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A note on singular Hermitian Yang-Mills connections

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We give an example of a homogeneous reflexive sheaf over $\mathbb{C}^3$ which admits a non-conical Hermitian Yang-Mills connection. This is expected to model bubbling phenomenon along complex codimension 2 submanifolds when the Fueter section takes zero value.

The guiding wisdom in the study of Hermitian Yang-Mills (HYM) connections is the principle that *metric properties of HYM connections translate into algebro-geometric properties of holomorphic vector bundles and vice versa*. For instance, the celebrated Donaldson-Uhlenbeck-Yau theorem [20][5][23] states that over compact Kähler manifolds, a holomorphic vector bundle admits a unique HYM connection if and only if it is polystable. This has been extended to reflexive sheaves by Bando and Siu [1], and by now there is a developed theory comparing the compactified moduli spaces of HYM connections versus the stable vector bundles [11]. In a more local setting, recent works [14][2][3] give an algebro-geometric characterization of analytic tangent cones of admissible Hermitian-Yang-Mills connections over any reflexive sheaves.

Remark. In our terminology, a HYM connection A over $\mathbb{C}^n$ has *tangent cone connection* A_∞ at infinity, if for some rescaling sequence $x \mapsto \lambda_i x$ with $\lambda_i \to \infty$, the pullback connections $\lambda_i^* A_i$ converge up to gauge transforms to A_∞ in C^∞_{loc}-topology on the complement of a (possibly empty) real codimension 4 subset in $\mathbb{C}^n$. By the higher dimensional generalisation of Uhlenbeck compactness [23][22], such subsequential smooth convergence away from codimension four holds as soon as we have a sequence of admissible

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Yang-Mills connections on a fixed Euclidean ball with a fixed uniform L^2-energy bound; taking the rescaling into account, the existence of at least one tangent cone is guaranteed if $\limsup_{r\to\infty} \frac{1}{r^{2n-4}} \int_{B(r)} |F|^2 < \infty$. For a short summary of known results on compactness, see [15].

The purpose of this paper is to provide a surprising example which shows that in the noncompact setting the holomorphic structre alone *does not* need to capture everything about HYM connections:

Theorem 0.1. *There is a HYM connection on the homogeneous reflexive sheaf*[1] $\ker(\underline{\mathbb{C}}^3 \xrightarrow{(x,y,z)} \underline{\mathbb{C}})$ *over the Euclidean space* $\mathbb{C}^3$ *with locally finite* L^2 *curvature, whose tangent cone at infinity is flat.*

Complex geometrically, the Euler sequence shows that the reflexive sheaf is isomorphic over $\mathbb{C}^3 \setminus \{0\}$ to the pullback of the cotangent bundle of $\mathbb{CP}^2$ via $\mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ (and therefore is homogeneous). Indeed, by [14][2] the local tangent cone at the origin must be the pullback of the Levi-Civita connection on $\Omega_{\mathbb{P}^2}$ up to twisting by a central $U(1)$-connection. The surprise is that the HYM metric in our example, and also its tangent cone at infinity, do not agree with this naïve conical HYM connection as one would predict from the complex geometry. Thus the roles of the tangent cone at infinity and the local tangent cone must be fundamentally asymmetrical. This has a similar flavour to the recent discovery of exotic Calabi-Yau metrics on $\mathbb{C}^n$ [15][4][21].

Despite the appearance our main result does not violate the Price monotonicity formula, which states that $r^{4-\dim_{\mathbb{R}}} \int_{B(r)} |F|^2$ must be an increasing function in the radius r. Geometrically, the curvature has faster than quadratic decay in the generic region near infinity, resulting in a flat tangent cone at infinity, but has slower than quadratic decay close to the z-axis, transverse to which the HYM connection is modelled on scaled copies of the standard one-instanton. Thus on large spheres the curvature becomes concentrated in a very small solid angle, and when we take the tangent cone the L^2 curvature is lost in the limit.

Our strategy is to produce an ansatz from the *monad construction*, and then use some fairly standard nonlinear existence machinery to find a HYM connection asymptotic to the ansatz near infinity. The monad construction is motivated by studying a family of HYM connections bubbling along a complex curve S inside a Calabi-Yau 3-fold [8]. Near a local patch of S, the

[1] The notation $\underline{\mathbb{C}}$ means the trivial line bundle over $\mathbb{C}^3$.

HYM connections restricted to the normal directions to S are modelled on (framed) ASD instantons, whose variation along S is governed by a holomorphic map (called the *Fueter map*) from S to the framed instanton moduli space. We are interested in the simplest case of instantons with rank 2 and charge 1, so the framed moduli space is $\mathbb{C}^2/\mathbb{Z}_2$. Now the most generic kind of zero for a holomorphic map $\mathbb{C} \to \mathbb{C}^2/\mathbb{Z}_2$ is up to linear change of coordinates given by $z \mapsto (z^{1/2}, 0)$ to leading order. Via the ADHM construction, this particular Fueter map translates into a monad ansatz for the HYM connection. In our actual construction we take a regularized version of the monad ansatz with better curvature decay properties.

This Fueter section viewpoint is an essential part of Donaldson and Segal's proposal [8] concerning enumerative invariants from gauge theory in higher dimensions. In a number of contexts related to special holonomy (Calabi-Yau 3-folds, G_2-manifolds, $Spin(7)$ manifolds), there are generalisations to the 4-manifold ASD instanton equation (HYM connections, G_2-instantons, $Spin(7)$-instantons), and one would like to define a weighted count of the number of solutions, which relies on certain compactness conjectures on the moduli space of such instantons [19]. Conjecturally, the main phenomenon one needs to account for is bubbling of these instantons along some codimension 4 locus S, and in each of these contexts, there is a version of the Fueter equation that governs how the transverse ASD instanton bubbles varies over a large scale on S (*cf.* [12][19]). Our construction complements this program, by providing a local model for codimension 6 singularities to be naturally embedded inside the codimension 4 bubbling locus, due to the most generic type of Fueter section singularity. This viewpoint is developed further in [16], where we use this local model to construct a sequence of HYM connections over the unit ball in $\mathbb{C}^3$, such that the L^2-energy is uniformly bounded, but the number of interior singularities inside a shrinked ball can be arbitrarily large.

Notation. The symbol $f \lesssim g$ means $f \leq Cg$ for some constant C. The symbol $f \sim g$ denotes uniform equivalence $f \lesssim g \lesssim f$. A dyadic scale refers to the region $\{\vec{x} \in \mathbb{C}^3 : 2^k \lesssim |\vec{x}| \lesssim 2^{k+1}\}$ for some k.

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1. An ansatz from monad construction

We begin with a general curvature formula for the cohomology of a monad over a complex manifold. It follows readily from standard curvature formulae for subbundles and quotient bundles.

Lemma 1.1. *Consider a monad $E_0 \xrightarrow{\alpha} E_1 \xrightarrow{\beta} E_2$, namely a complex of Hermitian holomorphic vector bundles with α injective fibrewise and β surjective fibrewise. Let $E = \ker\beta/im(\alpha)$ be the cohomology bundle. Then the curvature F_E of the natural induced connection on E satisfies*

$$\langle F_E s, s'\rangle = \langle F_{E_1} s, s'\rangle - \langle (\beta\beta^\dagger)^{-1}(\nabla\beta)s, (\nabla\beta)s'\rangle - \langle (\alpha^\dagger\alpha)^{-1}(\nabla\alpha^\dagger)s, (\nabla\alpha^\dagger)s'\rangle,$$

where F_{E_1} is the Chern connection on E_1, and s, s' are representing smooth sections of E satisfying $\alpha^\dagger s = \alpha^\dagger s' = \beta s = \beta s' = 0$, and $\nabla\alpha^\dagger$, $\nabla\beta$ are covariant derivatives computed on the Hom bundles.

Remark. The special cases of $E_0 = 0$ or $E_2 = 0$ reduce to the standard formula for the curvature of subbundles and quotient bundles [7, section 3.1.3]. In general, the above lemma follows by applying the quotient bundle curvature formula to $E_1 \to \ker\beta$.

Example 1.2. [7, Chapter 3] (ADHM construction of one-instantons) Start from the monads over Euclidean[2] $\mathbb{C}^2_{x,y}$, written in matrix notation

$$\underline{\mathbb{C}} \xrightarrow{\alpha=(x,y,a_1,a_2)^t} \underline{\mathbb{C}^4} \xrightarrow{\beta=(-y,x,b_1,b_2)} \underline{\mathbb{C}},$$

where the underlines signify trivial vector bundles, and the parameters a_1, a_2, b_1, b_2 satisfy the ADHM equation

$$a_1b_1 + a_2b_2 = 0, \quad |a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2 > 0.$$

The natural connections on cohomology bundles $E_{a,b}$ are ASD instantons on $\mathbb{C}^2$ with rank 2, charge 1 and curvature scale[3] $\sim \sqrt{|a_1|^2 + |a_2|^2}$. The situation with $a_1 = a_2 = b_1 = b_2 = 0$ is viewed as a degenerate case. As Hermitian vector bundles $E_{a,b}$ are identified as $\ker\beta \cap \ker\alpha^\dagger \subset \underline{\mathbb{C}^4} = \underline{\mathbb{C}^2} \oplus \underline{\mathbb{C}^2}$, and the monad construction provides a natural projection map into the second $\underline{\mathbb{C}^2}$

[2] The notation $\mathbb{C}^2_{x,y}$ means $\mathbb{C}^2$ with coordinates x, y.

[3] We say the curvature scale is of order $O(r)$ if in a ball of radius r around the given point, the curvature $|F| = O(r^{-2})$.

factor, giving a trivialisation of $E_{a,b}$ near infinity known as *framing data.* Notice the framed instantons are isomorphic under the $U(1)$-symmetry

$$(a_1, a_2, b_1, b_2) \mapsto (a_1 e^{i\theta}, a_2 e^{i\theta}, b_1 e^{-i\theta}, b_2 e^{-i\theta}).$$

The *moduli space of framed instantons* centred at the origin is

$$\begin{aligned}\{a_1 b_1 + a_2 b_2 = 0, |a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2\}/U(1) \\ \simeq \{a_1 b_1 + a_2 b_2 = 0\}/\mathbb{C}^* \simeq \mathbb{C}^2/\mathbb{Z}_2,\end{aligned}$$

by identifying the symplectic quotient with the GIT quotient.

We now describe our *main ansatz.* Take the monad over Euclidean $\mathbb{C}^3_{x,y,z}$

$$\underline{\mathbb{C}} \xrightarrow{\alpha=(x,y,1,0)^t} \underline{\mathbb{C}^4} \xrightarrow{\beta=(-y,x,0,z)} \underline{\mathbb{C}}, \tag{1}$$

and equip the trivial bundle $\underline{\mathbb{C}^4}$ with the *nonstandard Hermitian structure* given by the diagonal matrix $h_{\mathbb{C}^4} = \operatorname{diag}((|\vec{x}|^2+1)^{-1/2}, (|\vec{x}|^2+1)^{-1/2}, 1, 1)$ where $|\vec{x}|^2 = |x|^2 + |y|^2 + |z|^2$. The monad has a unique singular point at the origin in $\mathbb{C}^3$ where β fails to be surjective; in fact simple linear algebra shows the cohomology sheaf E is isomorphic to the homogeneous coherent sheaf $\ker(\underline{\mathbb{C}^3} \xrightarrow{(x,y,z)} \underline{\mathbb{C}})$,[4] so is in particular *reflexive* by [9, Prop. 39, chapter 2]. Using the Euler sequence over $\mathbb{P}^2$

$$0 \to \Omega_{\mathbb{P}^2} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0,$$

E is isomorphic as vector bundles over $\mathbb{C}^3 \setminus \{0\}$ to the pullback of $\Omega_{\mathbb{P}^2}$ via $\mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$.

Remark. (Heuristics) The way we arrive at this ansatz (1) is as follows. The goal is to write down a HYM connection which when restricted to each $z = \text{const}$ slice is approximately a rank 2 charge 1 ASD instanton, and the way these instantons vary with z is controlled by a Fueter map from the z plane to the moduli space $\mathbb{C}^2/\mathbb{Z}_2$ of framed instantons. The Fueter map we choose is $z \to (z^{1/2}, 0)$, so the curvature scale of these ASD instantons

[4] Any element in a fibre of $E = \ker\beta/im(\alpha)$ has a unique representative modulo $im(\alpha)$ of the form $(s_1, s_2, 0, s_4)^t$ which lies in the kernel of β. Thus E is isomorphic to $\ker(\underline{\mathbb{C}^3} \xrightarrow{(-y,x,z)} \underline{\mathbb{C}})$, which after linear transformations is the same as claimed.

is $O(|z|^{1/2})$. The most naïve construction directly motivated by the ADHM construction is

$$\underline{\mathbb{C}} \xrightarrow{\alpha=(x,y,z^{1/2},0)^t} \underline{\mathbb{C}^4} \xrightarrow{\beta=(-y,x,0,z^{1/2})} \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}^4}$ carries the Euclidean metric. We need to remove the ambiguity due to the choice of square root. This can be done via fibrewise linear transformations to the 3rd and 4th coordinates of $\underline{\mathbb{C}}^4$, after which the monad complex becomes (1), and the Hermitian metric on $\underline{\mathbb{C}}^4$ becomes $\mathrm{diag}(1, 1, |z|, |z|)$. This ansatz still suffers from the degeneracy on the plane $z = 0$, so we take the noncanonical step to replace $|z|$ by $\sqrt{|\vec{x}|^2 + 1}$, which now makes the metric smooth, and since within the curvature scale $O(|z|^{1/2})$ of the ASD instantons $\sqrt{|\vec{x}|^2 + 1}$ is only a small perturbation of $|z|$ when $|z| \gg 1$, we expect the ASD instanton bubbling picture to be preserved. Finally, to make the connection almost flat when we approach infinity far away from $x = y = 0$, we conformally rescale the Hermitian metric on $\underline{\mathbb{C}^4}$ by a factor of $(|\vec{x}|^2 + 1)^{-1}$, which amounts to twisting by a $U(1)$-connection.

The key point is that the mean curvature ΛF_E has fast enough decay at infinity.

Lemma 1.3. *The curvature F_E of the natural connection on E induced by the monad construction admits the estimate $|\Lambda F_E| \le C\ell$, where ℓ is a fixed function defined up to uniform equivalence by*

$$\ell \sim \begin{cases} \frac{1}{(|x|^2+|y|^2+|z|)|\vec{x}|}, & |\vec{x}| \gtrsim 1, \\ \frac{1}{|x|^2+|y|^2+|z|^2}, & |\vec{x}| \lesssim 1. \end{cases} \tag{2}$$

Proof. We shall compute the curvature F_E. Taking into account the nonstandard Hermitian structure, the adjoint maps are given by

$$\begin{aligned} \alpha^\dagger &= (\bar{x}(|\vec{x}|^2 + 1)^{-1/2}, \bar{y}(|\vec{x}|^2 + 1)^{-1/2}, 1, 0), \\ \beta^\dagger &= (-\bar{y}\sqrt{|\vec{x}|^2 + 1}, \bar{x}\sqrt{|\vec{x}|^2 + 1}, 0, \bar{z})^t, \end{aligned}$$

hence

$$\begin{aligned} \alpha^\dagger\alpha &= (|x|^2 + |y|^2)(|\vec{x}|^2 + 1)^{-1/2} + 1, \\ \beta\beta^\dagger &= (|x|^2 + |y|^2)\sqrt{|\vec{x}|^2 + 1} + |z|^2. \end{aligned} \tag{3}$$

Since α, β are holomorphic, $\alpha^\dagger, \beta^\dagger$ are antiholomorphic,

$$(4)\qquad \begin{cases} \nabla\alpha^\dagger = \bar\partial\alpha^\dagger = (d\bar x(|\vec x|^2+1)^{-1/2} - \frac{\bar x(xd\bar x+yd\bar y+zd\bar z)}{2(|\vec x|^2+1)^{3/2}}, \\ \qquad\qquad d\bar y(|\vec x|^2+1)^{-1/2} - \frac{\bar y(xd\bar x+yd\bar y+zd\bar z)}{2(|\vec x|^2+1)^{3/2}}, 0, 0), \\ \nabla\beta = (\nabla\beta^\dagger)^\dagger = (\bar\partial\beta^\dagger)^\dagger = (-dy - \frac{y(\bar xdx+\bar ydy+\bar zdz)}{2(|\vec x|^2+1)}, \\ \qquad\qquad dx + \frac{x(\bar xdx+\bar ydy+\bar zdz)}{2(|\vec x|^2+1)}, 0, dz). \end{cases}$$

Let $s = (s_1, s_2, s_3, s_4)^t$ be a smooth local section of E represented as a section of $\underline{\mathbb{C}^4}$ with $\beta s = \alpha^\dagger s = 0$, so by expressing s_1, s_2 in terms of s_3, s_4,

$$\begin{cases} s_1 = \frac{-x(|\vec x|^2+1)^{1/2}s_3+z\bar y s_4}{|x|^2+|y|^2}, \\ s_2 = \frac{-y(|\vec x|^2+1)^{1/2}s_3+z\bar x s_4}{|x|^2+|y|^2}, \end{cases}$$

we see

$$|s_1| + |s_2| \leq \frac{(|s_3| + |s_4|)(|\vec x| + 1)}{|x| + |y|}.$$

Under the Hermitian structure

$$|s|_h^2 = (|s_1|^2 + |s_2|^2)(|\vec x|^2 + 1)^{-1/2} + (|s_3|^2 + |s_4|^2),$$

we have

$$(5)\qquad |s_1| + |s_2| \leq C\min\{(|\vec x| + 1)^{1/2}, \frac{|\vec x| + 1}{|x| + |y|}\}|s|_h.$$

Now the Chern curvature on $\underline{\mathbb{C}^4}$ is given by $F_{E_1} = \bar\partial(\partial h_{\mathbb{C}^4} h_{\mathbb{C}^4}^{-1})$, so

$$\langle\sqrt{-1}F_{E_1}s, s\rangle = O(\frac{|s_1|^2 + |s_2|^2}{(|\vec x|^2 + 1)^{3/2}}).$$

Substituting (3)(4)(5) into the curvature formula in Lemma 1.1,[5]

$$(6)\qquad \begin{aligned} &\langle\sqrt{-1}\Lambda F_E s, s\rangle \\ &= -(\alpha^\dagger\alpha)^{-1}(|\vec x|^2 + 1)^{-1}\sqrt{-1}\Lambda(s_1 d\bar x + s_2 d\bar y) \wedge (\bar s_1 dx + \bar s_2 dy) \\ &\quad - (\beta\beta^\dagger)^{-1}\sqrt{-1}\Lambda(-s_1 dy + s_2 dx) \wedge (-\bar s_1 d\bar y + \bar s_2 d\bar x) + O(\ell|s|_h^2). \end{aligned}$$

[5]This is a tedious but unenlightening calculation, in which one tries to absorb most terms into the $O(\ell|s|_h^2)$ error term, such as the effect from F_{E_1}. The only terms that we keep in $\nabla\beta$ is $(-dy, dx, 0, 0)$, and the only terms we keep in $\nabla\alpha^\dagger$ is $((|\vec x|^2 + 1)^{-1/2}d\bar x, |\vec x|^2 + 1)^{-1/2}d\bar y, 0, 0)$. Terms such as $\frac{y(\bar xdx+\bar ydy+\bar zdz)}{2(|\vec x|^2+1)}$ in $\nabla\beta$ are only significant if $|x| + |y|$ is large, but since this term is multiplied by s_1 it is suppressed by the factor $\frac{|\vec x|+1}{|x|+|y|}$ from (5), and in the end can be ignored.

Combined with a cancellation effect expressed by the inequality

$$(|\vec{x}|^2+1)^{-1}(\alpha^\dagger\alpha)^{-1}-(\beta\beta^\dagger)^{-1}=O(\frac{1}{(\beta\beta^\dagger)\sqrt{|\vec{x}|^2+1}}), \tag{7}$$

this implies $\langle\sqrt{-1}\Lambda F_E s, s\rangle = O(\ell|s|_h^2)$, or equivalently $|\Lambda F_E| = O(\ell)$ as required. □

Remark. From the proof one can readily extract estimates on $|F_E|$. In particular near the origin $|F_E| = O(|\vec{x}|^{-2})$ is *locally in* L^2. For $|x|+|y| \gtrsim |z|^{1/2}+1$, the curvature decays like $|F_E| = O(\frac{|\vec{x}|}{(|x|^2+|y|^2)^2})$. In particular when $|x|+|y| \gtrsim |\vec{x}| \gg 1$, the curvature has cubic decay. Thus if one takes a scaling sequence $\lambda_i^*\nabla_E$ with $\lambda_i \to \infty$, then on any given compact subset $K \subset \mathbb{C}^3 \setminus \{x=y=0\}$, the curvature $|\lambda_i^* F_E| \leq C(K)\lambda_i^{-1} \to 0$ as $\lambda_i \to \infty$, so up to gauge the connections $\lambda_i^*\nabla_E$ converge in C^1_{loc} to a flat connection away from the real codim 4 subset $\{x=y=0\}$. Higher derivative computation shows this convergence in fact holds in C^∞_{loc}, so we say *the tangent cone at infinity is flat.* The curvature becomes concentrated near the z-axis near the spatial infinity of $\mathbb{C}^3$.

For later usage, we estimate the potential integral

$$G(x,y,z) = \int_{\mathbb{C}^3} \frac{\ell(\vec{x}')}{|(x-x', y-y', z-z')|^4} d\mathrm{Vol}(\vec{x}'). \tag{8}$$

Lemma 1.4. *The function $G>0$ is well defined on $\mathbb{C}^3\setminus\{0\}$, and satisfies*

$$\begin{cases} \Delta G = const\cdot \ell, \\ |G| \leq C|\vec{x}|^{-1}\max(\log\frac{|\vec{x}|}{|x|+|y|+|z|^{1/2}}, 1), \quad |\vec{x}| \gtrsim 1. \end{cases}$$

Proof. The general strategy is to break up the integral over $\mathbb{C}^3$ into various dyadic scales, estimate each contribution to the potential integral $G(x,y,z)$, and then sum up the estimates. The sources fall into several characteristic regions: the small ball $|\vec{x}'| \lesssim 1$, the dyadic scale $|\vec{x}'| \sim |\vec{x}|$, and all the other dyadic scales where $|\vec{x}-\vec{x}'| \sim \max\{|\vec{x}|, |\vec{x}'|\}$. Within each dyadic scales, the quantity $|\vec{x}'|$ can be replaced by constants up to uniform equivalence, thus making the integral estimate elementary.

The singularity of the source ℓ at the origin is $O(\frac{1}{|\vec{x}|^2})$, which is mild enough to guarantee the potential integral is well defined and has the correct Laplacian. This contribution to G is bounded by $O(|\vec{x}|^{-1})$.

We focus on the estimates for $|\vec{x}| \gtrsim 1$. The L^1 integral of the source ℓ at a scale $|\vec{x}'| \sim 2^k$ is bounded by

$$\int_{|\vec{x}'|\sim 2^k} \frac{1}{(|x'|^2+|y'|^2)|\vec{x}'|} d\text{Vol}(\vec{x}') \lesssim 2^{3k},$$

so by summing over contributions to G from all dyadic scale except $|\vec{x}'| \sim |\vec{x}|$,

$$\begin{aligned}|G - \int_{|\vec{x}|\sim|\vec{x}'|} &\frac{\ell(\vec{x}')d\text{Vol}(\vec{x}')}{|(x-x', y-y', z-z')|^4}| \\ &\lesssim \sum_{1}^{\log_2|x|} 2^{3k}|x|^{-4} + \sum_{\log_2|x|}^{\infty} 2^{3k}2^{-4k} \lesssim |\vec{x}|^{-1}.\end{aligned}$$

Thus the only important contribution comes from the dyadic scale $|\vec{x}| \sim |\vec{x}'|$. The strategy is then to divide and conquer all regions $|\vec{x}-\vec{x}'| \sim 2^k$ for all $k \lesssim \log_2|x|$. For $|x|^2+|y|^2 \gtrsim |z|$, the $|\vec{x}| \sim |\vec{x}'|$ contribution to G is controlled by

$$\begin{aligned}\int_{|\vec{x}|\sim|\vec{x}'|} &\frac{1}{|(x-x', y-y', z-z')|^4} \frac{d\text{Vol}(\vec{x}')}{(|x'|^2+|y'|^2)|\vec{x}|} \\ &\lesssim |\vec{x}|^{-1}\max(1, \log\frac{|\vec{x}|}{|x|+|y|}),\end{aligned}$$

and for $|x|^2+|y|^2 \lesssim |z|$ is controlled by $O(|\vec{x}|^{-1}|\log\frac{|\vec{x}|}{|z|^{1/2}}|)$. Combining the above shows the claim. □

1.1. Asymptotic geometry near infinity

First we examine the asymptotic geometry for $|x|+|y| \gg |z|^{1/2}+1$, namely the generic region near spatial infinity, suitably away from the z-axis to ensure fast curvature decay. Consider the case $|x| \lesssim |y|$, so $|y| \gg |\vec{x}|^{1/2}+1$. A basis of holomorphic sections on E can be represented by sections of $\ker\beta$:

$$s_{(1)} = (0,0,1,0)^t, \quad s_{(2)} = (z/y, 0, 0, 1)^t.$$

The projections of s_1, s_2 to the orthogonal complement of $\text{Im}(\alpha)$ are respectively

$$\begin{cases} s_1' = s_{(1)} - \alpha(\alpha^\dagger\alpha)^{-1}\alpha^\dagger s_{(1)} = s_{(1)} - \frac{1}{(|x|^2+|y|^2)(|\vec{x}|^2+1)^{-1/2}+1}(x,y,1,0)^t, \\ s_2' = s_{(2)} - \alpha(\alpha^\dagger\alpha)^{-1}\alpha^\dagger s_{(2)} = s_{(2)} - \frac{\bar{x}z/y}{|x|^2+|y|^2+(|\vec{x}|^2+1)^{1/2}}(x,y,1,0)^t. \end{cases}$$

The Hermitian metric H_0 on the cohomology bundle is represented by the matrix

$$H_0(s_{(i)}, s_{(j)}) = h_{\mathbb{C}^4}(s'_i, s'_j) = \delta_{ij} + O(\frac{|\vec{x}|}{|y|^2}).$$

By repeatedly differentiating[6] H_0 in the x, y, z variables,

$$|\partial^k H_0| = O(\frac{|\vec{x}|}{|y|^{2+k}}), \quad k \geq 1,$$

where ∂^k refers to the k-th partial derivatives. The natural connection ∇_E on E is just the Chern connection induced by the Hermitian structure H_0, so in particular $|F_E| = O(\frac{|\vec{x}|}{|y|^4})$ compatible with Remark 1. The mean curvature ΛF_E has better decay properties. For this, we can apply the more accurate formula in Lemma 1.1 to derive an explict expression for the matrix $(\langle \Lambda F_E s'_i, s'_j \rangle)$ as formula (6). The same cancellation effect as in (7) happens. The higher order version of Lemma 1.3 then follows from inductively taking derivatives, each differentiation improving the decay by a factor of $O(|y|^{-1})$:

$$|\partial^k \langle \Lambda F_E s'_i, s'_j \rangle| \leq C(k) |\vec{x}|^{-1} |y|^{-2-k},$$

or equivalently $|\nabla_E^k (\Lambda F_E)| \leq C(k)|\vec{x}|^{-1}|y|^{-2-k}$.

Similarly in the case $|x| \gtrsim |y|$, we can find another basis of holomorphic sections on E, with

$$H_0 = \delta_{ij} + O(\frac{|\vec{x}|}{|x|^2}), \quad |\partial^k H_0| \leq C(k) \frac{|\vec{x}|}{|x|^{2+k}}, \quad k \geq 1,$$

and $|\nabla_E^k(\Lambda F_E)| \leq C(k)|\vec{x}|^{-1}|x|^{-2-k}$. To summarize, the Hermitian structure on E in the generic region $|x| + |y| \gg |z|^{1/2} + 1$ is *approximately flat.*

[6]Each time a derivative hits an expression such as $\frac{1}{|x|^2+|y|^2}$ or $\frac{1}{(|\vec{x}|^2+1)^{1/2}}$ it brings down the decay by an extra factor $O(\frac{1}{|y|})$, and each time it hits x, y, z, the homogeneity is lowered by degree one. Since the complicated expressions are built from such basic factors, one can proceed by induction on the order of derivatives. When $|x| < \frac{1}{2}|y|$, there is an additional trick of a Taylor expansion in $\frac{x}{y}$, which allows one to estimate away all appearance of x. Such tricks allow one to restrict only to the case where $|x| \sim |y|$, and ignore all terms with subleading homogeneity such as 1 in $\sqrt{|\vec{x}|^2 + 1}$, which simplifies the induction. We will use these lines of reasoning several times below.

Next we turn to the vicinity of the z-axis $1 \ll |z|^{1/2} \lesssim |x| + |y|$. Observe that if the ambient Hermitian structure on $\underline{\mathbb{C}^4}$ is changed from $h_{\mathbb{C}^4}$ to

$$(|\vec{x}|^2+1)^{1/2} h_{\mathbb{C}^4} = \mathrm{diag}(1, 1, (|\vec{x}|^2+1)^{1/2}, (|\vec{x}|^2+1)^{1/2}),$$

then the induced connection on E is *twisted by a* $U(1)$ *connection* with curvature $\frac{1}{2}\bar{\partial}\partial \log(|\vec{x}|^2+1)$. We shall focus on this twisted situation around a given point $(0,0,\zeta) \in \mathbb{C}^3$ with $|\zeta| \gg 1$, and choose a square root $\zeta^{1/2}$. After rescaling the basis vectors on $\underline{\mathbb{C}^4}$, the twisted monad can be written as

$$\underline{\mathbb{C}} \xrightarrow{(x,y,\zeta^{1/2},0)^t} \underline{\mathbb{C}^4} \xrightarrow{(-y,x,0,\zeta^{1/2})} \underline{\mathbb{C}}, \tag{9}$$

where the Hermitian structure on $\underline{\mathbb{C}^4}$ is

$$\tilde{h}_{\mathbb{C}^4} = \mathrm{diag}(1, 1, \frac{(|\vec{x}|^2+1)^{1/2}}{|\zeta|}, \frac{|\zeta|(|\vec{x}|^2+1)^{1/2}}{|z|^2}).$$

For $|x|+|y|+|z-\zeta| \lesssim |\zeta|^{1/2}$, repeated differentiation shows

$$\begin{cases} \tilde{h}_{\mathbb{C}^4} = \mathrm{diag}(1, 1, 1+O(\frac{1+|z-\zeta|}{|\zeta|}), 1+O(\frac{1+|z-\zeta|}{|\zeta|})), \\ |\partial^k \partial_x \tilde{h}_{\mathbb{C}^4}| + |\partial^k \partial_y \tilde{h}_{\mathbb{C}^4}| \leq C(k)\frac{1}{|\zeta|^{3/2+k/2}}, \quad k \geq 0. \\ |\partial^k \partial_z \tilde{h}_{\mathbb{C}^4}| \leq C(k)\frac{1}{|\zeta|^{1+k/2}}, \quad |\partial^k \partial_z \partial_{\bar{z}} \tilde{h}_{\mathbb{C}^4}| \leq C(k)\frac{1}{|\zeta|^{2+k/2}}, \quad k \geq 0. \end{cases}$$

To leading order, the ambient Hermitian metric on $\underline{\mathbb{C}^4}$ is Euclidean, and the twisted monad (9) dimensionally reduces to the monad in the ADHM construction with parameters $(a_1, a_2, b_1, b_2) = (\zeta^{1/2}, 0, 0, \zeta^{1/2})$ (*cf.* Example 1.2). In particular, the connection on the cohomology bundle of the twisted monad is approximately a framed instanton whose moduli parameter is identified as $(\zeta^{1/2}, 0) \in \mathbb{C}^2/\mathbb{Z}_2$. From a more global viewpoint, the twisted monad connection in the normal direction to the z-axis is described by the *Fueter map* into the moduli space of framed instantons:

$$\mathbb{C} \to \mathbb{C}^2/\mathbb{Z}_2, \quad \zeta \mapsto (\zeta^{1/2}, 0). \tag{10}$$

Notice the Fueter map is independent of the choice of square root $\pm\zeta^{1/2}$.

One can then estimate the difference between ∇_E and the instanton connection ∇_ζ associated with the ADHM monad[7], in the region $\{|x|+|y| \lesssim$

[7]The connection ∇_ζ is viewed as a connection over $\mathbb{C}^3$, which is ASD in the x, y direction and trivial in the z direction.

$|\zeta|^{1/2}, z = \zeta\}$, by taking into account both the control on $\tilde{h}_{\mathbb{C}^4}$, and the effect of $U(1)$-twisting:

$$|\nabla_\zeta^k (F_E - F_{\nabla_\zeta})| \leq C(k)|z|^{-2-k/2}, \quad |\nabla_\zeta^k (\Lambda F_E)| \leq C(k)|z|^{-2-k/2},$$

where we use that ADHM instantons are HYM. The regularity scale of ∇_E in this region is comparable to the regularity scale of ∇_ζ which is $\sim |\zeta|^{1/2}$, meaning that when we rescale these connections from balls of size $O(|\zeta|^{1/2})$ to unit balls, they have uniform C^k-bounds.

Combining the estimates on ΛF_E in all the regions, we have a unified higher order estimate for ΛF_E:

Corollary 1.5. In the region $|x| \gtrsim 1$,

$$|\nabla_E^k (\Lambda F_E)| = O(|\vec{x}|^{-1}(|x| + |y| + |z|^{1/2})^{-2-k}).$$

The estimates on the full curvature itself can be summarized as

Corollary 1.6. In the region $|\vec{x}| \gtrsim 1$,

$$|\nabla_E^k F_E| = O(\frac{|\vec{x}|}{(|x| + |y| + |z|^{1/2})^{4+k}}).$$

2. Perturbation into HYM metric

We seek a nonlinear perturbation of the ansatz to a genuine HYM connection. The strategy is to solve Dirichlet boundary value problems on larger and larger domains exhausting $\mathbb{C}^3$, obtain uniform estimates and then extract limits as in [17]. The analysis involved is by now fairly standard.

Theorem 2.1. *Let E be a reflexive sheaf over a compact Kähler manifold $(\overline{Z}, \omega)$ with nonempty boundary ∂Z, which is locally free near the boundary. For any Hermitian metric f on the restriction of E to ∂Z there is a unique Hermitian H on E, which is smooth on the locally free locus, has finite L^2 curvature, and*

$$\sqrt{-1}\Lambda F_H = 0 \text{ in } Z, \quad H = f \text{ over } \partial Z.$$

Proof. (Sketch) Donaldson [6] proved the special case when E is a vector bundle using the heat flow method. The key point is that $|\Lambda F|$ is a subsolution to the heat equation with zero boundary data, which forces $|\Lambda F|$ to decay exponentially in time. This together with uniform C^k control on the connection in the flow leads to long time existence and convergence to HYM connection at infinite time. Here the issue of stability does not appear.

The singularity problem can be addressed by the continuity method due to Bando and Siu [1]; we follow the exposition in [13], Section 6. The idea is to take repeated blow ups $\tilde{Z}$ in the interior of $\bar{Z}$ so that $E|_{\bar{Z}\setminus \mathrm{Sing}(E)}$ extends as a vector bundle $\tilde{E}$ across the exceptional locus. Equip $\tilde{E}$ with a fixed reference Hermitian metric $\tilde{H}_1$ admitting the given boundary data. One can find a sequence of degenerating Kähler metrics $\omega_\epsilon \to \omega$ on $\tilde{Z}$ as $\epsilon \to 0$, such that

- ω_ϵ agrees with ω on the $O(\epsilon)$ neighbourhood of the exceptional locus.
- Near the blow up loci ω_ϵ is locally modelled on a rescaling of the standard metric on $\mathrm{Bl}_0\mathbb{C}^k \times \mathbb{C}^{n-k}$, where k is the complex codimension of the blow up centre.
- The curvature of $\tilde{H}_1$ has uniform L^2 curvature bound as $\epsilon \to 0$.
- The metrics ω_ϵ have uniform Dirichlet-Sobolev constants.

Applying Donaldson's result to solve the Dirichlet problem, we obtain HYM connections $\tilde{H}_\epsilon$ on $\tilde{E}$ for the background metrics ω_ϵ. Using the almost subharmonicity estimate in the analyst's Laplacian convention (*cf.* Lemma 2.5 in [17]), and the HYM condition on $\tilde{H}_\epsilon$,

$$\Delta \log \mathrm{Tr}(\tilde{H}_\epsilon \tilde{H}_1^{-1}) \geq -|\Lambda F_{\tilde{H}_1}|, \quad \Delta \log \mathrm{Tr}(\tilde{H}_1 \tilde{H}_\epsilon^{-1}) \geq -|\Lambda F_{\tilde{H}_1}|,$$

and the uniform Dirichlet-Sobolev inequality, there is a uniform L^2 bound on $\log \mathrm{Tr}(\tilde{H}_\epsilon \tilde{H}_1^{-1})$ and $\log \mathrm{Tr}(\tilde{H}_1 \tilde{H}_\epsilon^{-1})$. On any compact subset of the locally free locus of E, the almost subharmonicty implies furthermore L^∞ estimates on $\log \mathrm{Tr}(\tilde{H}_\epsilon \tilde{H}_1^{-1})$ and $\log \mathrm{Tr}(\tilde{H}_1 \tilde{H}_\epsilon^{-1})$, so that $\tilde{H}_\epsilon$ is locally uniformly equivalent to $\tilde{H}_1$ independent of ϵ. Then the Bando-Siu interior estimate (*cf.* Appendix C, D in [13]) gives C^k_{loc}-estimates on $\tilde{H}_\epsilon$ over the locally free locus of E, uniform in ϵ. Furthermore, there are uniform L^2 curvature bounds on $\tilde{H}_\epsilon$ because of the topological energy formula for HYM connections

$$\begin{aligned} \int_{\bar{Z}} |F_{\tilde{H}_\epsilon}|^2 \omega_\epsilon^n &= -\mathrm{const} \cdot \int_{\bar{Z}} \mathrm{Tr}(F_{\tilde{H}_\epsilon} \wedge F_{\tilde{H}_\epsilon}) \omega_\epsilon^{n-2} \\ &= -\mathrm{const} \int_{\partial Z} \mathrm{Tr}(\partial \tilde{H}_\epsilon \tilde{H}_\epsilon^{-1} \wedge F_{\tilde{H}_\epsilon}) \wedge \omega_\epsilon^{n-2}. \end{aligned} \tag{11}$$

Now taking a subsequential weak limit as $\epsilon \to 0$, we obtain a HYM metric H over the locally free locus of E, which must have bounded L^2 curvature. The uniqueness statement follows from the fact that if H, H' are two solutions to the Dirichlet problems with finite L^2-curvature, then $\log \mathrm{Tr}(HH'^{-1})$ and $\log \mathrm{Tr}(H'H^{-1})$ are both subharmonic. $\square$

Theorem 2.2. *There is a HYM connection H on E over Euclidean $\mathbb{C}^3$ with locally finite L^2 curvature, which admits the decay estimates on $\{|x| \gtrsim 1\}$:*

$$|\nabla_E^k \log(HH_0^{-1})| \leq C(k)(|x|+|y|+|z|^{1/2})^{-k}|\vec{x}|^{-1} \times \max(1, \log \frac{|\vec{x}|}{|x|+|y|+|z|^{1/2}}), \quad k \geq 0. \tag{12}$$

In particular, the tangent cone at infinity is flat. The asymptotic estimate (12) and the HYM condition $\Lambda F_H = 0$ determine H uniquely.

Proof. Recall H_0 is the natural Hermitian metric for the cohomology sheaf E of our monad (1), whose curvature is F_E. We solve the Dirichlet problem on large balls $B(R) \subset \mathbb{C}^3$ with boundary data H_0 on $\partial B(R)$, and denote the solution as H_R. Crucially we need the almost subharmonicity estimate (*cf.* Lemma 2.5 in [17], and Lemma 1.3):

$$\begin{cases} \Delta \log \mathrm{Tr}(H_R H_0^{-1}) \geq -|\Lambda F_E| \geq -C\ell, \\ \Delta \log \mathrm{Tr}(H_0 H_R^{-1}) \geq -C\ell, \end{cases} \tag{13}$$

Notice near the origin (13) continues to hold in the distributional sense, since the L^2 curvature of H_0 and H_R are finite, and the singularity has complex codimension 3 (*cf.* proof of Proposition 3.1 in [14]). Using the boundary condition $\mathrm{Tr}(H_R H_0^{-1}) = \mathrm{Tr}(H_0 H_R^{-1}) = \mathrm{rank}(E) = 2$, we apply Lemma 1.4 and the comparison principle for the Laplacian to get

$$\log \frac{\mathrm{Tr}(H_R H_0^{-1})}{2} \geq -CG, \quad \log \frac{\mathrm{Tr}(H_0 H_R^{-1})}{2} \geq -CG,$$

or equivalently there is a *pointwise estimate* on $B(R)$:

$$e^{-CG} H_0 \leq H_R \leq e^{CG} H_0, \tag{14}$$

which is *uniform* as $R \to \infty$. Using the upper bound for $|G|$ in $\{|x| \gtrsim 1\}$ provided by Lemma 1.4,

$$|\log(H_R H_0^{-1})| \leq C|\vec{x}|^{-1} \max(1, \log \frac{|\vec{x}|}{|x|+|y|+|z|^{1/2}}).$$

Recall also the higher order control on the mean curvature of the ansatz in Lemma 1.5. Applying Bando and Siu's interior regularity estimate (*cf.* Appendix C, D in [13]) to rescaled balls, we derive the higher order estimate on $\{1 \lesssim |\vec{x}| < R/2\}$:

$$|\nabla_E^k \log(H_R H_0^{-1})| \leq C(k)(|x| + |y| + |z|^{1/2})^{-k}|\vec{x}|^{-1} \\ \times \max(1, \log \frac{|\vec{x}|}{|x| + |y| + |z|^{1/2}}), \quad k \geq 0.$$

Using the same topological energy formula as (11) for H_R instead of $\tilde{H}_\epsilon$, and noticing all boundary terms are already controlled, the L^2-curvature inside the unit ball is controlled. All estimates are uniform in R. Taking a subsequential limit as $R \to \infty$, we obtain the HYM connection H with estimates (12). Since the deviation is so small that the leading order asymptotic geometry at infinity is unchanged, and in particular $|F_H| = O(\frac{|\vec{x}|}{(|x|^2+|y|^2+|z|)^2})$ as $|\vec{x}| \to \infty$, we see the tangent cone at infinity is flat, as in the Remark of section 1.

To see the uniqueness, notice if H' is another HYM metric satifying (12), then both $\log \frac{\mathrm{Tr}(H'H^{-1})}{2}$ and $\log \frac{\mathrm{Tr}(HH'^{-1})}{2}$ are subharmonic in the distributional sense, and are asymptotic to zero at infinity. $\square$

2.1. Further discussions

The *tangent cone connection at the origin* is determined a priori by complex geometry of the reflexive sheaf $E \simeq \ker(\mathbb{C}^3 \xrightarrow{(x,y,z)} \mathbb{C})$ (*cf.* [2][14]). Complex geometrically, it is isomorphic to E, or equivalently the pullback of the cotangent bundle on $\mathbb{CP}^2$. The Levi-Civita connection on $\Omega_{\mathbb{P}^2}$ is HYM with respect to the integral Fubini-Study metric ω_{FS}:

$$\Lambda_{\omega_{FS}} F_{\Omega_{\mathbb{P}^2}} = 4\pi\mu \cdot \mathrm{Id}_{\Omega_{\mathbb{P}^2}}, \quad \mu = \frac{\text{degree}}{\text{rank}}(\Omega_{\mathbb{P}^2}) = -\frac{3}{2}.$$

The tangent cone at the origin is the pullback of the Levi-Civita connection, up to a conformal change of the Hermitian structure by a factor $|\vec{x}|^{-3}$ which cancels out the Einstein constant. Equivalently, the tangent cone is the natural connection on the kernel of $\underline{\mathbb{C}^3} \xrightarrow{(x,y,z)} \underline{\mathbb{C}}$, but $\underline{\mathbb{C}^3}$ is equipped with the *nonstandard Hermitian structure* $\mathrm{diag}(\frac{1}{|\vec{x}|}, \frac{1}{|\vec{x}|}, \frac{1}{|\vec{x}|})$. An application of [14] shows that our HYM connection on E is asymptotic to the tangent cone at the origin with polynomial decay rate.

Some additional insights can be gained by studying the *growth rate of holomorphic sections* as in [2]. On a reflexive sheaf with a conical HYM connection, there is a convexity estimate (*cf.* Proposition 3.5 in [2])

$$(\int_{B_{1/4}} |s|^2)(\int_{B(1)} |s|^2) \geq (\int_{B(1/2)} |s|^2)^2.$$

A basic heuristic in [2] is that if a singular HYM connection is sufficiently close to being conical on a certain scale, then the convexity behaviour transfers to the HYM connection, so that $\log \int_{B(r)} |s|^2 / \log r$ has some monotonicity property, and one can define the *local growth degree*

$$d(s) = \frac{1}{2} \lim_{r \to 0} \frac{\log \int_{B(r)} |s|^2}{\log r} - \dim_{\mathbb{C}}$$

which induces a *filtration on the germ of holomorphic sections*, intimately related to a Harder-Narasimhan-Seshardri filtration.

In the setting of our example, this motivates us to define the growth degree at infinity

$$d_\infty(s) = \frac{1}{2} \lim_{r \to \infty} \frac{\log \int_{B(r)} |s|^2}{\log r} - \dim_{\mathbb{C}} = \frac{1}{2} \lim_{r \to \infty} \frac{\log \int_{B(r)} |s|^2}{\log r} - 3,$$

inducing a filtration on the holomorphic sections with *at most polynomial growth*. It is then easy to check explicitly that there are holomorphic sections whose growth degree at infinity is larger than the growth degree at the origin, so *the two filtration structures are different*. We expect that there are large classes of examples generalizing our construction, and these filtration structures should play a major role in a more systematic theory. In particular, it would be interesting to relate the filtration and the tangent cone at infinity.

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