_{\mathbf{k}}(f_1, \ldots, f_s)$ it is enough to take $g(t)$ to be a 1-parameter subgroup of G defined over $\mathbf{k}$. Also, since G-semistability does not change under the base field extension, we deduce the following

Corollary 2.5. *Let $\overline{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$. Then one has*

$$r^G_{\mathbf{k}}(f_1, \ldots, f_s) = r^G_{\overline{\mathbf{k}}}(f_1, \ldots, f_s).$$

Let $T \subset G$ denote the maximal torus, i.e., the group of diagonal matrices with respect to a $\mathbf{k}$-basis (e_i) of V. Replacing G everywhere by T we get a notion of T-rank, $r^T_{\mathbf{k}}(f_1, \ldots, f_s)$. From Hilbert-Mumford-Kempf criterion we get

$$r^G_{\mathbf{k}}(f_1, \ldots, f_s) = \inf_{g \in G(\mathbf{k})} r^T_{\mathbf{k}}(g \cdot (f_1, \ldots, f_s)).$$

The reason we introduced the factor ds in the definition of $r^G_{\mathbf{k}}(f_1, \ldots, f_s)$ is so as to have the following normalization property.

Lemma 2.6. *One has* $r^G_{\mathbf{k}}(f_1, \ldots, f_s) \geq 1$.

Proof. It is enough to check that for any $g(t) \in T(\mathbf{k}[\![t]\!])$ and any distinct monomials $M_1, \ldots, M_s$ of (e_i) in S^dV, one has

$$\mathrm{val}_t(g(t) \cdot M_1 \wedge \ldots \wedge M_s) \leq ds \cdot \mathrm{val}_t(\det(g(t))).$$

Let $c_1, \ldots, c_n \geq 0$ be the valuations of the diagonal entries of $g(t)$, so that

$$\mathrm{val}_t(\det(g(t))) = c_1 + \ldots + c_n.$$

Then for a monomial $M = e_1^{a_1} \ldots e_n^{a_n}$, we have

$$\begin{aligned}\mathrm{val}_t(g(t) \cdot M) = a_1c_1 + \ldots + a_nc_n &\leq (a_1 + \ldots + a_n)(c_1 + \ldots + c_n) \\ &= d(c_1 + \ldots + c_n).\end{aligned}$$

Hence, $\mathrm{val}_t(g(t) \cdot M_1 \wedge \ldots \wedge M_s) \leq ds$, which gives the required inequality. □

2.2. Triangle inequality

Proposition 2.7. *For* $f_1, f_2 \in S^dV$ *one has* $r^G_{\mathbf{k}}(f_1 + f_2) \leq r^G_{\mathbf{k}}(f_1) + r^G_{\mathbf{k}}(f_2)$.

Proof. This is proved exactly as [5, Prop. 3.6]. Starting with $g_1(t), g_2(t) \in G(\mathbf{k}[\![t]\!])$ such that $\mathrm{val}_t(g_i(t) \cdot f_i) > 0$, one has to produce $u(t) \in G(\mathbf{k}[\![t]\!])$ with $\mathrm{val}_t(u(t) \cdot (f_1 + f_2)) > 0$ and

$$\mu(u(t), f_1 + f_2) \leq \mu(g_1(t), f_1) + \mu(g_2(t), f_2).$$

Making changes of variables $t \mapsto t^i$ if necessary, we can assume that

$$\mathrm{val}_t(g_1(t) \cdot f_1) = \mathrm{val}_t(g_2(t) \cdot f_2) = s > 0.$$

By [5, Lem. 3.5], there exists $u(t) \in G(\mathbf{k}[\![t]\!])$ such that $u(t) = u_1(t)g_1(t) = u_2(t)g_2(t)$ with $u_i(t) \in G(\mathbf{k}[\![t]\!])$ and

$$\mathrm{val}_t(\det u(t)) \leq \mathrm{val}_t(\det g_1(t)) + \mathrm{val}_t(\det g_2(t)).$$

Then

$$\mathrm{val}_t(u(t) \cdot (f_1 + f_2)) \geq \min(\mathrm{val}_t(u_1(t)g_1(t) \cdot f_1), \mathrm{val}_t(u_2(t)g_2(t) \cdot f_2)) \geq s,$$

and

$$\frac{1}{d}\mu(u(t), f_1+f_2) = \frac{\mathrm{val}_t(\det u(t))}{\mathrm{val}_t(u(t)\cdot(f_1+f_2))}$$
$$\le \frac{\mathrm{val}_t(\det g_1(t)) + \mathrm{val}_t(\det g_2(t))}{s} = \frac{1}{d}(\mu(g_1(t), f_1) + \mu(g_2(t), f_2)).$$

□

2.3. Relation to the slice rank and to the sums of powers

In this section we will prove Theorem 2.3. We always assume that $f \in S^dV$ (resp., $f_i \in S^dV$), where V is an n-dimensional space over a field $\mathbf{k}$.

Proposition 2.8. *(i) Let $f = v^d$ for some $v \in V \setminus 0$. Then $r^G_{\mathbf{k}}(f) = 1$.*
(ii) One has $r^G_{\mathbf{k}}(f) \le d \cdot \mathrm{srk}_{\mathbf{k}}(f)$.
(iii) If there exists a nontrivial linear combination $c_1 f_1 + \ldots + c_s f_s$ that has slice rank r then $r^G_{\mathbf{k}}(f_1, \ldots, f_s) \le dsr$.

Proof. (i) By Lemma 2.6, $r^G_{\mathbf{k}}(f) \ge 1$, so it is enough to find $g(t) \in G(\mathbf{k}[t])$ such that $\mu(g(t), v^d) = 1$. We can assume that $v = e_1$, and take

$$g(t) = \mathrm{diag}(t, 1, \ldots, 1).$$

Then $\mathrm{val}_t(g(t) \cdot e_1^d) = d$ and $\mathrm{val}_t(\det(g)) = 1$. (Alternatively, we can use Lemma 2.2 to reduce to the easy case $d = 1$.)
(ii) We can assume that $f = e_1 \cdot f_1 + \ldots + e_r \cdot f_r$. Then for $g(t) = \mathrm{diag}(\underbrace{t, \ldots, t}_{r}, 1, \ldots, 1)$, we have $\mathrm{val}_t(g(t) \cdot f) \ge 1$, while $\mathrm{val}_t(\det(g)) = r$, so

$$\mu(g(t), f) \le d \cdot \frac{r}{\mathrm{val}_t(g(t)\cdot f)} \le dr.$$

(iii) If this is the case then $f_1 \wedge \ldots \wedge f_s$ has form $(e_1h_1 + \ldots + e_rh_r) \wedge \ldots$, hence, for the same $g(t)$ as in (ii), we have $\mathrm{val}_t(g(t) \cdot f_1 \wedge \ldots \wedge f_s) \ge 1$. □

Proposition 2.9. *One has*

$$\mathrm{srk}_{\mathbf{k}}(f_1, \ldots, f_s) \le r^G_{\mathbf{k}}(f_1, \ldots, f_s).$$

Proof. Suppose $r^G_{\mathbf{k}}(f_1,\ldots,f_s)<r$. Then there exists a 1-parameter subgroup $g(t)$ such that

$$r\cdot \mathrm{val}_t(g(t)\cdot f_1\wedge\ldots\wedge f_s)>ds\cdot \mathrm{val}_t(\det(g(t))).$$

We can assume that $g(t)$ is diagonal with respect to some basis $(e_1,\ldots,e_n)$ of V. Now consider the set

$$S:=\{i\in[1,n]\mid \mathrm{val}_t(g(t)\cdot e_i)\ge \frac{\mathrm{val}_t(g(t)\cdot f_1\wedge\ldots\wedge f_s)}{ds}\}.$$

Note that

$$\mathrm{val}_t(\det(g(t)))\ge\sum_{i\in S}\mathrm{val}_t(g(t)\cdot e_i)\ge|S|\cdot\frac{\mathrm{val}_t(g(t)\cdot f_1\wedge\ldots\wedge f_s)}{ds},$$

hence,

$$|S|<r.$$

We claim that there exists a nontrivial linear combination $f=c_1f_1+\ldots+c_sf_s$ such that all the monomials appearing in f are divisible by some e_i with $i\in S$. Indeed, otherwise, the projection

$$\langle f_1,\ldots,f_s\rangle\to\mathbf{k}[e_1,\ldots,e_n]\to\mathbf{k}[e_1,\ldots,e_n]/(e_i\mid i\in S)\simeq\mathbf{k}[e_i\ |i\notin S]$$

is injective, so there exist s distinct monomials $M_1,\ldots,M_s$ of degree d in $\mathbf{k}[e_i\ |i\notin S]$ such that $M_1\wedge\ldots\wedge M_s$ appears with a nonzero coefficient in $f_1\wedge\ldots\wedge f_s$. But then by the choice of S,

$$\begin{aligned}\mathrm{val}_t(g(t)\cdot f_1\wedge\ldots\wedge f_s)&\le\mathrm{val}_t(g(t)\cdot M_1\wedge\ldots\wedge M_s)\\&<\mathrm{val}_t(g(t)\cdot f_1\wedge\ldots\wedge f_s)\end{aligned}$$

which is a contradiction, proving our claim. Now for the obtained linear combination f we have

$$\mathrm{srk}_{\mathbf{k}}(f)\le|S|<r.$$

□

Proof of Theorem 2.3. (i) This follows from Proposition 2.8(iii) and Proposition 2.9.
(ii) This follows from Proposition 2.7 (the triangle inequality), Lemma 2.2 and Proposition 2.8(i) (for the part concerning the Waring rank). □

Proof of Theorem A. We combine Theorem 2.3(i) with Corollary 2.5. □

2.4. Example of a calculation of G-rank

As we have seen before, for any linear form form l one has $r^G_{\mathbf{k}}(l^d) = 1$. Here is the next simplest case.

Proposition 2.10. *Let V be a 2-dimensional vector space with a basis x_1, x_2. For any $m > 0$, one has*

$$r^G_{\mathbf{k}}(x_1^{2m} x_2^m) = \frac{3}{2}.$$

Proof. By Lemma 2.2, it is enough to prove that

$$r^G_{\mathbf{k}}(x_1^2 x_2) = \frac{3}{2}.$$

Considering $g(t) = \mathrm{diag}(t, 1)$, we immediately see that $r^G_{\mathbf{k}}(x_1^2 x_2) \le 3/2$.

Now consider any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{k}[\![t]\!]).$$

It is enough to prove that $\mu(g, x_1^2 x_2) \le 3/2$. We have

$$g \cdot x_1^2 x_2 = a^2 c \cdot x_1^3 + a(ad + 2bc) \cdot x_1^2 x_2 + b(bc + 2ad) \cdot x_1 x_2^2 + b^2 d \cdot x_2^3.$$

Let us abbreviate $v(\cdot) = \mathrm{val}_t(\cdot)$, etc. Set $s := v(g \cdot x_1^2 x_2)$. Then we have

$$2v(a) + v(c) \ge s, \quad v(a) + v(ad + 2bc) \ge s,$$
$$v(b) + v(bc + 2ad) \ge s, \quad 2v(b) + v(d) \ge s.$$

We consider three cases.

Case $v(ad) > v(bc)$.

Then we have $v(\det(g)) = v(bc)$ and $v(bc + 2ad) = v(bc)$. Hence, from the above inequalities we get $v(b) + v(bc) \ge s$, hence, $2v(bc) \ge s$, so $v(\det(g)) = v(bc) \ge s/2$, and so $\mu(g, x_1^2 x_2) \ge 3/2$.

Case $v(ad) < v(bc)$.

Then we have $v(\det(g)) = v(ad)$ and $v(ad + 2bc) = v(ad)$. Hence, $2v(ad) \ge v(a) + v(ad) \ge s$, and we again get $v(\det(g)) \ge s/2$.

Case $v(ad) = v(bc)$.

Set $t = v(ad) = v(bc)$. Then we have $v(\det(g)) \geq t$. Now by the above inequalities,

$$4t = 2v(ad) + 2v(bc) \geq (2v(a) + v(c)) + (2v(b) + v(d)) \geq 2s,$$

which again implies $v(\det(g)) \geq s/2$. □

3. Linear subspaces of minimal codimension in cubics

In this section we will prove Theorem C(i) and its improved version for cubics of slice rank 2 (with $c(2,3) = 6$).

3.1. Some general observations

Let $f \in \mathbf{k}[V]$ be a nonzero homogeneous polynomial of slice rank r, and let $X \subset \mathbb{P}V$ be the corresponding projective hypersurface. We are interested in the intersection

$$L_f := \cap_{L \subset X, \operatorname{codim}_{\mathbb{P}V} L = r} L \subset \mathbb{P}V.$$

Recall that we are looking for an estimate for the codimension of L_f. The case $r = 1$ is straightforward:

Lemma 3.1. *Let f be a homogeneous polynomial of degree d and slice rank 1. Then there are at most d hyperplanes contained in X, so* $\operatorname{codim}_{\mathbb{P}V} L_f \leq d$.

Since the slice rank is determined in terms of ideals $(P) \subset \mathbf{k}[V]$ generated by subspaces P of linear forms, we record some easy observations about such ideals.

Lemma 3.2. *Let $A \subset B$ be an extension of commutative rings, such that B is flat as A-algebra. Then for any pair of ideals $J_1, J_2 \subset A$, one has*

$$(J_1 \cdot B) \cap (J_2 \cdot B) = (J_1 \cap J_2) \cdot B.$$

In particular, for a collection of linear subspaces $P_i \subset W$, $i = 1, \ldots, s$, where $W \subset V^*$ is a subspace, we have

$$P_1\mathbf{k}[V] \cap \ldots \cap P_s\mathbf{k}[V] = (P_1 S(W) \cap \ldots \cap P_s S(W)) \cdot \mathbf{k}[V].$$

Proof. Since for any ideal $J \subset A$ the natural map $J \otimes_A B \to J \cdot B$ is an isomorphism in this case, the assertion follows by applying the exact functor

$? \otimes_A B$ to the exact sequence

$$0 \to J_1 \cap J_2 \to J_1 \oplus J_2 \to A.$$

For the last statement we apply this to the flat extension of rings $S(W) \subset S(V^*) = \mathbf{k}[V]$. □

Lemma 3.3. *Let $P_1, \ldots, P_s \subset V^*$ be subspaces such that the ideal $(P_1)^{a_1} \cap \ldots \cap (P_s)^{a_s}$ contains no nonzero homogeneous polynomials of degree m, for some powers $a_i \geq 1$. Then we have an inclusion of ideals in $\mathbf{k}[V]$,*

$$(P_1)^{a_1} \cap \ldots \cap (P_s)^{a_s} \subset (W)^{m+1}.$$

where $W = P_1 + \ldots + P_s$. In particular, if $P_1 \cap \ldots \cap P_s = 0$ then

$$(P_1)^m \cap \ldots \cap (P_s)^m \subset (W)^{m+1}.$$

Proof. Applying Lemma 3.2 to the extension of rings $S(W) \subset S(V^*) = \mathbf{k}[V]$, we reduce to the case when $W = V^*$. But then the first statement reduces to the fact that if the ideal $I = (P_1)^{a_1} \cap \ldots \cap (P_s)^{a_s}$ does not contain polynomials of degree m then $I \subset (x_1, \ldots, x_n)^{m+1}$.

To prove the second statement we need to check that $(P_1)^m \cap \ldots \cap (P_s)^m$ does not contain any homogeneous polynomials of degree $\leq m$. This is clear in degrees $< m$ and in degree m follows from the statement that

$$0 = S^m(P_1 \cap \ldots \cap P_s) = S^m(P_1) \cap \ldots \cap S^m(P_s) \subset S^m W,$$

since $P_1 \cap \ldots \cap P_s = 0$. □

Definition 3.4. We say that a polynomial $f \in \mathbf{k}[V] = S(V^*)$ is a *pullback from a space of dimension m* if there exists a linear subspace $W \subset V^*$ of dimension m such that $f \in S(W) \subset S(V^*)$. In this case, if $f \in (P)$, where $P \subset V^*$ is a subspace of linear forms, then $f \in (W \cap P)$. In particular, the slice rank of f in $S(V^*)$ can be calculated within $S(W)$.

3.2. Proof of Theorem C(i)

Theorem C(i) is a consequence of the following more precise theorem.

Theorem 3.5. *Let f be a cubic of rank r, $X \subset \mathbb{P}V$ the corresponding hypersurface. Set*

$$c(r) := \frac{1}{2}\Big(\frac{(r+1)^2}{4} + r + 3\Big) \cdot \Big(\frac{(r+1)^2}{4} + r\Big).$$

Then

- *either all linear subspaces $L \subset X$ with $\operatorname{codim}_{\mathbb{P}V} L = r$ are contained in a fixed hyperplane,*
- *or f is a pullback from a space of dimension $c(r)$.*

In either case $\operatorname{codim} L_f \leq c(r)$.

Lemma 3.6. *Let $P_1, \ldots, P_s \subset V^*$ be an irredundant collection of subspaces such that $P_1 \cap \ldots \cap P_s = 0$ (i.e., the intersection of any proper subcollection is nonzero). Assume that $\dim P_i \leq r$ for every i. Then*

$$\dim(P_1 + \ldots + P_s) \leq r + \frac{(r+1)^2}{4}.$$

Proof. Let a be the minimal dimension of intersections $P_i \cap P_j$. Then we claim that $s \leq a + 2$. Indeed, without loss of generality we can assume that $\dim P_1 \cap P_2 = a$. Then for each $i \geq 2$ we should have

$$\dim P_1 \cap P_2 \cap \ldots \cap P_i \leq a + 2 - i,$$

due to irredundancy of the collection, which proves the claim for $i = s$.

On the other hand, since $\dim P_i/(P_i \cap P_1) \leq r - a$ for $i > 1$, we get that

$$N := \dim(P_1 + \ldots + P_s) \leq r + (s-1)(r-a)$$
$$\leq r + (a+1)(r-a) \leq r + \frac{(r+1)^2}{4}.$$

□

Proof of Theorem 3.5. We use induction on r. For $r = 1$ the assertion is clear. Assume $r > 1$ and the assertion holds for $r - 1$. Let $\mathcal{P}_f$ denote the set of r-dimensional subspaces $P \subset V^*$ such that $f|_{P^\perp} = 0$, or equivalently, $f \subset (P)$.

If all $P \in \mathcal{P}_f$ contain the same line (v^*) then we can apply the induction assumption to the restriction of f to the hyperplane $v^* = 0$ in V, which has slice rank $r-1$. Then the induction assumption implies that

$$\operatorname{codim}_{\mathbb{P}V} L_f \leq c(r-1) + 1 \leq c(r).$$

Otherwise, there exist $P_1, \dots, P_s \in \mathcal{P}_f$ such that $P_1 \cap \dots \cap P_s = 0$. Choosing a minimal such collection of subspaces and using Lemma 3.6, we get

$$N := \dim(P_1 + \dots + P_s) \leq r + \frac{(r+1)^2}{4}.$$

Now by Lemma 3.3, f belongs to $(W) \cdot (W)$, where $W = P_1 + \dots + P_s$. Hence, f can be written in the form

$$f = \sum_{1 \leq i \leq j \leq N} w_i w_j l_{ij},$$

for some linear forms l_{ij}, where (w_i) is a basis of W. Hence, f is a pullback from a space of dimension $\leq \frac{N(N+1)}{2} + N \leq c(r)$. □

3.3. Cubics of slice rank 2

The bound of Theorem C(i) may be far from optimal. Here we study in more detail the case of cubics of slice rank 2, proving in this case Conjecture B with $c(2,3) = 6$ and partially classifying such cubics.

Theorem 3.7. *Let f be a cubic of rank 2. Then*

- *either all $L \subset X$ with $\operatorname{codim}_{\mathbb{P}V} = 2$ are contained in a fixed hyperplane, or*
- *f is a pullback from a 6-dimensional space, or*
- *f can be written in the form*

$$f = x_1 y_1 z_1 + x_1 y_2 z_2 + x_2 y_1 z_3,$$

  where $x_1, x_2, y_1, y_2, z_1, z_2, z_3$ are linearly independent, or
- *f is a pullback from an 8-dimensional space and $\operatorname{codim}_{\mathbb{P}V} L_f \leq 4$, or*
- *f is a pullback from a 9-dimensional space and $\operatorname{codim}_{\mathbb{P}V} L_f \leq 3$.*

In either case $\operatorname{codim}_{\mathbb{P}V} L_f \leq 6$.

From now on we fix a cubic $f \in \mathbf{k}[V]$ of slice rank 2. As in the proof of Theorem 3.5 we denote by $\mathcal{P}_f$ the set of 2-dimensional subspaces $P \subset V^*$ such that $f|_{P^\perp} = 0$, or equivalently, $f \subset (P)$, where $(P) \subset \mathbf{k}[V]$ denotes the ideal generated by P.

The following result is well known but we include the (simple) proof for reader's convenience.

Lemma 3.8. *Let $\mathcal{S}$ be a set of 2-dimensional subspaces in V^* such that for any $P_1, P_2 \in \mathcal{S}$ we have $P_1 \cap P_2 \neq 0$. Then either there exists a line $L \subset V^*$ such that $L \subset P$ for all $P \in \mathcal{S}$, or there exists a 3-dimensional subspace $W \subset V^*$ such that $P \subset W$ for all $P \in \mathcal{S}$.*

Proof. We can think of $\mathcal{S}$ as a family of projective lines in the projective space such that any two intersect. Our statement is that either they all pass through one point, or they are contained in a plane. Indeed, assume they do not all pass through one point. Pick a pair of lines ℓ_1, ℓ_2 intersecting at a point p. There exists a line ℓ_3, not passing through p. Then ℓ_1, ℓ_2, ℓ_3 form a triangle in a plane. Now given any other line ℓ from $\mathcal{S}$, we can pick a vertex of the triangle such that ℓ does not pass through it. Say, assume ℓ does not pass through p. Then $\ell \cap \ell_1$ and $\ell \cap \ell_2$ are two distinct points of ℓ, so ℓ is contained in the plane of the triangle. □

Lemma 3.9. *Assume that for any pair $P_1, P_2 \in \mathcal{P}_f$ we have $P_1 \cap P_2 \neq 0$. Then either there exists a nonzero linear form $v^* \in V^*$, such that $v^* \in P$ for all $P \in \mathcal{P}_f$, in which case $\operatorname{codim}_{\mathbb{P}V} L_f \leq 4$, or f is a pullback from a 9-dimensional space and $\operatorname{codim}_{\mathbb{P}V} L_f \leq 3$.*

Proof. By Lemma 3.8, either all planes in $\mathcal{P}_f$ span at most 3-dimensional subspace $W \subset V^*$, or there exists a nonzero linear form $v^* \in V^*$ such that $v^* \in P$ for all $P \in \mathcal{P}_f$. In the latter case let us consider the restriction $\widetilde{f}$ of our cubic to the hyperplane $H_{v^*} \subset V$. Then $\widetilde{f}$ has rank 1 and $\mathcal{P}_f$ can be identified with $\mathcal{P}_{\widetilde{f}}$. So by Lemma 3.1, L_f has codimension 3 in H_{v^*}, hence, it has codimension 4 in V.

Now let us consider the case when all planes in $\mathcal{P}_f$ are contained in a 3-dimensional subspace W, and have zero intersection. Then by Lemma 3.3, $f \in (W)^2$. Hence, as in the proof of Theorem 3.5, we deduce that f depends on ≤ 9 variables. □

Lemma 3.10. *Assume there exist linearly independent linear forms $x_1, x_2, y_1, y_2 \in V^*$ such that $\operatorname{span}(x_1, x_2) \in \mathcal{P}_f$ and $\operatorname{span}(y_1, y_2) \in \mathcal{P}_f$. Then*

f is a pullback from an 8-dimensional space, and one of the following possibilities hold:

1) *f is a pullback from a 6-dimensional space;*
2) *for all $P \in \mathcal{P}_f$ one has $P \subset \mathrm{span}(x_1, x_2, y_1, y_2)$;*
3) *f can be written in the form*

$$f = x_1y_1z_1 + x_1y_2z_2 + x_2y_1z_3,$$

where $x_1, x_2, y_1, y_2, z_1, z_2, z_3$ are linearly independent.

Proof. Note that we can write

$$f = x_1y_1l_{11} + x_1y_2l_{12} + x_2y_1l_{21} + x_2y_2l_{22},$$

for some linear forms $l_{ij} \in V^*$. This immediately implies that f depends on ≤ 8 variables.

Let $P = \mathrm{span}(l_1, l_2)$ be in $\mathcal{P}_f$. First, we claim that if $P \cap \mathrm{span}(x_1, x_2) = 0$ and $P \cap \mathrm{span}(y_1, y_2) = 0$ then either $P \subset \mathrm{span}(x_1, x_2, y_1, y_2)$ or f is a pullback from a 6-dimensional space. Indeed, assume that P is not contained in $\mathrm{span}(x_1, x_2, y_1, y_2)$. First, we observe that for generic $x \in \mathrm{span}(x_1, x_2)$ and generic $y \in \mathrm{span}(y_1, y_2)$ we should have $P \cap \mathrm{span}(x, y_1, y_2) = 0$ and $P \cap \mathrm{span}(y, x_1, x_2) = 0$. Indeed, otherwise we could pick generic $x, x' \in \mathrm{span}(x_1, x_2)$ such that there exist nonzero vectors $v \in P \cap \mathrm{span}(x, y_1, y_2)$ and $v' \in P \cap \mathrm{span}(x', y_1, y_2)$. But then, since $P \cap \mathrm{span}(y_1, y_2) = 0$, we would have that v and v' are linearly independent, and so $P = \mathrm{span}(v, v') \subset (x_1, x_2, y_1, y_2)$. Hence, changing bases of $\mathrm{span}(x_1, x_2)$ and $\mathrm{span}(y_1, y_2)$ if necessary, we can assume that

$$\begin{aligned} P \cap \mathrm{span}(x_1, y_1, y_2) = P \cap \mathrm{span}(x_2, y_1, y_2) &= P \cap \mathrm{span}(y_1, x_1, x_2) \\ &= P \cap \mathrm{span}(y_2, x_1, x_2) = 0. \end{aligned}$$

Now the fact that $f \in (P)$ implies that

$$x_1(y_1l_{11} + y_2l_{12}) \in (x_2, P).$$

Note that any ideal generated by linear forms is prime (as the quotient is a domain), so (x_2, P) is a prime ideal. Since $x_1 \notin (x_2, P)$, we get that

$y_1 l_{11} + y_2 l_{12} \in (x_2, P)$. Hence,

$$y_1 l_{11} \in (x_2, y_2, P).$$

We know that $y_1 \notin (x_2, y_2, P)$ since otherwise we would get a nonzero intersection $P \cap (x_2, y_1, y_2)$. Hence $l_{11} \in (x_2, y_2, P)$. Similarly, we get $l_{12} \in (x_2, y_1, P)$, $l_{21} \in (x_1, y_2, P)$, and $l_{22} \in (x_1, y_1, P)$. But this implies that f is a pull-back from a 6-dimensional space.

It remains to consider the case when there exists P in $\mathcal{P}_f$, such that

$$P \cap \operatorname{span}(y_1, y_2) = 0 \ \text{ and } P \cap \operatorname{span}(x_1, x_2) = \operatorname{span}(x_1).$$

Then the condition $f \in (P)$ gives

$$x_2(y_1 l_{21} + y_2 l_{22}) \in (P).$$

Hence, $y_1 l_{21} + y_2 l_{22} \in (P)$, which implies that

$$y_1 l_{21} \in (y_2, P).$$

Since $y_1 \notin (y_2, P)$, we get $l_{21} \in (y_2, P)$. Similarly, we get $l_{22} \in (y_1, P)$. Let $P = \operatorname{span}(x_2, l)$, where $l \in V^*$. Then we can write

$$l_{21} = a_1 x_1 + b_1 y_2 + c_1 l, \quad l_{22} = a_2 x_1 + b_2 y_1 + c_2 l,$$

so we can rewrite f in the form

$$\begin{aligned} f = {} & x_1 y_1 (l_{11} + a_1 x_2) + x_1 y_2 (l_{12} + a_2 x_2) \\ & + x_2 (c_1 y_1 + c_2 y_2) l + (b_1 + b_2) x_2 y_1 y_2. \end{aligned}$$

The condition $f \in (x_1, l)$ gives $(b_1 + b_2) x_2 y_1 y_2 \in (x_1, l)$, which is possible only if $b_1 + b_2 = 0$. This easily implies that either f is a pullback from a 6-dimensional space, or can be written in the form (3). □

Proof of Theorem 3.7. Taking into account Lemmas 3.9 and 3.10, it remains to prove that in the situation of Lemma 3.10 one has $\operatorname{codim} L_f \leq 6$. This is clear in cases (1) and (2). In case (3), it is easy to check that $\mathcal{P}_f$ consists of 4 elements:

$$(x_1, x_2), (y_1, y_2), (x_1, z_3), (y_1, z_2).$$

The corresponding intersection has codimension 6. □

4. Hypersurfaces of higher degree

4.1. Proof of Theorem C(ii)

We use induction on $d \geq 1$. The case $d = 1$ is clear, so assume that $d \geq 2$ and the assertion holds for degrees $< d$. Assume that $\dim \sum_{P \in \mathcal{P}_f} P > d^2 - 1$ (otherwise we are done), and let $\{P_1, \ldots, P_n\}$ be a minimal subset of $\mathcal{P}_f$ such that

$$\dim \sum_{i=1}^{n} P_i > d^2 - 1.$$

Note that by minimality, $\dim \sum_{i=1}^{n-1} P_i \leq d^2 - 1$, so

$$\dim \sum_{i=1}^{n} P_i \leq d^2 + 1.$$

Claim. There are no nonzero homogeneous polynomials of degree $d - 1$ in the ideal $(P_1) \cap \ldots \cap (P_n)$.

Indeed, suppose $g \in (P_1) \cap \ldots \cap (P_n)$ is such a polynomial. We have one of the two cases:

Case 1. $g = l_1 \ldots l_k \cdot h$, where $\deg l_i = 1$, $0 \leq k < d - 2$, $\mathrm{srk}(h) \geq 2$.

Case 2. $g = l_1 \ldots l_{d-1}$, where $\deg l_i = 1$.

Let us consider Case 1 first. Since each (P_i) is a prime ideal, we should have a decomposition

$$\{1, \ldots, n\} = S_1 \cup \ldots \cup S_k \cup S,$$

where $l_j \in P_i$ for all $i \in S_j$ and $h \in (P_i)$ for $i \in S$ (and $S = \emptyset$ if $\mathrm{srk} h > 2$).

Let us fix j such that $S_j \neq \emptyset$. Then $f \mod (l_i)$ has slice rank 1, hence

$$\dim \sum_{i \in S_j} P_i/(l_i) \leq d$$

(by Lemma 3.1). In other words,

$$\dim \sum_{i \in S_j} P_i \leq d + 1.$$

On the other hand, assuming that $S \neq \emptyset$ and applying the induction hypothesis to h, we get

$$\dim \sum_{i\in S} P_i \leq (d-1-k)^2 + 1.$$

Hence, we obtain

$$\dim \sum_{i=1}^{n} P_i \leq k(d+1) + (d-1-k)^2 + 1 \leq d^2 - 1,$$

which is a contradiction.

Similarly, in Case 2 we get

$$\dim \sum_{i=1}^{n} P_i \leq (d-1)(d+1) = d^2 - 1,$$

which is a contradiction. This proves the Claim.

Combining the Claim with Lemma 3.3, we get the inclusion

$$f \in (P_1) \cap \ldots \cap (P_n) \subset (P_1 + \ldots + P_n)^d.$$

Hence, f is a pullback from a space of dimension $\leq d^2 + 1$. This finishes the proof. □

4.2. Proof of Theorem D

Let us dualize the recursive procedure described in Definition 1.1. For a collection $\mathcal{P} = (P_1, \ldots, P_s)$ of subspaces of V^* we set $P^{(1)} = \cap_{i=1}^{s} P_i$, and for each minimal subset $J \subset [1, s]$ such that $\cap_{j\in J} P_j = P^{(1)}$, we set $P_J := \sum_{j\in J} P_j$. We denote by $\mathcal{P}^{(1)}$ the collection of all subspaces P_J of V^* obtained in this way. Iterating this procedure we get collections of subspaces $\mathcal{P}^{(i)}$ for $i \geq 0$, where $\mathcal{P}^{(0)} = \mathcal{P}$. Let us also set $P^{(0)} = 0$ and for $i \geq 0$,

$$P^{(i+1)} := \cap_{P\in\mathcal{P}^{(i)}} P.$$

Note that $P^{(i)} \subset P^{(i+1)}$.

Step 1. If $\dim P_i \leq r$ for all i then $\dim P_J \leq r^2$. Indeed, let $a = \dim P^{(1)}$. Then $\dim P_i/P^{(1)} \leq r - a$ and applying Lemma 3.6 we see that for every minimal subset J with $\cap_{j\in J}P_j = P^{(1)}$, one has

$$\dim P_J = a + \dim P_J/P^{(1)} \le a + (r-a) + \frac{(r-a+1)^2}{4} \le r + \frac{(r+1)^2}{4}.$$

Since

$$\lfloor r + \frac{(r+1)^2}{4} \rfloor \le r^2$$

for $r \ge 2$, the assertion follows.

Step 2. Suppose f is a homogeneous polynomial such that $f \in (P_i)$ for $i = 1, \dots, s$. Let us prove by induction on $i \ge 0$ that

$$f \in (P^{(i)}) + (P)^{i+1}$$

for any $P \in \mathcal{P}^{(i)}$. Indeed, for $i = 0$ this is true by assumption. Assume that $i > 0$ and the assertion holds for $i-1$. Let us apply Lemma 3.3 to a collection of subspaces $\{Q_1, \dots, Q_p\} \subset \mathcal{P}^{(i-1)}$ such that $Q_1 \cap \ldots \cap Q_p = P^{(i)}$, or rather to the corresponding subspaces $\overline{Q}_i = Q_i/P^{(i)}$ of $V^*/P^{(i)}$. We get the inclusion of ideals

$$(\overline{Q}_1)^i \cap \ldots \cap (\overline{Q}_p)^i \subset (\sum \overline{Q}_j)^{i+1}$$

in the symmetric algebra of $V^*/P^{(i)}$. Let us consider the polynomial

$$\overline{f} = f \mod (P^{(i)})$$

in this algebra. By assumption, $\overline{f} \in (\overline{Q}_j)^i$ for $j = 1, \dots, p$. Hence, we deduce that $\overline{f} \in (\sum \overline{Q}_j)^{(i+1)}$, i.e.,

$$f \in (P^{(i)}) + (\sum Q_j)^{(i+1)}.$$

Since every subspace in $\mathcal{P}^{(i)}$ has form $\sum Q_j$, with $(Q_1, \dots, Q_p)$ as above, this proves the induction step.

Step 3. For $i = d$, since f is homogeneous of degree d, the result of the previous step gives

$$f \in (P^{(d)}).$$

Recall that $P^{(d)}$ is the intersection of all subspaces in $\mathcal{P}^{(d-1)}$. Iterating the result of Step 1, we see that the dimension of any subspace in $\mathcal{P}^{(d-1)}$, and hence of $P^{(d)}$, is $\le r^{2^{d-1}}$. This ends the proof of Theorem D. □

Remark 4.1. Suppose we have an s-dimensional subspace F of homogeneous polynomials of the same degree d, defined over $\mathbf{k}$, such that there exists a nonzero $f \in F_{\overline{\mathbf{k}}}$ and a subspace of linear forms of dimension r over $\overline{\mathbf{k}}$ such that $f \in (L)$. One can ask how to produce an element $f_0 \in F \setminus 0$ and a subspace of linear forms L_0, both defined over $\mathbf{k}$, such that $f_0 \in (L_0)$ and dimension of L_0 is $\leq c(sr)$ (by Theorem A, we know that such an element exists).

Let $F_0 \subset F$ denote the subspace spanned by all the Galois conjugates of f. Then F_0 is defined over $\mathbf{k}$. As f_0 we will take any nonzero element of F_0.

Since $\dim F_0 \leq \dim F \leq s$, we can choose a set of elements of the Galois group $\sigma_1, \ldots, \sigma_s$, such that $(\sigma_1 f, \ldots, \sigma_s f)$ span F_0. Hence, f_0 is a linear combination of $(\sigma_1 f, \ldots, \sigma_s f)$, and so,

$$f_0 \in (\sigma_1 L + \ldots + \sigma_s L).$$

Now applying our algorithm from Theorem D for f_0, we find a subspace L_0 of dimension $\leq c(sr)$ defined over $\mathbf{k}$, with $f_0 \in (L_0)$.

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