

Affine symmetries in quantum cohomology: corrections and new results

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In [CMP09] a general formula was given for the multiplication by some special Schubert classes in the quantum cohomology of any homogeneous space. Although this formula is true in the non equivariant setting, the stated equivariant version is wrong. We provide correction for the equivariant formula, thus giving a correct argument for the non equivariant formula.

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1. Introduction

In [CMP09] a general formula was given for the multiplication in the quantum cohomology of any homogeneous space by some special Schubert classes coming from cominuscule weights. Although this formula is true in the non

equivariant setting, the stated equivariant version is wrong. We provide correction for the equivariant formula, thus giving a correct argument for the non equivariant formula. We also provide new product formulas in the equivariant homology of the affine grassmannian.

Let G be a semisimple simply connected algebraic group and fix $T \subset B$ a maximal torus and a Borel subgroup containing it. Denote by P^\vee and Q^\vee be the coweight and coroot lattices. A dominant coweight $\lambda^\vee \in P^\vee$ is minuscule if $\langle \lambda^\vee, \alpha \rangle \in \{0, 1\}$ for any positive root α . A minuscule dominant coweight is a fundamental coweight. Denote by I_m the subset of the set I of vertices of the Dynkin diagram of G parametrising minuscule coweights.

We consider a finite group Z which has several interpretation. Define Z has

$$Z := P^\vee / Q^\vee.$$

Representatives for this quotient are for example the opposites of the minuscule fundamental coweights $(-\varpi_i^\vee)_{i \in I_m}$. The group Z is also the center of G and if G^{ad} the the adjoint group associated to G , then $Z = \pi_1(G^{\text{ad}})$.

The group Z can be realised as a subgroup of the Weyl group W of G as follows. Let w_0 be the longest element in W . For $i \in I_m$ define $v_i \in W$ to be the smallest element in W such that $v_i \varpi_i = w_0 \varpi_i$. Then the family $(v_i)_{i \in I_m}$ forms a finite subgroup of W isomorphic to Z . Finally Z can be realised as a subgroup of the extended affine Weyl group $W_{\text{aff}} = W \ltimes P^\vee$ (see Section 2.2 below) by $i \mapsto \tau_i := v_i t_{-\varpi_i}$.

For $P \subset G$ a parabolic subgroup, let I_P be the set of vertices in the Dynkin diagram such that, for $i \in I$, the simple root α_i is a root of P if and only if $i \in I_P$. For $w \in W$, denote by $\sigma^P(w)$ the Schubert class in $H^{2\ell(w)}(G/P, \mathbb{Z})$ defined by w . Denote by Q_P^\vee the coroot lattice of P and consider $\eta_P : Q^\vee \rightarrow Q^\vee / Q_P^\vee$ the quotient map. We define an action of the Weyl group W of G on the equivariant cohomology $H_T^*(G/P)$ using, for $w \in W$, the pull-back in cohomology of the left multiplication by w (see Subsection 6.1). We denote this action by w^* . This action is trivial in non-equivariant cohomology and extends to an action on equivariant quantum cohomology $\text{QH}_T^*(G/P)$. In this paper we obtain the following formula in the quantum equivariant cohomology $\text{QH}_T^*(G/P)$ for any parabolic subgroup $P \subset G$ (see Theorem 6.9).

Theorem 1.1. *Let i be a cominuscule node. In $\text{QH}_T^*(G/P)$ we have*

$$\sigma^P(v_i) \times v_i^*(\sigma^P(w)) = q_{\eta_P(\varpi_i^\vee - w^{-1}(\varpi_i^\vee))} \sigma^P(v_i w).$$

This result corrects our formula in [CMP09, Theorem 1] which was wrong in the equivariant setting (the action v_i^* on the second factor on the LHS was missing). The error in [CMP09] comes from an incorrect description of the ring structure of $H_*^T(\Omega K^{\text{ad}})$ the equivariant homology of the adjoint affine grassmannian (see Section 5). If ΩK is the affine grassmannian for G , the incorrect claim ([CMP09, Page 12]) was that $H_*^T(\Omega K)$ should be isomorphic to $Z \otimes H_*^T(\Omega K)$. This is not true as explained in Section 5 (see Remark 5.6). This is corrected in the present paper. Especially, in Proposition 5.2, we prove the S -algebra isomorphism (here $S = H_T^*(\text{pt})$):

$$H_*^T(\Omega K^{\text{ad}}) \simeq S[P^\vee] \otimes_{S[Q^\vee]} H_*^T(\Omega K).$$

The incorrect product formula was then used only once in [CMP09, Proposition 3.16]. We give a correct version of Proposition 3.16 in [CMP09] in Proposition 5.14.

We tried to write this paper as independently from [CMP09] as possible and included many preliminary results on the algebra and the module structure of the extended affine Hecke algebra $\widetilde{\mathbb{A}}_{\text{aff}}$ (see Section 3) and on its module structure \widetilde{M} which is isomorphic to $H_*^T(\Omega K^{\text{ad}})$ the homology of the adjoint affine grassmannian. We also added new results. Especially we provide a generalization of a formula in [Lam08, Proposition 5.4] to coweights for the map $j^{\text{ad}} : H_*^T(\Omega K^{\text{ad}}) \rightarrow Z_{\widetilde{\mathbb{A}}_{\text{aff}}}^-(S)$ (see Proposition 7.1).

Proposition 1.2. *Let $\mu^\vee \in P^\vee$ be antidominant. Set*

$$W_{\mu^\vee} = \langle s_{\alpha_i} \mid i \in [1, r] \text{ and } \langle \alpha_i, \mu^\vee \rangle = 0 \rangle = \{w \in W \mid w(\mu^\vee) = \mu^\vee\}.$$

Then

$$j^{\text{ad}}(\xi_{t_{\mu^\vee}}) = \sum_{w \in W/W_{\mu^\vee}} \widetilde{A}_{t_{w(\mu^\vee)}}.$$

Finally, we use this formula to give an explicit formula for the image of the map $j : H_*^T(\Omega K) \rightarrow Z_{\widetilde{\mathbb{A}}_{\text{aff}}}^-(S)$ for the special elements $\tau_i(v_i) = \tau_i v_i \tau_i^{-1}$ (see Proposition 7.4). Here $\widetilde{W}_{\text{aff}}^-$ denotes the set of minimal representatives of the quotient $\widetilde{W}_{\text{aff}}/W$.

Proposition 1.3. *We have $\tau_i(v_i) \in \widetilde{W}_{\text{aff}}^-$ and*

$$j(\xi_{\tau_i(v_i)}) = \sum_{w \leq_L v_i} \sum_{v \leq_{v_i}^{-1}} \tau_i(\xi^v(v_i^{-1})) A_{\tau_i(w)v_i w^{-1}} A_v,$$

where \leq is the Bruhat order and \leq_L the weak left Bruhat order.

We hope to use the above formula to prove Pieri type formulas in $H_*^T(\Omega K)$ in the spirit of what Lam, Lapointe, Morse and Shimozono [LLMS10] did in type A .

2. Notations

In this section, we fix notation for affine Kac-Moody Lie algebras, we introduce the finite group Z and define the extended affine Weyl group $\widetilde{W}_{\text{aff}}$.

2.1. Affine Lie algebras

We denote by \mathfrak{g} a simple finite-dimensional Lie algebra of rank r , and by \mathfrak{h} a Cartan subalgebra. We denote by G the simply-connected group corresponding to \mathfrak{g} and by G^{ad} the adjoint group. The affine Kac-Moody group corresponding to G will be denoted by \mathcal{G} and $\mathcal{P} \subset \mathcal{G}$ is the parabolic subgroup such that \mathcal{G}/\mathcal{P} is the affine Grassmannian.

The corresponding affine Lie algebra will be denoted by $\mathfrak{g}_{\text{aff}}$, with Cartan subalgebra $\mathfrak{h}_{\text{aff}}$. The simple roots are denoted $(\alpha_i)_{i \in [1,r]}$ and the null-root, orthogonal to all the simple coroots $(\alpha_i^\vee)_{i \in [1,r]}$, will be denoted by ϵ . Recall that we have the equality $\epsilon = \Theta + \alpha_0$, where Θ is the highest root of \mathfrak{g} . As in [Kac90, p.82] we will use the decompositions $\mathfrak{h}_{\text{aff}}^\vee = \mathfrak{h}^\vee \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\epsilon$ and $\mathfrak{h}_{\text{aff}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We denote by R_{aff} the set of roots of $\mathfrak{g}_{\text{aff}}$ and by R those of \mathfrak{g} .

We denote by Q, P, Q^\vee, P^\vee the root, weight, coroot, coweight lattices of \mathfrak{g} .

2.2. Affine Weyl groups

Let W be the Weyl group of \mathfrak{g} and let $W_{\text{aff}} = Q^\vee \rtimes W$ be the affine Weyl group. For $\lambda^\vee \in Q^\vee$, the corresponding element in W_{aff} will be denoted by t_{λ^\vee} . The reflection associated to a root α will be denoted by s_α . The group W_{aff} is a Coxeter group with Coxeter generators s_i for $1 \leq i \leq r$ and $s_0 = t_{\Theta^\vee} s_\Theta$ ([Kum02, Prop 13.1.7], see also Lemma 2.7).

Define the extended affine Weyl group $\widetilde{W}_{\text{aff}} := P^\vee \rtimes W \supset W_{\text{aff}}$. The group W_{aff} acts on the affine weight lattice inside the dual of $\mathfrak{h}_{\text{aff}}$ and restricts to an action on the sublattice of weights of level 0, defined by $\langle \lambda, K \rangle = 0$. This lattice identifies with $P \oplus \mathbb{Z}\epsilon$. Explicitly we have [Kac90, 6.5.5]:

$$(1) \quad wt_{\lambda^\vee} \cdot (\mu + n\epsilon) = w(\mu) + (n - \langle \mu, \lambda^\vee \rangle)\epsilon$$

for $\lambda^\vee \in Q^\vee$ and $\mu \in P$. We may also define an action of $\widetilde{W}_{\text{aff}}$ on $Q \oplus \mathbb{Z}\epsilon$ by a similar formula:

$$(2) \quad wt_{\mu^\vee} \cdot (\lambda + n\epsilon) = w(\lambda) + (n - \langle \lambda, \mu^\vee \rangle)\epsilon$$

for $\mu^\vee \in P^\vee$ and $\lambda \in Q$. Note that in general $\widetilde{W}_{\text{aff}}$ does not act on $P \oplus \mathbb{Z}\epsilon$ since $\langle P, P^\vee \rangle \not\subset \mathbb{Z}$.

The action of W_{aff} on $P \oplus \mathbb{Z}\epsilon$ stabilizes $\mathbb{Z}\epsilon$, so that we may mod out by $\mathbb{Z}\epsilon$ and obtain the action of W_{aff} on P defined by $wt_{\lambda^\vee}(\mu) = w(\mu)$ for $w \in W$, $\lambda^\vee \in Q^\vee$ and $\mu \in P$, which readily extends to an action of $\widetilde{W}_{\text{aff}}$ (letting $\lambda^\vee \in P^\vee$). Similarly, we may define an action of $\widetilde{W}_{\text{aff}}$ on P^\vee by $wt_{\lambda^\vee}(\mu^\vee) = w(\mu^\vee)$ for $w \in W$ and $\lambda^\vee, \mu^\vee \in P^\vee$.

Notation 2.1. Since an element in Q^\vee is also an element in $Q^\vee \oplus \mathbb{Z}\epsilon$, we will denote by $w \cdot \lambda^\vee$ the result of the action (2) of $w \in \widetilde{W}_{\text{aff}}$ on λ^\vee as an element in $Q^\vee \oplus \mathbb{Z}\epsilon$, and by $w(\lambda^\vee)$ the element in Q^\vee .

Recall the definition of the fundamental alcove

$$A_\circ = \{\lambda \in \mathfrak{h}_\mathbb{R}^\vee \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in [1, r] \text{ and } \langle \lambda, \Theta^\vee \rangle \leq 1\}.$$

The stabiliser of A_\circ in $\widetilde{W}_{\text{aff}}$ will be denoted by Z ; it is a subgroup of $\widetilde{W}_{\text{aff}}$ isomorphic to P^\vee/Q^\vee [LS10, §10.1]. In *loc. cit.*, the authors also prove the following result.

Lemma 2.2. *Let $\tau \in Z$. The conjugation by τ is an automorphism of the Coxeter group W_{aff} . In fact, there exists an automorphism f_τ of the affine Dynkin diagram such that*

$$\forall i \in I \cup \{0\}, \tau s_{\alpha_i} \tau^{-1} = s_{\tau \cdot \alpha_i} = s_{\alpha_{f_\tau(i)}}.$$

In particular, we have $\tau \cdot \epsilon = \epsilon$.

Notation 2.3. For $\widehat{x} \in W_{\text{aff}}$, set $\tau(\widehat{x}) := \tau \widehat{x} \tau^{-1} \in W_{\text{aff}}$. We have $\ell(\tau(\widehat{x})) = \ell(\widehat{x})$.

Lemma 2.4. *An element τ in Z permutes the positive real roots.*

Proof. According to Lemma 2.2, we have $\tau \cdot (\sum n_j \alpha_j + n\delta) = \sum n_j \alpha_{f_\tau(j)} + n\delta$. Since a real root $\alpha + n\delta$ is positive if and only if $n > 0$ or $n = 0$ and $\alpha > 0$, τ indeed permutes positive roots. □

As explained in [CMP09], $\widetilde{W}_{\text{aff}}$ is not a Coxeter group, but we have a well defined length function.

Definition 2.5. Every element $x \in \widetilde{W}_{\text{aff}}$ can be uniquely written as $\tau\widehat{x}$ with $\tau \in Z$ and $\widehat{x} \in W_{\text{aff}}$

- 1) Define the length function by $\ell(x) := \ell(\widehat{x})$.
- 2) Define a partial order on $\widetilde{W}_{\text{aff}}$ by $\tau\widehat{x} \leq \sigma\widehat{y} \iff \tau = \sigma$ and $\widehat{x} \leq \widehat{y}$.

Covering relations in $\widetilde{W}_{\text{aff}}$ for the above partial order are defined by $x \leq y$ if $x \leq y$ and $\ell(x) = \ell(y) - 1$.

Remark 2.6. The length of $x \in \widetilde{W}_{\text{aff}}$ is also the number of inversions, namely the cardinal of the set $I(x) = \{\alpha \in R_{\text{aff}} \mid \alpha > 0, \alpha \text{ is real and } x(\alpha) < 0\}$. Indeed, for $x = \tau\widehat{x}$, by Lemma 2.4, we have $I(x) = I(\widehat{x})$.

2.3. Translations

We will need the following lemma.

Lemma 2.7. *Let $\alpha \in R$. We have $t_{\alpha^\vee} = s_{\epsilon - \alpha}s_\alpha$.*

Proof. Set $K^\perp = \{\mu \in \mathfrak{h}_{\text{aff}}^\vee \mid \langle \mu, K \rangle = 0\}$. By [Kac90, p.87], it is enough to compute $s_{\epsilon - \alpha}s_\alpha(\mu)$ for $\mu \in K^\perp$. We have

$$\begin{aligned} s_{\epsilon - \alpha}s_\alpha(\mu) &= \mu - \langle \mu, (\epsilon - \alpha)^\vee \rangle (\epsilon - \alpha) - \langle \mu, \alpha^\vee \rangle \alpha \\ &\quad + \langle \mu, \alpha^\vee \rangle \langle \alpha, (\epsilon - \alpha)^\vee \rangle (\epsilon - \alpha). \end{aligned}$$

Now, for any $\beta \in R_{\text{aff}}$, we have by [Kac90, §2.3.5 and §6.2.3]:

$$\langle \mu, (\epsilon + \beta)^\vee \rangle = \frac{2(\epsilon + \beta, \mu)}{(\epsilon + \beta, \epsilon + \beta)} = \frac{2(\beta, \mu)}{(\beta, \beta)} = \langle \mu, \beta^\vee \rangle.$$

Therefore,

$$\begin{aligned} s_{\epsilon - \alpha}s_\alpha(\mu) &= \mu + \langle \mu, \alpha^\vee \rangle (\epsilon - \alpha) - \langle \mu, \alpha^\vee \rangle \alpha - 2\langle \mu, \alpha^\vee \rangle (\epsilon - \alpha) \\ &= \mu - \langle \mu, \alpha^\vee \rangle \epsilon = t_{\alpha^\vee}(\mu), \end{aligned}$$

where the last equality follows from the definition of t_{α^\vee} in [Kac90, §6.5.5].

□

Corollary 2.8. *For $\alpha \in R, k \in \mathbb{Z}$ and $\mu^\vee \in P^\vee$, we have $s_{\alpha + k\epsilon}(\mu^\vee) = s_\alpha(\mu^\vee)$.*

Proof. We have $s_{\alpha+\epsilon}(\mu^\vee) = s_{\alpha+\epsilon}t_{-\alpha}(\mu^\vee) = s_{\alpha+\epsilon}s_{\alpha+\epsilon}s_{-\alpha}(\mu^\vee) = s_{-\alpha}(\mu^\vee) = s_\alpha(\mu^\vee)$. The result follows by induction. \square

3. extended nil-Hecke ring

The goal of this section is to extend the notion of the nil-Hecke ring defined by Kostant and Kumar [KK86]. This ring was used in [LS10] to compare the quantum cohomology of G/P and the homology of affine Grassmannians ΩK . We need a refined version of this nil-Hecke ring that enables dealing with ΩK^{ad} the adjoint affine Grassmannian (see Section 5).

3.1. Definition

We denote by S resp. \mathbf{S} the symmetric algebra on P resp. $P \oplus \mathbb{Z}\epsilon \oplus \mathbb{Z}\Lambda_0$. Moreover, we denote by S_{loc} resp. \mathbf{S}_{loc} the localization of S resp. \mathbf{S} where we invert all the roots in R resp. all the real roots in R_{aff} . Moreover, we consider inside \mathbf{S} the subalgebra \mathbf{S}' generated by P and ϵ , and its localization at real affine roots $\mathbf{S}'_{\text{loc}} \subset \mathbf{S}_{\text{loc}}$. Note that modding out by ϵ yields isomorphisms $\mathbf{S}'/(\epsilon) \simeq S$ and $\mathbf{S}'_{\text{loc}}/(\epsilon) \simeq S_{\text{loc}}$, since any real affine root in R_{aff} can be written as $\gamma + n\epsilon$ with $\gamma \in R$ and $n \in \mathbb{Z}$. This is the reason why we are using \mathbf{S}_{loc} , where ϵ is not invertible, whereas $\text{Frac}(\mathbf{S})$ is used in [Kum02].

We extend several classical objects, in particular the affine nil-Hecke algebra, to the adjoint setting. Our reference for these classical objects is Kumar’s book [Kum02]. However, there is a little subtlety, since as in [LS10], we restrict the scalars from \mathbf{S} to S . Thus, Kostant and Kumar’s objects will be denoted with bold letters, whereas Lam and Shimozono’s restricted objects will be denoted with usual letters.

Definition 3.1. Recall that the rings \mathbf{Q}_{aff} and Q_{aff} are

$$(3) \quad \mathbf{Q}_{\text{aff}} = \bigoplus_{w \in W_{\text{aff}}} \mathbf{S}_{\text{loc}} \delta_w \text{ and } Q_{\text{aff}} = \bigoplus_{w \in W_{\text{aff}}} S_{\text{loc}} \delta_w$$

We define the following extended version:

$$(4) \quad \tilde{\mathbf{Q}}_{\text{aff}} = \bigoplus_{w \in \tilde{W}_{\text{aff}}} \mathbf{S}_{\text{loc}} \delta_w \text{ and } \tilde{Q}_{\text{aff}} = \bigoplus_{w \in \tilde{W}_{\text{aff}}} S_{\text{loc}} \delta_w$$

In both cases, the ring structure is defined by the equations $\delta_u \delta_v = \delta_{uv}$, $\delta_u s = u(s) \delta_u$, and $\delta_u \mathbf{s} = (u \cdot \mathbf{s}) \delta_u$ for $u, v \in W_{\text{aff}}$ resp. $u, v \in \tilde{W}_{\text{aff}}$, $s \in S$ and $\mathbf{s} \in \mathbf{S}$.

Remark 3.2. A subring of \mathbf{Q}_{aff} is $\mathbf{Q}'_{\text{aff}} := \bigoplus_{w \in W_{\text{aff}}} \mathbf{S}'_{\text{loc}} \delta_w$, ϵ is central in \mathbf{Q}_{aff} , and we have $\mathbf{Q}'_{\text{aff}}/(\epsilon) \simeq Q_{\text{aff}}$. We denote by $\pi: \mathbf{Q}'_{\text{aff}} \rightarrow \mathbf{Q}'_{\text{aff}}/(\epsilon) \simeq Q_{\text{aff}}$ the projection morphism. Applying π , any relation proved in [Kum02] for \mathbf{Q}_{aff} , that involves only elements in \mathbf{Q}'_{aff} , is valid in Q_{aff} .

Definition 3.3. As in [Kum02], we consider particular elements in \mathbf{Q}_{aff} and Q_{aff} :

- 1) For $i \in I$, set $\mathbf{A}_i = A_i = \frac{1}{\alpha_i}(\delta_e - \delta_{s_i})$. For $i = 0$, set $\mathbf{A}_0 = \frac{1}{\alpha_0}(\delta_e - \delta_{s_0})$ and $A_0 = \frac{-1}{\Theta}(\delta_e - \delta_{s_0})$ (note that this is coherent with the forthcoming Definition 3.7).
- 2) For $w \in W_{\text{aff}}$ and for $w = s_{i_1} \cdots s_{i_\ell}$ a reduced expression, we set:

$$\mathbf{A}_w = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_\ell} \in \mathbf{Q}'_{\text{aff}} \text{ and } A_w = A_{i_1} \cdots A_{i_\ell} \in Q_{\text{aff}}.$$

By [Kum02, Theorem 11.1.2], the definition of \mathbf{A}_w does not depend on the chosen reduced expression. Since $\pi(\mathbf{A}_i) = A_i$, the same holds for A_w and $\pi(\mathbf{A}_w) = A_w$.

Recall that for $x \in \widetilde{W}_{\text{aff}}$, there is a unique decomposition $x = \tau \widehat{x}$ with $\tau \in Z$ and $\widehat{x} \in W_{\text{aff}}$.

Definition 3.4. Let $x = \tau \widehat{x} \in \widetilde{W}_{\text{aff}}$, we set $\widetilde{A}_w = \delta_\tau A_{\widehat{x}}$.

By [Kum02, 11.1.2(e)], the matrix expressing each \mathbf{A}_w as a linear combination of elements δ_v is invertible in \mathbf{S}_{loc} . Applying π , the matrix expressing each A_w as a linear combination of elements δ_v is invertible in S_{loc} . In particular, the elements A_w in Q_{aff} are linearly independent over S_{loc} , and the following definition makes sense:

Definition 3.5. As in [KK86], the (extended) nil-Hecke ring is generated over S by the elements A_w .

- 1) The nil-Hecke ring is

$$\mathbb{A}_{\text{aff}} = \bigoplus_{w \in W_{\text{aff}}} S \cdot A_w \subset Q_{\text{aff}}.$$

- 2) The extended nil-Hecke ring is

$$\widetilde{\mathbb{A}}_{\text{aff}} = \bigoplus_{w \in \widetilde{W}_{\text{aff}}} S \cdot \widetilde{A}_w \subset \widetilde{Q}_{\text{aff}}.$$

Remark 3.6. We will see below that both are indeed subrings of $\widetilde{Q}_{\text{aff}}$.

3.2. Definition and properties of A_α

It will be helpful to generalize the definition of A_i in the following way.

Definition 3.7. For a real root $\alpha = \gamma + k\epsilon$ with $\gamma \in R$, set $A_\alpha = \frac{1}{\gamma}(\delta_e - \delta_{s_\alpha})$.

Recall [LS10, §6.1] that we have the equation

$$(5) \quad A_i \lambda = s_i(\lambda) + \langle \lambda, \alpha_i^\vee \rangle 1.$$

We generalize this formula to the elements A_α as follows:

Proposition 3.8. Let $w \in \widetilde{W}_{\text{aff}}$, $\alpha \in R_{\text{aff}}$ a real root and $\lambda \in Q$. Then we have:

- 1) $\delta_w A_\alpha \delta_{w^{-1}} = A_{w(\alpha)}$.
- 2) $A_\alpha \lambda = s_\alpha(\lambda) A_\alpha + \langle \lambda, \alpha^\vee \rangle 1$.

Proof. Let $w = ut_{\mu^\vee}$, $\alpha = \gamma + k\epsilon$, $\lambda \in P$ be as in the proposition. Then,

$$\begin{aligned} \delta_w A_\alpha \delta_{w^{-1}} &= \delta_w \frac{1}{\gamma} (\delta_e - \delta_{s_\alpha}) \delta_{w^{-1}} = \frac{1}{u(\gamma)} \delta_w (\delta_e - \delta_{s_\alpha}) \delta_{w^{-1}} \\ &= \frac{1}{u(\gamma)} (\delta_e - \delta_{ws_\alpha w^{-1}}) = \frac{1}{u(\gamma)} (\delta_e - \delta_{s_{w(\alpha)}}) = A_{w(\alpha)}. \end{aligned}$$

For the second point, we use the formula (5) and the above conjugation relation. Let $w \in W_{\text{aff}}$ and $i \in I \cup \{0\}$ be such that $\alpha = w(\alpha_i)$. We have

$$\begin{aligned} A_\alpha \lambda &= \delta_w A_i \delta_{w^{-1}} \lambda \\ &= \delta_w A_i w^{-1}(\lambda) \delta_{w^{-1}} \\ &= \delta_w (s_i w^{-1}(\lambda) A_i + \langle w^{-1}(\lambda), \alpha_i^\vee \rangle) \delta_{w^{-1}} \\ &= w s_i w^{-1}(\lambda) \delta_w A_i \delta_{w^{-1}} + \langle \lambda, w(\alpha_i)^\vee \rangle \\ &= s_\alpha(\lambda) A_\alpha + \langle \lambda, \alpha^\vee \rangle. \end{aligned}$$

□

Corollary 3.9. For any real root α , we have $A_\alpha \in \mathbb{A}_{\text{aff}}$.

Corollary 3.10. *The (extended) nil Hecke rings \mathbb{A}_{aff} and $\tilde{\mathbb{A}}_{\text{aff}}$ are subrings of \tilde{Q}_{aff} .*

Proof. The second formula above shows that for $s, s' \in S$ and $u, v \in W_{\text{aff}}$, the product $sA_us'A_v$ lies in \mathbb{A}_{aff} therefore \mathbb{A}_{aff} is a ring. The first formula proves that for $\tau, \sigma \in Z$ and for $u, v \in W_{\text{aff}}$, we have $\delta_\tau A_u \delta_\sigma A_v = \delta_\tau \delta_\sigma A_{\sigma^{-1}(u)} A_v \in \tilde{\mathbb{A}}_{\text{aff}}$ proving that $\tilde{\mathbb{A}}_{\text{aff}}$ is a ring. \square

4. Module and ring structures of $\tilde{\mathbb{A}}_{\text{aff}}$

In this section we present three different descriptions of $\tilde{\mathbb{A}}_{\text{aff}}$ and describe its S -module structure and its ring structure in each case.

4.1. S -module structure of $\tilde{\mathbb{A}}_{\text{aff}}$

Recall that we have an injection of W_{aff} in the group of invertibles of \mathbb{A}_{aff} , given by $w \mapsto \delta_w$: in fact $\delta_{s_i} = 1 - \alpha_i A_i \in \mathbb{A}_{\text{aff}}$ thus $\delta_w \in \mathbb{A}_{\text{aff}}$ for all $w \in W_{\text{aff}}$. Therefore the subgroup $Q^\vee \subset W_{\text{aff}}$ also injects in \mathbb{A}_{aff} , and since \mathbb{A}_{aff} is a ring we have an injection of the Laurent polynomial algebra $\mathbb{Z}[Q^\vee]$ inside \mathbb{A}_{aff} . Thus \mathbb{A}_{aff} is $\mathbb{Z}[Q^\vee]$ -module via left multiplication. The natural \mathbb{Z} -module basis of $\mathbb{Z}[Q^\vee]$ will be denoted by $(h_{\lambda^\vee})_{\lambda^\vee \in Q^\vee}$.

We now introduce two new algebraic models of $\tilde{\mathbb{A}}_{\text{aff}}$.

Definition 4.1. Let φ_1, φ_2 be the following morphisms of \mathbb{Z} -modules:

$$\begin{aligned} \varphi_1 & : \mathbb{Z}[P^\vee] \otimes_{\mathbb{Z}[Q^\vee]} \mathbb{A}_{\text{aff}} & \rightarrow & \tilde{Q}_{\text{aff}} \\ & h_{\lambda^\vee} \otimes A_w & \mapsto & \delta_{t_{\lambda^\vee}} A_w, \end{aligned}$$

$$\begin{aligned} \varphi_2 & : \mathbb{Z}[Z] \otimes_{\mathbb{Z}} \mathbb{A}_{\text{aff}} & \rightarrow & \tilde{Q}_{\text{aff}} \\ & \tau \otimes A_w & \mapsto & \delta_\tau A_w. \end{aligned}$$

Note that \mathbb{A}_{aff} has a structure of S -bimodule, thus also the two tensor products in this definition. Both maps φ_1 and φ_2 are S -linear on the right, moreover φ_1 is also S -linear on the left whereas φ_2 is not.

Proposition 4.2. *With the above notations, $\text{Im}(\varphi_1) = \text{Im}(\varphi_2) = \tilde{\mathbb{A}}_{\text{aff}}$. Moreover, if $J \subset \mathbb{A}_{\text{aff}}$ is a left ideal, then*

$$\varphi_1(\mathbb{Z}[P^\vee] \otimes_{\mathbb{Z}[Q^\vee]} J) = \varphi_2(\mathbb{Z}[Z] \otimes_{\mathbb{Z}} J).$$

Proof. Observe that φ_1 is well-defined: $\varphi_1(h_{\lambda^\vee} \otimes 1) = \varphi_1(1 \otimes \delta_{t_{\lambda^\vee}}) = \delta_{t_{\lambda^\vee}}$ for $\lambda^\vee \in Q^\vee$. We now prove that $\varphi_1(\mathbb{Z}[P^\vee] \otimes J) \subset \varphi_2(\mathbb{Z}[Z] \otimes J)$. Let $\lambda^\vee \in P^\vee$: there exists $\tau \in Z$ and $\widehat{w} \in W_{\text{aff}}$ such that $t_{\lambda^\vee} = \tau\widehat{w}$. Then for $a \in J$, we have $\varphi_1(h_{\lambda^\vee} \otimes a) = \delta_{t_{\lambda^\vee}} a = \delta_\tau \delta_{\widehat{w}} a \in \varphi_2(\mathbb{Z}[Z] \otimes J)$ since $\delta_{\widehat{w}} a \in J$.

The reverse inclusion $\varphi_2(\mathbb{Z}[Z] \otimes J) \subset \varphi_1(\mathbb{Z}[P^\vee] \otimes J)$ follows similarly from the fact that any element in Z can be written as a product $t_{\lambda^\vee} u$ for some $\lambda^\vee \in P^\vee$ and $u \in W$. Finally, the equality $\text{Im}(\varphi_2) = \widetilde{\mathbb{A}}_{\text{aff}}$ follows from the definition of $\widetilde{\mathbb{A}}_{\text{aff}}$ (Definition 3.5). □

4.2. Ring structure of $\widetilde{\mathbb{A}}_{\text{aff}}$

We give the description of the ring structure of $\widetilde{\mathbb{A}}_{\text{aff}}$ according to the given three equivalent definitions of this module.

Proposition 4.3. *Let $x, y \in \widetilde{W}_{\text{aff}}$, then we have*

$$\widetilde{A}_x \widetilde{A}_y = \begin{cases} \widetilde{A}_{xy} & \text{if } \ell(xy) = \ell(x) + \ell(y) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $x = \sigma\widehat{x}$ and $y = \tau\widehat{y}$ with $\sigma, \tau \in Z$ and $\widehat{x}, \widehat{y} \in W_{\text{aff}}$. Recall that for $u, v \in W_{\text{aff}}$, we have:

$$A_u A_v = \begin{cases} A_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2, we have

$$\begin{aligned} \widetilde{A}_x \widetilde{A}_y &= \delta_\sigma A_{\widehat{x}} \delta_\tau A_{\widehat{y}} \\ &= \delta_\sigma \delta_\tau A_{\tau^{-1}(\widehat{x})} A_{\widehat{y}} \\ &= \begin{cases} \delta_{\sigma\tau} A_{\tau^{-1}(\widehat{x})\widehat{y}} & \text{if } \ell(\tau^{-1}(\widehat{x})\widehat{y}) = \ell(\tau^{-1}(\widehat{x})) + \ell(\widehat{y}) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

But $\ell(\tau^{-1}(\widehat{x})\widehat{y}) = \ell(xy)$ since $xy = \sigma\widehat{x}\tau\widehat{y} = \sigma\tau\tau^{-1}(\widehat{x})\widehat{y}$, and $\ell(\tau^{-1}(\widehat{x})) + \ell(\widehat{y}) = \ell(\widehat{x}) + \ell(\widehat{y}) = \ell(x) + \ell(y)$. The result follows. □

We now express the product in $\widetilde{\mathbb{A}}_{\text{aff}} = \varphi_1(\mathbb{Z}[P^\vee] \otimes_{\mathbb{Z}[Q^\vee]} \mathbb{A}_{\text{aff}})$. Note that we need to compute the product $(\delta_{t_{\lambda^\vee}} A_u)(\delta_{t_{\mu^\vee}} A_v)$. We therefore need to “move” $\delta_{t_{\mu^\vee}}$ to the left of A_u . The following proposition gives formulas for this.

Proposition 4.4. *Let $\lambda^\vee \in P^\vee$ and let $\alpha = \gamma + k\epsilon \in R_{\text{aff}}$. Then:*

- 1) $A_\alpha \delta_{t_{\lambda^\vee}} = \delta_{s_\alpha(t_{\lambda^\vee})} A_\alpha + \frac{1}{\gamma} (\delta_{t_{\lambda^\vee}} - \delta_{t_{s_\alpha(\lambda^\vee)}})$.
- 2) $\delta_{t_{\lambda^\vee}} - \delta_{t_{s_\alpha(\lambda^\vee)}} = \delta_{t_{\lambda^\vee}} (1 - \delta_{t_{-(\alpha, \lambda^\vee)_{\alpha^\vee}}})$,
- 3) $1 - \delta_{t_{n\alpha^\vee}} = (1 + \delta_{t_{\alpha^\vee}} + \dots + \delta_{t_{(n-1)\alpha^\vee}}) (1 - \delta_{t_{\alpha^\vee}})$ for $n \in \mathbb{N}$,
- 4) $1 - \delta_{t_{-\alpha^\vee}} = \gamma (A_\alpha - \gamma A_\alpha A_{\epsilon-\alpha} + A_{\epsilon-\alpha})$.

Proof. (1) From the equality $s_\alpha t_{\lambda^\vee} = t_{s_\alpha(\lambda^\vee)} s_\alpha$, we get $\delta_{s_\alpha} \delta_{t_{\lambda^\vee}} = \delta_{t_{s_\alpha(\lambda^\vee)}} \delta_{s_\alpha}$ in $\widetilde{\mathbb{A}}_{\text{aff}}$. By definition of A_α (Definition 3.7), this relation implies

$$(1 - \gamma A_\alpha) \delta_{t_{\lambda^\vee}} = \delta_{t_{s_\alpha(\lambda^\vee)}} (1 - \gamma A_\alpha).$$

Thus we get $\gamma A_\alpha \delta_{t_{\lambda^\vee}} = \gamma \delta_{t_{s_\alpha(\lambda^\vee)}} A_\alpha + \delta_{t_{\lambda^\vee}} - \delta_{t_{s_\alpha(\lambda^\vee)}}$.

(2) and (3) are easy consequences of the product formulas in $\widetilde{Q}_{\text{aff}}$.

(4) By Lemma 2.7, we have

$$\begin{aligned} 1 - \delta_{t_{-\alpha^\vee}} &= 1 - \delta_{s_\alpha s_{\epsilon-\alpha}} \\ &= 1 - (1 - \gamma A_\alpha)(1 + \gamma A_{\epsilon-\alpha}) \\ &= \gamma A_\alpha - \gamma A_{\epsilon-\alpha} + \gamma A_\alpha \gamma A_{\epsilon-\alpha} \\ &= \gamma A_\alpha - \gamma A_{\epsilon-\alpha} + \gamma (s_\alpha(\gamma) A_\alpha + \langle \alpha^\vee, \gamma \rangle) A_{\epsilon-\alpha} \\ &= \gamma A_\alpha + \gamma A_{\epsilon-\alpha} - \gamma^2 A_\alpha A_{\epsilon-\alpha}, \end{aligned}$$

where we used Proposition 3.8 on the fourth line. □

The ring structure in $\widetilde{\mathbb{A}}_{\text{aff}} = \varphi_2(\mathbb{Z}[Z] \otimes_{\mathbb{Z}} \mathbb{A}_{\text{aff}})$ is easy to describe:

Proposition 4.5. *Let $\sigma, \tau \in Z$ and let $a, b \in \mathbb{A}_{\text{aff}}$. Then:*

$$\varphi_2(\sigma \otimes a) \cdot \varphi_2(\tau \otimes b) = \varphi_2(\sigma\tau \otimes \tau^{-1}(a)b).$$

Proof. This follows from the fact that in $\widetilde{\mathbb{A}}_{\text{aff}}$, we have

$$\delta_\sigma a \delta_\tau b = \delta_\sigma \delta_\tau \tau^{-1}(a)b. \quad \square$$

In the next proposition, we give an explicit formula for the commuting relation of the elements \widetilde{A}_x and $\lambda \in P$, generalizing [KK86, Proposition 4.3.b]:

Proposition 4.6. *Let $x \in \widetilde{W}_{\text{aff}}$ and let $\lambda \in P$. We have:*

$$\widetilde{A}_x \lambda = x(\lambda) \widetilde{A}_x + \sum_{\alpha: x s_\alpha \prec x} \langle \lambda, \alpha^\vee \rangle \widetilde{A}_{x s_\alpha},$$

where the sum runs over positive real roots α such that $x s_\alpha \prec x$.

Proof. Let $x = \tau\hat{x} \in \widetilde{W}_{\text{aff}}$ with $\tau \in Z$ and $\hat{x} \in W_{\text{aff}}$. Let $\lambda \in P$. According to Definition 3.4, we have $A_x = \delta_\tau A_{\hat{x}}$. Using [KK86, Proposition 4.3.b], we get (sums always run over positive real roots):

$$\begin{aligned} \widetilde{A}_x \lambda &= \delta_\tau A_{\hat{x}} \lambda = \delta_\tau \widehat{x}(\lambda) A_{\hat{x}} + \delta_\tau \sum_{\alpha: \widehat{x}s_\alpha < \widehat{x}} \langle \lambda, \alpha^\vee \rangle A_{\widehat{x}s_\alpha} \\ &= \tau \widehat{x}(\lambda) \delta_\tau A_{\hat{x}} + \sum_{\alpha: \widehat{x}s_\alpha < \widehat{x}} \langle \lambda, \alpha^\vee \rangle \delta_\tau A_{\widehat{x}s_\alpha} \\ &= x(\lambda) \widetilde{A}_x + \sum_{\alpha: \widehat{x}s_\alpha < \widehat{x}} \langle \lambda, \alpha^\vee \rangle \widetilde{A}_{xs_\alpha}. \end{aligned}$$

Since, by Definition 3.4, the relation $\widehat{x}s_\alpha < \widehat{x}$ holds if and only if the relation $xs_\alpha < x$ holds, we get the result. \square

4.3. Module over $\widetilde{\mathbb{A}}_{\text{aff}}$

We now define a natural module over $\widetilde{\mathbb{A}}_{\text{aff}}$ which will be identified in the next section with the homology of the adjoint affine Grassmannian ΩK^{ad} .

Definition 4.7. Let W_{aff}^- resp. $\widetilde{W}_{\text{aff}}^-$ be the set of minimal length representatives of the quotient W_{aff}/W resp. $\widetilde{W}_{\text{aff}}/W$. By [LS10, Lemma 3.3], W_{aff}^- is the set of elements $w = ut_{\lambda^\vee}$ such that $\forall i \in I, \langle \lambda^\vee, \alpha_i \rangle \leq 0$ and $\langle \lambda^\vee, \alpha_i \rangle = 0 \implies u(\alpha_i) > 0$.

We generalize the characterization of W_{aff}^- as follows:

Lemma 4.8. *We have $ut_{\lambda^\vee} \in \widetilde{W}_{\text{aff}}^-$ if and only if $\lambda^\vee \leq 0$ and for all i in I it holds*

$$\langle \lambda^\vee, \alpha_i \rangle = 0 \implies u(\alpha_i) > 0.$$

Proof. Recall that we have a length formula in $\widetilde{W}_{\text{aff}}$ similar to the one in W_{aff} :

$$\ell(ut_{\lambda^\vee}) = \sum_{\alpha \in R^+} |\langle \lambda^\vee, \alpha \rangle + \chi(u(\alpha) < 0)|,$$

where $\chi(\mathcal{P}) = 1$ if \mathcal{P} is true and $\chi(\mathcal{P}) = 0$ if \mathcal{P} is false. This is proved in [CMP09, Corollary 3.13]. It follows that

$$\ell(ut_{\lambda^\vee s_i}) - \ell(ut_{\lambda^\vee}) = |\langle \lambda^\vee, -\alpha_i \rangle + \chi(u(\alpha_i) > 0)| - |\langle \lambda^\vee, \alpha_i \rangle + \chi(u(\alpha_i) < 0)|.$$

This is non-negative for all i in I if and only if for all $i, \langle \lambda^\vee, \alpha_i \rangle \leq 0$, and $\langle \lambda^\vee, \alpha_i \rangle = 0$ implies $u(\alpha_i) > 0$. \square

Definition 4.9. For each $w \in W_{\text{aff}}^-$, we define a variable ξ_w and we set

$$M = \bigoplus_{w \in W_{\text{aff}}^-} S \cdot \xi_w.$$

Recall [LS10, §6.2] that we may define a left \mathbb{A}_{aff} -module structure on M via:

$$A_w \cdot \xi_u = \begin{cases} \xi_{wu} & \text{if } \ell(wu) = \ell(w) + \ell(u) \text{ and } wu \in W_{\text{aff}}^-, \\ 0 & \text{otherwise.} \end{cases}$$

As left \mathbb{A}_{aff} -module, we have an isomorphism

$$M \simeq \mathbb{A}_{\text{aff}}/J, \text{ where } J = \bigoplus_{w \notin W_{\text{aff}}^-} S \cdot A_w$$

is a left ideal in \mathbb{A}_{aff} .

Using Proposition 4.2, we define similarly a left ideal in $\widetilde{\mathbb{A}}_{\text{aff}}$.

Definition 4.10. Let

$$\widetilde{J} = \varphi_1(\mathbb{Z}[P^\vee] \otimes_{\mathbb{Z}[Q^\vee]} J) = \varphi_2(\mathbb{Z}[Z] \otimes_{\mathbb{Z}} J) = \bigoplus_{w \notin \widetilde{W}_{\text{aff}}^-} S \cdot \widetilde{A}_w.$$

Definition 4.11. We introduce the following three modules.

- Let \widetilde{M}_1 be the S -module $\mathbb{Z}[P^\vee] \otimes_{\mathbb{Z}[Q^\vee]} M$.
- Let $\widetilde{M}_2 = \mathbb{Z}[Z] \otimes_{\mathbb{Z}} M$. This is an $\widetilde{\mathbb{A}}_{\text{aff}}$ -module with the action given by $(\sigma \otimes a) \cdot (\tau \otimes \xi) = \sigma\tau \otimes \tau^{-1}(a) \cdot \xi$, for $\sigma \otimes a \in \mathbb{Z}[Z] \otimes_{\mathbb{Z}} \mathbb{A}_{\text{aff}} = \widetilde{\mathbb{A}}_{\text{aff}}$.
- Let $\widetilde{M}_3 = \bigoplus_{w \in \widetilde{W}_{\text{aff}}^-} S \cdot \widetilde{\xi}_w$. This is an $\widetilde{\mathbb{A}}_{\text{aff}}$ -module with the action given by

$$\widetilde{A}_w \cdot \widetilde{\xi}_u = \begin{cases} \widetilde{\xi}_{wu} & \text{if } \ell(wu) = \ell(w) + \ell(u) \text{ and } wu \in \widetilde{W}_{\text{aff}}^-, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } \widetilde{A}_w \in \widetilde{\mathbb{A}}_{\text{aff}} = \bigoplus_{w \in \widetilde{W}_{\text{aff}}^-} S \cdot \widetilde{A}_w.$$

Proposition 4.12. *With the above definitions,*

- 1) *Moding out by \widetilde{J} , the morphism φ_1 induces an S -module isomorphism $\widetilde{M}_1 \rightarrow \widetilde{\mathbb{A}}_{\text{aff}}/\widetilde{J}$.*

- 2) *Moding out by $\tilde{\mathcal{J}}$, the morphism φ_2 induces an $\tilde{\mathbb{A}}_{\text{aff}}$ -module isomorphism $\tilde{M}_2 \rightarrow \tilde{\mathbb{A}}_{\text{aff}}/\tilde{\mathcal{J}}$ (which is not S -linear if we give \tilde{M}_2 the tensor product S -module structure).*
- 3) *The left $\tilde{\mathbb{A}}_{\text{aff}}$ -modules $\tilde{\mathbb{A}}_{\text{aff}}/\tilde{\mathcal{J}}$ and \tilde{M}_3 are isomorphic.*

Proof. This proposition follows easily from Propositions 4.3 and 4.5. □

Remark 4.13. The $\tilde{\mathbb{A}}_{\text{aff}}$ -module structure induced on \tilde{M}_1 by the isomorphism in Proposition 4.12.(1) can also be described via Proposition 4.4.

Definition 4.14. The $\tilde{\mathbb{A}}_{\text{aff}}$ -module defined by one of the above equivalent definitions will be denoted by \tilde{M} .

5. Homology of the adjoint affine Grassmannian ΩK^{ad}

In this section, we recall the adjoint affine Grassmannian ΩK^{ad} , we prove that the $\tilde{\mathbb{A}}_{\text{aff}}$ -module \tilde{M} is isomorphic to the homology of ΩK^{ad} , we define a ring structure on this module and study the compatibility of these two structures.

5.1. Cohomology of the finite-dimensional flag manifold G/B

Recall, see for example [Kum02, Chapter 11], that $H_T^*(G/B)$ has an S -basis consisting of the elements $\xi^w = \sigma^B(w)$ indexed by the Weyl group. The pull-back along the map $(G/B)^T \rightarrow G/B$ induces an inclusion

$$H_T^*(G/B) \rightarrow H_T^*((G/B)^T) = S^W.$$

Viewing ξ^w as a function on W , Kumar [Kum02, 11.1.6.(3)] sets $d_{u,v} = \xi^u(v) = \langle \xi^u, v \rangle$ and $D = (d_{u,v})_{u,v \in W}$. If $(f^w)_{w \in W}$ is the basis of S^W given by $\langle f^u, v \rangle = f^u(v) = \delta_{u,v}$, then we have $(\xi^u)_u = D(f^u)_u$. Given the identification [Kum02, 11.1.4(2)], we also have $\langle f^u, \delta_v \rangle = \delta_{u,v}$.

The dual of $H_T^*(G/B)$ is $H_*^T(G/B)$ and identifies as an S -module with the S -subalgebra \mathbb{A} of \mathbb{A}_{aff} generated by $(A_w)_{w \in W}$:

$$(6) \quad H_*^T(G/B) \simeq \bigoplus_{w \in W} S \cdot A_w$$

Note that $(A_w)_{w \in W}$ is the dual basis to $(\xi^w)_{w \in W}$ i.e. $\langle \xi^u, A_v \rangle = \xi^u(A_v) = \delta_{u,v}$ (see [Kum02, 11.1.5], were A_u is denoted by x_u). Over $F = \text{Frac}(S)$ we

also have the basis $(\delta_w)_{w \in W}$ for $H_*^T(G/B)$. Kumar, in [Kum02, 11.1.2.(e)], describes the base change:

$$A_u = \sum_v c_{u,v} \delta_v$$

with $C = (c_{u,v})_{u,v \in W}$ a matrix with coefficients in S , in particular, we have $(A_v)_v = C(\delta_v)_v$. We have the following relation between the matrices C and D .

Fact 5.1 (See [Kum02, 11.1.7.(a)]). *We have $D^{-1} = C^T$. Thus,*

$$\delta_v = \sum_{w \leq v} \xi^w(v) A_w .$$

Proof. In fact, from the identity $(A_v)_v = C(\delta_v)_v$, we deduce that $(\delta_v)_v = C^{-1}(A_v)_v = D^T(A_v)$. Since $D_{v,w}^T = \xi^w(v)$ and the matrix D is triangular, we get the result. □

Note also that an explicit formula for the coefficients $\xi^w(v)$ is known: see [Kum02, Proposition 11.1.11].

5.2. Affine Grassmannian and the Pontryagin ring structure

Let G be the simply-connected almost simple group associated to \mathfrak{g} , and let G^{ad} be the adjoint quotient of this group. Let K resp. K^{ad} be a maximal compact subgroup in G resp. G^{ad} . Let ΩK resp. ΩK^{ad} be the group of loops l with values in K resp. K^{ad} such that $l(0)$ is the unit element in K resp. K^{ad} . By a loop we mean a map $l : \mathbb{S}^1 \rightarrow K^{(\text{ad})}$ that extends to a meromorphic map $\mathbb{D}^\circ \rightarrow G^{(\text{ad})}$, where \mathbb{D}° denotes the pointed disk. Modding out a loop by the center of K yields an inclusion $\Omega K \subset \Omega K^{\text{ad}}$. The action of $T \cap K$ on ΩK resp. ΩK^{ad} is given by conjugation.

This implies that the equivariant homology of ΩK and ΩK^{ad} have a natural structure of an algebra, given by the Pontryagin product which is also $(T \cap K)$ -equivariantly homotopy equivalent to the point-wise product of loops. In this section, we will recall an algebraic model for $H_*^{T \cap K}(\Omega K)$ and give one for $H_*^{T \cap K}(\Omega K^{\text{ad}})$. In particular we will describe the ring structure as well as an \mathbb{A}_{aff} -module structure on $H_*^{T \cap K}(\Omega K^{\text{ad}})$ extending the ring structure and the \mathbb{A}_{aff} -module structure on $H_*^{T \cap K}(\Omega K)$.

5.3. Geometry of fixed points in ΩK^{ad}

Since $K \rightarrow K^{\text{ad}}$ is the universal cover of K^{ad} , the connected components of ΩK^{ad} are isomorphic to ΩK and are indexed by $\pi_1(G^{\text{ad}}) = \pi_1(K^{\text{ad}}) = Z$. We now describe the T -fixed points in ΩK^{ad} . We have, in the loop space picture

$$(\Omega K^{\text{ad}})^T = \{\tilde{\psi}_{t_{\lambda^\vee}} : \mathbb{S}^1 \rightarrow K^{\text{ad}} \mid \lambda^\vee \in P^\vee\},$$

where $\tilde{\psi}_{t_{\lambda^\vee}}(t) = \exp(2i\pi t \lambda^\vee)$ is the loop induced by the one-parameter subgroup λ^\vee of T^{ad} (the maximal torus of K^{ad}). For $\lambda^\vee \in P^\vee$, let $[\lambda^\vee]$ be its class in $P^\vee/Q^\vee = \pi_1(K^{\text{ad}})$ and denote by $\Omega K_{[\lambda^\vee]}^{\text{ad}}$ be the connected component of ΩK^{ad} containing $\tilde{\psi}_{t_{\lambda^\vee}}$. We have

$$\Omega K^{\text{ad}} = \coprod_{[\lambda^\vee] \in P^\vee/Q^\vee} \Omega K_{[\lambda^\vee]}^{\text{ad}}.$$

Let $m_{\lambda^\vee} : \Omega K \rightarrow \Omega K_{[\lambda^\vee]}^{\text{ad}}$ be the left multiplication by $\tilde{\psi}_{t_{\lambda^\vee}}$. Since T and $\tilde{\psi}_{t_{\lambda^\vee}}$ commute, this is a T -equivariant isomorphism. Thus, $H_*^T(\Omega K_{[\lambda^\vee]}^{\text{ad}}) \simeq H_*^T(\Omega K)$.

5.4. Reminder on $H_*^T(\Omega K)$

Recall from [KK86] that ΩK has a cellular decomposition whose cells are indexed by W_{aff}^- . This implies that, as S -module, we have

$$H_*^T(\Omega K) = \bigoplus_{w \in W_{\text{aff}}^-} S \cdot \xi_w \simeq M.$$

Furthermore, according to [Lam08, (3.1) and (3.2)], \mathbb{A}_{aff} acts on $H_*^T(\mathcal{G}/\mathcal{P})$ by

$$A_v \cdot \xi_w = \begin{cases} \xi_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w) \text{ and } vw \in W_{\text{aff}}^- \\ 0 & \text{otherwise} \end{cases}$$

and \mathbb{A}_{aff} acts on $H_T^*(\mathcal{G}/\mathcal{P})$ by

$$A_v \cdot \xi^w = \begin{cases} \xi^{vw} & \text{if } \ell(vw) = \ell(w) - \ell(v) \text{ and } vw \in W_{\text{aff}}^- \\ 0 & \text{otherwise} \end{cases}$$

5.5. S -algebra structure on $H_*^T(\Omega K^{\text{ad}})$

We use the $T \cap K$ -equivariant homology of the $T \cap K$ -space ΩK^{ad} , where $T \cap K$ acts on ΩK^{ad} via $T \cap K \rightarrow (T \cap K)^{\text{ad}} \rightarrow G^{\text{ad}}$. The inclusion $T \cap K \rightarrow T$ induces an isomorphism in equivariant cohomology $H_T^*(pt) \rightarrow H_{T \cap K}^*(pt)$. Note that we have $H_{T \cap K}^*(\Omega K) \simeq H_{T \cap K}^*(\mathcal{G}/\mathcal{P}) \simeq H_T^*(\mathcal{G}/\mathcal{P})$, where \mathcal{G}/\mathcal{P} is the affine Grassmannian. Abusing notations slightly, we will denote in the following $H_{T \cap K}^*(\Omega K)$ simply by $H_T^*(\Omega K)$, and similarly for $H_T^*(\Omega K^{\text{ad}})$. The T -equivariant cohomology of the point is the symmetric algebra on P , namely S , see [Br98, p.5], so that the homology $H_*^T(\Omega K^{\text{ad}})$ will be an S -module and even an S -algebra. We are not considering T^{ad} -equivariant homology.

Proposition 5.2. *As S -algebras, we have: $H_*^T(\Omega K^{\text{ad}}) \simeq S[P^\vee] \otimes_{S[Q^\vee]} H_*^T(\Omega K)$.*

Proof. We have the following inclusions that are compatible with pointwise multiplication and T -equivariant inducing S -algebra morphisms

$$\begin{array}{ccc} \Omega K^T & \longrightarrow & \Omega K \\ \downarrow & & \downarrow \\ (\Omega K^{\text{ad}})^T & \longrightarrow & \Omega K^{\text{ad}}, \end{array} \quad \begin{array}{ccc} H_*^T(\Omega K^T) & \longrightarrow & H_*^T(\Omega K) \\ \downarrow & & \downarrow \\ H_*^T((\Omega K^{\text{ad}})^T) & \longrightarrow & H_*^T(\Omega K^{\text{ad}}). \end{array}$$

Recall that we have bijections $(\Omega K^{\text{ad}})^T \simeq P^\vee$ and $\Omega K^T \simeq Q^\vee$ that are group homomorphisms since $\tilde{\psi}_{t_{\lambda^\vee}} \psi_{t_{\mu^\vee}} = \tilde{\psi}_{t_{\lambda^\vee + \mu^\vee}}$. We thus have $H_*^T((\Omega K^{\text{ad}})^T) \simeq S[P^\vee]$ and $H_*^T(\Omega K^T) \simeq S[Q^\vee]$. In particular, the above diagram induces an S -algebra morphism $S[P^\vee] \otimes_{S[Q^\vee]} H_*^T(\Omega K) \rightarrow H_*^T(\Omega K^{\text{ad}})$. The restriction of this map to $\tilde{\psi}_{t_{\lambda^\vee}} \otimes H_*^T(\Omega K^T) \rightarrow H_*^T((\Omega K^{\text{ad}})^T)$ is the multiplication m_{λ^\vee} . The above decomposition of ΩK^{ad} in connected components gives an isomorphism of S -modules

$$H_*^T(\Omega K^{\text{ad}}) = \bigoplus_{[\lambda^\vee] \in P^\vee/Q^\vee} H_*^T(\Omega K_{[\lambda^\vee]}^{\text{ad}})$$

proving that the map $S[P^\vee] \otimes_{S[Q^\vee]} H_*^T(\Omega K) \rightarrow H_*^T(\Omega K^{\text{ad}})$ is surjective.

To prove injectivity, first note that, since $H_*^T(\Omega K)$ is a free S -module and $S[P^\vee]$ is a free $S[Q^\vee]$ -module, the S -module $S[P^\vee] \otimes_{S[Q^\vee]} H_*^T(\Omega K)$ is free. We therefore only need to prove the injectivity of the map after base extension to $F = \text{Frac}(S)$ the field of fractions of S . Now recall the following

general result (see [Kum02, C.8 Theorem]: on the level of T -equivariant cohomology we have isomorphisms $H_T^*(\Omega K) \otimes_S F \simeq H_T^*(\Omega K^T) \otimes_S F$ and $H_T^*(\Omega K^{\text{ad}}) \otimes_S F \simeq H_T^*((\Omega K^{\text{ad}})^T) \otimes_S F$. This induces isomorphisms in T -equivariant homology:

$$H_*^T(\Omega K^T) \otimes_S F \simeq H_*^T(\Omega K) \otimes_S F$$

and $H_*^T((\Omega K^{\text{ad}})^T) \otimes_S F \simeq H_*^T(\Omega K^{\text{ad}}) \otimes_S F$.

After base change to F , since $H_*^T(\Omega K^T) \simeq F[Q^\vee]$ and $H_*^T((\Omega K^{\text{ad}})^T) \otimes_S F \simeq F[P^\vee]$, our map is given by

$$F[P^\vee] \otimes_{F[Q^\vee]} F[Q^\vee] \rightarrow F[P^\vee]$$

and is therefore injective. □

Recall that, as S -module, we have an isomorphism $H_*^T(\Omega K) = M$. In particular the above results identifies $H_*^T(\Omega K^{\text{ad}})$ with the S -module \widetilde{M} of Definition 4.14:

Corollary 5.3. *As S -modules, we have: $H_*^T(\Omega K^{\text{ad}}) \simeq \widetilde{M}$.*

Corollary 5.4. *There exists an $\widetilde{\mathbb{A}}_{\text{aff}}$ -module structure on $H_*^T(\Omega K^{\text{ad}})$ compatible with the \mathbb{A}_{aff} -module structure on $H_*^T(\Omega K)$. Furthermore, for this structure, we have an isomorphism of $\widetilde{\mathbb{A}}_{\text{aff}}$ -modules $H_*^T(\Omega K^{\text{ad}}) \simeq \widetilde{M}$.*

Proof. We define the $\widetilde{\mathbb{A}}_{\text{aff}}$ -module structure on $H_*^T(\Omega K^{\text{ad}})$. Since we have the isomorphism of S -modules $H_*^T(\Omega K^{\text{ad}}) \simeq \widetilde{M}_1 = S[P^\vee] \otimes_{S[Q^\vee]} M$, we may extend the \mathbb{A}_{aff} -module structure on M to the $\widetilde{\mathbb{A}}_{\text{aff}}$ -module structure \widetilde{M} . □

Remark 5.5. The above $\widetilde{\mathbb{A}}_{\text{aff}}$ -module structure on $H_*^T(\Omega K^{\text{ad}})$ also has a geometric description, see [CMP09, Proposition 3.3].

Remark 5.6. The above result shows that our claim on [CMP09, Page 12] that $H_*^T(\Omega K^{\text{ad}})$ is the tensor product ring $\mathbb{Z}[Z] \otimes_{\mathbb{Z}} H_*^T(\Omega K)$ is wrong: by localization $H_*^T(\Omega K^{\text{ad}})$ is a subring of $F[P^\vee]$ and this Laurent polynomial algebra contains no roots of unity, whereas $\mathbb{Z}[Z] \otimes_{\mathbb{Z}} H_*^T(\Omega K)$ does.

Recall that $\widetilde{W}_{\text{aff}}$ can be embedded in $\widetilde{\mathbb{A}}_{\text{aff}}$ via $w \mapsto \delta_w$. The induced action is denoted by $x \cdot \widetilde{\xi} := \delta_x \cdot \widetilde{\xi}$ for $x \in \widetilde{W}_{\text{aff}}$ and $\widetilde{\xi} \in H_*^T(\Omega K^{\text{ad}})$.

Corollary 5.7. *Let $w \in W$ and $\lambda^\vee, \mu^\vee \in P^\vee$, we have*

$$wt_{\lambda^\vee} \cdot \widetilde{\psi}_{t_{\mu^\vee}} := \delta_{wt_{\lambda^\vee}} \cdot \widetilde{\psi}_{t_{\mu^\vee}} = \widetilde{\psi}_{t_{w(\lambda^\vee + \mu^\vee)}}.$$

Proof. As already explained in the proof of Proposition 5.2, we have $\delta_{t_{\lambda^\vee}} \cdot \tilde{\psi}_{t_{\mu^\vee}} = \tilde{\psi}_{t_{\lambda^\vee + \mu^\vee}}$. We need to check that $\delta_w \cdot \tilde{\psi}_{t_{\mu^\vee}} = \tilde{\psi}_{t_w(\mu^\vee)}$. But our identification of $H_*^T(\Omega K^{\text{ad}})$ with \widetilde{M} identifies $\tilde{\psi}_{t_{\mu^\vee}}$ with $h_{\mu^\vee} \otimes 1$. Recall that $1 = [\delta_e] \in \widetilde{\mathbb{A}}_{\text{aff}} / \widetilde{J}$ (e being the neutral element of W), so that $h_{\mu^\vee} \otimes 1 = [\delta_{t_{\mu^\vee}}]$ and $\delta_w \cdot \tilde{\psi}_{t_{\mu^\vee}} = \delta_w \cdot [\delta_{t_{\mu^\vee}}] = [\delta_w \delta_{t_{\mu^\vee}}] = [\delta_w \delta_{t_{\mu^\vee}} \delta_{w^{-1}}]$ since $\delta_{w^{-1}} \in \widetilde{J}$. We get

$$\delta_w \cdot \tilde{\psi}_{t_{\mu^\vee}} = [\delta_w \delta_{t_{\mu^\vee}} \delta_{w^{-1}}] = [\delta_{t_w(\mu^\vee)}] = \tilde{\psi}_{t_w(\mu^\vee)}$$

proving the result. □

5.6. Compatibility between the ring and the $\widetilde{\mathbb{A}}_{\text{aff}}$ -module structure

The above description of $H_*^T(\Omega K^{\text{ad}})$ as ring and as $\widetilde{\mathbb{A}}_{\text{aff}}$ -module is not enough for our purposes: we need to be able to multiply two classes of the form $\sigma \otimes \xi_x$ and $\tau \otimes \xi_y$, see also Remark 5.6. To this end, we recall the definition and properties of j^{ad} given in [CMP09, §3.3].

Proposition 5.8. *There is an S -algebra isomorphism $j^{\text{ad}} : H_*^T(\Omega K^{\text{ad}}) \rightarrow Z_{\widetilde{\mathbb{A}}_{\text{aff}}}(S)$. It satisfies:*

- 1) $j^{\text{ad}}(\xi) \cdot \xi' = \xi \xi'$ for $\xi, \xi' \in H_*^T(\Omega K^{\text{ad}})$;
- 2) $j^{\text{ad}}(\tilde{\psi}_{t_{\lambda^\vee}}) = \delta_{t_{\lambda^\vee}}$ for $\lambda^\vee \in P^\vee$.

For $w \in \widetilde{W}_{\text{aff}}^-$, $j^{\text{ad}}(\tilde{\xi}_w)$ is characterized by the two following properties:

- (a) $j^{\text{ad}}(\tilde{\xi}_w)$ is congruent to A_w modulo $\sum_{x \in W \setminus \{e\}} \widetilde{\mathbb{A}}_{\text{aff}} \cdot A_x$;
- (b) $j^{\text{ad}}(\tilde{\xi}_w)$ belongs to $Z_{\widetilde{\mathbb{A}}_{\text{aff}}}(S)$.

The map j^{ad} has the following equivariance property:

Proposition 5.9. *Let $u \in W, \lambda^\vee \in P^\vee, \tilde{\xi} \in H_*^T(\Omega K^{\text{ad}})$. Then*

- 1) $j^{\text{ad}}(ut_{\lambda^\vee} \cdot \tilde{\xi}) = \delta_{t_{u(\lambda^\vee)}} \delta_u j^{\text{ad}}(\tilde{\xi}) \delta_{u^{-1}} = \delta_{ut_{\lambda^\vee}} j^{\text{ad}}(\tilde{\xi}) \delta_{u^{-1}}$;
- 2) $\delta_{t_{\lambda^\vee}} j^{\text{ad}}(\tilde{\xi}) = j^{\text{ad}}(\tilde{\xi}) \delta_{t_{\lambda^\vee}}$.

Proof. (1) Let $s \in S$ be a scalar, we have:

- $j^{\text{ad}}(ut_{\lambda^\vee} \cdot s\tilde{\xi}) = j^{\text{ad}}(u(s)ut_{\lambda^\vee} \cdot \tilde{\xi}) = u(s)j^{\text{ad}}(ut_{\lambda^\vee} \cdot \tilde{\xi})$;
- $\delta_{t_{u(\lambda^\vee)}} \delta_u j^{\text{ad}}(s\tilde{\xi}) \delta_{u^{-1}} = \delta_{t_{u(\lambda^\vee)}} \delta_u s j^{\text{ad}}(\tilde{\xi}) \delta_{u^{-1}} = u(s) \delta_{t_{u(\lambda^\vee)}} \delta_u j^{\text{ad}}(\tilde{\xi}) \delta_{u^{-1}}$.

Thus, by semi-linearity, it is enough to prove the result for $\tilde{\xi} = \tilde{\psi}_{t_{\mu^\vee}}$. For $\tilde{\xi} = \tilde{\psi}_{t_{\mu^\vee}}$, we have $j^{\text{ad}}(ut_{\lambda^\vee} \cdot \tilde{\psi}_{t_{\mu^\vee}}) = j^{\text{ad}}(u \cdot \tilde{\psi}_{t_{\lambda^\vee + \mu^\vee}}) = j^{\text{ad}}(\tilde{\psi}_{t_{u(\lambda^\vee) + u(\mu^\vee)}}) = \delta_{t_{u(\lambda^\vee) + u(\mu^\vee)}}$. We also have $\delta_{t_{u(\lambda^\vee)}} \delta_u j^{\text{ad}}(\tilde{\psi}_{t_{\mu^\vee}}) \delta_{u^{-1}} = \delta_{t_{u(\lambda^\vee)}} \delta_u \delta_{t_{\mu^\vee}} \delta_{u^{-1}} = \delta_{t_{u(\lambda^\vee) + u(\mu^\vee)}}$. Thus the result is proved.

(2) Both terms are S -linear so we only need to check this for $\tilde{\xi} = \tilde{\psi}_{t_{\mu^\vee}}$ but we have $\delta_{t_{\lambda^\vee}} j^{\text{ad}}(\tilde{\psi}_{t_{\mu^\vee}}) = \delta_{t_{\lambda^\vee}} \delta_{t_{\mu^\vee}} = \delta_{t_{\mu^\vee}} \delta_{t_{\lambda^\vee}} = j^{\text{ad}}(\tilde{\psi}_{t_{\mu^\vee}}) \delta_{t_{\lambda^\vee}}$. \square

In particular, the previous Proposition allows computing j^{ad} in terms of j :

Example 5.10. Let $\tau_i = v_i t_{-\varpi_i} \in Z$ and let $\xi \in H_*^T(\Omega K)$. Then

$$j^{\text{ad}}(\tau_i \cdot \xi) = \delta_{\tau_i} j(\xi) \delta_{v_i^{-1}}.$$

We deduce a formula allowing reducing products in the homology of ΩK^{ad} to products in the homology of ΩK :

Corollary 5.11. Let $\sigma = ut_{\lambda^\vee}, \tau = vt_{\mu^\vee}$ be elements in Z . Let $\tilde{\xi}, \tilde{\xi}' \in H_*^T(\Omega K^{\text{ad}})$. Then

$$(\sigma \cdot \tilde{\xi}) \times (\tau \cdot \tilde{\xi}') = \sigma \tau \cdot (\psi_{\sigma, \tau} \times (v_*^{-1} \tilde{\xi}) \times (u_*^{-1} \tilde{\xi}')),$$

where $\psi_{\sigma, \tau} = \psi_{u^{-1}(\mu^\vee) - \mu^\vee} = \psi_{v^{-1}(\lambda^\vee) - \lambda^\vee}$.

Proof. Since Z is abelian, $\sigma \tau = \tau \sigma$. We have $\sigma \tau = ut_{\lambda^\vee} vt_{\mu^\vee} = uvt_{v^{-1}(\lambda^\vee) + \mu^\vee}$ and $\tau \sigma = vt_{\mu^\vee} ut_{\lambda^\vee} = vut_{u^{-1}(\mu^\vee) + \lambda^\vee}$. We get $v^{-1}(\lambda^\vee) + \lambda^\vee = u^{-1}(\mu^\vee) + \mu^\vee$ so $u^{-1}(\mu^\vee) - \mu^\vee = v^{-1}(\lambda^\vee) - \lambda^\vee$, so that $\psi_{\sigma, \tau}$ is well defined. We also get $uv = vu$.

Using Proposition 5.9, we compute:

$$\begin{aligned} (\sigma \cdot \tilde{\xi}) \times (\tau \cdot \tilde{\xi}') &= j^{\text{ad}}(\sigma \cdot \tilde{\xi}) \cdot (\tau \cdot \tilde{\xi}') \\ &= \delta_u \delta_{t_{\lambda^\vee}} j^{\text{ad}}(\tilde{\xi}) \delta_{u^{-1}} \delta_v \delta_{t_{\mu^\vee}} \cdot \tilde{\xi}' \\ &= \delta_u \delta_{t_{\lambda^\vee}} j^{\text{ad}}(\tilde{\xi}) \delta_v \delta_{u^{-1}} \delta_{t_{\mu^\vee}} \cdot \tilde{\xi}' \\ &= \delta_u \delta_{t_{\lambda^\vee}} \delta_v j^{\text{ad}}(v^{-1} \cdot \tilde{\xi}) \delta_{t_{u^{-1}(\mu^\vee)}} \delta_{u^{-1}} \cdot \tilde{\xi}' \\ &= \delta_u \delta_{t_{\lambda^\vee}} \delta_v \delta_{t_{u^{-1}(\mu^\vee)}} j^{\text{ad}}(v^{-1} \cdot \tilde{\xi}) \delta_{u^{-1}} \cdot \tilde{\xi}' \\ &= \delta_u \delta_{t_{\lambda^\vee}} \delta_v \delta_{t_{\mu^\vee}} \delta_{t_{u^{-1}(\mu^\vee) - \mu^\vee}} j^{\text{ad}}(v^{-1} \cdot \tilde{\xi}) \delta_{u^{-1}} \cdot \tilde{\xi}' \\ &= \sigma \tau \cdot (\psi_{\sigma, \tau} \times (v_*^{-1} \tilde{\xi}) \times (u_*^{-1} \tilde{\xi}')). \end{aligned}$$

\square

Remark 5.12. In [CMP09, p.12], it is claimed that $H_*^T(\Omega K^{\text{ad}})$ is the tensor product ring $\mathbb{Z}[Z] \otimes_{\mathbb{Z}} H_*^T(\Omega K)$. As explained in Remark 5.6, this is not true. However as the next corollary shows, this is true in the non equivariant homology.

Corollary 5.13. *In non equivariant homology, let $\tilde{\xi}, \tilde{\xi}' \in H_*(\Omega K^{\text{ad}})$ and $\sigma, \tau \in Z$, then*

$$(\sigma \cdot \tilde{\xi}) \times (\tau \cdot \tilde{\xi}') = \sigma\tau \cdot (\tilde{\xi} \times \tilde{\xi}').$$

Proof. Indeed, push-forwards u_*^{-1} and v_*^{-1} are trivial in non equivariant homology. Moreover, the equivariant classes $\tilde{\psi}_{\lambda^\vee}$ restrict to the class of a point in $H_*(\Omega K^{\text{ad}})$, which is the unit in $H_*(\Omega K^{\text{ad}})$. □

5.7. Translations modulo P

We use [LS10, Lemma 10.1] and [CMP09, Corollary 3.15] as a definition:

$$(7) \quad (W^P)_{\text{aff}} = \left\{ ut_{\nu^\vee} \mid \forall \gamma \in R_P^+, \langle \nu^\vee, \gamma \rangle = \begin{cases} 0 & \text{if } u(\gamma) > 0 \\ -1 & \text{if } u(\gamma) < 0 \end{cases} \right\}$$

$$(8) \quad (\widetilde{W}^P)_{\text{aff}} = \left\{ ut_{\nu^\vee} \in \widetilde{W}_{\text{aff}} \mid \forall \gamma \in R_P^+, \langle \nu^\vee, \gamma \rangle = \begin{cases} 0 & \text{if } u(\gamma) > 0 \\ -1 & \text{if } u(\gamma) < 0 \end{cases} \right\}.$$

Following [LS10, §10.2 and 10.3], we also define $(W_P)_{\text{aff}} = \{wt_{\lambda^\vee} \mid w \in W_P, \lambda^\vee \in Q_P^\vee\}$. Recall, from [CMP09, Section 3.4] that any element $w \in \widetilde{W}_{\text{aff}}$ can be uniquely factorized as $w_1 w_2$ with $w_1 \in (\widetilde{W}^P)_{\text{aff}}$ and $w_2 \in (W_P)_{\text{aff}}$ and $\ell(w) = \ell(w_1) + \ell(w_2)$. We denote $w_1 = \pi_P(w)$. Thus $(\widetilde{W}^P)_{\text{aff}}$ is a set of representatives for the quotient $\widetilde{W}_{\text{aff}} / (W_P)_{\text{aff}}$ which will be relevant for Peterson’s isomorphism (9).

Following [LS10, Section 10.4] and [CMP09, Section 3.4], define the ideals $J_P \subset M$ and $\tilde{J}_P \subset \tilde{M}$ as follows:

$$J_P = \sum_{x \in W_{\text{aff}}^- \setminus (W^P)_{\text{aff}}} S\xi_x \quad \text{and} \quad \tilde{J}_P = \sum_{x \in \widetilde{W}_{\text{aff}}^- \setminus (\widetilde{W}^P)_{\text{aff}}} S\tilde{\xi}_x.$$

The following result corrects [CMP09, Proposition 3.16] which used the wrong product structure, see Remark 5.6.

Proposition 5.14. *Let $x \in \widetilde{W}_{\text{aff}}^- \cap (\widetilde{W}^P)_{\text{aff}}$ and let $\nu^\vee \in P_-^\vee$. Then $x\pi_P(t_{\nu^\vee}) \in \widetilde{W}_{\text{aff}}^- \cap (\widetilde{W}^P)_{\text{aff}}$. Let us write as usual $x = \sigma\widehat{x}$ and $\pi_P(t_{\nu^\vee}) = \tau\pi_P(t_{\nu^\vee})$ with $\sigma = ut_{\lambda^\vee}, \tau = vt_{\mu^\vee}$. Then*

$$(v_*^{-1}\xi_{\widehat{x}}) \times (u_*^{-1}\xi_{\pi_P(t_{\nu^\vee})}) = \psi_{\sigma,\tau}^{-1}\xi_{x\pi_P(t_{\nu^\vee})} \text{ modulo } J_P.$$

Proof. The proof follows the arguments in [CMP09]. In particular, we get

$$(\sigma \cdot \xi_{\widehat{x}}) \times (\tau \cdot \xi_{\pi_P(t_{\nu^\vee})}) = \sigma\tau \cdot \xi_{x\pi_P(t_{\nu^\vee})} \text{ modulo } \widetilde{J}_P.$$

Using the correct product formula given in Corollary 5.11, the left hand side is

$$\sigma\tau \cdot (\psi_{\sigma,\tau}(v_*^{-1}\xi_{\widehat{x}}) \times (u_*^{-1}\xi_{\pi_P(t_{\nu^\vee})})).$$

This proves the result since $\widetilde{J}_P \cap M = J_P$ (as $\sigma\widehat{x} \in \widetilde{W}_{\text{aff}}^- \Leftrightarrow \widehat{x} \in W_{\text{aff}}^-$). □

In particular, the case $P = B$ yields:

Corollary 5.15. *Let $x \in \widetilde{W}_{\text{aff}}^-$ and let $\nu^\vee \in P_-^\vee$. Then $xt_{\nu^\vee} \in \widetilde{W}_{\text{aff}}^-$. Let us write as usual $x = \sigma\widehat{x}$ and $t_{\nu^\vee} = \tau t_{\nu^\vee}$ with $\sigma = ut_{\lambda^\vee}, \tau = vt_{\mu^\vee}$. Then*

$$(v_*^{-1}\xi_{\widehat{x}}) \times (u_*^{-1}\xi_{t_{\nu^\vee}}) = \psi_{\sigma,\tau}^{-1}\xi_{xt_{\nu^\vee}}.$$

6. Affine symmetries

In this section, we correct [CMP09, Section 3.5], see Remark 5.6 using the correct product formula given in Corollary 5.11 and Proposition 5.14. In particular we prove that the formulas given in [CMP09] are correct in the non equivariant setting.

6.1. Peterson’s isomorphism

Proposition 5.14 is our needed result in the equivariant homology of the affine Grassmannian. Translating this formula in the quantum cohomology of G/P , we prove our main theorem. We use Peterson’s isomorphism [Pet98] proved in [LS10] to relate $H_*^T(\Omega K)$ and $QH_T^*(G/P)$.

Let $\eta_P : Q^\vee \rightarrow Q_P^\vee$ be the projection on the coroot subspace generated by simple roots α_i with $\alpha_i \notin R_P$. Peterson’s isomorphism is the map

$$(9) \quad \begin{aligned} \psi_P : \quad H_*^T(\Omega K)_P &\rightarrow \mathbb{Z}[Q_P^\vee] \otimes_{\mathbb{Z}} H_T^*(G/P) \\ \xi_{w\pi_P(t_{\lambda^\vee})} \xi_{\pi_P(t_{\mu^\vee})}^{-1} &\mapsto q_{\eta_P(\lambda^\vee - \mu^\vee)} \sigma^P(w) \end{aligned}$$

where $w \in W^P$ and $\lambda^\vee, \mu^\vee \in Q_-^\vee$ with Q_-^\vee the set of antidominant elements in Q^\vee .

Remark 6.1. In the above statement we have:

- 1) The space $H_*^T(\Omega K)_P$ is a quotient and a localization of $H_*^T(\Omega K)$ defined in [CMP09, §2.2]. The family $\{\sigma^P(w), w \in W/W_P\}$ is the Schubert base of $H_T^*(G/P)$, and the element in $\mathbb{Z}[Q_P^\vee]$ corresponding to $\nu^\vee \in Q_-^\vee$ is denoted by q_{ν^\vee} . We have for $\nu^\vee \in Q_-^\vee$ the formula

$$\deg(q_{\nu^\vee}) = \sum_{\alpha \in R^+ \setminus R_P^+} \langle \nu^\vee, \alpha \rangle = -\ell(t_{\nu^\vee}).$$

- 2) This isomorphism is graded. In fact, for very negative coweights λ^\vee, μ^\vee , the element $\xi_{w\pi_P(t_{\lambda^\vee})} \xi_{\pi_P(t_{\mu^\vee})}^{-1}$ has homological degree $\ell(\pi_P(t_{\lambda^\vee})) - \ell(w) - \ell(\pi_P(t_{\mu^\vee}))$, by [LS10, Lemma 3.3]. On the other hand, in quantum cohomology, the element $q_{\eta_P(\lambda^\vee - \mu^\vee)} \sigma^P(w)$ has degree $-\ell(\pi_P(t_{\lambda^\vee})) + \ell(\pi_P(t_{\mu^\vee})) + \ell(w)$.

6.2. A Weyl group action on $\mathrm{QH}_T^*(G/P)$.

In this subsection, we recall that left multiplication in the group G induces an action of the Weyl group on $H_T^*(G/P)$ and on $\mathrm{QH}_T^*(G/P)$. This action is very natural and appears eg in [MNS21, Section 3.1]. We will prove the compatibility of this action with Peterson’s isomorphism in the next subsection. Since this action is different from the action defined in Kumar [Kum02, 11.3.4] we define it carefully. We start with the action on G/B and then deal with the general situation for G/P .

We define an algebraic and a geometric action of the Weyl group W on $H_T^*(G/B)$. We then prove that these actions coincide.

Let $n \in G$ be in the normalizer of T and let w be the corresponding element of the Weyl group. Define the left action $L_n : G/B \rightarrow G/B$ by left multiplication: $L_n \cdot [x] = [n^{-1}x]$. This action is T -equivariant if we consider the w -twisted action of T on G/B given by $t \cdot [x] = [w(t)x]$. It therefore

induces a w -semilinear map $H_T^*(G/B) \rightarrow H_T^*(G/B)$, denoted L_n^* : $L_n^*(s\xi) = w(s)L_n^*(\xi)$ for $s \in S$ and $\xi \in H_T^*(G/B)$.

Fact 6.2. *The above action L_n^* satisfies the following properties:*

- 1) L_n^* depends on w and not on n itself; it will be denoted by w^* in the sequel.
- 2) Via the inclusion $H_T^*(G/P) \subset H_T^*(G/B)$ given by pulling back the projection $G/B \rightarrow G/P$, we have $w^*H_T^*(G/P) \subset H_T^*(G/P)$.
- 3) The induced action of w^* on the non equivariant cohomology $H^*(G/B)$ is trivial.

Proof. (1) Let N denote the normalizer of T . The map $N \times G/B \rightarrow G/B, (n, [x]) \mapsto L_n \cdot [x]$ is continuous and therefore for $\xi \in H_T^*(G/B)$, the map $N \mapsto L_n^*\xi$ is locally constant.

(2) For n in the normalizer of T , we have a commutative diagram:

$$\begin{CD} G/B @>L_n^B>> G/B \\ @VVV @VVV \\ G/P @>L_n^P>> G/P \end{CD}$$

Here we made a difference between the action of n on G/B and G/P using superscripts. It follows that for $\xi \in H_T^*(G/P)$, we have $w^*\xi = (L_n^B)^*\xi = (L_n^P)^*\xi \in H_T^*(G/P)$.

(3) For $g \in G$, we can consider the action of left translation L_g^* on non equivariant cohomology $H^*(G/B)$. By the same argument as in (1), this action is trivial. In particular, for $g = n$ in N , we obtain that the action L_n^* on non equivariant cohomology is trivial. □

Recall that W can be embedded in \mathbb{A}_{aff} via $v \mapsto \delta_v$.

Definition 6.3. Let $w \in W$. Consider $H_T^*(G/B)$ as the dual of $H_*^T(G/B) \subset \mathbb{A}_{\text{aff}}$ and set

$$(w \bullet f)(x) = f(\delta_{w^{-1}x}) \text{ for } x \in H_*^T(G/B) = \mathbb{A}.$$

Proposition 6.4. For $f \in H_T^*(G/B)$ and $w \in W$, we have $w \bullet f = w^*f$.

Proof. Using $\text{Frac}(S)$ -linearity, we only need to compare these actions on the elements ξ^v . We have $(w \bullet \xi^v)(\delta_u) = \xi^v(\delta_{w^{-1}\delta_u}) = \xi^v(\delta_{w^{-1}u}) = \delta_{v, w^{-1}u} = \xi^v(w^{-1}u) = (w^*\xi^v)(u)$, proving the result. □

Corollary 6.5. *Let α be a simple root and $w \in W^P$. We have*

$$(s_\alpha)^* \sigma^P(w) = \begin{cases} \sigma^P(w) & \text{if } s_\alpha w > w; \\ \sigma^P(w) - \alpha \sigma^P(s_\alpha w) & \text{if } s_\alpha w < w. \end{cases}$$

Proof. We compute $((s_\alpha)^* \sigma^P(w))(A_u) = \sigma^P(w)(\delta_{s_\alpha} A_u) = \sigma^P(w)((1 - \alpha A_\alpha)A_u) = \sigma^P(w)(A_u) - \alpha \sigma^P(w)(A_\alpha A_u)$. Now we have

$$A_\alpha A_u = \begin{cases} 0 & \text{if } s_\alpha u < u \\ A_{s_\alpha u} & \text{if } s_\alpha u > u. \end{cases}$$

Since $\sigma^P(w)(A_v) = \delta_{v,w}$, we get

$$((s_\alpha)^* \sigma^P(w))(A_u) = \begin{cases} \delta_{u,w} & \text{if } s_\alpha u < u \\ \delta_{u,w} - \alpha \delta_{s_\alpha u,w} & \text{if } s_\alpha u > u. \end{cases}$$

This in turn gives the result. □

Remark 6.6. 1) Note that, for α simple, the two conditions $w \in W^P$ and $s_\alpha w < w$ imply the inclusion $s_\alpha w \in W^P$ since the inversion set of $s_\alpha w$ is contained in the inversion set of w . In particular, in the second case of the above formula, the class $\sigma^P(s_\alpha w)$ is well defined.

2) This formula also shows that the action w^* is trivial in the non equivariant setting (indeed, in that case we set $\alpha = 0$).

The action w^* is extended to $\text{QH}_T^*(G/P)$ by linearity on quantum parameters.

6.3. Compatibility of Peterson’s isomorphism

In this subsection we prove that Peterson’s isomorphism is compatible with the actions u_* in homology and u^* in cohomology. We start with a useful lemma.

Lemma 6.7. *Let $w \in W^P$ and let $\lambda^\vee \in Q_-^\vee$ be such that $x = w\pi_P(t_{\lambda^\vee}) \in (W^P)_{\text{aff}}$. Write $x = w\pi_P(t_{\mu^\vee}) = vt_{\mu^\vee}$ with $v \in W$ and $\mu^\vee \in Q^\vee$.*

Let α be a simple root and let $\beta = w^{-1}(\alpha)$, $\beta' = v^{-1}(\alpha)$.

- 1) *We have $w^{-1}v \in W_P$ and $\mu^\vee \in Q_-^\vee$.*
- 2) *We have $\beta \in R_P \iff \beta' \in R_P$.*
- 3) *We have $s_\alpha x \in W_{\text{aff}}^- \iff \langle \mu^\vee, \beta \rangle \neq 0$.*

4) We have $s_\alpha x \in (W^P)_{\text{aff}} \iff \beta \notin R_P \iff s_\alpha w \in W^P$.

5) We have the equivalence:

$$\begin{aligned} & (s_\alpha x \in W_{\text{aff}}^- \cap (W^P)_{\text{aff}} \text{ and } \ell(s_\alpha x) > \ell(x)) \\ & \iff (s_\alpha w \in W^P \text{ and } \ell(s_\alpha w) < \ell(w)). \end{aligned}$$

Proof. (1) By [LS10, Lemma 10.7], we have $\pi_P(t_{\lambda^\vee}) = ut_{\mu^\vee}$ with $u \in W_P$. This give $w^{-1}v = u \in W_P$. Since $vt_{\mu^\vee} \in W_{\text{aff}}^-$ we have $\mu^\vee \in Q_-^\vee$.

(2) Since $u = w^{-1}v \in W_P$ and $\beta = u(\beta')$, we have $\beta \in R_P \iff \beta' \in R_P$.

(3) We have $vt_{\mu^\vee} \in W_{\text{aff}}^-$ therefore $\mu^\vee \in Q_-^\vee$ and for $\gamma > 0$, we have the implication ($\langle \mu^\vee, \gamma \rangle = 0 \Rightarrow v(\gamma) > 0$). The condition $s_\alpha vt_\mu \in W_{\text{aff}}^-$ is thus equivalent to ($\langle \mu^\vee, \gamma \rangle = 0 \Rightarrow s_\alpha v(\gamma) > 0$) for $\gamma > 0$. But since for $\gamma = \beta'$, the roots $v(\gamma)$ and $s_\alpha v(\gamma)$ have opposite signs, the condition $s_\alpha vt_{\mu^\vee} \in W_{\text{aff}}^-$ is equivalent to $\langle \mu^\vee, \beta' \rangle = 0$.

(4) We have $vt_{\mu^\vee} \in (W^P)_{\text{aff}}$ therefore, for $\gamma \in R_P^+$, we have the equivalences ($\langle \mu^\vee, \gamma \rangle = 0 \iff v(\gamma) > 0$) and ($\langle \mu^\vee, \gamma \rangle = -1 \iff v(\gamma) < 0$). The condition $s_\alpha vt_\mu \in (W^P)_{\text{aff}}$ is equivalent to having the equivalences ($\langle \mu^\vee, \gamma \rangle = 0 \iff s_\alpha v(\gamma) > 0$) and ($\langle \mu^\vee, \gamma \rangle = -1 \iff s_\alpha v(\gamma) < 0$). Since for $\gamma = \beta'$, the roots $v(\gamma)$ and $s_\alpha v(\gamma)$ have opposite signs, the last equivalences occur if and only if $\beta' \notin R_P$. This in turn is equivalent to $\beta \notin R_P$ by (2).

For the last equivalence, note that by definition, the conditions $s_\alpha w \in W^P$ and $s_\alpha w(R_P^+) \subset R^+$ are equivalent. Since $w \in W^P$, we have $w(R_P^+) \subset R^+$. Since the inversion sets of w and $s_\alpha w$ only differ by β (or its opposite, depending on the sign of β) we get the last equivalence.

(5) Note that we have the equivalence ($\ell(s_\alpha w) < \ell(w) \iff \beta < 0$). We therefore need to prove that the left hand side of the equivalence is equivalent to $\beta \notin R_P$ and $\beta < 0$. Note that since $w \in W^P$, this is equivalent to $\beta < 0$.

First assume that $s_\alpha x \in W_{\text{aff}}^- \cap (W^P)_{\text{aff}}$ and $\ell(s_\alpha x) > \ell(x)$. By [LS10, Lemma 3.3], since $vt_{\mu^\vee}, s_\alpha vt_{\mu^\vee} \in W_{\text{aff}}$, we have $\ell(s_\alpha x) = \ell(t_{\mu^\vee}) - \ell(s_\alpha v)$ and $\ell(x) = \ell(t_{\mu^\vee}) - \ell(v)$. In particular, we have $\ell(s_\alpha v) < \ell(v)$, thus $\beta' < 0$. Since $s_\alpha x \in (W^P)_{\text{aff}}$, we also have $\beta \notin R_P$ thus $\beta' \notin R_P$. Now, since $u \in W_P$, this implies $\beta < 0$.

Conversely, assume $\beta < 0$. By the above arguments, this implies $\beta \notin R_P$ and thus $s_\alpha x \in (W^P)_{\text{aff}}$. This also implies $\beta' \notin R_P$ and since $u \in W_P$ and $\beta' = u^{-1}(\beta)$, we get $\beta' < 0$. Since $vt_\mu \in W_{\text{aff}}^-$, $-\beta' > 0$ and $v(-\beta') = -\alpha < 0$, we must have $\langle \mu^\vee, \beta' \rangle \neq 0$ and by (3), this implies $s_\alpha x \in W_{\text{aff}}^-$. \square

We have the following equivariance property of ψ_P .

Proposition 6.8. For $\xi \in H_*^T(\Omega K)_P$, we have $\psi_P(u_*\xi) = u^*\xi$.

Proof. Peterson [Pet98] proved that ψ_P is an isomorphism of nilHecke modules, which implies this Proposition. Since [Pet98] is not published, we include details.

We may assume that $u = s_i$, with α_i a simple root. Then $u_*\xi_{w\pi_P(t_{\lambda^\vee})} = \delta_u \cdot \xi_{w\pi_P(t_{\lambda^\vee})} = (1 - \alpha_i A_i) \cdot \xi_{w\pi_P(t_{\lambda^\vee})}$. If $\ell(s_i w \pi_P(t_{\lambda^\vee})) > \ell(w \pi_P(t_{\lambda^\vee}))$ and $s_i w \pi_P(t_{\lambda^\vee}) \in (W^P)_{\text{aff}} \cap W_{\text{aff}}^-$, then this is equal to $\xi_{w\pi_P(t_{\lambda^\vee})} - \alpha_i \xi_{s_i w \pi_P(t_{\lambda^\vee})}$. Otherwise, this is equal to $\xi_{w\pi_P(t_{\lambda^\vee})}$.

The action $s_i^* \sigma^P(w)$ is computed in Corollary 6.5. If $\ell(s_i w) < \ell(w)$ and $s_i w \in W^P$, then this is equal to $\sigma^P(w) - \alpha_i \sigma^P(s_i w)$. Otherwise, this is equal to $\sigma^P(w)$.

Let $\beta = w^{-1}(\alpha_i)$. The condition $s_i w \in W^P$ and $\ell(s_i w) < \ell(w)$ is equivalent to the condition $\ell(s_i w \pi_P(t_{\lambda^\vee})) > \ell(w \pi_P(t_{\lambda^\vee}))$ and $s_i w \pi_P(t_{\lambda^\vee}) \in (W^P)_{\text{aff}} \cap W_{\text{aff}}^-$ by Lemma 6.7.(5). This proves the result. \square

6.4. The result

We now prove our main result. For i a cominuscle node, *i.e.* such that ϖ_i^\vee is a minuscule coweight, we let v_i be the smallest element in W such that $v_i(\varpi_i^\vee) = w_0(\varpi_i^\vee)$ (w_0 is the longest element in W). The coweight $v_i(\varpi_i^\vee) = w_0(\varpi_i^\vee)$ is the opposite of a fundamental coweight: there exists $f(i) \in I$ such that $v_i(\varpi_i^\vee) = -\varpi_{f(i)}^\vee$. Actually we have $\alpha_{f(i)} = -w_0(\alpha_i)$ and $v_{f(i)} = v_i^{-1}$.

Theorem 6.9. *Let i be a cominuscle node. In $QH_T^*(G/P)$ we have*

$$\sigma^P(v_i) \times v_i^*(\sigma^P(w)) = q_{\eta_P(\varpi_i^\vee - w^{-1}(\varpi_i^\vee))} \sigma^P(v_i w).$$

Proof. Let $w \in W^P$, we have $\pi_P(w) = w$. Let ϖ_i^\vee be the minuscule coweight associated to i and let μ^\vee and ν^\vee be in Q^\vee and dominant enough. As in [CMP09, §3.5], we get

$$\begin{aligned} \pi_P(t_{-\varpi_i^\vee - \mu}) &= \tau_i \pi_P(v_{f(i)}) \pi_P(t_{-(\varpi_i^\vee + \varpi_{f(i)}^\vee) + \mu}) \quad \text{and} \\ \pi_P(w t_{-\nu} t_{-\varpi_i^\vee - \mu}) &= \tau_i \pi_P(v_{f(i)} w) \pi_P(t_{-(\varpi_i^\vee + w^{-1}(\varpi_{f(i)}^\vee) + \mu + \nu)}. \end{aligned}$$

For μ and ν dominant enough, the elements $w t_{-\nu}$, $t_{-\varpi_i^\vee - \mu}$ and $w t_{-\nu} t_{-\varpi_i^\vee - \mu}$ are in $\widetilde{W}_{\text{aff}}^-$ and their image by π_P are in $(\widetilde{W}^P)_{\text{aff}} \cap \widetilde{W}_{\text{aff}}^-$. We may therefore apply Proposition 5.14 to the elements $w t_{-\nu}$ and $t_{-\varpi_i^\vee - \mu}$ to get:

$$\begin{aligned} (v_{f(i)})_* \xi_{w\pi_P(t_{-\nu^\vee})} \times \xi_{\pi_P(v_{f(i)})\pi_P(t_{-(\varpi_i^\vee + \varpi_{f(i)}^\vee) + \mu^\vee})} \\ \equiv \xi_{\pi_P(v_{f(i)} w)\pi_P(t_{-(\varpi_i^\vee + w^{-1}(\varpi_{f(i)}^\vee) + \mu^\vee + \nu^\vee)})}, \end{aligned}$$

where \equiv means equality in $H_*^T(\Omega K)_P$ or equivalently equality modulo J_P . Applying Peterson’s map (9), we get thanks to Proposition 6.8 the corresponding formula in the quantum cohomology ring:

$$\begin{aligned} v_{f(i)}^* \sigma^P(w) q_{-\eta_P(\nu^\vee)} * \times \sigma^P(v_{f(i)}) q_{-\eta_P((\varpi_i^\vee + \varpi_{f(i)}^\vee + \mu^\vee))} \\ = \sigma^P(v_{f(i)} w) q_{-\eta_P((\varpi_i^\vee + w^{-1}(\varpi_{f(i)}^\vee) + \mu^\vee + \nu^\vee))}, \end{aligned}$$

hence finally:

$$v_{f(i)}^* \sigma^P(w) * \sigma^P(v_{f(i)}) = q_{\eta_P(\varpi_{f(i)}^\vee - w^{-1}(\varpi_{f(i)}^\vee))} \sigma^P(v_{f(i)} w).$$

This concludes the proof of the theorem. □

Corollary 6.10. *Let i be such that ϖ_i^\vee is a minuscule coweight. In $QH^*(G/P)$, we have*

$$\sigma^P(v_i) \times \sigma^P(w) = q_{\eta_P(\varpi_i^\vee - w^{-1}(\varpi_i^\vee))} \sigma^P(v_i w).$$

Example 6.11. Let G be of type A_1 , so that $G/B = \mathbb{P}^1$. Let s be the non trivial element of W and α the simple root. We have

$$\sigma^B(s) * (\sigma^B(s) - \alpha) = q.$$

Proof. Let i be the unique node of the Dynkin diagram of G . Then $v_i = s$. To apply Theorem 6.9, we also set $w = s$. Let x resp. y be the B -stable resp. B^- -stable point in \mathbb{P}^1 . The class $\sigma^B(s)$ is the T -equivariant class of x , and $v_i^* \sigma^B(s)$ is the T -equivariant class of y . Since $[x] - [y] = \alpha$, we have $v_i^* \sigma^B(s) = \sigma^B(s) - \alpha$. Denoting $h = \sigma^B(s)$, the theorem yields $h \times (h - \alpha) = q$, as claimed. Note that $h^2 = q + \alpha h$ is also predicted eg by [Mi07, Theorem 1]. □

7. Pieri formulas

We now give another application of Proposition 5.8 to prove a formula for $j(\xi_{\tau_i(v_i)})$, see Proposition 7.4. This gives the multiplication in $H_*^T(\Omega K)$ by the class $\xi_{\tau_i(v_i)}$. We hope in subsequent work to deduce Pieri formulas for the non-equivariant multiplication by classes generating $H_*(\Omega K)$ in all classical types.

We first provide a generalization of [Lam08, Proposition 5.4] to coweights. For $\mu^\vee \in P^\vee$, set $W_{\mu^\vee} = \langle s_{\alpha_i} \mid i \in [1, r] \text{ and } \langle \alpha_i, \mu^\vee \rangle = 0 \rangle = \{w \in W \mid w(\mu^\vee) = \mu^\vee\}$.

Proposition 7.1. *Let $\mu^\vee \in P^\vee$ be antidominant. Then*

$$j^{\text{ad}}(\xi_{t_{\mu^\vee}}) = \sum_{w \in W/W_{\mu^\vee}} \tilde{A}_{t_{w(\mu^\vee)}}.$$

Proof. We follow the idea of proof given in [Lam08, Proposition 5.4]. Using Lemma 4.8, we see that for $w \in W/W_{\mu^\vee}$ non trivial, $t_{w(\mu^\vee)} \notin \widetilde{W}_{\text{aff}}^-$, so that $A_{t_{w(\mu^\vee)}}$ belongs to the ideal $\sum_{x \in W \setminus \{e\}} \tilde{\mathbb{A}}_{\text{aff}} \cdot A_x$ of Proposition 5.8. Thus, using Proposition 5.8, we only need to prove that $\sum_{w \in W/W_{\mu^\vee}} A_{t_{w(\mu^\vee)}} \in Z_{\tilde{\mathbb{A}}_{\text{aff}}}(S)$.

To prove that $c := \sum_{w \in W/W_{\mu^\vee}} \tilde{A}_{t_{w(\mu^\vee)}}$ centralizes S , or equivalently commutes with any λ in Q , we use Proposition 4.6 to compute $\tilde{A}_{t_{\nu^\vee}} \lambda$. In this formula, the term $t_{\nu^\vee}(\lambda)$ is equal to λ by (2) in §2.2. Let \mathcal{P} be the set of pairs (ν^\vee, β) where $\nu^\vee \in W \cdot \mu^\vee$, β is a positive real root, and $t_{\nu^\vee} s_\beta \leq t_{\nu^\vee}$. We have:

$$(10) \quad c\lambda - \lambda c = \sum_{(\nu^\vee, \beta) \in \mathcal{P}} \langle \lambda, \beta^\vee \rangle \tilde{A}_{t_{\nu^\vee} s_\beta},$$

so our concern now is to prove that this sum vanishes.

We consider the map $\iota : \mathcal{P} \rightarrow \mathcal{P}$ defined by $\iota(\nu^\vee, \beta) = (s_\beta(\nu^\vee), -t_{\nu^\vee}(\beta))$. Let $(\nu^\vee, \beta) \in \mathcal{P}$. We have

$$\begin{aligned} t_{\nu^\vee} s_\beta &= t_{\nu^\vee} s_\beta t_{-t_{\nu^\vee}(\beta)} t_{\nu^\vee} &= s_{t_{\nu^\vee}(\beta)} t_{\nu^\vee} \\ & &= t_{s_{t_{\nu^\vee}(\beta)}(\nu^\vee)} s_{t_{\nu^\vee}(\beta)} &= t_{s_\beta(\nu^\vee)} s_{-t_{\nu^\vee}(\beta)}, \end{aligned}$$

where the last equality follows from (2) and Lemma 2.8. By the length formula in [CMP09, Corollary 3.13], $\ell(t_{s_\beta(\nu^\vee)}) = \ell(t_{\nu^\vee})$ and by definition of \mathcal{P} , $\ell(t_{\nu^\vee} s_\beta) = \ell(t_{\nu^\vee}) - 1$. Thus, $\ell(t_{s_\beta(\nu^\vee)} s_{-t_{\nu^\vee}(\beta)}) = \ell(t_{s_\beta(\nu^\vee)}) - 1$. Moreover, by [BB05, Proposition 4.4.6], $t_{\nu^\vee}(\beta) < 0$, which implies $t_{s_\beta(\nu^\vee)} s_{-t_{\nu^\vee}(\beta)} \leq t_{s_\beta(\nu^\vee)}$ and $-t_{\nu^\vee}(\beta) > 0$, so $(s_\beta(\nu^\vee), -t_{\nu^\vee}(\beta)) \in \mathcal{P}$ as claimed.

We also observe that $\langle \lambda, -t_{\nu^\vee}(\beta)^\vee \rangle = \langle \lambda, -\beta^\vee \rangle = -\langle \lambda, \beta^\vee \rangle$. Finally,

$$-t_{s_\beta(\nu^\vee)}(-t_{\nu^\vee}(\beta)) = t_{s_\beta(\nu^\vee)} t_{\nu^\vee}(\beta) = s_\beta t_{\nu^\vee} s_\beta t_{\nu^\vee}(\beta).$$

One can check that this root is equal to β , so that ι is an involution and the terms in (10) cancel pairwise. □

We now prove some preliminary lemmas.

Lemma 7.2. *Let $i \in I_{\text{aff}}$. We have $j^{\text{ad}}(\tilde{\xi}_{w_0(\varpi_i^\vee)}) = \delta_{\tau_i}^{-1} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}}$.*

Proof. Since $w_0(\varpi_i^\vee) \leq 0$ we may apply Proposition 7.1 and get

$$j^{\text{ad}}(\xi_{w_0(\varpi_i^\vee)}) = \sum_{\mu^\vee \in W \cdot w_0(\varpi_i^\vee)} A_{t_{\mu^\vee}}.$$

Thus,

$$\begin{aligned} j^{\text{ad}}(\xi_{w_0(\varpi_i^\vee)}) &= \sum_{\mu^\vee \in W \cdot \varpi_i^\vee} A_{t_{\mu^\vee}} = \sum_{w \leq_L v_i} A_{wt_{\varpi_i^\vee} w^{-1}} \\ &= \sum_{w \leq_L v_i} A_{w\tau_i^{-1}v_i w^{-1}} = \sum_{w \leq_L v_i} A_{\tau_i^{-1}\tau_i(w)v_i w^{-1}} \\ &= \delta_{\tau_i^{-1}} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}}. \end{aligned}$$

□

Lemma 7.3. *Let $s \in S$ and $i \in I_{\text{aff}}$. We have*

$$\left(\sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \right) s = \tau_i(s) \left(\sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \right).$$

Proof. Let $i \in I_{\text{aff}}$. Since $j^{\text{ad}}(\xi_{w_0(\varpi_i^\vee)}) = \delta_{\tau_i^{-1}} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}}$, we deduce that

$$\delta_{\tau_i^{-1}} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \in Z_{\mathbb{A}_{\text{aff}}}(S).$$

Let $s \in S$, we have:

$$\begin{aligned} \delta_{\tau_i^{-1}} \left(\sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \right) s &= s \delta_{\tau_i^{-1}} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \\ &= \delta_{\tau_i^{-1}} \tau_i(s) \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}}, \end{aligned}$$

which proves the lemma. □

Proposition 7.4. *Let $i \in I_{\text{aff}}$, let as above v_i the maximal element in W^{P_i} and τ_i the automorphism of the affine Dynkin diagram defined by i . Then $\tau_i(v_i) \in W_{\text{aff}}^-$ and we have:*

$$j(\xi_{\tau_i(v_i)}) = \sum_{w \leq_L v_i} \sum_{v \leq v_i^{-1}} \tau_i(\xi^v(v_i^{-1})) A_{\tau_i(w)v_i w^{-1}} A_v.$$

Proof. We first prove that $\tau_i(v_i) \in W_{\text{aff}}^-$. We know that $\tau_i(\alpha_i) = \alpha_0$. Since $v_i \in W^{P_i}$, we have for $1 \leq j \leq n$ with $j \neq i$, $\ell(v_i s_j) > \ell(v_i)$. Since $v_i \in W$, $\ell(v_i s_0) > \ell(v_i)$. Applying τ_i , we deduce that for all $k > 0$, $\ell(\tau_i(v_i) s_k) > \ell(\tau_i(v_i))$. Thus, $\tau_i(v_i) \in W_{\text{aff}}^-$.

Moreover, we know that $\tau_i = v_i t_{-\varpi_i^\vee}$. Therefore, $v_i = \tau_i t_{\varpi_i^\vee} = t_{w_0(\varpi_i^\vee)} \tau_i$, so that $t_{w_0(\varpi_i^\vee)} = v_i \tau_i^{-1} = \tau_i^{-1} \tau_i(v_i)$. By Proposition 5.9, we deduce that $j^{\text{ad}}(\xi_{t_{w_0(\varpi_i^\vee)}}) = \delta_{\tau_i^{-1}} j(\xi_{\tau_i(v_i)}) \delta_{v_i}$.

By Lemma 7.2, we deduce that

$$\delta_{\tau_i^{-1}} j(\xi_{\tau_i(v_i)}) \delta_{v_i} = \delta_{\tau_i^{-1}} \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}}.$$

Therefore, using Fact 5.1 and then Lemma 7.3, we find

$$\begin{aligned} j(\xi_{\tau_i(v_i)}) &= \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \delta_{v_i^{-1}} \\ &= \sum_{w \leq_L v_i} A_{\tau_i(w)v_i w^{-1}} \sum_{v \leq v_i^{-1}} \xi^v(v_i^{-1}) A_v \\ &= \sum_{w \leq_L v_i} \sum_{v \leq v_i^{-1}} \tau_i(\xi^v(v_i^{-1})) A_{\tau_i(w)v_i w^{-1}} A_v. \end{aligned}$$

□

Remark 7.5. Let $x \in W_{\text{aff}}^-$. In the non equivariant homology, we thus have

$$\xi_{\tau_i(v_i)} \cdot \xi_x = \sum \xi_{\tau_i(w)v_i w^{-1}x},$$

where the sum is over $w \leq_L v_i$ such that we have $\ell(\tau_i(w)v_i w^{-1}x) = \ell(v_i) + \ell(x)$ and $\tau_i(w)v_i w^{-1}x \in W_{\text{aff}}^-$.

By Corollary 5.15, we know that there is only one Schubert class in the product $\xi_{\tau_i(v_i)} \cdot \xi_x$, from which we deduce that there is exactly one $w \leq_L v_i$ such that $\ell(\tau_i(w)v_i w^{-1}x) = \ell(v_i) + \ell(x)$ and $\tau_i(w)v_i w^{-1}x \in W_{\text{aff}}^-$.

Example 7.6. Let us assume we are in type \tilde{A}_3 and let us write for short A_{210} instead of $A_{s_2 s_1 s_0}$ and similarly for ξ_{210} and δ_{210} . Let $i = 1$ so that $v_i = s_3 s_2 s_1$ and $\tau_i(v_i) = s_2 s_1 s_0$. First we observe that

$$\begin{aligned} \delta_{\tau_i^{-1}} &= \delta_{123} = (1 - \alpha_1 A_1)(1 - \alpha_2 A_2)(1 - \alpha_3 A_3) \\ &= 1 - \alpha_1 A_1 - (\alpha_1 + \alpha_2) A_2 - (\alpha_1 + \alpha_2 + \alpha_3) A_3 \\ &\quad + \alpha_1(\alpha_1 + \alpha_2) A_{12} + \alpha_1(\alpha_1 + \alpha_2 + \alpha_3) A_{13} \\ &\quad + (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3) A_{23} \\ &\quad - \alpha_1(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3) A_{123}. \end{aligned}$$

Since $\tau_i(\alpha_1) = -\theta = -(\alpha_1 + \alpha_2 + \alpha_3)$, $\tau_i(\alpha_2) = \alpha_1$ and $\tau_i(\alpha_3) = \alpha_2$, we get:

$$\begin{aligned}
 j(\xi_{210}) = & A_{210} + A_{321} + A_{032} + A_{103} \\
 & + \alpha_3(A_{2103} + A_{3213} + A_{0323}) \\
 & + (\alpha_2 + \alpha_3)(A_{2102} + A_{3212}A_{1032}) \\
 & + (\alpha_1 + \alpha_2 + \alpha_3)(A_{2101} + A_{0321} + A_{1031}) \\
 & + \alpha_3(\alpha_2 + \alpha_3)(A_{21023} + A_{32123} + A_{10323}) \\
 & + \alpha_3(\alpha_1 + \alpha_2 + \alpha_3)(A_{21031} + A_{03231}) \\
 & + (\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)(A_{21012} + A_{03212} + A_{10312}) \\
 & + \alpha_3(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)(A_{210123} + A_{032123} + A_{103123}).
 \end{aligned}$$

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