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Derived categories of Quot schemes of locally free quotients via categorified Hall products

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We prove Qingyuan Jiang's conjecture on semiorthogonal decompositions of derived categories of Quot schemes of locally free quotients. The author's result on categorified Hall products for Grassmannian flips is applied to prove the conjecture.

1. Introduction

1.1. Quot formula

Let X be a smooth quasi-projective variety over $\mathbb{C}$, $\mathscr{G}$ a coherent sheaf on X and $d \geq 0$ be an integer. The relative Quot scheme

$$\mathrm{Quot}_{X,d}(\mathscr{G}) \to X \tag{1.1}$$

parametrizes rank d locally free quotients of $\mathscr{G}$. All the fibers of the above morphism are Grassmannian varieties, whose dimensions are different in general. Here we remark that $\mathrm{Quot}_{X,0}(\mathscr{G}) = X$.

Let us take a right exact sequence

$$\mathscr{E}^{-1} \xrightarrow{\phi} \mathscr{E}^0 \to \mathscr{G} \to 0 \tag{1.2}$$

where $\mathscr{E}^0$ and $\mathscr{E}^{-1}$ are locally free sheaves on X. Let $\delta := \mathrm{rank}(\mathscr{E}^0) - \mathrm{rank}(\mathscr{E}^{-1})$. By taking its dual, we obtain the right exact sequence

$$\mathscr{E}_0 \xrightarrow{\phi^\vee} \mathscr{E}_1 \to \mathscr{H} \to 0$$

where $\mathcal{E}_i := (\mathscr{E}^{-i})^\vee$ and $\mathscr{H}$ is the cokernel of $\phi^\vee$. Note that $\mathrm{rank}(\mathcal{E}_1) - \mathrm{rank}(\mathcal{E}_0) = -\delta$. As a dual side of (1.1), we also consider the relative Quot scheme $\mathrm{Quot}_{X,d}(\mathscr{H}) \to X$.

We will see that there exist quasi-smooth derived schemes over X (see Section 2.1)

$$\mathbf{Quot}_{X,d}(\mathscr{G}) \to X \leftarrow \mathbf{Quot}_{X,d}(\mathscr{H}) \tag{1.3}$$

which depend on a sequence (1.2) and with classical truncations $\mathrm{Quot}_{X,d}(\mathscr{G})$ and $\mathrm{Quot}_{X,d}(\mathscr{H})$ that have virtual dimensions $\dim X + \delta d - d^2$ and $\dim X - \delta d - d^2$ respectively. The following is the main result in this paper:

Theorem 1.1. (Theorem 2.16) *Suppose that $\delta \geq 0$. There is a semiorthogonal decomposition of the form*

$$D^b(\mathbf{Quot}_{X,d}(\mathscr{G})) \\ = \left\langle \binom{\delta}{i}\text{-copies of } D^b(\mathbf{Quot}_{X,d-i}(\mathscr{H})) : 0 \leq i \leq \min\{d, \delta\} \right\rangle.$$

The above result is a generalization of the conjecture by Qingyuan Jiang [Jia, Conjecture A.5] when X is smooth (see Corollary 1.2). The case of $d = 1$ is called *projectivization formula* and proved in [Kuz07, Theorem 5.5], [JL, Theorem 3.4], [Todb, Theorem 4.6.11]. The $d = 2$ case is proved in [Jia, Theorem 6.19]. The Quot formula in Theorem 1.1 recovers several known formulas (see [Jia, Section 1.4.2] for details), e.g. Kapranov exceptional collection for Grassmannian [Kap84] (by setting X to be a point), the projectivization formula [Kuz07, JL, Todb] (by setting $d = 1$). The proof involves semiorthogonal decomposition of Grassmannian flip [BCF$^+$21, Todc], which itself generalizes Bondal-Orlov standard flip formula [BO].

Suppose that $\mathscr{G}$ has homological dimension less than or equal to one. Then there is a sequence (1.2) so that ϕ is injective, and in that case $\mathscr{H} = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathscr{G}, \mathcal{O}_X)$ (which is independent of a choice of (1.2) with ϕ injective), and $\delta = \mathrm{rank}(\mathscr{G})$. In [Jia, Conjecture A.5], the conjecture is stated for derived categories of the classical Quot schemes $\mathrm{Quot}_{X,d}(\mathscr{G})$, $\mathrm{Quot}_{X,d}(\mathscr{H})$, when $\mathscr{G}$ has homological dimension less than or equal to one, $\mathscr{H} = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathscr{G}, \mathcal{O}_X)$, and under some Tor-independence condition. The Tor-independence condition implies that the dimensions of the above classical Quot schemes coincide with the virtual dimensions if they are non-empty (see [Jia, Lemma 6.7]). So in this case, they are equivalent to $\mathbf{Quot}_{X,d}(\mathscr{G})$, $\mathbf{Quot}_{X,d}(\mathscr{H})$ respectively, where we take a sequence (1.2) so that ϕ is injective. We also note that, if $\mathrm{Quot}_{X,d}(\mathscr{G}) = \emptyset$ then $\mathbf{Quot}_{X,d}(\mathscr{G})$ is equivalent to $\emptyset$ regardless of the virtual dimension, and the same is true for $\mathrm{Quot}_{X,d}(\mathscr{H})$. Therefore we obtain the following corollary, which proves [Jia, Conjecture A.5] when X is smooth:

Corollary 1.2. *Suppose that* $\mathscr{G}$ *has homological dimension less than or equal to one, and* $\mathscr{H} := \mathcal{E}xt^1_{\mathcal{O}_X}(\mathscr{G}, \mathcal{O}_X)$*. Assume that* $\dim \mathrm{Quot}_{X,d}(\mathscr{G}) = \dim X + \delta d - d^2$ *and* $\dim \mathrm{Quot}_{X,d}(\mathscr{H}) = \dim X - \delta d - d^2$*, where* $\delta = \mathrm{rank}(\mathscr{G}) \geq 0$*. Then we have a semiorthogonal decomposition of the form*

$$D^b(\mathrm{Quot}_{X,d}(\mathscr{G})) \\ = \left\langle \binom{\delta}{i}\text{-copies of } D^b(\mathrm{Quot}_{X,d-i}(\mathscr{H})) : 0 \leq i \leq \min\{d, \delta\} \right\rangle.$$

Example 1.3. *When* $\mathscr{G}$ *is locally free, then* $\mathscr{H} = 0$*. Suppose that* $\delta \geq d$*. Then* $\mathrm{Quot}_{X,d}(\mathscr{G})$ *is a Grassmannian bundle over* X *with fiber* $\mathrm{Gr}(d, \delta)$*, and* $\mathrm{Quot}_{X,d}(\mathscr{H}) = X$ *for* $d = 0$*,* $\emptyset$ *for* $d > 0$*. In this case, Corollary 1.2 gives*

$$D^b(\mathrm{Quot}_{X,d}(\mathscr{G})) = \left\langle \binom{\delta}{d}\text{-copies of } D^b(X) \right\rangle.$$

When X *is a point, the above semiorthogonal decomposition gives Kapranov exceptional collection of Grassmannian variety [Kap84].*

Remark 1.4. *In [Jia, Conjecture A.5], the conjecture is formulated in a more general assumption on* X*. We focus on the case that* X *is a smooth quasi-projective variety over* $\mathbb{C}$ *in order to avoid some technical subtleties. This assumption is enough for applications in [Jia, Section 1.5].*

Remark 1.5. *Each fully-faithful functor* $D^b(\mathbf{Quot}_{X,d-i}(\mathscr{H})) \hookrightarrow D^b(\mathbf{Quot}_{X,d}(\mathscr{G}))$ *in Theorem 1.1 can be shown to be of Fourier-Mukai type, though we will not discuss its details. However the proof of Theorem 1.1 does not give any information about the kernel objects.*

We prove Theorem 1.1 by interpreting (-1)-shifted cotangent derived schemes in (1.3) (see Section 2.2 for (-1)-shifted cotangent derived schemes or stacks) as d-critical Grassmannian flip in the sense of [Toda] (see Remark 2.8), and then use Koszul duality together with categorified Hall products for families of Grassmannian flips. The categorified Hall products for Grassmannian flip are used in [Todc] as an intermediate step toward the categorification of wall-crossing formula of Donaldson-Thomas invariants on the resolved conifold.

1.2. Applications

The Quot formula in Corollary 1.2 has lots of applications on derived categories of classical moduli spaces (see [Jia, Section 1.5]). Here we mention two examples: one is a generalization of [Tod21, Corollary 5.11] and [Jia, Corollary 1.3] on semiorthogonal decompositions of varieties associated with Brill-Noether loci for curves, and the other one is a categorical blow-up formula of Hilbert schemes of points on surfaces obtained by Koseki [Kos].

Let C be a smooth projective curve over $\mathbb{C}$ with genus g. We denote by $\mathrm{Pic}^d(C)$ the Picard variety parameterizing degree d line bundles on C, which is a g-dimensional complex torus and (non-canonically) isomorphic to the Jacobian $\mathrm{Jac}(C)$ of C. The Brill-Noether locus on $\mathrm{Pic}^d(C)$ is defined by

$$W_d^r(C) := \{L \in \mathrm{Pic}^d(C) : h^0(L) \geq r+1\}.$$

There is a scheme $G_d^r(C)$ parameterizing g_d^r's which appears in the classical study of Brill-Noether loci (see [ACGH85, Chapter 4, Section 3]). It is set theoretically given by

$$G_d^r(C) = \{(L, W) : L \in W_d^r(C), W \subset H^0(C, L), \dim W = r+1\}$$

where W is a vector subspace. If C is a general curve, then $G_d^r(C)$ is a smooth projective variety of expected dimension $g-(r+1)(g-d+r)$. As explained in [Jia, Section 1.5.1], for any $\delta \geq 0$ there is a coherent sheaf $\mathscr{G}$ on $X = \mathrm{Pic}^{g-1+\delta}(C)$ of rank δ that has homological dimension less than or equal to 1 and such that

$$\mathrm{Quot}_{X,r+1}(\mathscr{G}) = G^r_{g-1+\delta}(C),\ \ \mathrm{Quot}_{X,r+1}(\mathscr{H}) = G^r_{g-1-\delta}(C).$$

Here $\mathscr{H} = \mathscr{E}xt^1_{\mathcal{O}_X}(\mathscr{G}, \mathcal{O}_X)$. By applying Corollary 1.2, we have the following:

Corollary 1.6. *Let C be a general smooth projective curve with genus g. Then for any $r \in \mathbb{Z}_{\geq 0}$ and $\delta \geq 0$, there is a semiorthogonal decomposition*

$$D^b(G^r_{g-1+\delta}(C)) = \left\langle \binom{\delta}{i}\text{-copies of } D^b(G^{r-i}_{g-1-\delta}(C)) : 0 \leq i \leq \min\{\delta, r+1\} \right\rangle.$$

Here for $i = r+1$, we have $G^{-1}_{g-1-\delta}(C) = \mathrm{Pic}^{g-1-\delta}(C)$.

The case of $r = 0$ gives the semiorthogonal decomposition of symmetric products

$$D^b(\mathrm{Sym}^{g-1+\delta}(C)) = \left\langle D^b(\mathrm{Sym}^{g-1-\delta}(C)), \delta\text{-copies of } D^b(\mathrm{Jac}(C)) \right\rangle$$

proved in [Tod21, Corollary 5.11]. The case of $r = 1$ is given in [Jia, Corollary 1.3], and it is

$$\begin{aligned} D^b(G^1_{g-1+\delta}(C)) = \Big\langle & D^b(G^1_{g-1-\delta}(C)), \delta\text{-copies of } D^b(\mathrm{Sym}^{g-1-\delta}(C)), \\ & \binom{\delta}{2}\text{-copies of } D^b(\mathrm{Jac}(C)) \Big\rangle. \end{aligned}$$

The result of Corollary 1.6 extends the above results to an arbitrary $r \in \mathbb{Z}_{\geq 0}$.

Another application is on semiorthogonal decompositions of Hilbert schemes of points on surfaces under blow-up. Let S be a smooth projective surface and $\widehat{S} \to S$ be a blow-up at a point. Then the Göttsche formula [Gö90] for the Euler numbers of Hilbert schemes of points $\mathrm{Hilb}^n(S)$ in particular implies the blow-up formula

$$\sum_{n\geq 0} e(\mathrm{Hilb}^n(\widehat{S}))q^n = \sum_{n \geq 0} e(\mathrm{Hilb}^n(S))q^n \cdot \prod_{d \geq 1} \frac{1}{(1-q^d)}. \tag{1.4}$$

Note that if we define $p(j)$ to be the number of partitions of j, we have the formula

$$\sum_{j \geq 0} p(j)q^j = \prod_{d\geq 1} \frac{1}{(1-q^d)}.$$

On the other hand, Nakajima-Yoshioka [NY11] proved that $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(\widehat{S})$ are related by wall-crossing diagrams. One can show that each wall-crossing diagram fits into the framework of Quot formula in Theorem 1.1, see [NY11, Theorem 4.1], [Kos, Theorem 4.1]. Based on this observation and using Theorem 1.1, the following blow-up formula is obtained in [Kos]:

Theorem 1.7. (Koseki [Kos]) *There is a semiorthogonal decomposition of the form*

$$D^b(\mathrm{Hilb}^n(\widehat{S})) = \left\langle p(j)\text{-}copies\ of\ D^b(\mathrm{Hilb}^{n-j}(S)) : j = 0, \dots, n \right\rangle.$$

The semiorthogonal decomposition in Theorem 1.7 categorifies the blow-up formula (1.4).

1.3. Notation and convention

In this paper, all the varieties or (derived) stacks are defined over $\mathbb{C}$. For an introduction to derived algebraic geometry, we refer to [Toë14]. For a locally free sheaf $\mathscr{E}$ on a stack X, we often regard it as a total space of its associated vector bundle, i.e. $\operatorname{Spec}\operatorname{Sym}(\mathscr{E}^\vee) \to X$. For a derived Artin stack $\mathfrak{M}$, we denote by $t_0(\mathfrak{M})$ its classical truncation. Explicitly if $\mathfrak{M} = [\operatorname{Spec} A^\bullet/G]$ for a commutative dg-algebra $A^\bullet$ with non-positive degrees and an algebraic group G acting on $A^\bullet$, we have $t_0(\mathfrak{M}) = [\operatorname{Spec}\mathcal{H}^0(A^\bullet)/G]$, also see Remark 2.4. For a complex of vector bundles $\mathscr{E}^\bullet$ with differential $d_{\mathscr{E}^\bullet}$ on X, we denote by $\operatorname{Sym}(\mathscr{E}^\bullet)$ the sheaf of dg-algebras on X, whose underlying graded sheaf is the super-symmetric product of $\mathscr{E}^\bullet$, and the differential $d_{\operatorname{Sym}(\mathscr{E}^\bullet)}$ is uniquely determined by the condition that $d_{\operatorname{Sym}(\mathscr{E}^\bullet)}|_{\mathscr{E}^\bullet} = d_{\mathscr{E}^\bullet}$ and the Leibniz rule.

For a derived stack $\mathfrak{M}$, the triangulated category $D^b(\mathfrak{M})$ is defined to be the homotopy category of the ∞-category of quasi-coherent sheaves on $\mathfrak{M}$ with bounded coherent cohomologies. The tangent complex of $\mathfrak{M}$ is denoted by $\mathbb{T}_{\mathfrak{M}}$ (see [Toë14, Section 3.1]), and the cotangent complex $\mathbb{L}_{\mathfrak{M}}$ is defined to be its dual. A derived stack $\mathfrak{M}$ is called quasi-smooth if its cotangent complex $\mathbb{L}_{\mathfrak{M}}$ is perfect and $\mathbb{L}_{\mathfrak{M}}|_{t_0(\mathfrak{M})}$ has cohomological amplitude contained in $[-1, 1]$. The rank of $\mathbb{L}_{\mathfrak{M}}|_{t_0(\mathfrak{M})}$ is called the virtual dimension of $\mathfrak{M}$. For example if $\mathcal{Y}$ is a smooth (classical) Artin stack, $\mathscr{E} \to \mathcal{Y}$ is a vector bundle with a section s, the derived fiber product $\mathcal{Y} \times_{0,\mathscr{E},s} \mathcal{Y}$ is quasi-smooth with virtual dimension $\dim \mathcal{Y} - \operatorname{rank}(\mathscr{E})$, which is called derived zero locus of s. When $\mathcal{Y} = \operatorname{Spec} A$ for a commutative $\mathbb{C}$-algebra and $\mathscr{E}$ is determined by a projective A-module M, then the derived zero locus is $\operatorname{Spec} K(A, M, s)$, where $K(A, M, s)$ is the Koszul complex $\cdots \to \wedge M^\vee \xrightarrow{s} M^\vee \xrightarrow{s} A \to 0$, see [Toë14, Last paragraph of Section 2.2].

2. Proof of Theorem 1.1

2.1. Derived structures of Quot schemes

Let X be a smooth quasi-projective variety over $\mathbb{C}$, $\mathscr{G}$ a coherent sheaf on it. Recall that the Quot scheme $\operatorname{Quot}_{X,d}(\mathscr{G})$ represents the functor

$$\mathcal{Q}uot_{X,d}(\mathscr{G}) \colon (Sch/X)^{op} \to (Set)$$

which sends $T \to X$ to the equivalence classes of $\mathscr{G}_T \twoheadrightarrow \mathscr{P}$ where $\mathscr{P}$ is a locally free sheaf on T of rank d and $\mathscr{G}_T$ is the pull-back of $\mathscr{G}$ to T.

Let us take a right exact sequence

$$\mathscr{E}^{-1} \xrightarrow{\phi} \mathscr{E}^{0} \to \mathscr{G} \to 0 \tag{2.1}$$

such that $\mathscr{E}^i$ are locally free sheaves of finite rank on X. The surjection $\mathscr{E}^0 \twoheadrightarrow \mathscr{G}$ induces the closed immersion

$$\mathrm{Quot}_{X,d}(\mathscr{G}) \hookrightarrow \mathrm{Quot}_{X,d}(\mathscr{E}^0). \tag{2.2}$$

Below we fix a vector space V of dimension d, and denote by $\mathrm{GL}_X(V) := \mathrm{GL}(V) \times X \to X$ the group scheme over X. We also set

$$\mathfrak{C}(\mathscr{E}^0) := [\mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X)/\,\mathrm{GL}_X(V)].$$

Here we have identified the locally free sheaf $\mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X)$ with the associated vector bundle over X, i.e. $\operatorname{Spec}\operatorname{Sym}(\mathscr{E}^0 \otimes V^\vee) \to X$.

Lemma 2.1. *There is an open immersion* $\mathrm{Quot}_{X,d}(\mathscr{E}^0) \subset \mathfrak{C}(\mathscr{E}^0)$.

Proof. For $T \to X$, the T-valued points of the stack $\mathfrak{C}(\mathscr{E}^0)$ consist of $(\mathscr{P}, s)$ where $\mathscr{P}$ is a vector bundle on T of rank d and $s\colon \mathscr{E}^0_T \to \mathscr{P}$ is a morphism. Indeed giving a X-morphism $T \to \mathfrak{C}(\mathscr{E}^0)$ is equivalent to giving a $\mathrm{GL}_T(V)$-torsor $\mathscr{F} \to T$ and a $\mathrm{GL}_T(V)$-equivariant morphism $\mathscr{F} \to \mathcal{H}om(\mathscr{E}^0_T, V \otimes \mathcal{O}_T)$. The $\mathrm{GL}_T(V)$-torsor $\mathscr{F}$ corresponds to a vector bundle $\mathscr{P}$ on T such that $\mathscr{F}$ is isomorphic to the local framing of $\mathscr{P}$, i.e. the set of sections of $\mathscr{F}$ over an étale morphism $U \to T$ is the set of isomorphisms $\mathscr{P}_U \xrightarrow{\cong} V \otimes \mathcal{O}_U$. Then the $\mathrm{GL}_T(V)$-equivariant morphism $\mathscr{F} \to \mathcal{H}om(\mathscr{E}^0_T, V \otimes \mathcal{O}_T)$ corresponds to a vector bundle morphism $\mathscr{E}^0_T \to \mathscr{P}$ on T.

From the definition of $\mathrm{Quot}_{X,d}(\mathscr{E}^0)$, it is isomorphic to the open substack of $\mathfrak{C}(\mathscr{E}^0)$ whose T-valued points correspond to $(\mathscr{P}, s)$ such that s is surjective. $\square$

We have the following vector bundle over $\mathfrak{C}(\mathscr{E}^0)$ with a section s

$$\begin{tikzcd} \big[\big(\mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X) \oplus \mathcal{H}om(\mathscr{E}^{-1}, V \otimes \mathcal{O}_X)\big)\,/\,\mathrm{GL}_X(V)\big] \arrow[r] & \mathfrak{C}(\mathscr{E}^0). \arrow[l, bend right, "s"'] \end{tikzcd} \tag{2.3}$$

The section s is induced by the $\mathrm{GL}_X(V)$-equivariant morphism

$$\begin{aligned} s\colon \mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X) &\to \mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X) \oplus \mathcal{H}om(\mathscr{E}^{-1}, V \otimes \mathcal{O}_X), \\ f &\mapsto (f, f \circ \phi). \end{aligned} \tag{2.4}$$

We denote by $\mathscr{E}^\bullet$ the two term complex $(\mathscr{E}^{-1} \xrightarrow{\phi} \mathscr{E}^0)$ such that $\mathscr{E}^0$ is of degree zero. Let $\mathfrak{C}(\mathscr{E}^\bullet)$ be the derived zero locus of s. The Koszul complex associated with s is

$$\begin{aligned}\cdots \to \wedge^2\mathscr{E}^{-1} &\otimes \mathrm{Sym}(\mathscr{E}^0 \otimes V^\vee) \\ &\xrightarrow{s} \mathscr{E}^{-1} \otimes \mathrm{Sym}(\mathscr{E}^0 \otimes V^\vee) \xrightarrow{s} \mathrm{Sym}(\mathscr{E}^0 \otimes V^\vee) \to 0\end{aligned}$$

which coincides with $\mathrm{Sym}(\mathscr{E}^\bullet \otimes V^\vee)$, see Subsection 1.3 for the dg-algebra structure on $\mathrm{Sym}(\mathscr{E}^\bullet \otimes V^\vee)$ over X. Therefore $\mathfrak{C}(\mathscr{E}^\bullet)$ is written as

$$\mathfrak{C}(\mathscr{E}^\bullet) := \left[\mathrm{Spec}\,\mathrm{Sym}(\mathscr{E}^\bullet \otimes V^\vee)/\,\mathrm{GL}_X(V)\right]. \tag{2.5}$$

Note that $\mathfrak{C}(\mathscr{E}^\bullet)$ is a derived closed substack of $\mathfrak{C}(\mathscr{E}^0)$. We set

$$\mathbf{Quot}_{X,d}(\mathscr{G}) := \mathrm{Quot}_{X,d}(\mathscr{E}^0) \cap \mathfrak{C}(\mathscr{E}^\bullet),$$

in other word $\mathbf{Quot}_{X,d}(\mathscr{G})$ is the derived zero locus of s restricted to the open substack $\mathrm{Quot}_{X,d}(\mathscr{E}^0) \subset \mathfrak{C}(\mathscr{E}^0)$.

Lemma 2.2. *The derived stack* $\mathbf{Quot}_{X,d}(\mathscr{G})$ *has virtual dimension* $\dim X + \delta d - d^2$*, with classical truncation* $\mathrm{Quot}_{X,d}(\mathscr{G})$*.*

Proof. The derived stack $\mathfrak{C}(\mathscr{E}^\bullet)$ is a derived zero locus of s, so it is quasi-smooth with virtual dimension

$$\begin{aligned}\dim \mathfrak{C}(\mathscr{E}^0) &- \mathrm{rank}(V \otimes \mathscr{E}^{-1\vee}) \\ &= \dim X + d\,\mathrm{rank}(\mathscr{E}^0) - \dim \mathrm{GL}(V) - d\,\mathrm{rank}(\mathscr{E}^{-1}) \\ &= \dim X + \delta d - d^2.\end{aligned}$$

The derived stack $\mathbf{Quot}_{X,d}(\mathscr{G})$ is an open substack of $\mathfrak{C}(\mathscr{E}^\bullet)$, so it also has virtual dimension $\dim X + \delta d - d^2$.

For a X-scheme $T \to X$, a T-valued point of the classical truncation of $\mathbf{Quot}_{X,d}(\mathscr{G})$ consists of a surjection $\mathscr{E}^0_T \twoheadrightarrow \mathscr{P}$ such that the composition $\mathscr{E}^{-1}_T \to \mathscr{E}^0_T \twoheadrightarrow \mathscr{P}$ is zero. This is equivalent to giving a surjection $\mathscr{G}_T \twoheadrightarrow \mathscr{P}$, i.e. a T-valued point of $\mathrm{Quot}_{X,d}(\mathscr{G})$. $\square$

By taking the dual of the sequence (2.1), we obtain the right exact sequence

$$\mathscr{E}_0 \xrightarrow{\phi^\vee} \mathscr{E}_1 \to \mathscr{H} \to 0.$$

Here we have set $\mathscr{E}_i := (\mathscr{E}^{-i})^\vee$, and $\mathscr{H}$ is defined to be the cokernel of $\phi^\vee$. We apply the above construction for the quotient $\mathscr{E}_1 \twoheadrightarrow \mathscr{H}$. By replacing V with $V^\vee$ and noting $\mathrm{GL}_X(V) = \mathrm{GL}_X(V^\vee)$, we have the closed immersion and an open immersion

$$\mathrm{Quot}_{X,d}(\mathscr{H}) \hookrightarrow \mathrm{Quot}_{X,d}(\mathscr{E}_1) \subset \mathfrak{C}(\mathscr{E}_1) := [\mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X)/\,\mathrm{GL}_X(V)].$$

We also have the vector bundle with a section $s^\vee$

(2.6)

$$\begin{tikzcd}
{[(\mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X) \oplus \mathcal{H}om(\mathscr{E}_0, V^\vee \otimes \mathcal{O}_X))\,/\,\mathrm{GL}_X(V)]} \arrow[r] & \mathfrak{C}(\mathscr{E}_1). \arrow[l, bend right, "s^\vee"']
\end{tikzcd}$$

The section $s^\vee$ is induced by the morphism

$$s^\vee \colon \mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X) \to \mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X) \oplus \mathcal{H}om(\mathscr{E}_0, V^\vee \otimes \mathcal{O}_X)$$

given by $f \mapsto (f, f \circ \phi^\vee)$. Similarly to (2.5), the derived zero locus of $s^\vee$ is written as

$$\mathfrak{C}(\mathscr{E}_\bullet[1]) := [\mathrm{Spec}\,\mathrm{Sym}(\mathscr{E}_\bullet[1] \otimes V)/\,\mathrm{GL}_X(V)]\,.$$

Here $\mathscr{E}_\bullet[1]$ is the complex $(\mathscr{E}_0 \xrightarrow{\phi^\vee} \mathscr{E}_1)$ such that $\mathscr{E}_1$ is of degree zero. We set

$$\mathbf{Quot}_{X,d}(\mathscr{H}) := \mathrm{Quot}_{X,d}(\mathscr{E}_1) \cap \mathfrak{C}(\mathscr{E}_\bullet[1]). \tag{2.7}$$

The same proof of Lemma 2.2 shows that $\mathbf{Quot}_{X,d}(\mathscr{H})$ has virtual dimension $\dim X - \delta d - d^2$, and its classical truncation is $\mathrm{Quot}_{X,d}(\mathscr{H})$.

2.2. (-1)-shifted cotangent derived stacks

For a derived Artin stack $\mathfrak{M}$, its (-1)-shifted cotangent is defined by (see [Cal19])

$$\Omega_{\mathfrak{M}}[-1] := \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_{\mathfrak{M}}}(\mathbb{T}_{\mathfrak{M}}[1]).$$

Here $\mathbb{T}_{\mathfrak{M}}$ is the tangent complex of $\mathfrak{M}$.

In the case that $\mathfrak{M}$ is a derived zero locus, the classical truncation of $\Omega_{\mathfrak{M}}[-1]$ has the following critical locus description. Let $\mathcal{Y} = [Y/G]$ for a smooth quasi-projective scheme Y and G is an affine algebraic group acting on Y. Let $\mathcal{F} \to \mathcal{Y}$ be a vector bundle on it with a section s, which is identified with a G-equivariant vector bundle $F \to Y$ together with a G-invariant section $\widetilde{s}$ of $F \to Y$. Suppose that $\mathfrak{M}$ is a derived zero locus of s, that is $\mathfrak{M} = [\widetilde{M}/G]$ where $\widetilde{M}$ is the derived zero locus of $\widetilde{s}$. Let w be the function

$$w \colon \mathcal{F}^{\vee} \to \mathbb{A}^1, \ w(y, v) = \langle s(y), v \rangle \tag{2.8}$$

for $y \in \mathcal{Y}$ and $v \in \mathcal{F}^{\vee}|_y$, which is identified with a G-invariant function $\widetilde{w}$ on $F^{\vee}$. We set

$$\mathrm{Crit}(w) := [\mathrm{Crit}(\widetilde{w})/G] \subset \mathcal{F}^{\vee}$$

which is a closed substack of $\mathcal{F}^{\vee}$. Here $\mathrm{Crit}(\widetilde{w}) \subset F^{\vee}$ is the scheme theoretic critical locus of $\widetilde{w}$, defined by the ideal generated by the image of $d\widetilde{w} \colon T_{F^{\vee}} \to \mathcal{O}_{F^{\vee}}$. (Alternatively $\mathrm{Crit}(w)$ is the closed substack of $\mathcal{F}^{\vee}$ defined by the ideal generated by the image of $dw \colon \mathcal{H}^0(\mathbb{T}_{\mathcal{F}^{\vee}}) \to \mathcal{O}_{\mathcal{F}^{\vee}}$).

Lemma 2.3. *Suppose that $\mathfrak{M}$ is the derived zero locus of a section s of a vector bundle $\mathcal{F} \to \mathcal{Y}$ for a quotient stack $\mathcal{Y} = [Y/G]$ as above. Then the classical truncation $t_0(\Omega_{\mathfrak{M}}[-1])$ of $\Omega_{\mathfrak{M}}[-1])$ is isomorphic to* $\mathrm{Crit}(w)$.

Proof. We denote by $M \subset Y$ the classical truncation of $\widetilde{M}$, that is the classical zero locus of $\widetilde{s}$. Note that $M \subset Y$ is a G-invariant closed subscheme, and we have $\mathcal{M} := t_0(\mathfrak{M}) = [M/G]$, see Remark 2.4. The shifted tangent complex $\mathbb{T}_{\mathfrak{M}}[1]$ restricted to $\mathcal{M}$ is given by

$$\mathbb{T}_{\mathfrak{M}}[1]|_{\mathcal{M}} = (\mathfrak{g} \otimes \mathcal{O}_{\mathcal{M}} \to T_{\mathcal{Y}}|_{\mathcal{M}} \stackrel{ds}{\to} \mathcal{F}|_{\mathcal{M}})$$

where $T_{\mathcal{Y}} = [T_Y/G]$ which is a vector bundle on $\mathcal{Y}$, $\mathcal{F}$ is located in degree zero. In particular $\mathfrak{M}$ is quasi-smooth, see Subsection 1.3 for the definition of quasi-smoothness. Let us take a distinguished triangle in $D^b(\mathfrak{M})$

$$\mathcal{R} \to \mathbb{T}_{\mathfrak{M}}[1] \to \mathbb{T}_{\mathfrak{M}}[1]|_{\mathcal{M}}.$$

Here we regarded the last term as an object in $D^b(\mathfrak{M})$ by the push-forward of the closed immersion $\mathcal{M} \hookrightarrow \mathfrak{M}$. Then $\mathcal{R}$ is concentrated in negative degrees, $\mathbb{T}_{\mathfrak{M}}[1]$ and $\mathbb{T}_{\mathfrak{M}}[1]|_{\mathcal{M}}$ are concentrated on non-positive degrees. Therefore by taking the symmetric products and the zero-th cohomology, we have

$$\mathcal{H}^0(\mathrm{Sym}_{\mathcal{O}_{\mathfrak{M}}}(\mathbb{T}_{\mathfrak{M}}[1])) \stackrel{\cong}{\to} \mathcal{H}^0(\mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathbb{T}_{\mathfrak{M}}|_{\mathcal{M}}[1])).$$

We also have the distinguished triangle

$$\mathfrak{g} \otimes \mathcal{O}_{\mathcal{M}}[1] \to (T_{\mathcal{Y}}|_{\mathcal{M}} \stackrel{ds}{\to} \mathcal{F}|_{\mathcal{M}}) \to \mathbb{T}_{\mathfrak{M}}[1]$$

where in the middle term $\mathcal{F}|_{\mathcal{M}}$ is located in degree zero. Again by taking the symmetric products and the zero-th cohomology, we obtain

$$\mathcal{H}^0(\mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(T_{\mathcal{Y}}|_{\mathcal{M}} \stackrel{ds}{\to} \mathcal{F}|_{\mathcal{M}})) \stackrel{\cong}{\to} \mathcal{H}^0(\mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(\mathbb{T}_{\mathfrak{M}}|_{\mathcal{M}}[1])).$$

Therefore the stack $t_0(\Omega_{\mathfrak{M}}[-1])$ is isomorphic to the classical truncation of

$$\mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_{\mathcal{M}}}(T_{\mathcal{Y}}|_{\mathcal{M}} \stackrel{ds}{\to} \mathcal{F}|_{\mathcal{M}}) = [\mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_M}(T_Y|_M \stackrel{d\widetilde{s}}{\to} F|_M)/G].$$

The classical truncation of the derived scheme $\mathrm{Spec}\,\mathrm{Sym}(T_Y|_M \stackrel{d\widetilde{s}}{\to} F|_M)$ is isomorphic to $\mathrm{Crit}(\widetilde{w})$ (see [JT17, Proposition 2.8], [Todb, Section 2.1.1]), therefore the lemma holds. $\square$

Remark 2.4. *We use the fact taking the classical truncation $t_0(-)$ commutes with taking the quotient stack. Indeed let $\mathfrak{Y}$ be a derived scheme with a G-action, and $Y = t_0(\mathfrak{Y})$. The quotient stack $[\mathfrak{Y}/G]$ is obtained as a colimit of the simplicial derived scheme that is equal to $G^{\times n} \times \mathfrak{Y}$ in degree n. As $t_0(-)$ commutes with taking colimits, see [TV08, Paragraph after Definition 2.2.4.3], we see that $t_0([\mathfrak{Y}/G]) = [Y/G]$.*

The above construction is summarized in the following diagram

(2.9)
$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "0"] & \mathcal{F} \arrow[d] \\ \mathfrak{M} \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\square"] & \mathcal{Y}, \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} t_0(\Omega_{\mathfrak{M}}[-1]) \arrow[r, "\cong"] \arrow[d] & \mathrm{Crit}(w) \arrow[r, hook] \arrow[d] & \mathcal{F}^{\vee} \arrow[d] \arrow[dr, "w"] & \\ \mathfrak{M} \arrow[r, equal] & \mathfrak{M} \arrow[r, hook] & \mathcal{Y} & \mathbb{A}^1. \end{tikzcd}$$

Here the left square is a derived Cartesian.

We return to the setting of the previous subsections. Let V be a d-dimensional vector space. We set $Y(d)$ and $\mathcal{Y}(d)$ to be

$$\begin{aligned} Y(d) &:= \mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X) \oplus \mathcal{H}om(V \otimes \mathcal{O}_X, \mathscr{E}^{-1}), \\ \mathcal{Y}(d) &:= [Y(d)/\operatorname{GL}_X(V)]. \end{aligned} \tag{2.10}$$

Again we have regarded $Y(d)$ as the total space of a vector bundle over X. For $T \to X$, the T-valued points of the stack $\mathcal{Y}(d)$ consist of tuples

$$(\mathscr{P}, \alpha, \beta), \ \alpha\colon \mathscr{E}^0_T \to \mathscr{P}, \ \beta\colon \mathscr{P} \to \mathscr{E}^{-1}_T \tag{2.11}$$

where $\mathscr{P}$ is a locally free sheaf on T of rank d. Note that the projection

$$\mathcal{Y}(d) \to [\mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X)/\operatorname{GL}_X(V)] = \mathfrak{C}(\mathscr{E}^0)$$

identifies $\mathcal{Y}(d)$ with the dual vector bundle of (2.3). We define the super-potential

$$w\colon \mathcal{Y}(d) \to \mathbb{A}^1, \ (\mathscr{P}, \alpha, \beta) \mapsto \langle s(\alpha), \beta \rangle = \operatorname{Tr}(\alpha \circ \phi_T \circ \beta). \tag{2.12}$$

Here over the T-valued points, the last expression is given by taking the trace of the composition

$$\alpha \circ \phi_T \circ \beta\colon \mathscr{P} \xrightarrow{\beta} \mathscr{E}^{-1}_T \xrightarrow{\phi_T} \mathscr{E}^0_T \xrightarrow{\alpha} \mathscr{P}.$$

From the diagram (2.9), Lemma 2.3 (applied for $\mathcal{Y} = \mathfrak{C}(\mathscr{E}^0)$, $\mathcal{F}$ is the vector bundle (2.3) so that $\mathcal{F}^\vee = \mathcal{Y}(d)$, the section s is (2.4)) implies that we have the isomorphism

$$\operatorname{Crit}(w) \xrightarrow{\cong} t_0(\Omega_{\mathfrak{C}(\mathscr{E}^\bullet)}[-1]). \tag{2.13}$$

Remark 2.5. *Let $a = \operatorname{rank}(\mathscr{E}^0)$ and $b = \operatorname{rank}(\mathscr{E}^{-1})$, and denote by $Q_{a,b}$ the quiver with two vertices $\{0, 1\}$, the a-arrows from 0 to 1 and b-arrows from 1 to 0. We denote by*

$$\mathcal{R}_{Q_{a,b}}(d) := [(V^{\oplus a} \oplus V^{\vee \oplus b})/\operatorname{GL}(V)],$$

the moduli stack of representations of $Q_{a,b}$ with dimension vector $(1, d)$ for $d = \dim V$. If X is a point, then $\mathcal{Y}(d)$ is isomorphic to $\mathcal{R}_{Q_{a,b}}(d)$. In general there is a projection $h\colon \mathcal{Y}(d) \to X$ whose fiber is isomorphic to $\mathcal{R}_{Q_{a,b}}(d)$. Moreover $\mathcal{Y}(d) \cong \mathcal{R}_{Q_{a,b}}(d) \times X$ if $\mathscr{E}^0$ and $\mathscr{E}^{-1}$ are free $\mathcal{O}_X$-modules.

We have the isomorphism

$$\mathcal{Y}(d) \xrightarrow{\cong} \left[\left(\mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X) \oplus \mathcal{H}om(V^\vee \otimes \mathcal{O}_X, \mathscr{E}_0)\right) / \operatorname{GL}_X(V)\right] \tag{2.14}$$

by the correspondence over T-valued points

$$(\mathscr{P}, \alpha, \beta) \mapsto (\mathscr{P}^\vee, \beta^\vee, \alpha^\vee),\ \alpha^\vee \colon \mathscr{P}^\vee \to (\mathscr{E}_0)_T,\ \beta^\vee \colon (\mathscr{E}_1)_T \to \mathscr{P}^\vee.$$

Under the isomorphism (2.14), the projection

$$\mathcal{Y}(d) \to [\mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X)/\operatorname{GL}_X(V)] = \mathfrak{C}(\mathscr{E}_1)$$

identifies the stack $\mathcal{Y}(d)$ with the dual vector bundle of (2.6). Moreover under the isomorphism (2.14), the super-potential (2.12) is also identified with

$$w(\alpha, \beta) = \langle s^\vee(\beta^\vee), \alpha^\vee \rangle = \operatorname{Tr}(\beta^\vee \circ \phi^\vee \circ \alpha^\vee),$$

where over the T-valued points, the last expression is the trace for the composition

$$\beta^\vee \circ \phi^\vee \circ \alpha^\vee \colon \mathscr{P}^\vee \xrightarrow{\alpha^\vee} (\mathscr{E}_0)_T \xrightarrow{\phi^\vee} (\mathscr{E}_1)_T \xrightarrow{\beta^\vee} \mathscr{P}^\vee.$$

Therefore again by Lemma 2.3, we also have the isomorphism

$$\operatorname{Crit}(w) \xrightarrow{\cong} t_0(\Omega_{\mathfrak{C}(\mathscr{E}_\bullet[1])}[-1]). \tag{2.15}$$

Let χ_0 be the determinant character of $\operatorname{GL}(V)$

$$\chi_0 \colon \operatorname{GL}(V) \to \mathbb{C}^*,\ g \mapsto \det g, \tag{2.16}$$

which naturally determines a line bundle on $\mathcal{Y}(d)$, denoted by the same symbol χ_0.

Lemma 2.6. *The GIT semistable locus*

$$\mathcal{Y}(d)^{\chi_0\text{-ss}} \subset \mathcal{Y}(d),\ \mathcal{Y}(d)^{\chi_0^{-1}\text{-ss}} \subset \mathcal{Y}(d)$$

consists of $(\mathscr{P}, \alpha, \beta)$ *in (2.11) such that* α *is surjective,* $\beta^\vee$ *is surjective, respectively.*

Proof. We only prove the case of $\mathcal{Y}(d)^{\chi_0\text{-ss}}$. By the Hilbert-Mumford criterion in terms of the Θ-stack $\Theta := [\mathbb{A}^1/\mathbb{C}^*]$ (see [HL]), the semistable locus

$\mathcal{Y}(d)^{\chi_0\text{-ss}}$ consists of $p \in \mathcal{Y}(d)$ such that for any $g\colon \Theta \to \mathcal{Y}(d)$ with $g(1) = p$, we have $\mathrm{wt}(g(0)^*\chi_0) \geq 0$. Since $\Theta \to \operatorname{Spec}\mathbb{C}$ is the good moduli space for Θ, see [Alp13, Example 8.2], any map $g\colon \Theta \to \mathcal{Y}(d)$ composed with the projection $\mathcal{Y}(d) \to X$ factors through $\Theta \to \operatorname{Spec}\mathbb{C}$ by the universal property of the good moduli space, see [Alp13, Theorem 6.6]. Therefore any map $g\colon \Theta \to \mathcal{Y}(d)$ is contained in a fiber of $\mathcal{Y}(d) \to X$. Moreover α is surjective if and only if $\alpha|_x$ is surjective for any $x \in X$. Therefore we may assume that X is a point. In this case, the lemma follows from [Todb, Lemma 5.1.9]. $\square$

Let us take the GIT quotient

$$\mathcal{Y}(d) \to Y(d)/\!\!/ \mathrm{GL}_X(V) := \operatorname{Spec}(h_*\mathcal{O}_{Y(d)})^{\mathrm{GL}_X(V)}$$

where $h\colon Y(d) \to X$ is the projection. The above morphism is a good moduli space morphism for $\mathcal{Y}(d)$ in the sense of [Alp13], see [Alp13, Theorem 13.2]. We have the commutative diagram

$$\begin{tikzcd} \mathcal{Y}(d)^{\chi_0\text{-ss}} \arrow[r] \arrow[dr, "w^+"'] & Y(d)/\!\!/ \mathrm{GL}_X(V) \arrow[d, "w"] & \mathcal{Y}(d)^{\chi_0^{-1}\text{-ss}} \arrow[l] \arrow[dl, "w^-"] \\ & \mathbb{A}^1. & \end{tikzcd} \tag{2.17}$$

Lemma 2.7. *The equivalences (2.13), (2.15) restrict to isomorphisms*

$$\operatorname{Crit}(w^+) \xrightarrow{\cong} t_0(\Omega_{\mathbf{Quot}_{X,d}(\mathscr{G})}[-1]),\ \operatorname{Crit}(w^-) \xrightarrow{\cong} t_0(\Omega_{\mathbf{Quot}_{X,d}(\mathscr{H})}[-1]). \tag{2.18}$$

Proof. We only prove the first isomorphism. Lemma 2.6 implies that the following diagram is Cartesian

$$\begin{tikzcd} \mathcal{Y}(d)^{\chi_0\text{-ss}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}(d) \arrow[d] \\ \mathrm{Quot}_{X,d}(\mathscr{E}^0) \arrow[r, hook] & \mathfrak{C}(\mathscr{E}^0) \end{tikzcd} \tag{2.19}$$

where each horizontal arrow is an open immersion. Note that $\operatorname{Crit}(w) \cap \mathcal{Y}(d)^{\chi_0\text{-ss}} = \operatorname{Crit}(w^+)$ as $\mathcal{Y}(d)^{\chi_0\text{-ss}} \subset \mathcal{Y}(d)$ is an open immersion. Therefore we obtain the Cartesian square

$$\begin{tikzcd}
\mathrm{Crit}(w^+) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Crit}(w) \arrow[d] \\
\mathrm{Quot}_{X,d}(\mathscr{E}^0) \arrow[r, hook] & \mathfrak{C}(\mathscr{E}^0).
\end{tikzcd} \tag{2.20}$$

We also have the following Cartesian diagrams from the definition of (-1)-shifted cotangents and $\mathbf{Quot}_{X,d}(\mathscr{G})$

(2.21)

$$\begin{tikzcd}
t_0(\Omega_{\mathbf{Quot}_{X,d}(\mathscr{G})}[-1]) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & t_0(\Omega_{\mathfrak{C}(\mathscr{E}^\bullet)}[-1]) \arrow[d] \\
\mathbf{Quot}_{X,d}(\mathscr{G}) \arrow[r, hook] & \mathfrak{C}(\mathscr{E}^\bullet),
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Quot}_{X,d}(\mathscr{G}) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathfrak{C}(\mathscr{E}^\bullet) \arrow[d, hook] \\
\mathrm{Quot}_{X,d}(\mathscr{E}^0) \arrow[r, hook] & \mathfrak{C}(\mathscr{E}^0).
\end{tikzcd}$$

The lemma follows from the Cartesian squares (2.20), (2.21) together with the isomorphism (2.13). $\square$

When X is a point, the top row in (2.17) is a Grassmannian flip considered in [Todc, (4.5)]. In general, it is a family of Grassmannian flips parametrized by X.

Remark 2.8. *The diagram*

$$\begin{tikzcd}
t_0(\Omega_{\mathbf{Quot}_{X,d}(\mathscr{G})}[-1]) \arrow[dr] & & t_0(\Omega_{\mathbf{Quot}_{X,d}(\mathscr{H})}[-1]) \arrow[dl] \\
& Y(d)/\!\!/\mathrm{GL}_X(V) &
\end{tikzcd}$$

is a d-critical flip in [Toda]. If $\mathrm{Quot}_{X,d}(\mathscr{G})$ *and* $\mathrm{Quot}_{X,d}(\mathscr{H})$ *are smooth of expected dimensions, then the above diagram is identified with*

$$\begin{tikzcd}
\mathrm{Quot}_{X,d}(\mathscr{G}) \arrow[dr] & & \mathrm{Quot}_{X,d}(\mathscr{H}) \arrow[dl] \\
& Y(d)/\!\!/\mathrm{GL}_X(V). &
\end{tikzcd}$$

In general, the above diagram is not necessary a d-critical flip since the relative Quot schemes are not necessary written as critical loci.

2.3. Koszul duality

We apply Koszul duality equivalences to relate derived categories of relative Quot schemes with triangulated categories of $\mathbb{C}^*$-equivariant factorizations. Below we use the convention in [KT21, Section 2.2, 2.3].

Let $\widetilde{\mathrm{GL}}(V)$ be defined by

$$\widetilde{\mathrm{GL}}(V) := \mathrm{GL}(V) \times_{\mathrm{PGL}(V)} \times \mathrm{GL}(V).$$

There is a natural exact sequence

$$1 \to \mathrm{GL}(V) \stackrel{\Delta}{\to} \widetilde{\mathrm{GL}}(V) \stackrel{\tau}{\to} \mathbb{C}^* \to 1 \tag{2.22}$$

where Δ is the diagonal embedding, and τ is the character defined by $\tau(g_1, g_2) = g_1 g_2^{-1}$. The above exact sequence splits non-canonically. Indeed for each $k \in \mathbb{Z}$, $t \mapsto (t^k, t^{k-1})$ gives a splitting of τ. So for each $k \in \mathbb{Z}$, there is an isomorphism

$$\iota_k \colon \mathrm{GL}(V) \times \mathbb{C}^* \stackrel{\cong}{\to} \widetilde{\mathrm{GL}}(V),\ (g, t) \mapsto (t^k g, t^{k-1} g). \tag{2.23}$$

Once we fix a splitting as above, giving a $\widetilde{\mathrm{GL}}(V)$-action is equivalent to giving a $\mathrm{GL}(V)$-action together with an auxiliary $\mathbb{C}^*$-action which commutes with the above $\mathrm{GL}(V)$-action. The $\mathrm{GL}_X(V)$-action on $Y(d)$ over X naturally extends to an action of $\widetilde{\mathrm{GL}}_X(V) := \widetilde{\mathrm{GL}}(V) \times X$ over X. Indeed $\mathrm{GL}_X(V) \times_X \mathrm{GL}_X(V)$ naturally acts on $Y(d)$, where the first factor of $\mathrm{GL}_X(V) \times_X \mathrm{GL}_X(V)$ acts on $\mathcal{H}om(\mathscr{E}^0, V \otimes \mathcal{O}_X)$ and the second factor acts on $\mathcal{H}om(V \otimes \mathcal{O}_X, \mathscr{E}^{-1})$, and the $\widetilde{\mathrm{GL}}_X(V)$-action is given by its restriction. For $k = 0$ in (2.23), the auxiliary $\mathbb{C}^*$-action is given by the weight one action on the second factor of $Y(d)$, for $k = 1$ it is the weight one action on the first factor of $Y(d)$.

The triangulated category of $\mathbb{C}^*$-equivariant factorizations

$$\mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w) \tag{2.24}$$

is defined to be the category whose objects consist of

$$P_0 \stackrel{f}{\to} P_1 \stackrel{g}{\to} P_0\langle\tau\rangle \tag{2.25}$$

where P_0, P_1 are $\widetilde{\mathrm{GL}}_X(V)$-equivariant coherent sheaves on $Y(d)$, f, g are $\widetilde{\mathrm{GL}}_X(V)$-equivariant morphisms such that $f \circ g = \cdot w$, $g \circ f = \cdot w$. Here $\langle\tau\rangle$ means the twist by the $\widetilde{\mathrm{GL}}(V)$-character τ. The category (2.24) is defined to be the localization of the homotopy category of the factorizations (2.25) by its subcategory of acyclic factorizations (see [EP15]). The categories $\mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\chi_0^{\pm 1}\text{-ss}}, w^{\pm})$ are also defined in a similar way.

We now state the Koszul duality equivalence in [Hir17, Proposition 4.8] (also see [Isi13, Shi12, Todb]) in the setting of the diagram (2.9):

Theorem 2.9. ([Hir17, Isi13, Shi12, Todb]) *Let $\mathcal{Y} = [Y/G]$ for a smooth quasi-projective scheme Y and G is an affine algebraic group acting on Y. Let $\mathcal{F} \to \mathcal{Y}$ be a vector bundle on it with a section s, and $\mathfrak{M}$ the derived zero locus of s. Then there is an equivalence*

$$D^b(\mathfrak{M}) \xrightarrow{\sim} \mathrm{MF}^{\mathbb{C}^*}(\mathcal{F}^\vee, w)$$

where $\mathbb{C}^*$ acts on fibers of $\mathcal{F}^\vee \to Y$ with weight one, and w is the function (2.8).

By applying Theorem 2.9, we obtain the following:

Proposition 2.10. *We have the equivalences*

$$\begin{aligned} D^b(\mathbf{Quot}_{X,d}(\mathscr{G})) &\xrightarrow{\sim} \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\chi_0\text{-ss}}, w^+), \\ D^b(\mathbf{Quot}_{X,d}(\mathscr{H})) &\xrightarrow{\sim} \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\chi_0^{-1}\text{-ss}}, w^-). \end{aligned} \tag{2.26}$$

Proof. We apply Theorem 2.9 for $\mathcal{Y} = \mathrm{Quot}_{X,d}(\mathscr{E}^0)$ and the vector bundle $\mathcal{F} \to \mathcal{Y}$ with section s given by the pull-back of (2.3) by the open immersion $\mathrm{Quot}_{X,d}(\mathscr{E}^0) \subset \mathfrak{C}(\mathscr{E}^0)$. Then from the Cartesian square (2.19), we obtain the first equivalence in (2.26) by Theorem 2.9. Here we have used the choice of splitting (2.23) for $k = 0$ in order to specify the auxiliary $\mathbb{C}^*$-action. The second equivalence in (2.26) is similarly proved using another splitting (2.23) for $k = 1$. $\square$

2.4. Window subcategories

We fix a basis of V and a Borel subgroup $B \subset \mathrm{GL}(V)$ to be consisting of upper triangular matrices, and set roots of B to be negative roots. Let $M = \mathbb{Z}^d$ be the character lattice for $\mathrm{GL}(V)$, and $M_{\mathbb{R}}^+ \subset M_{\mathbb{R}}$ the dominant chamber. By the above choice of negative roots, we have

$$M_{\mathbb{R}}^+ = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 \le x_2 \le \cdots \le x_d\}.$$

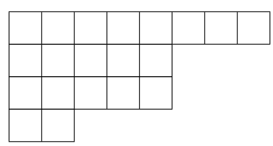
We set $M^+ := M^+_{\mathbb{R}} \cap M$. For $c \in \mathbb{Z}$, we set

$$\mathbb{B}_c(d) := \{(x_1, x_2, \ldots, x_d) \in M^+ : 0 \le x_i \le c - d\}. \tag{2.27}$$

Here $\mathbb{B}_c(d) = \emptyset$ if $c < d$.

Remark 2.11. *For $\chi \in \mathbb{B}_c(d)$, we have the associated Young diagram whose number of boxes at the i-th row is x_{d-i+1}. The above assignment identifies $\mathbb{B}_c(d)$ with the set of Young diagrams with height less than or equal to d, width less than or equal to $c-d$. For example, the following picture illustrates the case of $(2, 5, 5, 8) \in \mathbb{B}_c(d)$ for $d = 4$ and $c \ge 12$:*

Figure 1: $(2, 5, 5, 8) \in \mathbb{B}_c(d), d = 4, c \ge 12$



By fixing a splitting (2.23), we define the triangulated subcategory

$$\mathbb{W}_c(d) \subset \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w) \tag{2.28}$$

to be split generated by factorizations whose entries are of the form $V(\chi) \otimes_{\mathcal{O}_X} \mathscr{P}\langle \tau^i \rangle$ for $\chi \in \mathbb{B}_c(d)$, $i \in \mathbb{Z}$ and $\mathscr{P} \in D^b(X)$. Here $V(\chi)$ is the irreducible $\mathrm{GL}(V)$-representation with highest weight χ (i.e. the Schur power of V associated with the Young diagram determined by χ), and $\tau \colon \widetilde{\mathrm{GL}}(V) \to \mathbb{C}^*$ is the character in (2.22). Note that the subcategory (2.28) does not depend on a choice of a splitting (2.23), since a different splitting only affects on $V(\chi) \otimes_{\mathcal{O}_X} \mathscr{P}\langle \tau^i \rangle$ by a power of τ. We also set

$$a := \mathrm{rank}(\mathscr{E}^0), \ b := \mathrm{rank}(\mathscr{E}^{-1}), \ \delta = a - b. \tag{2.29}$$

Lemma 2.12. *The following compositions are equivalences*

$$\mathbb{W}_a(d) \subset \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w) \twoheadrightarrow \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\chi_0\text{-ss}}, w^+),$$
$$\mathbb{W}_b(d) \subset \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w) \twoheadrightarrow \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\chi_0^{-1}\text{-ss}}, w^-).$$

Proof. We only prove the first equivalence. The lemma is proved in [Todc, Proposition 4.3] when X is a point and there is no super-potential and an auxiliary $\mathbb{C}^*$-action, i.e. $D^b(\mathcal{Y}(d))$ instead of $\mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w)$. Namely let $\mathbb{W}'_a(d) \subset D^b(\mathcal{Y}(d))$ be the triangulated subcategory generated by $V(\chi) \otimes_{\mathcal{O}_X} \mathscr{P}$ for $\chi \in \mathbb{B}_a(d)$ and $\mathscr{P} \in D^b(X)$. If X is a point, then the composition functor

$$\mathbb{W}'_a(d) \subset D^b(\mathcal{Y}(d)) \twoheadrightarrow D^b(\mathcal{Y}^{\chi_0\text{-ss}}(d)) \tag{2.30}$$

is an equivalence by [Todc, Proposition 4.3]. If $\mathscr{E}^i$ are free $\mathcal{O}_X$-modules so that $\mathcal{Y}(d) \cong \mathcal{R}_{Q_{a,b}}(d) \times X$ (see Remark 2.5 for the notation $\mathcal{R}_{Q_{a,b}}(d)$), then (2.30) is an equivalence by taking the $\boxtimes$-product of the equivalence (2.30) in the case of $\mathcal{Y}(d) = \mathcal{R}_{Q_{a,b}}(d)$ with $D^b(X)$. For a general X, let us take the factorization

$$\mathcal{Y}(d) \xrightarrow{\pi} [X/\operatorname{GL}_X(V)] \to X$$

where $\operatorname{GL}_X(V)$ acts on X trivially (so $[X/\operatorname{GL}_X(V)] = X \times B\operatorname{GL}(V)$), and π is a natural morphism induced by the projection $\mathcal{H}om(\mathscr{E}_1, V^\vee \otimes \mathcal{O}_X) \oplus \mathcal{H}om(V^\vee \otimes \mathcal{O}_X, \mathscr{E}_0) \to X$, which is an affine space bundle. For $\chi, \chi' \in \mathbb{B}_a(d)$ and $\mathscr{P}, \mathscr{P}' \in D^b(X)$, we have the natural morphism in $D_{\mathrm{qcoh}}([X/\operatorname{GL}_X(V)])$

$$\begin{aligned} &\mathcal{H}om_{[X/\operatorname{GL}_X(V)]}(V(\chi) \otimes \mathscr{P}, V(\chi') \otimes \mathscr{P}' \otimes \pi_* \mathcal{O}_{\mathcal{Y}(d)}) \\ &\quad \to \mathcal{H}om_{[X/\operatorname{GL}_X(V)]}(V(\chi) \otimes \mathscr{P}, V(\chi') \otimes \mathscr{P}' \otimes \pi_* \mathcal{O}_{\mathcal{Y}(d)^{\chi_0\text{-ss}}}). \end{aligned} \tag{2.31}$$

For a Zariski open subset $U \subset X$, we write $\mathcal{Y}(d)_U := \pi^{-1}([U/\operatorname{GL}_U(V)])$ and denote by $\pi_U \colon \mathcal{Y}(d)_U \to [U/\operatorname{GL}_U(V)]$ the restriction of π to $\mathcal{Y}(d)_U$. Then we have

$$\begin{aligned} &\mathbf{R}\Gamma([U/\operatorname{GL}_U(V)], \mathcal{H}om_{[X/\operatorname{GL}_X(V)]}(V(\chi) \otimes \mathscr{P}, V(\chi') \otimes \mathscr{P}' \otimes \pi_* \mathcal{O}_{\mathcal{Y}(d)})) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{Y}(d)_U}(\pi_U^*(V(\chi) \otimes \mathscr{P}|_U), \pi_U^*(V(\chi') \otimes \mathscr{P}'|_U)), \\ &\mathbf{R}\Gamma([U/\operatorname{GL}_U(V)], \mathcal{H}om_{[X/\operatorname{GL}_X(V)]}(V(\chi) \otimes \mathscr{P}, V(\chi') \otimes \mathscr{P}' \otimes \pi_* \mathcal{O}_{\mathcal{Y}(d)^{\chi_0\text{-ss}}})) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{Y}(d)_U^{\chi_0\text{-ss}}}(\pi_U^*(V(\chi) \otimes \mathscr{P}|_U)|_{\mathcal{Y}(d)_U^{\chi_0\text{-ss}}}, \pi_U^*(V(\chi') \otimes \mathscr{P}'|_U)|_{\mathcal{Y}(d)_U^{\chi_0\text{-ss}}}). \end{aligned}$$

Therefore the morphism (2.31) is an isomorphism Zariski locally on X (by the equivalence (2.30) when $\mathscr{E}^i$ are free), hence (2.31) is an isomorphism. By taking derived global section $\mathbf{R}\Gamma([X/\operatorname{GL}_X(V)], -)$ of the isomorphism (2.31), we see that the functor (2.30) is fully-faithful. For the essential surjectivity of (2.30), we modify the argument of [Todc, Proposition 4.3] by replacing Kapranov exceptional collections on Grassmannians with relative exceptional collections of Grassmannian bundles in [Jia, Theorem 3.70].

The above argument applies verbatim with an auxiliary $\mathbb{C}^*$-action. Namely let $\mathbb{C}^*$ acts on $\mathcal{Y}(d)$ with weight one on the second factor, and $\mathbb{W}''_a(d) \subset D^b([\mathcal{Y}(d)/\mathbb{C}^*])$ the triangulated subcategory generated by $V(\chi) \otimes_{\mathcal{O}_X} \mathscr{P}\langle \tau^i \rangle$ for $\chi \in \mathbb{B}_a(d)$, $\mathscr{P} \in D^b(X)$ and $i \in \mathbb{Z}$ where τ is the weight one character for the auxiliary $\mathbb{C}^*$-action. Then the composition functor

$$\mathbb{W}''_a(d) \subset D^b([\mathcal{Y}(d)/\mathbb{C}^*]) \twoheadrightarrow D^b([\mathcal{Y}(d)^{\chi_0\text{-ss}}/\mathbb{C}^*])$$

is an equivalence. Then the lemma holds by applying the super-potential w to the above equivalence (e.g. applying [Pǎd, Proposition 2.1] for $\mathcal{X} = [\mathcal{Y}(d)/\mathbb{C}^*]$, $I = \{1\}$, $\mathcal{A}_1 = \mathbb{W}''_a(d)$). $\square$

2.5. Categorified Hall products

For a one parameter subgroup $\lambda \colon \mathbb{C}^* \to \mathrm{GL}(V)$, let $V^{\lambda \geq 0} \subset V$ be the subspace of non-negative λ-weights, and $V^{\lambda=0} \subset V$ the λ-fixed subspace. We have the Levi and parabolic subgroups

$$\mathrm{GL}(V)^{\lambda=0} \subset \mathrm{GL}(V)^{\lambda \geq 0} \subset \mathrm{GL}(V)$$

where $\mathrm{GL}(V)^{\lambda=0}$ is the centralizer of λ and $\mathrm{GL}(V)^{\lambda \geq 0}$ is the subgroup of $g \in \mathrm{GL}(V)$ such that there is a limit of $\lambda(t)g\lambda(t)^{-1} \in \mathrm{GL}(V)$ for $t \to 0$. We set

$$\mathcal{Y}(d)^{\lambda \geq 0} := \left[\left(\mathcal{H}om(\mathscr{E}^0, V^{\lambda \geq 0} \otimes \mathcal{O}_X) \oplus \mathcal{H}om(V^{\lambda \leq 0} \otimes \mathcal{O}_X, \mathscr{E}^{-1}) \right) / \, \mathrm{GL}_X(V)^{\lambda \geq 0} \right]$$
$$\mathcal{Y}(d)^{\lambda = 0} := \left[\left(\mathcal{H}om(\mathscr{E}^0, V^{\lambda = 0} \otimes \mathcal{O}_X) \oplus \mathcal{H}om(V^{\lambda = 0} \otimes \mathcal{O}_X, \mathscr{E}^{-1}) \right) / \, \mathrm{GL}_X(V)^{\lambda = 0} \right].$$

Here the right hand sides make sense since the $\mathrm{GL}(V)$-action on V restricts to the $\mathrm{GL}(V)^{\lambda \geq 0}$-action on $V^{\lambda \geq 0}$. We have the following diagram

$$\begin{tikzcd} \mathcal{Y}(d)^{\lambda \geq 0} \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] \arrow[dr, "w^{\lambda \geq 0}"] & \mathcal{Y}(d) \arrow[d, "w"] \\ \mathcal{Y}(d)^{\lambda = 0} \arrow[r, "w^{\lambda=0}"'] & \mathbb{A}^1. \end{tikzcd} \tag{2.32}$$

Here p_λ is induced by the natural inclusion $V^{\lambda \geq 0} \subset V$ and surjection $V \twoheadrightarrow V^{\lambda \leq 0}$, and q_λ is given by taking the $t \to 0$ limit of the λ-action.

Remark 2.13. *The morphisms p_λ, q_λ are morphisms of algebraic stacks. Indeed the diagram $\mathcal{Y}(d)^{\lambda=0} \leftarrow \mathcal{Y}(d)^{\lambda\geq 0} \to \mathcal{Y}(d)$ is identified with some components of the diagram*

$$\mathrm{Map}(B\mathbb{C}^*, \mathcal{Y}(d)) \leftarrow \mathrm{Map}(\Theta, \mathcal{Y}(d)) \to \mathcal{Y}(d)$$

where $\Theta = [\mathbb{A}^1/\mathbb{C}^*]$, and the above arrows are induced by $\{0\}/\mathbb{C}^* \in \Theta$, $1 \in \Theta$, respectively (see [HL, Theorem 1.4.8]).

In the diagram (2.32), the function $w^{\lambda\geq 0}$ is a defined to be the pull-back of w by p_λ, which uniquely descends to a function $w^{\lambda=0}$. Since p_λ is proper (as any fiber of p_λ is a closed subscheme of the partial flag variety $\mathrm{GL}(V)/\mathrm{GL}(V)^{\lambda\geq 0}$), the following functor is well-defined

$$p_{\lambda*}q_\lambda^* \colon \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\lambda=0}, w^{\lambda=0}) \to \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w). \tag{2.33}$$

See [BFK14, Section 3] for the above functors of the categories of factorizations.

We take the following special choice for λ

$$\lambda(t) = (t, 1, \dots, 1).$$

Then $\dim V^{\lambda=0} = d-1$ and $\mathrm{GL}(V)^{\lambda=0} = \mathbb{C}^* \times \mathrm{GL}(V^{\lambda=0})$, so that we have

$$\mathcal{Y}(d)^{\lambda=0} = B\mathbb{C}^* \times \mathcal{Y}(d-1).$$

We have the decomposition

$$\mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d)^{\lambda=0}, w^{\lambda=0}) = \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{B\mathbb{C}^*}(j) \boxtimes \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d-1), w)$$

where $\mathcal{O}_{B\mathbb{C}^*}(j)$ is the $\mathbb{C}^*$-representation of weight j, and each direct summand is equivalent to $\mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d-1), w)$.

It is easy to see that, when X is a point, the stack $\mathcal{Y}(d)^{\lambda\geq 0}$ is isomorphic to the moduli stack of short exact sequences of $Q_{a,b}$-representations (see Remark 2.5)

$$0 \to R'' \to R \to R' \to 0$$

where R has dimension vector $(1, d)$ and R'' has dimension vector $(0, 1)$. It is straightforward to extend the above statement for an arbitrary X. Here we give some more details:

Lemma 2.14. *For $T \to X$, the T-valued points of the stack $\mathcal{Y}(d)^{\lambda \geq 0}$ consist of diagrams*

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathscr{E}_T^0 \arrow[r, equal] \arrow[d, "\alpha"] & \mathscr{E}_T^0 \arrow[d, "\alpha'"] & \\
0 \arrow[r] & \mathscr{P}'' \arrow[r] \arrow[d, equal] & \mathscr{P} \arrow[r] \arrow[d, "\beta"] & \mathscr{P}' \arrow[r] \arrow[d, "\beta'"] & 0 \\
 & 0 \arrow[r] & \mathscr{E}_T^{-1} \arrow[r, equal] & \mathscr{E}_T^{-1} &
\end{tikzcd} \tag{2.34}
$$

where the middle horizontal sequence is an exact sequence of vector bundles on T such that $\operatorname{rank}(\mathscr{P}'') = 1$, $\operatorname{rank}(\mathscr{P}') = d - 1$. *The morphisms p_λ, q_λ sends the above diagram to $(\mathscr{P}, \alpha, \beta)$, $(\mathscr{P}'', (\mathscr{P}', \alpha', \beta'))$ respectively.*

Proof. We set $\mathcal{Z}(d)^{\lambda \geq 0} = [Y(d)/\operatorname{GL}_X(V)^{\lambda \geq 0}]$ where $Y(d)$ is given in (2.10). We have the factorization of the projection $\mathcal{Y}(d)^{\lambda \geq 0} \to X$

$$\mathcal{Y}(d)^{\lambda \geq 0} \hookrightarrow \mathcal{Z}(d)^{\lambda \geq 0} \to [X/\operatorname{GL}_X(V)^{\lambda \geq 0}] \to [X/\operatorname{GL}_X(V)] \to X.$$

Here $\operatorname{GL}_X(V)^{\lambda \geq 0}$ and $\operatorname{GL}_X(V)$ act on X trivially. For $T \to X$, giving its lift to $[X/\operatorname{GL}_X(V)]$ is equivalent to giving a vector bundle $\mathscr{P} \to X$ of rank d. The fiber of $[X/\operatorname{GL}_X(V)^{\lambda \geq 0}] \to [X/\operatorname{GL}_X(V)]$ is $[\operatorname{GL}(V)/\operatorname{GL}(V)^{\lambda \geq 0}]$. Since $\operatorname{GL}(V)^{\lambda \geq 0}$ is the subgroup of $\operatorname{GL}(V)$ which preserves the one dimensional subspace $V^{\lambda > 0} \subset V$, the stack $[\operatorname{GL}(V)/\operatorname{GL}(V)^{\lambda \geq 0}]$ is isomorphic to the projective space $\mathbb{P}(V)$ which parametrizes one dimensional subspaces in V. Therefore giving a lift of $T \to [X/\operatorname{GL}_X(V)]$ to $[X/\operatorname{GL}_X(V)^{\lambda \geq 0}]$ is equivalent to giving a rank one vector subbundle $\mathscr{P}'' \subset \mathscr{P}$. By taking its cokernel, we obtain the exact sequence $0 \to \mathscr{P}'' \to \mathscr{P} \to \mathscr{P}' \to 0$ of the middle horizontal sequence in (2.14). Then giving its lift to $T \to \mathcal{Z}(d)^{\lambda \geq 0}$ is equivalent to giving morphisms $\mathscr{E}_T^0 \to \mathscr{P} \to \mathscr{E}_T^{-1}$. Since $V^{\lambda \geq 0} = V$ and $V^{\lambda \leq 0} = V/V^{\lambda > 0}$, the above lift $T \to \mathcal{Z}(d)^{\lambda \geq 0}$ factors through $T \to \mathcal{Y}(d)^{\lambda \geq 0}$ if and only if $\mathscr{P} \to \mathscr{E}_T^{-1}$ factors through $\mathscr{P} \twoheadrightarrow \mathscr{P}' \to \mathscr{E}_T^{-1}$. Therefore we obtain the lemma. □

The functor (2.33) gives the categorified Hall product

$$*\colon \mathcal{O}_{B\mathbb{C}^*}(j) \boxtimes \operatorname{MF}^{\mathbb{C}^*}(\mathcal{Y}(d-1), w) \to \operatorname{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w)$$

which is in fact induced by the stack of the diagrams (2.34). By the iteration, we also have the functor

(2.35)
$$*\colon \mathcal{O}_{B\mathbb{C}^*}(j_1) \boxtimes \cdots \boxtimes \mathcal{O}_{B\mathbb{C}^*}(j_l) \boxtimes \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d-l), w) \to \mathrm{MF}^{\mathbb{C}^*}(\mathcal{Y}(d), w).$$

In the case that X is a point, the above product is a special case of categorical Hall products for quivers with super-potential (see [Păd, Section 3]). The above product is their generalization to the family of moduli stacks of representations of quivers.

2.6. Semiorthogonal decomposition

We take a lexicographical order on $\mathbb{Z}^d$, i.e. for $m_\bullet = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $m'_\bullet = (m'_1, \ldots, m'_d) \in \mathbb{Z}^d$, we write $m_\bullet \succ m'_\bullet$ if $m_i = m'_i$ for $1 \le i \le k$ for some $k \ge 0$ and $m_{k+1} > m'_{k+1}$. For $j_\bullet = (j_1, j_2, \ldots, j_l)$ and $j'_\bullet = (j'_1, j'_2 \ldots, j'_{l'})$ with $l, l' \le d$, we define $j_\bullet \succ j'_\bullet$ if we have $\widetilde{j}_\bullet \succ \widetilde{j}'_\bullet$, where $\widetilde{j}_\bullet$ is defined by

$$\widetilde{j}_\bullet = (j_1, j_2, \ldots, j_l, -1, \ldots, -1) \in \mathbb{Z}^d. \tag{2.36}$$

We recall that (a, b, δ) is defined in (2.29), and χ_0 is the determinant character (2.16) which determines a line bundle on $\mathcal{Y}(d)$. Below, we also assume that $\delta \ge 0$, i.e. $a \ge b$. By abuse of notation, we use the same symbol χ_0 for the line bundle on $\mathcal{Y}(d')$ for any other d' defined by the determinant character on $\mathrm{GL}(\mathbb{C}^{d'})$.

Proposition 2.15. *For $0 \le j_1 \le \cdots \le j_l \le \delta - l$, the categorified Hall product (2.35) restricts to the fully-faithful functor*

$$*\colon \mathcal{O}_{B\mathbb{C}^*}(j_1) \boxtimes \cdots \boxtimes \mathcal{O}_{B\mathbb{C}^*}(j_l) \boxtimes (\mathbb{W}_b(d-l) \otimes \chi_0^{j_l}) \to \mathbb{W}_a(d) \tag{2.37}$$

such that, by setting $\mathcal{C}_{j_\bullet}$ to be the essential image of the above fully-faithful functor, we have the semiorthogonal decomposition

$$\mathbb{W}_a(d) = \langle \mathcal{C}_{j_\bullet} : 0 \le l \le d, j_\bullet = (0 \le j_1 \le \cdots \le j_l \le \delta - l) \rangle.$$

Here $\mathrm{Hom}(\mathcal{C}_{j_\bullet}, \mathcal{C}_{j'_\bullet}) = 0$ *for* $j_\bullet \succ j'_\bullet$.

Proof. The proposition is given in [Todc, Corollary 4.22] when X is a spectrum of a complete local ring and there is no auxiliary $\mathbb{C}^*$-action. The argument applies verbatim with an auxiliary $\mathbb{C}^*$-action. The categorified Hall products are defined globally, and they have right adjoints by the same proof in [Todc, Lemma 6.6]. Therefore in order to show that (2.37) is fully-faithful and forms a semiorthogonal decomposition, it is enough to check these properties formally locally on X (see the arguments of [Todd, Proposition 6.9, Theorem 6.11] or the last part of [Todc, Theorem 5.16]). Therefore the proposition holds. □

The following is the main result in this paper:

Theorem 2.16. *Suppose that* $\delta \geq 0$*. Then there is a semiorthogonal decomposition of the form*

$$\begin{aligned} &D^b(\mathbf{Quot}_{X,d}(\mathscr{G})) \\ &\qquad = \left\langle \binom{\delta}{i}\text{-copies of } D^b(\mathbf{Quot}_{X,d-i}(\mathscr{H})) : 0 \leq i \leq \min\{d, \delta\} \right\rangle. \end{aligned}$$

Proof. In Proposition 2.15, each semiorthogonal summand $\mathcal{C}_{j_\bullet}$ is equivalent to $\mathbb{W}_b(d-l)$. Therefore the corollary follows from Proposition 2.15 together with Lemma 2.12 and equivalences (2.26). □

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