Math. Res. Lett.
Volume 30, Number 2, 375–412, 2023

Positive currents on non-kählerian surfaces

Ionuţ Chiose and Matei Toma

To the memory of Marco Brunella

We propose a classification of non-kählerian surfaces from a dynamical point of view and show how the known non-kählerian surfaces fit into it.

1. Introduction

Since Kodaira's foundational work on the classification of compact complex surfaces, non-kählerian surfaces have been a subject of interest for many complex geometers. Beside the elliptic non-kählerian surfaces and the Hopf surfaces which were studied by Kodaira, two further series of examples appeared in the seventies: the Inoue surfaces [Ino74] and the Kato surfaces [Kat77]. According to the Global Spherical Shell Conjecture [Nak84] these classes should exhaust all non-kählerian compact complex surfaces up to bimeromorphic equivalence. Some recent progress towards a solution of this conjecture was achieved by Andrei Teleman in [Tel05], [Tel10], [Tel18]. His approach is to study a certain moduli space of stable rank two vector bundles on a given surface X and deduce the existence of a compact analytic curve on X.

In this paper we look at objects on X of a different nature, namely at positive d-exact currents. It is known by [HL83] and [Lam99] that every non-kählerian surface admits non-trivial such currents. Extending our approach from [CT13] we introduce an invariant $I(T)$ of a positive d-exact current T on a non-kählerian compact complex surface and investigate its behaviour for the known classes of surfaces. This analysis leads us to a rough classification of non-kählerian surfaces into *parabolic* and *hyperbolic* surfaces, see Definition 3.5. Note that the commonly used invariants such as the Kodaira dimension, the algebraic dimension or the Kähler rank do not adapt well to the historical partition of non-kählerian surfaces into elliptic, Hopf, Inoue and Kato surfaces, or to Kodaira's partition into classes. (An example is Kodaira's class VII which was given a slightly restricted area in the monograph [BHPVdV04].) We show that the results of Marco Brunella's papers [Bru13b], [Bru13a], [Bru14] fit perfectly into our classification. These papers were a source of motivation for our investigation and we therefore dedicate this work to the memory of Marco Brunella.

We start by presenting some preliminary facts in Section 2 on positive pluriharmonic currents and on Green functions on compact complex surfaces. In Section 3 we propose a classification and show how the known classes of non-kählerian surfaces fit into it. This is followed by a short section presenting three conjectures inspired by this classification and by our previous work on the Kähler rank of surfaces [CT13]. The paper ends with an appendix on nef (pluri)closed currents and on the corresponding positive cones in Bott-Chern and in Aeppli cohomology.

2. Preparations

2.1. Positive pluriharmonic $(1,1)$-currents on non-kählerian surfaces

In this section X will always stand for a non-kählerian compact complex surface. It is known that any compact complex surface admits some Gauduchon metric, that is a hermitian metric whose associated Kähler form is $i\partial\bar\partial$-closed. We shall call such forms *Gauduchon forms* and we shall fix one

Gauduchon form ω on X. We use the following definition following Lamari, [Lam99].

Definition 2.1. *A* $(1,1)$*-current on* X *will be said to be* nef pluriharmonic *if it is a weak limit of positive* $i\partial\bar\partial$*-closed* $(1,1)$*-forms on* X *(or equivalently a weak limit of Gauduchon forms).*

Nef pluriharmonic currents are clearly positive and $i\partial\bar\partial$-closed. In the case of surfaces, extending the characterization of compact non-Kähler manifolds given by Harvey and Lawson in [HL83], Lamari shows that any non-kählerian surface admits some non-trivial nef pluriharmonic current which is d-exact, [Lam99, Theorem 7.1]. Since its evaluation on the Gauduchon form ω is positive, it follows that its Bott-Chern cohomology class is non-zero. Moreover, up to a positive multiplicative constant there is only one such class in $H^{1,1}_{BC}(X,\mathbb{R})$. In the sequel we shall denote by τ a smooth representative of such a class. We fix the class $\{\tau\}$ by requiring $\int_X \tau\wedge\omega = 1$.

Note also that the intersection form $H^{1,1}_{BC}(X,\mathbb{R}) \times H^{1,1}_{BC}(X,\mathbb{R}) \to \mathbb{R}$, $(\{\alpha\},\{\beta\}) \to \int_X \alpha\wedge\beta$ is negative semi-definite with totally isotropic space spanned by the class of τ, see Section 5. Therefore, since $H^{1,1}_{BC}(X,\mathbb{R})$ and $H^{1,1}_{A}(X,\mathbb{R})$ are dual under the natural pairing, it follows that the kernel of the morphism $j : H^{1,1}_{BC}(X,\mathbb{R}) \to H^{1,1}_{A}(X,\mathbb{R})$ is of dimension 1 and is generated by τ, and the cokernel is also of dimension 1, the image of j consisting of the classes that vanish on τ. Here we have denoted by $H^{1,1}_{BC}(X,\mathbb{R})$ and by $H^{1,1}_{A}(X,\mathbb{R})$ the corresponding Bott-Chern and Aeppli cohomology groups; see Section 5 for definitions and further facts on these topics.

Proposition 2.2. *Let* T *be a positive,* $\partial\bar\partial$*-closed* $(1,1)$*-current on* X*. Then* T *has a decomposition*

$$T = \sum_{j\in J} c_j[E_j] + T' \tag{1}$$

where J *is a countable set,* $c_j \geq 0$ *are non-negative real numbers,* E_j *are irreducible compact curves on* X *and* T' *is a nef pluriharmonic current.*

Proof. Given an irreducible compact curve $E \subset X$, Bassanelli in Theorem 4.10 in [Bas94] proved that $\chi_E T$ is a current of the form $f[E]$, where f is a weakly plurisubharmonic function on E. Since E is compact, it follows that f is a constant c, and, therefore, for a given irreducible compact curve $E \subset X$, the current T can be written as $c[E] + T'$, where T' is a positive pluriharmonic current such that $\chi_E T' = 0$.

If X is non-elliptic, then there are finitely many compact curves E_j on X and from Bassanelli's Theorem it follows that T can be written

$$T = \sum_j c_j[E_j] + T' \tag{2}$$

where $\chi_{E_j}T' = 0, \forall j$.

If X is elliptic, denote by $\mathcal{C}$ the set of all compact complex curves in X. If ω is a fixed Gauduchon form on X, then there exists $c > 0$ such that $\int_E \omega \geq c$, $\forall E \in \mathcal{C}$, see Remark 2.3. Now if $n \in \mathbb{N}$, denote by

$$\mathcal{C}_n = \left\{ E \in \mathcal{C} \middle| \chi_E T \geq \frac{1}{n}[E] \right\}$$

and by $E_n = \cup_{E\in\mathcal{C}_n} E$. We claim that $\mathcal{C}_n$ is finite. Indeed, we have $T \geq \chi_{E_n}T = \sum_{E\in\mathcal{C}_n} \chi_E T \geq \sum_{E\in\mathcal{C}_n} \frac{1}{n}[E]$ and therefore

$$\int_X \omega \wedge T \geq \sum_{E\in\mathcal{C}_n} \frac{1}{n} \int_E \omega \geq \frac{1}{n} \cdot c \cdot \operatorname{card} \mathcal{C}_n.$$

and this inequality proves the claim stated above. Here we used the fact that, if A and B are analytic subsets of pure dimension 1, such that $C := A \cap B$ has zero dimension, then $\chi_{A\cup B}T = \chi_A T + \chi_B T$. Indeed, $\chi_{A\cup B}T + \chi_C T = \chi_A T + \chi_B T$, and, since C is a finite set of points, it follows that $\chi_C T = 0$ [AB93] (see below for a more detailed proof of $\chi_C T = 0$).

Denote by T_n the d-closed current $\sum_{E\in\mathcal{C}_n} \chi_E T$. It is d-closed because each $\chi_E T$ is of the form $c[E]$, with c a non-negative constant. Clearly $\mathcal{C}_n \subset \mathcal{C}_{n+1}$, and therefore $T_{n+1} \geq T_n$. Denote by $\mathcal{C}_+ = \cup_n \mathcal{C}_n = \{E \in \mathcal{C} | \chi_E T \neq 0\}$. Since $\mathcal{C}_n$ are finite sets, it follows that $\mathcal{C}_+$ is a countable set $\{E_j | j \in \mathbb{N}\}$. Note that $\int_X \omega \wedge \sum_j \chi_{E_j} T \leq \int_X \omega \wedge T$. Indeed, if $\int_X \omega \wedge \sum_{j=1}^N \chi_{E_j} T > \int_X \omega \wedge T$, pick n so that $\{E_j | 1 \leq j \leq N\} \subset \mathcal{C}_n$, and then $\sum_{j=1}^N \chi_{E_j} T \leq T_n \leq T$, contradiction. Therefore $\sum_j \chi_{E_j} T$ is a closed positive current of the form $\sum_j c_j[E_j]$ and $\sum_j c_j[E_j] \leq T$.

The current $T' := T - \sum_j c_j[E_j]$ is a positive $i\partial\bar{\partial}$-closed current. From the construction of $\sum_j c_j[E_j]$, it follows that $\chi_E T' = 0$, $\forall E \in \mathcal{C}$. Indeed, if

$\chi_E T' = c[E]$ with $c > 0$, then from $T' \leq T$ it follows that $\chi_E T = d[E]$ with some $d > 0$, therefore E is in $\mathcal{C}_+$ and there exists $k \in \mathbb{N}$ such that $d = c_k$ but then $\chi_{E_k} T' = \chi_{E_k} T - \chi_{E_k} \sum_j c_j[E_j] = c_k[E_k] - \chi_{E_k} c_k[E_k] = c_k[E_k] - c_k[E_k] = 0$, contradiction.

Therefore, on any non-Kähler compact surface, the positive $i\partial\bar\partial$-closed $(1,1)$-currents admit a Siu decomposition.

We have to prove that T' is a nef pluriharmonic current, i.e., that it belongs to $\overline{\mathcal{G}}$, the weak closure of the cone of Gauduchon metrics $\mathcal{G}$ in $\mathcal{D}'^{1,1}(X, \mathbb{R})$ the space of $(1,1)$-currents.

Suppose that $T' \notin \overline{\mathcal{G}}$; then let $K = \{G \in \overline{\mathcal{G}} | \langle \omega, G \rangle = 1\}$ where ω is our fixed Gauduchon form and $L = \mathbb{R}T' \subset \mathcal{D}'^{1,1}(X, \mathbb{R})$. Since $L \cap K = \emptyset$, K is weakly compact and L is closed, they can be separated by a $\mathcal{C}^\infty$ $(1,1)$-form θ such that $\langle \theta, G \rangle \geq \varepsilon_0 > 0, \forall G \in K$ and $\langle \theta, G \rangle \leq 0, \forall G \in L$. We obtain $\langle \theta, T' \rangle = 0$ and further from Lemme 1.4 in [Lam99] applied to the form $\theta - \varepsilon_0 \omega$ that there exists φ a distribution such that

$$\theta + i\partial\bar\partial\varphi \geq \varepsilon_0 \omega. \tag{3}$$

It follows that φ is actually quasi-plurisubhamonic, and from Proposition 3.7 in [De92], we can approximate φ by another quasi-plurisubharmonic function φ' which has logarithmic poles (in particular the set $E_+ = \{x \in X | \nu(\varphi', x) > 0\}$ is an analytic subset of X), and such that

$$i\partial\bar\partial\varphi' \geq \frac{\varepsilon_0}{2}\omega - \theta. \tag{4}$$

Note that $\chi_{E_+} T' = 0$. Indeed, $\chi_C T' = 0$ for any irreducible compact curve in X, and, if Y is a finite set in X, then the fact that $\chi_Y T' = 0$ follows from [AB93]: denote by $(T')^\circ$ the simple extension of $T'|_{X \setminus Y}$. Its existence is guaranteed by Theorem 5.4 in [AB93]. It is a positive pluriharmonic current (see also Remark 5.5 in [AB93]). Then $\chi_Y T' = T' - (T')^\circ$ is positive, pluriharmonic and supported in Y. From Theorem 5.1 in [AB93] it follows that $\chi_Y T' = 0$.

Apply Proposition 3.1 in [Lam99] with $T = T'$, $\chi = \varphi'$, $\alpha = 0$, $Y = E_+$, $\eta = 0$ and

$$\gamma = \frac{\varepsilon_0}{2}\omega - \theta. \tag{5}$$

It follows that

$$0 = \langle 0, T' \rangle \geq \frac{\varepsilon_0}{2}\langle \omega, T' \rangle - \langle \theta, T' \rangle = \frac{\varepsilon_0}{2}\langle \omega, T' \rangle \tag{6}$$

hence $T' = 0$, contradiction. □

In the above proof we made use of the following

Remark 2.3. *If (X, ω) is an n-dimensional compact complex manifold endowed with a hermitian metric, then there is a constant $c > 0$ such that for any positive (non-zero) k-cycle E on X we have*

$$\int_E \omega^k \geq c.$$

Indeed, the void cycle, which we denote by 0, is an isolated point in the cycle space $\mathcal{C}_k(X)$ of k-dimensional cycles on X, [BM14, Remarque after Lemme IV.2.2.3], the volume function with respect to ω is continuous on the cycle space, [BM14, Proposition IV.2.3.1], and the set of all cycles whose volume is bounded from above by some constant M is compact, [BM14, Théorème IV.2.7.20], which is a consequence of Bishop's Theorem. Thus the minimum of the volume function restricted to $\mathcal{C}_k(X) \setminus \{0\}$ is attained and is non-zero.

Proposition 2.4. *Let T be a positive $i\partial\bar{\partial}$-closed $(1,1)$-current such that $\int_X \tau \wedge T = 0$. Then T is closed. If, moreover, T is nef pluriharmonic, then it is d-exact.*

Proof. Since $\int_X \tau \wedge T = 0$ and τ is d-exact, it follows that $\int_X \tau \wedge T' = 0$, where T' is the nef pluriharmonic current that appears in Proposition 2.2. Thus T' is a weak limit of Gauduchon forms $T' = \lim \omega_n$. We noted above that the natural morphism $H^{1,1}_{BC}(X, \mathbb{R}) \to H^{1,1}_{A}(X, \mathbb{R})$ has a 1-dimensional kernel, therefore its image has codimension 1. So $H^{1,1}_{A}(X, \mathbb{R})$ is generated by the class of ω and the image of $H^{1,1}_{BC}(X, \mathbb{R})$ and therefore each ω_n can be written

$$\omega_n = \varepsilon_n \omega + \alpha_n + \partial \bar{\sigma}_n + \bar{\partial} \sigma_n, \tag{7}$$

where

$$\varepsilon_n = \int_X \tau \wedge \omega_n \to \int_X \tau \wedge T' = 0, \tag{8}$$

α_n are d-closed $(1,1)$-forms, and σ_n are $(1,0)$-forms. Indeed,

$$\int_X (\omega_n - \varepsilon_n \omega) \wedge \tau = 0$$

and, as explained above, it follows that the class of $\omega_n - \varepsilon_n\omega$ in $H_A^{1,1}(X,\mathbb{R})$ is in the image of the natural morphism $j : H_{BC}^{1,1}(X,\mathbb{R}) \to H_A^{1,1}(X,\mathbb{R})$, see also the Appendix.

Then

$$
\begin{aligned}
(9) \qquad 0 \geq \int_X \alpha_n^2 &= \int_X (\alpha_n + d(\sigma_n + \bar{\sigma}_n))^2 \\
&= \int_X (\omega_n - \varepsilon_n\omega + \partial\sigma_n + \bar{\partial}\bar{\sigma}_n)^2 \\
&= \int_X (\omega_n - \varepsilon_n\omega)^2 + 2\int_X \partial\sigma_n \wedge \bar{\partial}\bar{\sigma}_n \\
&= \int_X \omega_n^2 - 2\varepsilon_n \int_X \omega_n \wedge \omega + \varepsilon_n^2 \int_X \omega^2 + 2\int_X \partial\sigma_n \wedge \bar{\partial}\bar{\sigma}_n \\
&\geq -2\varepsilon_n \int_X \omega_n \wedge \omega + \varepsilon_n^2 \int_X \omega^2 + 2\int_X \partial\sigma_n \wedge \bar{\partial}\bar{\sigma}_n.
\end{aligned}
$$

Since $\int_X \omega_n \wedge \omega \to \int_X T' \wedge \omega$ and $\varepsilon_n \to 0$, it follows that

$$(10) \qquad \int_X \partial\sigma_n \wedge \bar{\partial}\bar{\sigma}_n \to 0$$

and therefore $\partial\sigma_n \to 0$ strongly in L^2, in particular as currents. So from (7)

$$(11) \qquad \partial T' = \lim \partial\omega_n = \lim(\varepsilon_n \partial\omega + \partial\bar{\partial}\sigma_n) = -\lim \bar{\partial}\partial\sigma_n = 0,$$

therefore T' is closed and hence T is closed as well.

If T is nef pluriharmonic and d-closed, let α be a $\mathcal{C}^\infty$ representative of T in the Bott-Chern cohomology class of T, i.e., $T = \alpha + i\partial\bar{\partial}\varphi$ where φ is a quasi-plurisubharmonic function on X. If $T = \lim \omega_n$, where ω_n are Gauduchon forms, then

$$(12) \qquad 0 \geq \int_X \alpha^2 = \lim \int_X \alpha \wedge \omega_n = \lim \int_X T \wedge \omega_n \geq 0$$

so $\int_X \alpha^2 = 0$ and α is d-exact and therefore T is d-exact. We have used the fact that the intersection form on $H_{BC}^{1,1}(X,\mathbb{R})$ is negative semi-definite with totally isotropic space spanned by the class of τ. □

2.2. Positive exact $(1,1)$-currents in $L^2_{-1}(X)$

We shall denote by $L^2(X)$ and by $L^2_{-1}(X)$ spaces of currents with coefficients in the corresponding spaces of functions without making their degrees precise. A closed positive current of bidegree $(1,1)$ is in $L^2_{-1}(X)$ if it admits local $\partial\bar{\partial}$-potentials which are square integrable along with their gradients.

Bedford and Taylor defined in [BT78] the self intersection of a closed positive $(1,1)$ current T in $L^2_{-1}(X)$ as follows: if $T = i\partial\bar{\partial}u$ on some open subset U of X and if ψ is a test function on U, then

$$< T \wedge T, \psi >:= \int \psi T \wedge T := -\int i\partial\bar{\partial}\psi \wedge i\partial u \wedge \bar{\partial} u.$$

A direct computation shows that this definition does not depend on the chosen $i\partial\bar{\partial}$-potential u and the definition is extended by linearity to define a current on X. By [BT78, Theorem 3.6] $T \wedge T$ is a positive $(2,2)$-current on X. This may also be seen in the following way. Let Ω be an open subset of $\mathbb{C}^2$. For a plurisubharmonic function u in $L^2_1(\Omega)$ we define a distribution $MA(u)$ on Ω by setting

$$MA(u)(\psi) := -\int i\partial\bar{\partial}\psi \wedge i\partial u \wedge \bar{\partial} u. \tag{13}$$

We regularize u in the usual way by means of a sequence of regularizing kernels $(\rho_\epsilon)_\epsilon$ converging to the Dirac distribution. The sequence of functions $u_\epsilon := u \star \rho_\epsilon$ decreases towards u. The functions u_ϵ are in $\mathcal{C}^\infty(\Omega)$ and plurisubharmonic on the smaller open sets Ω_ϵ. By the Meyers-Serrin theorem we also have $\lim_{\epsilon\to 0} u_\epsilon = u$ in $L^2_1(\Omega)$. Thus if ψ is a test function on Ω, then $\mathrm{Supp}(\psi) \subset \Omega_\epsilon$ for $0 < \epsilon << 1$ and $\lim_{\epsilon\to 0} MA(u_\epsilon)(\psi) = MA(u)(\psi)$ and on the other hand

$$MA(u_\epsilon)(\psi) := -\int_\Omega i\partial\bar{\partial}\psi \wedge i\partial u_\epsilon \wedge \bar{\partial} u_\epsilon = \int_\Omega \psi (i\partial\bar{\partial} u_\epsilon)^2$$

which will be positive if ψ is positive.

From the degeneration of the Frölicher spectral sequence at the E_1-level for compact complex surfaces [BHPVdV04, Theorem IV.2.8], it follows that if T is an exact positive $(1,1)$ current, then there exists a bidegree $(0,1)$ current S such that $T = \partial S$ and $\bar{\partial} S = 0$ (see Lemma 12.1 p.42 in [BHPVdV04]). (In fact it is not difficult to check that any bidegree $(0,1)$ current S' with $\partial S' = T$ has the property $\bar{\partial} S' = 0$.) We investigate the situation when T is in $L^2_{-1}(X)$.

Proposition 2.5. *Let T be a positive d-exact current of bidegree $(1,1)$ in $L^2_{-1}(X)$ and let $T = \partial S$ for some bidegree $(0,1)$-current S. Then S is in $L^2(X)$, $i\bar{S}\wedge S$ is $i\partial\bar{\partial}$-closed, and $\chi_Y i\bar{S}\wedge S = 0$ for any compact proper analytic subset Y of X. In particular, $i\bar{S}\wedge S$ is a nef pluriharmonic current.*

Moreover the value of the integral

$$\int_X \tau \wedge i\bar{S}\wedge S$$

depends only on T and not on the chosen potential S.

Proof. Note that if T is in $L^2_{-1}(X)$, then the ellipticity of ∂ on $(0,1)$ forms implies that S is in $L^2(X)$, hence $i\bar{S}\wedge S$ is well-defined and is a $(1,1)$-current with L^1 coefficients.

Locally we may write $T = i\partial\bar{\partial}u$ and $S = i\bar{\partial}u$. It follows that: $\int \psi T\wedge T = -\int i\partial\bar{\partial}\psi \wedge i\bar{S}\wedge S$ for any $\mathcal{C}^\infty$ function ψ on X and in particular estimating on $\psi = 1$ one gets $T\wedge T = 0$ and $i\partial\bar{\partial}(i\bar{S}\wedge S) = 0$.

If $\dim Y = 0$, the statement on the vanishing of $\chi_Y i\bar{S}\wedge S$ follows from [AB93] as in the proof of Proposition 2.2. If $\dim Y = 1$, the statement follows from the fact that $i\bar{S}\wedge S$ has L^1 coefficients, and an L^1-function cannot dominate a Dirac measure. Indeed, if $\chi_Y i\bar{S}\wedge S = c[Y]$ with $c > 0$, then $i\bar{S}\wedge S \geq c[Y]$ and if $(\varphi_n)_n$ is a sequence of positive functions that converges pointwise to χ_Y, then, from the dominated convergence theorem, it follows that

$$\int_X i\bar{S}\wedge S\wedge \varphi_n\omega \to 0$$

while $\int_Y \varphi_n\omega \to \int_Y \omega > 0$. Therefore the inequality $i\bar{S}\wedge S \geq c[Y]$ cannot be true with $c > 0$.

If S_1, S_2 are two primitive currents for T as above, then $\eta := \bar{S}_1 - \bar{S}_2$ is a holomorphic 1-form on X. If this form is non-zero then $i\eta\wedge\bar{\eta}$ is a non-trivial closed positive $(1,1)$-form such that $\int_X (i\eta\wedge\bar{\eta})^2 = 0$ hence as remarked in Section 2.1 $\{\tau\} = c\{i\eta\wedge\bar{\eta}\} \in H^{1,1}_{BC}(X,\mathbb{R})$ for some positive constant c. Thus

$$\int_X \tau\wedge i\bar{S}_1\wedge S_1 = c\int_X i\eta\wedge\bar{\eta}\wedge i(\bar{S}_2+\eta)\wedge(S_2+\bar{\eta}) = \int_X \tau\wedge i\bar{S}_2\wedge S_2.$$

□

Definition 2.6. *Under the above assumptions, i.e. for a positive d-exact current $T = \partial S$ of bidegree $(1,1)$ in $L^2_{-1}(X)$, we define a linear form $I(T)$*

on $Ker(H^{1,1}_{BC}(X,\mathbb{R}) \to H^{1,1}_A(X,\mathbb{R}))$ *by setting*

$$I(T)(\alpha) := \int_X \alpha \wedge i\bar{S} \wedge S.$$

Note that the linear form $I(T)$ is determined by its value on τ, which is exactly the integral appearing in Proposition 2.5.

2.3. Green functions

To our knowledge the notion of Green function for a non-kählerian surface appears first in the paper [DO99]. It was further used in [Bru13a] and in [Bru14].

Definition 2.7. *We say that a compact complex surface* X *admits a* Green function *if there exist a* $\mathbb{Z}$*-covering* $\pi : X' \to X$*, a divisor* $D \geq 0$ *on* X *and a negative plurisubharmonic function* $G : X' \to]-\infty, 0[$ *which is multiplicatively automorphic on* X' *and pluriharmonic on* $X' \setminus \pi^{-1}(D)$*. Being* multiplicatively automorphic *for* G *means that if* $g \in Aut(X')$ *generates the deck transformation group of* $\pi : X' \to X$*, there exists a positive constant* k *such that* $G \circ g = kG$*. We will always implicitely assume that Green functions are non-trivial in the sense that* X' *is connected and that* $k \neq 1$*. By interchanging* g *and* g^{-1} *we may further assume that* $k < 1$*.*

Proposition 2.8. *If* (π, D, G) *is a data system defining a Green function on a compact complex surface* X *and if* $u := -\log(-G)$*, then the following assertions hold:*

1) u *is plurisubharmonic and* additively automorphic*. The additive automorphy for* u *means that* $u \circ g = u + p$*, where* $p := -\log k$*.*
2) $i\partial\bar{\partial}u$ *has trivial automorphy and it defines a non-trivial exact positive current on* X*. In particular* X *is non-kählerian.*
3) X *is non-elliptic.*
4) $i\partial\bar{\partial}G = \sum_j a_j[D_j]$*, where* D_j *are the irreducible components of* $\pi^{-1}(D)$ *and* a_j *are non-negative constants.*
5) u *is in* $L^2_{1,loc}(X')$ *and*
$$i\partial\bar{\partial}u = i\partial u \wedge \bar{\partial}u.$$
6) $i\partial\bar{\partial}u$ *is in* $L^2_{-1}(X)$ *and* $I(i\partial\bar{\partial}u) = 0$*.*

7) *For any continuous p-periodic function* $h : \mathbb{R} \to \mathbb{R}$ *satisfying* $1 + h' + h'' \geq 0$ *as distributions, the function* $v := u + h \circ u$*, understood as being* $-\infty$ *on the polar locus of* u*, is plurisubharmonic, additively automorphic and defines an exact positive* $(1,1)$*-current* $T := i\partial\bar{\partial}v \in L^2_{-1}(X)$ *with* $I(T) = 0$*. Moreover, such a function* v *satisfies* $i\partial\bar{\partial}v = i\partial v \wedge \bar{\partial}v$ *if and only if it equals* u *up to an additive constant.*

Proof. The function $\psi :]-\infty, 0[\to \mathbb{R}, t \mapsto -\log(-t)$ is convex and increasing hence u is plurisubharmonic. The assertions on the additive automorphic behaviour and on the fact that $i\partial\bar{\partial}u$ descends to a non-trivial exact positive current on X are clear. (Note for later use that ∂u and $\bar{\partial}u$ also descend to X.)

Suppose now by contradiction that X is elliptic with elliptic fibration $f : X \to B$ over a compact complex curve B. By Liouville's theorem it follows that G is constant on the connected components of the general fibers of $f \circ \pi : X' \to B$. Indeed, Liouville's theorem says that any upper bounded subharmonic function on $\mathbb{C}$ is constant. In our case the connected components of the fibers of $f \circ \pi : X' \to B$ are compact or else isomorphic to $\mathbb{C}$ or to $\mathbb{C}^*$. Note also that a negative subharmonic function on $\mathbb{C}^*$ which has non-trivial multiplicative automorphy has a pole at one of the ends of $\mathbb{C}^*$ and extends to $\mathbb{C}$ as a subharmonic function. Thus, by the automorphic behaviour of G, the connected components of these general fibers are elliptic curves and π factorizes through a $\mathbb{Z}$-covering $\pi' : B' \to B$ of the base and a proper elliptic fibration $f' : X' \to B'$. Clearly G and u descend then to plurisubharmonic functions on B' with the corresponding automorphic behaviour. But as above this contradicts the fact that B is Kähler.

Thus X is non-kählerian of algebraic dimension zero and the considerations in [Bru14, pp. 252-253] apply to show that $\pi^{-1}(D)$ is a divisor with simple normal crossings and that $i\partial\bar{\partial}G = \sum_j a_j[D_j]$.

We now look at $u := \psi \circ G$. From [Bło09, Theorem 1] it follows that u is in $L^2_{1,loc}$. We now show that $i\partial\bar{\partial}u = i\partial u \wedge \bar{\partial}u$. Since $i\partial\bar{\partial}G = \sum_j a_j[D_j]$ and $\pi^{-1}(D)$ has simple normal crossings, we can assume that locally around of point of $\pi^{-1}(D)$, the Green function G can be written as $G_0(z_1, z_2) = 2a_1 \log|z_1| + 2a_2 \log|z_2|$, where $a_1, a_2 \geq 0$ and $a_1 + a_2 > 0$, where G_0 is defined on the polydisc $\{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$. Indeed, if there is a difference, $G - G_0$, between the two functions then it must be pluriharmonic and thus of the form $2\Re f$ for some holomorphic function f. But then we may change the coordinate function z_1, say, to $z_1 e^{\frac{f}{a_1}}$ and obtain the desired local expression for G. We consider next the smooth plurisubharmonic functions $G_\varepsilon(z_1, z_2) = a_1 \log(|z_1|^2 + \varepsilon) + a_2 \log(|z_2|^2 + \varepsilon)$, $\varepsilon > 0$. Clearly, $(G_\varepsilon)_{\varepsilon>0}$ is a family of functions that decreases to G. Denote by $u_\varepsilon = -\log(-G_\varepsilon)$ on $\left\{(z_1, z_2) \| z_1| \leq \frac{1}{2}, |z_2| \leq \frac{1}{2}\right\}$. Since $(u_\varepsilon)_{\varepsilon>0}$ decreases to u when $\varepsilon \to 0$, it follows that $(i\partial\bar{\partial}u_\varepsilon)_{\varepsilon>0}$ converges to $i\partial\bar{\partial}u$ weakly. At this point one may infer from [Ceg07] that $\partial u_\varepsilon \to \partial u$ in L^2. However in our case one can see this also by a local direct calculation as follows. Away from D one gets

$$\partial u_\varepsilon = -\frac{\partial G_\varepsilon}{G_\varepsilon} = -a_1 \frac{\bar{z}_1 \mathrm{d}z_1}{(|z_1|^2+\varepsilon)G_\varepsilon} - a_2 \frac{\bar{z}_2 \mathrm{d}z_2}{(|z_2|^2+\varepsilon)G_\varepsilon}$$

and when $a_1 \neq 0$ it is therefore enough to check that the sequence $\left(\frac{\bar{z}_1}{(|z_1|^2+\varepsilon)G_\varepsilon}\right)_\varepsilon$ converges towards $\frac{\bar{z}_1}{|z_1|^2 G_0}$ in L^2 locally around 0, the other component being similar. We have

$$\left|\frac{\bar{z}_1}{(|z_1|^2+\varepsilon)G_\varepsilon} - \frac{\bar{z}_1}{|z_1|^2 G_0}\right| \leq \frac{|\bar{z}_1|}{|G_0|}\left(\frac{1}{|z_1|^2} - \frac{1}{|z_1|^2+\varepsilon}\right) + \frac{|\bar{z}_1|}{|z_1|^2+\varepsilon}\left(\frac{1}{-G_\varepsilon} - \frac{1}{-G_0}\right).$$

Now the first term squared gives

$$\frac{\varepsilon^2}{|z_1|^2(|z_1|^2+\varepsilon)^2 G_0^2}$$

which is bounded by

$$\frac{\varepsilon^2}{a_1^2|z_1|^2(|z_1|^2+\varepsilon)^2\log^2(|z_1|^2)}$$

which in turn leads to an integral of the form $\int_0^b \frac{\varepsilon^2}{t(t+\varepsilon)^2\log^2 t}\mathrm{d}t$ for some fixed $b \in]0,1[$. The integrands are uniformly bounded by the function $\frac{1}{t\log^2 t}$ which is integrable on $]0,b[$ and we conclude by Lebesgue's dominated convergence theorem for this term. We now turn our attention to the second term. We have

$$\begin{aligned}
\frac{|\bar{z}_1|}{|z_1|^2+\varepsilon}\left(\frac{1}{-G_\varepsilon} - \frac{1}{-G_0}\right) &= \frac{|\bar{z}_1|}{|z_1|^2+\varepsilon}\cdot\frac{G_\varepsilon - G_0}{G_\varepsilon G_0} \\
&= \frac{|\bar{z}_1|}{|z_1|^2+\varepsilon}\cdot\frac{a_1\log(1+\frac{\varepsilon}{|z_1|^2}) + a_2\log(1+\frac{\varepsilon}{|z_2|^2})}{G_0 G_\varepsilon} \\
&\leq \frac{1}{|z_1|(-\log|z_1|^2)} + \frac{1}{|z_1|(-\log|z_1|^2)}\cdot C\log\left(1+\frac{\varepsilon}{|z_2|^2}\right)
\end{aligned}$$

for some constant C not depending on ε, where for the first term we used the inequalities

$$\frac{|\bar z_1| a_1 \log\left(1+\frac{\varepsilon}{|z_1|^2}\right)}{(|z_1|^2+\varepsilon)G_0G_\varepsilon} \le \frac{|\bar z_1| a_1 \frac{\varepsilon}{|z_1|^2}}{(|z_1|^2+\varepsilon)(-a_1\log|z_1|^2)(-a_1\log(|z_1|^2+\varepsilon))} \le \frac{1}{|z_1|(-\log|z_1|^2)}$$

which hold for z and ε small. We may again apply Lebesgue's dominated convergence theorem since the bounding functions are square-integrable.

It follows that $i\partial u_\varepsilon \wedge \bar\partial u_\varepsilon \to i\partial u \wedge \bar\partial u$ in L^1. It remains only to prove that $i\partial\bar\partial u_\varepsilon - i\partial u_\varepsilon \wedge \bar\partial u_\varepsilon \to 0$ weakly.

Straight forward computations show that

$$\begin{aligned} i\partial\bar\partial u_\varepsilon - i\partial u_\varepsilon \wedge \bar\partial u_\varepsilon &= -\frac{1}{G_\varepsilon} i\partial\bar\partial G_\varepsilon \\ &= -\frac{1}{G_\varepsilon}\left(a_1\frac{\varepsilon}{(|z_1|^2+\varepsilon)^2} idz_1\wedge d\bar z_1 + a_2\frac{\varepsilon}{(|z_2|^2+\varepsilon)^2} idz_2\wedge d\bar z_2\right)\end{aligned}$$

and for $\varepsilon < \frac12$, $|z_1| < \frac12$, $|z_2| < \frac12$ we have $G_\varepsilon \le a_1\log(|z_1|^2+\varepsilon)$ and $G_\varepsilon \le a_2\log(|z_2|^2+\varepsilon)$, hence

$$\begin{aligned} 0 &\le i\partial\bar\partial u_\varepsilon - i\partial u_\varepsilon \wedge \bar\partial u_\varepsilon \\ &\le -\frac{1}{\log(|z_1|^2+\varepsilon)}\cdot\frac{\varepsilon}{(|z_1|^2+\varepsilon)^2} idz_1\wedge d\bar z_1 \\ &\quad -\frac{1}{\log(|z_2|^2+\varepsilon)}\cdot\frac{\varepsilon}{(|z_2|^2+\varepsilon)^2} idz_2\wedge d\bar z_2 \end{aligned}$$

therefore, in order to show that $i\partial\bar\partial u_\varepsilon - i\partial u_\varepsilon \wedge \bar\partial u_\varepsilon \to 0$ weakly, it is enough to show that $\int_{B(0,\frac12)} -\frac{\varepsilon}{\log(|z|^2+\varepsilon)(|z|^2+\varepsilon)^2} dz d\bar z \to 0$ when $\varepsilon\to 0$, $\varepsilon > 0$. A polar change of coordinates leads to the condition $\lim_{\varepsilon\to 0}\int_0^{\frac12} -\frac{\varepsilon r}{\log(r^2+\varepsilon)(r^2+\varepsilon)^2} dr = 0$ and the substitution $s = r^2+\varepsilon$ to $\lim_{\varepsilon\to 0}\int_\varepsilon^{\frac14+\varepsilon} -\frac{\varepsilon}{\log s\cdot s^2} ds = 0$. Since $\varepsilon < \frac12$, it follows that

$$\int_\varepsilon^{\frac14+\varepsilon} -\frac{\varepsilon}{\log s\cdot s^2} ds < \int_\varepsilon^{\frac34} -\frac{\varepsilon}{\log s\cdot s^2} ds$$

and if n is any positive integer and $\varepsilon < e^{-n}$, we write

$$\int_{\varepsilon}^{\frac{3}{4}} -\frac{\varepsilon}{\log s \cdot s^2} ds = \int_{e^{-n}}^{\frac{3}{4}} -\frac{\varepsilon}{\log s \cdot s^2} ds + \int_{\varepsilon}^{e^{-n}} -\frac{\varepsilon}{\log s \cdot s^2} ds.$$

Now

$$\int_{\varepsilon}^{e^{-n}} -\frac{\varepsilon}{\log s \cdot s^2} ds \leq \int_{\varepsilon}^{e^{-n}} \frac{\varepsilon}{n \cdot s^2} \mathrm{d}s = \frac{1}{n} - \frac{\varepsilon e^n}{n} < \frac{1}{n}$$

and if we choose ε such that furthermore $\int_{e^{-n}}^{\frac{3}{4}} -\frac{\varepsilon}{\log s \cdot s^2} ds < \frac{1}{n}$, we obtain that $\int_{\varepsilon}^{\frac{3}{4}} -\frac{\varepsilon}{\log s \cdot s^2} ds < \frac{2}{n}$. Therefore

$$\lim_{\varepsilon \to 0} \int_{B(0,\frac{1}{2})} -\frac{\varepsilon}{\log(|z|^2+\varepsilon)(|z|^2+\varepsilon)^2} dz d\bar{z} = 0$$

and $i\partial\bar{\partial}u = i\partial u \wedge \bar{\partial}u$. Thus $i\partial u \wedge \bar{\partial}u$ is d-exact and hence $I(i\partial\bar{\partial}u) = 0$.

Let finally h be a p-periodic function satisfying $1 + h' + h'' \geq 0$ as distributions and let $v := u + h \circ u$. Away from the poles of u we have $i\partial\bar{\partial}v = ((1 + h' + h'') \circ u) i\partial\bar{\partial}u$ and the plurisubharmonicity of v here is a consequence of our assumption on h. By the mean value inequality v is plurisubharmonic around the poles of u as well. Since h is continuous and periodic it will be bounded by some positive constant C and we get $v \geq u - C$. Thus the singularities of v are no worse than those of u, by [Bło04, Theorem 3.3]. (Note that the negativity assumption present in [Bło04, Theorem 3.3] may be achieved locally by substracting some positive constant since plurisubharmonic functions are by definition upper semicontinuous and hence are locally bounded from above.)

We now check that $I(T) = 0$. For this, note first that the condition $1 + h' + h'' \geq 0$ is equivalent to $(e^t + e^t h')' \geq 0$ and thus we may define an increasing function $f : [0, \infty[\to \mathbb{R}$ by $f(x) := (e^t + e^t h')'([0, x])$, since $(e^t + e^t h')'$ is a positive measure. The function f is the (essentially unique) primitive of the positive mass $(e^t + e^t h')'$ on $[0, \infty[$. Since f is increasing, it is measurable and locally bounded. Taking into account periodicity one obtains that the distribution h' is represented by an L^∞ function. Thus we can write

$$i\partial v \wedge \bar{\partial}v = (1 + h' \circ u)^2 i\partial u \wedge \bar{\partial}u. \tag{14}$$

We shall exhibit a positive constant μ and a continuous p-periodic function g on $\mathbb{R}$ such that

$$(1 + h' \circ u)^2 i\partial u \wedge \bar{\partial} u = i\partial\bar{\partial}(\mu u + g \circ u), \tag{15}$$

from which we immediately see that $i\partial v \wedge \bar{\partial} v$ is d-closed and hence $I(T) = 0$. Put $H := (1 + h')^2$, $\mu := \frac{1}{p}\int_0^p H(s)\mathrm{d}s$, $C := \frac{1}{e^p - 1}\int_0^p (e^s H(s) - e^s\mu)\mathrm{d}s$ and $g(t) := \int_0^t (H(s) - \mu)\mathrm{d}s - e^{-t}\int_0^t (e^s H(s) - e^s\mu)\mathrm{d}s + C(1 - e^{-t})$. We next check that μ and g fulfill the desired conditions.

Since H is in L^∞ we get that μ, C and g are well defined and that g is continuous. To see that g is p-periodic we compute

$$\begin{aligned}
g(t+p) - g(t) &= \int_t^{t+p} (H(s) - \mu)\mathrm{d}s - e^{-t-p}\int_0^{t+p} (e^s H(s) - e^s\mu)\mathrm{d}s \\
&\quad + e^{-t}\int_0^t (e^s H(s) - e^s\mu)\mathrm{d}s + C(e^{-t} - e^{-t-p}) \\
&= \int_0^p (H(s) - \mu)\mathrm{d}s - e^{-t-p}\int_0^p (e^s H(s) - e^s\mu)\mathrm{d}s \\
&\quad - e^{-t-p}\int_p^{t+p} (e^s H(s) - e^s\mu)\mathrm{d}s \\
&\quad + e^{-t}\int_0^t (e^s H(s) - e^s\mu)\mathrm{d}s + Ce^{-t}(1 - e^{-p}) \\
&= -e^{-t-p}(e^p - 1)C - e^{-t-p}\int_0^t (e^{s'+p}H(s') - e^{s'+p}\mu)\mathrm{d}s' \\
&\quad + e^{-t}\int_0^t (e^s H(s) - e^s\mu)\mathrm{d}s + Ce^{-t}(1 - e^{-p}) \\
&= 0.
\end{aligned}$$

Moreover, direct computation also shows that as distributions

$$\begin{aligned}
g'(t) &= e^{-t}\int_0^t (e^s H(s) - e^s\mu)\mathrm{d}s + Ce^{-t} \\
&= -g(t) + \int_0^t (H(s) - \mu)\mathrm{d}s + C, \\
g''(t) &= g(t) - \int_0^t (H(s) - \mu)\mathrm{d}s + H(t) - C - \mu
\end{aligned}$$

and

$$g'' + g' + \mu = H.$$

In particular g' and g'' are represented by continuous and L^∞ functions, respectively. The equation (15) is established now by developping its right-hand member:

$$\begin{aligned} i\partial\bar\partial(\mu u + g \circ u) &= \mu i\partial\bar\partial u + i\partial((g' \circ u)\bar\partial u) \\ &= \mu i\partial\bar\partial u + (g'' \circ u) i\partial u \wedge \bar\partial u + (g' \circ u) i\partial\bar\partial u \\ &= (H \circ u) i\partial u \wedge \bar\partial u. \end{aligned}$$

Let finally $v = u + h \circ u$ be as above and such that $i\partial\bar\partial v = i\partial v \wedge \bar\partial v$. We shall show that in this case h must be constant. Put $\tilde f := e^t + e^t h'$. We have seen that it is an increasing function. Using the equation (14), the condition $i\partial\bar\partial v = i\partial v \wedge \bar\partial v$ on v may be reformulated as

$$\tilde f' = e^{-t} \tilde f^2,$$

which in particular implies that $\tilde f$ is differentiable and is either identically zero or vanishes nowhere. But the first situation cannot occur by the periodicity of h', so we may find $\tilde f$ by integrating

$$\frac{\tilde f'}{\tilde f^2} = e^{-t}.$$

We deduce

$$-\frac{1}{1 + h'} = -1 + ce^t$$

for some constant c which has to vanish by periodicity of h' again. Thus $h' = 0$, proving that h is constant as stated. □

In fact it will follow from the work of Brunella in [Bru13a], [Bru14] and from our Proposition 3.6 that if X admits a Green function then all exact positive $(1, 1)$-currents on X are up to a multiplicative factor of the form $i\partial\bar\partial v$ for an additively automorphic function v as above, see Corollary 3.9.

3. Classification of non-kählerian surfaces from a dynamical point of view

3.1. The known classes of non-kählerian surfaces

Recall that a compact complex surface is called minimal if it does not appear as a blowing up of a point of another smooth complex surface. Any compact complex surface X appears as the result of a finite sequence of blow-ups of points on a minimal surface Y. In this case Y is called a minimal model of X. Thus the classification of compact complex surfaces reduces itself to the biholomorphic classification of minimal models. Moreover a surface is kählerian if and only if it admits a kählerian minimal model. Note that any non-kählerian surface admits a unique minimal model [BHPVdV04, Theorem VI.1.1].

We also recall that the algebraic dimension $a(X)$ of a non-kählerian surface X can equal 0 or 1. It is 1 if and only if X is an elliptic surface. When $a(X) = 0$ there exist only finitely many compact complex curves on X. In this case we will denote by D_{max} the maximal effective reduced divisor on X. For these facts and more on compact complex surfaces we refer the reader to the monograph [BHPVdV04].

The known minimal non-kählerian surfaces may be divided into the following classes:

1) minimal elliptic non-kählerian surfaces,
2) non-elliptic Hopf surfaces,
3) Inoue surfaces,
4) Kato surfaces.

We will say that a non-kählerian surface is *unknown* if its minimal model does not belong to one of these classes.

Here we will give a short description of each class; see [Nak84] for a detailed exposition.

3.1.1. Minimal elliptic non-kählerian surfaces. These are by definition minimal surfaces X with odd first Betti number, admitting a fibration $\pi : X \to Y$ with elliptic general fibers onto a curve Y. It can be shown [Brî96, Proposition 3.17] that in this case the fibration π is a *quasi-bundle*, i.e. all its smooth fibers are pairwise isomorphic and its singular fibers are multiples of smooth elliptic curves. From loc. cit. it also follows that $h^{1,0}(X) = h^{1,0}(Y)$, i.e. all holomorphic 1-forms on X are pull-backs of holomorphic 1-forms on Y, see also the proof of the next proposition.

Proposition 3.1. *If X is a minimal elliptic non-kählerian surface, then the following assertions hold:*

1) *Every positive divisor D on X is a positive combination with rational coefficients of fibers of π and is homologically trivial over $\mathbb{Q}$. In particular there exist exact positive $(1,1)$-currents on X not in L^2_{-1}.*
2) *All exact positive $(1,1)$-currents T which are in L^2_{-1} necessarily have $I(T) \neq 0$.*

Proof. Let D be an irreducible curve on X and let F be a general fiber of the elliptic fibration $\pi : X \to Y$. If D is not contained in a fiber of π, then the self-intersection number of the divisor $D + nF$ will be positive for n large, implying that X is projective, cf. [BHPVdV04, Theorem IV.6.2]. Thus D must be contained in a fiber and, since $\pi : X \to Y$ is a quasi-bundle, a multiple of D is homologically equivalent to F and thus D is homologically trivial over $\mathbb{Q}$.

Let now $T = i\partial S$ be an exact positive $(1,1)$-current on X, with S a $(0,1)$-current with coefficients in $L^2(X)$. Let ω_Y a volume form on Y. Then $\omega_X := \pi^*\omega_Y$ is positive non-trivial and such that $\omega_X \wedge \omega_X = 0$. Thus $\{\omega_X\} = c\{\tau\} \in H^{1,1}_{BC}(X, \mathbb{R})$ for some positive real number c. Suppose that

$$0 = I(T)(\omega_X) := \int i\bar{S} \wedge S \wedge \omega_X. \tag{16}$$

We shall show that $T = 0$.

Let Y° be the set of regular values of π and set $X^\circ := \pi^{-1}(Y^\circ)$. We will begin by working on X°. Since $\pi : X \to Y$ is a quasi-bundle it follows that the fibration $\pi^\circ : X^\circ \to Y^\circ$ is locally trivial over Y°. For such a local trivialization we choose local coordinates (z, w) on X° where z is a local coordinate on Y° and w is a coordinate for the fiber direction. The formula (16) implies that $S = f\mathrm{d}\bar{z}$ where f is locally in L^2 on X°. Since T is real and $T = i\partial S$ we also get $\frac{\partial f}{\partial w} = 0$ as distributions. Since $\bar{\partial} S = 0$ we further get $\frac{\partial f}{\partial \bar{w}} = 0$. Thus the distribution f is independent of the w coordinate and it follows that f is a tensor product of the function 1 in the vertical direction with an L^2_{loc}-function f° on Y°, cf. [Sch66, IV.5.Exemple 1]. Setting $R^\circ = f^\circ \mathrm{d}\bar{z}$ on Y° we may say that S "comes from R° from the base", meaning by this that S is the tensor power of the function 1 in fiber direction with R° in horizontal direction. The form R° has coefficients in $L^2_{loc}(Y^\circ)$. Moreover,

T "comes from $i\partial R^\circ$ from the base", in particular $i\partial R^\circ$ is a positive $(1,1)$-current on Y°. We shall next show that it admits an extension to Y as a positive exact $(1,1)$-current. From this it will follow that $i\partial R^\circ = 0$.

We look at the situation around a singular fiber of π over some critical value $y_0 \in Y$. By [BHPVdV04, Proposition III.9.1 and p.207] we know that over a small neighbourhood V of y_0 in Y the restriction X_V of X may be seen as the quotient $p : \mathbb{T} \times \mathbb{D} \to X_V$ of $\mathbb{T} \times \mathbb{D}$ by the action of $\mathbb{Z}/n\mathbb{Z}$ generated by $(w, z) \mapsto (w + 1/n, \rho z)$ where $\mathbb{T}$ is a one dimensional complex torus given as $\mathbb{C}/\Lambda$, Λ is the lattice generated by 1 and some $\alpha \in \mathbb{H}$, and $\rho = \exp(\frac{2i\pi}{n})$. Supposing that V is biholomorphic to $\mathbb{D}$ we thus get a commutative diagram

$$\begin{array}{ccc} \mathbb{T} \times \mathbb{D} & \xrightarrow{p} & X_V \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \pi} \\ \mathbb{D} & \xrightarrow{\phi} & V, \end{array}$$

where $\phi(z) = z^n$. Note that p is an unramified covering map. Let ω be a parallel volume form on $\mathbb{T}$ with $\int_{\mathbb{T}} \omega = 1$. Then $\Omega := p_*((pr_1)^*\omega)$ is a closed positive $(1,1)$-form on X_V such that over $V^* := V \setminus \{y_0\}$ one has $\pi_*(\Omega_{X_{V^*}}) = 1_{V^*}$. Since it can be considered on the whole Y, this extension must be trivial. Then the currents $\pi_*(S \wedge \Omega)$ and $\pi_*(T \wedge \Omega)$ extend the currents R° and respectively $i\partial R^\circ$ over y_0 on V and $\pi_*(T \wedge \Omega) = i\partial\pi_*(S \wedge \Omega)$. Thus the current $i\partial R^\circ$ extends as a positive exact current on Y and is therefore trivial. Thus T itself is trivial on X°. But then T is concentrated on a finite number of fibers of π. Unless $T = 0$ this contradicts the assumption $T \in L^2_{-1}(X)$ and the proof is finished. □

3.1.2. Non-elliptic Hopf surfaces. A compact complex surface X is said to be a *Hopf surface* if its universal covering space is isomorphic to $\mathbb{C}^2 \setminus \{0\}$. A Hopf surface is called *primary* if its fundamental group is infinite cyclic, and *secondary* otherwise. The following facts on Hopf surfaces X and much more were shown by Kodaira in [Kod66]:

1) If X is a primary Hopf surface, then its fundamental group is generated by a *contraction* $g : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ which for suitable global holomorphic coordinates (z_1, z_2) on $\mathbb{C}^2$ has the following *normal form*

$$g(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \tag{17}$$

where $m \in \mathbb{Z}_{>0}$, $\alpha_1, \alpha_2, \lambda \in \mathbb{C}$ and

$$(\alpha_1 - \alpha_2^m)\lambda = 0,\ 0 < |\alpha_1| \le |\alpha_2| < 1.$$

2) A primary Hopf surface $X = (\mathbb{C}^2 \setminus \{0\})/\langle g \rangle$ with g as above is elliptic if and only if $\lambda = 0$ and $\alpha_1^{k_1} = \alpha_2^{k_2}$ for some positive integers k_1, k_2.

3) If $X = (\mathbb{C}^2 \setminus \{0\})/\pi_1(X)$ is a non-elliptic secondary Hopf surface then its fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/l\mathbb{Z})$ where the direct factor $\mathbb{Z}$ is generated by a contraction g of the form (17) and the finite cyclic group $\mathbb{Z}/l\mathbb{Z}$ is generated by an automorphism of $\mathbb{C}^2 \setminus \{0\}$ of the form

$$(z_1, z_2) \mapsto (\epsilon_1 z_1, \epsilon_2 z_2),$$

where ϵ_1, ϵ_2 are primitive l-th roots of unity satisfying.

$$(\epsilon_1 - \epsilon_2^m)\lambda = 0.$$

In particular X admits a finite unramified cyclic covering by the primary Hopf surface $(\mathbb{C}^2 \setminus \{0\})/\langle g \rangle$.

4) $b_1(X) = 1$ and $b_2(X) = 0$.

5) Non-elliptic Hopf surfaces contain one or at most two irreducible compact curves according to whether $\lambda \neq 0$ or $\lambda = 0$, for λ as in equation (17). These curves are elliptic.

In their study of closed positive $(1,1)$-currents on compact complex surfaces done in [HL83], Harvey and Lawson subdivide non-elliptic primary Hopf surfaces into two classes. Their definitions are immediately extended to secondary non-elliptic Hopf surfaces too as follows:

1) *Class* 1 contains those non-elliptic Hopf surfaces for which the coefficient λ in the above formulas vanishes. (Thus this class contains exactly those Hopf surfaces admitting precisely two elliptic curves.)

2) *Class* 0 contains those non-elliptic Hopf surfaces for which $\lambda \neq 0$. (These are the Hopf surfaces containing only one elliptic curve.)

Proposition 3.2. 1) *Up to a non-negative factor there exists exactly one closed positive* $(1,1)$*-current on a non-elliptic Hopf surface of class* 0*. This is the integration current along the elliptic curve of the surface.*

2) *Every non-elliptic Hopf surface* X *of class* 1 *admits non-trivial closed positive* $(1,1)$*-currents* T *in* $L^2_{-1}(X)$ *and for such currents one always has* $I(T) \neq 0$.

Note that on a Hopf surface X closed positive $(1,1)$-currents are exact since $b_2(X) = 0$, so in the above statement $I(T)$ is well defined.

Proof. The assertion on Hopf surface of class 0 was proved in [HL83, Theorem 69] for primary Hopf surfaces. The case of the secondary Hopf surfaces immediately follows from this by pull-back and push-forward through the finite covering map $(\mathbb{C}^2 \setminus \{0\})/\langle g\rangle \to (\mathbb{C}^2 \setminus \{0\})/(\mathbb{Z} \times (\mathbb{Z}/l\mathbb{Z}))$.

In the same way it will be enough to establish the second assertion only for primary non-elliptic Hopf surfaces of class 1. Let X be such a surface given by a contraction g of the form

$$g(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2),$$

with $0 < |\alpha_1| \leq |\alpha_2| < 1$. The existence of non-trivial closed positive $(1,1)$-currents in $L^2_{-1}(X)$ follows from [HL83, Theorem 58], where it is even proved that smooth such currents exist. More precisely in [HL83] Harvey and Lawson consider the following objects on $\mathbb{C}^2 \setminus \{0\}$ some of which obviously descend to X. Set

$$\begin{gathered} r = \frac{\log|\alpha_1|}{\log|\alpha_2|}, \\ \phi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R},\ \phi(z_1, z_2) := \log(|z_1|^2 + |z_2|^{2r}), \\ \eta := z_2 \mathrm{d}z_1 - r z_1 \mathrm{d}z_2. \\ \Omega := i\partial\bar{\partial}\phi = \frac{|z_2|^{2(r-1)}}{(|z_1|^2 + |z_2|^{2r})^2} i\eta \wedge \bar{\eta}, \\ V := r z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \\ \pi : X \to [0,1],\ \pi(z_1, z_2) := \frac{|z_1|^2}{|z_1|^2 + |z_2|^{2r}}. \end{gathered}$$

It is said in [HL83] that the form Ω is smooth on X but this might not be the case around the elliptic curve $E_1 := \{z_2 = 0\}$ when $r \notin \mathbb{N}$. To remedy to this one may consider

$$\begin{gathered} r' = \frac{1}{r}, \\ \phi' : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R},\ \phi'(z_1, z_2) := \log(|z_2|^2 + |z_1|^{2r'}), \\ \eta' := z_1 \mathrm{d}z_2 - r' z_2 \mathrm{d}z_1 = -r'\eta. \\ \Omega' := i\partial\bar{\partial}\phi' = \frac{|z_1|^{2(r'-1)}}{(|z_2|^2 + |z_1|^{2r'})^2} i\eta' \wedge \bar{\eta}', \\ \pi' : X \to [0,1],\ \pi'(z_1, z_2) := \frac{|z_2|^2}{|z_2|^2 + |z_1|^{2r'}}, \end{gathered}$$

and

$$\tilde{\Omega} := (\psi \circ \pi)\Omega + (\psi \circ \pi')\Omega',$$

where $\psi : [0,1] \to [0,1]$ is smooth and equals 1 in a neighbourhood of 0 and 0 in a neighbourhood of 1. Then $\tilde{\Omega}$ is a smooth positive d-closed $(1,1)$-form on X without zeroes on X. The d-closedness of $\tilde{\Omega}$ follows from the fact that $\partial\pi$ and $\partial\pi'$ are proportional to η on $X \setminus E_1$ and on $X \setminus E_2$, respectively.

Note further that the holomorphic vector field V defines a holomorphic foliation $\mathcal{F}$ on X, which coincides with the complex foliation defined by $\tilde{\Omega}$ (i.e. the foliation whose leaves are tangent to $\ker(\tilde{\Omega})$).

Let now T be a non-trivial closed positive $(1,1)$-current in $L^2_{-1}(X)$. By [Tom08, Proposition 4] there exists an additively automorphic $i\partial\bar{\partial}$-potential u of T in $L^2_{1,loc}(\mathbb{C}^2 \setminus \{0\})$. Supposing by contradiction that $I(T) = 0$, we infer that $i\partial u \wedge \bar{\partial} u \wedge \tilde{\Omega} = 0$ on X. This implies that $\partial u \wedge \tilde{\Omega} = 0$ and $\bar{\partial} u \wedge \tilde{\Omega} = 0$. Indeed, since at any point $x \in X$ the $(1,1)$-form $\tilde{\Omega}$ has rank one, it may be written locally as $\tilde{\Omega} = i\beta \wedge \bar{\beta}$ for some $(1,0)$-form β and thus $0 = i\partial u \wedge \bar{\partial} u \wedge i\beta \wedge \bar{\beta} = \partial u \wedge \beta \wedge \bar{\partial} u \wedge \beta = a\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \bar{a}\mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2 = |a|^2 i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge i\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2$, where $a\mathrm{d}z_1 \wedge \mathrm{d}z_2$ is the expression of the $(2,0)$-form $\partial u \wedge \beta$ in local coordinates around x. Hence $\partial u \wedge \beta = 0$ and thus the L^2-forms $\partial u \wedge \tilde{\Omega}$ and $\bar{\partial} u \wedge \tilde{\Omega}$ vanish identically. In fact since the foliation $\mathcal{F}$ is holomorphic, we may express $\tilde{\Omega}$ as $\tilde{\Omega} = if(\zeta_1, \zeta_2)\mathrm{d}\zeta_1 \wedge \mathrm{d}\bar{\zeta_1}$ for suitable local holomorphic coordinates (ζ_1, ζ_2). It follows that $\frac{\partial u}{\partial \zeta_2} = 0$, $\frac{\partial u}{\partial \bar{\zeta}_2} = 0$, so u is a function depending locally only on the ζ_1 variable. In particular the restriction of u to those leaves of $\mathcal{F}$ not contained in the polar set of u is constant. We now show that this implies that u has trivial additive automorphy and hence that $T = 0$. For suppose by contradiction that $u \circ g = u - p$ with $p > 0$ and let z be a point in $\mathbb{C}^* \times \mathbb{C}^*$ with $u(z) \neq -\infty$. Then $u(g(z)) = u(z) - p$ and by the upper semi-continuity of u its values in a neighbourhood of $g(z)$ cannot be larger than $u(z) - \frac{p}{2}$, say, on one hand. On the other hand it is shown in [HL83, Lemma 54] and its proof that the leaves of the foliation $\tilde{\mathcal{F}}$ pulled-back from $\mathcal{F}$ to $\mathbb{C}^2 \setminus \{0\}$ are dense in the fibers of $\pi \circ g$. Thus the leaf of $\tilde{\mathcal{F}}$ passing through z comes arbitrarily close to $g(z)$. Since u is constant equal to $u(z)$ on this leaf we get a contradiction. □

3.1.3. Inoue surfaces. In this paper by an *Inoue surface* we mean a compact complex surface X with $b_1(X) = 1$, $b_2(X) = 0$ and no compact complex curves. The construction of Inoue surfaces appears in [Ino74] and their classification was completed in [Tel94] and in [LYZ94]. Their universal cover is $\mathbb{H} \times \mathbb{C}$ and their fundamental group is generated by four affine transformations g_0, g_1, g_2, g_3 in such a way that $\pi_1(X)$ appears as a semidirect product

$\Gamma \rtimes \langle g_0 \rangle$ of Γ by $\langle g_0 \rangle$, where Γ is the subgroup generated by g_1, g_2, g_3, and g_0 acts on $\mathbb{H} \times \mathbb{C}$ by

$$g_0(w, z) = (\alpha w, \beta z + t),$$

for some positive real number $\alpha < 1$ and suitable complex numbers β and t. Moreover for $i = 1, 2, 3$ the elements g_i act on $\mathbb{H} \times \mathbb{C}$ by

$$g_i(w, z) = (w + a_i, z + b_i w + c_i),$$

for some real numbers a_i b_i and complex numbers c_i, see [Ino74]. Here w and z denote complex coordinates on $\mathbb{H}$ and on $\mathbb{C}$ respectively. Thus the quotient group $\pi_1(X)/\Gamma$ is infinite cyclic generated by the class $\hat{g}_0$ of g_0, defines a $\mathbb{Z}$-covering $\pi : X' \to X$ of X and the function $y := \Im m(w)$ defined on $\mathbb{H} \times \mathbb{C}$ descends to a function $\hat{y} : X' \to \mathbb{R}$.

Proposition 3.3. *If X is an Inoue surface, then under the above notations putting $G := -\hat{y}$ we get a Green function $G : X' \to \mathbb{R}$ without poles on X'. Moreover if $u := -\log(-G)$ and $p := -\log\alpha$, then, up to a multiplicative factor, any non-trivial closed positive $(1,1)$-current T on X is of the form $T = i\partial\bar{\partial}v$, where $v := u + h \circ u$ for some continuous p-periodic function $h : \mathbb{R} \to \mathbb{R}$ satisfying $1 + h' + h'' \geq 0$ as distributions. All such currents are in $L^2_{-1}(X)$ and have $I(T) = 0$.*

Proof. The fact that G is a Green function without poles is clear. By [HL83, Theorem 82] every closed positive $(1,1)$-current T on X is of the form $T = (\phi \circ u)i\partial\bar{\partial}u$, where ϕ is a positive p-periodic generalized function on $\mathbb{R}$. We may see ϕ as a p-periodic (positive) measure on $\mathbb{R}$ and we may assume that $\phi(]0, p]) = p$. In order to find the desired function h it suffices to solve the equation

$$1 + h' + h'' = \phi$$

on $\mathbb{R}$. For this, remark first that $1 + h' + h'' = \phi$ is equivalent to $(e^t + e^t h')' = e^t\phi$. Integrating once gives us a right-continuous increasing function $f : \mathbb{R} \to \mathbb{R}$ of bounded variation such that $f(x) := (e^t\phi)(]0, x])$ for all $x \in \mathbb{R}$, [oMA20]. We will now obtain the desired function h by integrating the equation $e^t + e^t h' = f + C$ for a suitable real constant C. The continuity of the resulting function h being clear, we only need to check that a constant C may be found so that the function $e^{-t}(f + C)$ become p-periodic, from which the periodicity of h will follow. The p-periodicity of $e^{-t}(f + C)$ means by definition that for all $x \in \mathbb{R}$ we have $e^{-x-p}(f(x + p) + C) - e^{-x}(f(x) + C) = 0$, which is equivalent to $e^{-p}f(x+p)-f(x)=C(1-e^{-p})$. But by periodicity of ϕ we have $e^{-p}f(x+p)-f(x)=e^{-p}(e^t\phi)(]0,x+p])-(e^t\phi)(]0,x])=e^{-p}(e^t\phi)(]0,p])+e^{-p}(e^t\phi)(]p,x+p])-(e^t\phi)(]0,x])=e^{-p}(e^t\phi)(]0,p])=e^{-p}f(p)$ for all $x\in\mathbb{R}$. By choosing now C such that $e^{-p}f(p)=C(1-e^{-p})$ we obtain the desired continuous p-periodic function h.

The assertion on the regularity of T and on $I(T)$ follows now from Proposition 2.8. □

3.1.4. Kato surfaces. A *Kato surface* is a minimal surface X with $b_1(X)=1$, $b_2(X)>0$ and admitting a *global spherical shell*, that is an open neighbourhood Σ of the 3-dimensional sphere S^3 in $\mathbb{C}^2\setminus\{0\}$ holomorphically embedded in X and such that $X\setminus\Sigma$ is connected. Their construction is due to Masahide Kato, [Kat77], and their properties have been studied by many authors.

Any Kato surface X admits exactly $b_2(X)$ rational curves. Conversely, if a minimal non-kählerian surface X admits $b_2(X)$ rational curves, then X is a Kato surface.

The class of Kato surfaces contains subclasses of previously constructed surfaces known as parabolic Inoue surfaces [Ino75] and Inoue-Hirzebruch surfaces, also called hyperbolic Inoue surfaces [Ino77]. We will not use the terminology "‘parabolic Inoue"’ and "‘hyperbolic Inoue"’in order not to create confusion with the already described class of Inoue surfaces. The reader may consult [Nak84] for an account of these surfaces. We prefer instead to consider the following subclassification of Kato surfaces:

1) *Enoki surfaces*, which are non-kählerian compactifications of affine line bunles over elliptic curves by cycles D of rational curves. Enoki shows that these surfaces are Kato surfaces, that $(D^2)=0$, and that, conversely, any minimal surface with $b_1=1$, $b_2>0$ and with a non-trivial divisor D with $(D^2)=0$ is in this subclass, [Eno81].
2) *Inoue-Hirzebruch surfaces*, which are Kato surfaces whose rational curves are organized into one or two homologically non-trivial cycles.
3) *Intermediate Kato surfaces*, which are Kato surfaces whose divisor of rational curves is a cycle with at least one branch attached.

Proposition 3.4. 1) *On an Enoki surface there exists exactly one exact positive* $(1,1)$*-current up to a positive multiplicative factor. This is the integration current along the reduced divisor of rational curves of the surface.*

2) *If X is an Inoue-Hirzebruch surface or an intermediate Kato surface, then X admits a Green function G. Moreover if $u := -\log(-G)$ is the associated additively automorphic plurisubharmonic function with $u \circ g = u + p$ and $\langle g \rangle = \pi_1(X)$, then up to a multiplicative factor any non-trivial exact positive $(1,1)$-current T on X is of the form $T = i\partial\bar{\partial}v$, where $v := u + h \circ u$ for some continuous p-periodic function $h : \mathbb{R} \to \mathbb{R}$ satisfying $1 + h' + h'' \geq 0$ as distributions. All such currents are in $L^2_{-1}(X)$ and have $I(T) = 0$.*

Proof. The first statement is part of [Tom08, Theorem 10]. The existence of Green functions on intermediate Kato and on Inoue-Hirzebruch surfaces was shown in [DO99]. A complete description of the exact positive $(1,1)$-currents on these surfaces was given in [Tom08, Theorem 11, Theorem 12]. □

3.2. Hyperbolic and parabolic non-kählerian surfaces

The next definition divides the known classes of non-kählerian surfaces into two groups: *parabolic* surfaces and *hyperbolic* ones. We will then show that members of each of these groups have many properties in common. One may speculate as to which of these properties are better suited to approach the Global Spherical Shell Conjecture.

Definition 3.5. *A non-kählerian compact complex surface X will be said to be* parabolic *if its minimal model belongs to one of the classes: Hopf surfaces, Enoki surfaces, non-kählerian elliptic surfaces. It will be said to be* hyperbolic *if its minimal model is either an Inoue surface, an Inoue-Hirzebruch surface, or an intermediate Kato surface.*

This terminology is first used by Inoue in the particular cases of the examples of non-kählerian surfaces that he constructs in [Ino75] and in [Ino77]. Note that non-kählerian surfaces that are hyperbolic in the above sense are not hyperbolic according to the standard terminology used in complex geometry, as they have many entire curves.

Proposition 3.6. *If X is a hyperbolic non-kählerian surface, then the following assertions hold:*

1) *X admits a Green function G such that the function $u = \psi(G) := -\log(-G)$ is in $L^2_{1,loc}$ and*

$$i\partial\bar{\partial}u = i\partial u \wedge \bar{\partial}u.$$

2) *All exact positive* $(1,1)$*-currents* T *are of the form* $T = \lambda i\partial\bar{\partial}(u + h \circ u)$ *with* $\lambda \geq 0$*,* $h : \mathbb{R} \to \mathbb{R}$ *a continuous* p*-periodic function satisfying* $1 + h' + h'' \geq 0$ *and* p *the automorphy summand of* u *as in Proposition 2.8. In particular all these currents are in* L^2_{-1} *and have* $I(T) = 0$*.*

3) *The only homologically trivial divisor on* X *is* 0*.*

4) $\widetilde{X \setminus D}_{max} \cong \mathbb{D} \times \mathbb{C}$*.*

Proof. The assertions on the Green functions and on the exact positive currents follow from Propositions 2.8, 3.3 and 3.4.

The assertion on the homologically trivial divisors follows from the knowledge of the structure of the reduced divisor of curves on these surfaces, cf. [Nak84].

Finally the facts on the universal cover of $X \setminus D_{max}$ are established in [Ino74], [Ino77] and [DOT03, Theorem 3.7]. □

Proposition 3.7. *If* X *is a parabolic non-kählerian surface, then the following assertions hold:*

1) X *admits no Green function.*

2) *All exact positive* $(1,1)$*-currents* T *in* $L^2_{-1}(X)$ *necessarily have* $I(T) \neq 0$*.*

3) *There exist homologically trivial divisors* D *on* X *with* $D > 0$*, and in particular there exist exact positive* $(1,1)$*-currents on* X *not in* L^2_{-1}*.*

4) *If the algebraic dimension of* X *is zero, then* $\widetilde{X \setminus D_{max}} \cong \mathbb{C}^2$ *and in particular there exists no divisor* D *on* X *such that* $\widetilde{X \setminus D} \cong \mathbb{D} \times \mathbb{C}$ *in this case.*

Proof. If a non-kählerian surface X admits a Green function then X is non-elliptic by Proposition 2.8 and thus of algebraic dimension zero. In this case it is shown by Brunella in [Bru13a] and [Bru14] that X is necessarily hyperbolic. In particular, parabolic surfaces will not admit Green functions.

The assertions on the exact positive currents and on the homologically trivial divisors follow from Propositions 3.1, 3.2 and 3.4.

Finally, for the two classes of parabolic surfaces of algebraic dimension zero, namely for non-elliptic Hopf surfaces and for Enoki surfaces, it follows almost from the definition that the universal cover of the complement of the union of compact complex curves is isomorphic to $\mathbb{C}^2$. □

For a compact complex surface X we consider the following properties:

- (P1) For every non-trivial divisor D on X one has $D^2 < 0$.
- (P2) All exact positive $(1,1)$-currents on X have L^2_{-1}-coefficients.
- (P3) All exact positive currents T on X with L^2_{-1}-coefficients have $I(T) = 0$
- (P4) X admits a Green function.
- (P5) $a(X) = 0$ and there exists an effective divisor D on X such that the universal cover of $X \setminus D$ is biholomorphic to $\mathbb{D} \times \mathbb{C}$.

Theorem 3.8. *Let X be a non-kählerian compact complex surface. Then the following hold:*

1) *If X has (P1), it is either a hyperbolic or an uknown non-kählerian surface. If X does not have (P1), it is a parabolic surface.*
2) *If X has (P2), it is either a hyperbolic or an uknown non-kählerian surface. If X does not have (P2), it is either a parabolic or an uknown non-kählerian surface.*
3) *If X has (P3), it is either a hyperbolic or an uknown non-kählerian surface. If X does not have (P3), it is either a parabolic or an uknown non-kählerian surface.*
4) *If X has (P4), it is a hyperbolic surface. If X does not have (P4), it is either a parabolic or an uknown non-kählerian surface.*
5) *If X has (P5), it is a hyperbolic surface. If X does not have (P5), it is either a parabolic or an uknown non-kählerian surface.*

The statement is summarized by the following table.

Criterion C	X satisfying C	X not satisfying C
$(D^2) < 0 \quad \forall D \in \mathrm{Div}(X) \setminus \{0\}$	hyperbolic, ?	parabolic
all positive exact $(1,1)$-currents on X are in L^2_{-1}	hyperbolic, ?	parabolic, ?
all positive exact currents $T \in L^2_{-1}(X)$ have $I(T) = 0$	hyperbolic, ?	parabolic, ?
X admits a Green function	hyperbolic	parabolic, ?
$a(X) = 0$ and $\exists D$ with $\widetilde{X \setminus D} \cong \mathbb{D} \times \mathbb{C}$	hyperbolic	parabolic, ?

The question marks signal that possibly not yet known surfaces may respond to the corresponding criteria.

Proof. The presence of hyperbolic and parabolic surfaces at the indicated places of the table is a consequence of the Propositions 3.6 and 3.7. We are left only with the task of explaining the absence of question marks at three places of the table.

The fact that parabolic surfaces are the only compact complex surfaces admitting homologically trivial divisors D with $D > 0$ is due to Enoki, [Eno81].

Non-kählerian non-elliptic surfaces admitting Green functions have been shown to be hyperbolic by Brunella in [Bru13a] and [Bru14]. The case of elliptic surfaces is settled by Proposition 2.8.

Finally, it is again Brunella who proved in [Bru13b] that hyperbolic surfaces are the only non-kählerian non-elliptic surfaces whose complement of the maximal divisor of curves is uniformized by $\mathbb{D} \times \mathbb{C}$. □

Combining the Theorem and Proposition 3.6 one immediately gets the following

Corollary 3.9. *If X admits a Green function G and if $u := -\log(-G)$ is the associated additively automorphic plurisubharmonic function with $u \circ g = u + p$ and $\langle g \rangle = \pi_1(X)$, then up to a multiplicative factor any non-trivial exact positive $(1,1)$-current T on X is of the form $T = i\partial\bar{\partial}v$, where $v := u + h \circ u$ for some continuous p-periodic function $h : \mathbb{R} \to \mathbb{R}$ satisfying $1 + h' + h'' \geq 0$ as distributions.*

4. Perspectives

In this section we wish to briefly discuss a number of conjectures and questions related to the degree of regularity of the d-closed positive $(1,1)$-currents on compact non-kählerian surfaces. The leading idea is the same which guided our approach to the study of the Kähler rank of surfaces in [CT13]. In that paper we worked under the assumption that a non-trivial positive smooth exact $(1,1)$-current T exists on a compact complex surface X and we aimed at a classification by distinguishing two cases according to whether $I(T)$ vanishes or not. In the second case we showed that X was necessarily elliptic or Hopf of class 1. In the first case we proved that X admitted a Green function without poles. This case was afterwards completely settled by Brunella in [Bru13a], who showed that such Green functions were only supported by Inoue surfaces. Trying to extend this type of strategy and in view of the striking similarities exhibited by Theorem 3.8 for the surfaces which are hyperbolic or respectively parabolic we are led to the following conjectures.

Conjecture 4.1. *If X is a non-kählerian surface all of whose exact positive $(1,1)$-currents T are in $L^2_{-1}(X)$ and satisfy $I(T) = 0$, then X admits a Green function, and in particular X is hyperbolic.*

Conjecture 4.2. *If X is a non-kählerian surface all of whose exact positive $(1,1)$-currents T are in $L^2_{-1}(X)$ but do not all satisfy $I(T)=0$, then X admits a cycle of rational curves.*

Conjecture 4.3. *If X is a non-kählerian surface admitting an exact positive $(1,1)$-current not in $L^2_{-1}(X)$, then there exists on X some exact positive current T with a non-vanishing Lelong number at at least one point of X.*

Note that in this case the surface X would be parabolic. Indeed, for such a current the Lelong-Siu level sets $E_c(T)$ cannot all be zero-dimensional, by [Tel08, Theorem A.1]. Thus supposing that X is non-elliptic, we would get $T=[C]+R$ with C a curve and R is residual, and R would be d-closed and nef pluriharmonic by Proposition 2.2 and thus d-exact. Therefore $[C]$ would also be d-exact, which implies the parabolicity of X.

5. Appendix

Since the notion of nef pluriharmonic current is not used frequently in the literature we present here some properties relating it to the more common notion of nef class.

As before, also in this section we denote by X a compact non-kählerian surface.

Let $\mathcal{E}^{p,q}$ and $\mathcal{D}'^{p,q}$ be the sheaves of germs of smooth (p,q)-forms and respectively of bidegree (p,q)-currents on X. We will write $\mathcal{E}^{p,q}_{\mathbb{R}}$ and $\mathcal{D}'^{p,q}_{\mathbb{R}}$ for the subsheaves of real forms, and respectively real currents. We will be interested in the real Bott-Chern and Aeppli cohomolgy groups of bidegree $(1,1)$ on X. They may be defined using either global forms or global currents. We recall their definition in terms of forms:

$$H^{1,1}_{BC}(X,\mathbb{R}) := \{\eta\in\mathcal{E}^{1,1}_{\mathbb{R}}(X)\mid \mathrm{d}\eta=0\}/i\partial\bar\partial\mathcal{E}^{0,0}_{\mathbb{R}}(X),$$

$$H^{1,1}_{A}(X,\mathbb{R}) := \{\eta\in\mathcal{E}^{1,1}_{\mathbb{R}}(X)\mid i\partial\bar\partial\eta=0\}/\{\bar\partial S+\partial\bar S\mid S\in\mathcal{E}^{1,0}(X)\}.$$

The evaluation of currents on forms gives a duality between these two spaces. We also get natural comparison morphisms to and from the second de Rham cohomology group:

$$H^{1,1}_{BC}(X,\mathbb{R})\to H^2_{dR}(X,\mathbb{R}),\ \ H^2_{dR}(X,\mathbb{R})\to H^{1,1}_{A}(X,\mathbb{R}).$$

We denote the image of the first one by $H^{1,1}_{dR}(X,\mathbb{R})$. We clearly have

$$H^{1,1}_{dR}(X,\mathbb{R}) = \{\eta\in\mathcal{E}^{1,1}_{\mathbb{R}}(X)\mid \mathrm{d}\eta=0\}/\{\eta\in\mathcal{E}^{1,1}_{\mathbb{R}}(X)\mid \eta=\mathrm{d}\phi,\ \phi\in\mathcal{E}^{1}_{\mathbb{R}}(X)\}.$$

It is known that $\mathrm{Ker}(H^{1,1}_{BC}(X,\mathbb{R}) \to H^{1,1}_{dR}(X,\mathbb{R}))$ is 1-dimensional, [Lam99, Proof of Theorem 7.1]. By duality one gets that $\mathrm{Coker}(H^{1,1}_{dR}(X,\mathbb{R}) \to H^{1,1}_{A}(X,\mathbb{R}))$ is also 1-dimensional. In fact it may be seen also more directly that the map $H^{1,1}_{dR}(X,\mathbb{R}) \to H^{1,1}_{A}(X,\mathbb{R})$ is injective. Indeed if a d-closed form $\eta \in \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ equals $\bar\partial S + \partial \bar S$ for some $S \in \mathcal{E}^{1,0}(X)$, then $0 = \mathrm{d}(\mathrm{d}(S+\bar S)) = \mathrm{d}(\eta + \partial S + \bar\partial \bar S) = \mathrm{d}(\partial S + \bar\partial \bar S) = \bar\partial(\partial S) + \partial(\bar\partial \bar S)$ and the form ∂S is holomorphic and ∂-exact and thus must vanish, see [BHPVdV04, Lemma IV(2.3)(ii)], hence $\eta = \mathrm{d}(S+\bar S)$ and its class vanishes in $H^{1,1}_{dR}(X,\mathbb{R})$. We clearly have that $\mathrm{Ker}(H^{1,1}_{BC}(X,\mathbb{R}) \to H^{1,1}_{dR}(X,\mathbb{R}))$ is generated by $\{\tau\}_{BC}$ and that $\mathrm{Coker}(H^{1,1}_{dR}(X,\mathbb{R}) \to H^{1,1}_{A}(X,\mathbb{R}))$ is generated by the image of $\{\omega\}_A$.

It is also known that the intersection form on $H^{1,1}_{dR}(X,\mathbb{R})$ is negative definite, [BHPVdV04, Theorem IV(2.14)]. It follows that the induced intersection form on $H^{1,1}_{BC}(X,\mathbb{R})$ by means of the map $H^{1,1}_{BC}(X,\mathbb{R}) \to H^{2}_{dR}(X,\mathbb{R})$ is negative semi-definite with totally isotropic space spanned by the class of τ.

We next define "positive" convex cones in $H^{1,1}_{BC}(X,\mathbb{R})$ and in $H^{1,1}_{A}(X,\mathbb{R})$ by

$$\mathrm{Psef}_{BC}(X) := \{\{T\}_{BC} \mid T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X),\ \mathrm{d}T = 0,\ T \geq 0\},$$

$$\mathrm{Psef}_{A}(X) := \{\{T\}_{A} \mid T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X),\ i\partial\bar\partial T = 0,\ T \geq 0\},$$

$$\mathrm{Nef}_{BC}(X) := \{\{T\}_{BC} \mid T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X),\ \mathrm{d}T = 0,\ T \text{ nef pluriharmonic}\},$$

$$\mathrm{Nef}_{A}(X) := \{\{T\}_{A} \mid T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X),\ i\partial\bar\partial T = 0,\ T \text{ nef pluriharmonic}\}.$$

We also denote by G the set of Aeppli cohomology classes of Gauduchon forms on X.

Proposition 5.1.

1) *$Psef_{BC}(X)$ and $Nef_{BC}(X)$ are closed in $H^{1,1}_{BC}(X,\mathbb{R})$.*
2) *$Nef_{BC}(X) = \{\alpha \in H^{1,1}_{BC}(X,\mathbb{R}) \mid \forall \epsilon > 0\ \exists \eta_\epsilon \in \alpha \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X)\ \eta_\epsilon \geq -\epsilon\omega\}$.*
3) *$Nef_{BC}(X) = \mathbb{R}_{\geq 0}\{\tau\}_{BC}$.*
4) *If the Bott-Chern cohomology class of a positive closed current T is in $Nef_{BC}(X)$, then T is nef pluriharmonic.*
5) *$Nef_{BC}(X) = Psef_A(X)^*$ and $Psef_{BC}(X) \setminus \{0\} = \{\alpha \in H^{1,1}_{BC}(X,\mathbb{R}) \mid \langle \alpha, \eta\rangle > 0 \quad \forall \eta \in G\}$. In particular $\overline{Psef_{BC}(X)} = Nef_A(X)^*$, $\overline{Psef_A(X)} = Nef_{BC}(X)^*$ and $\overline{Nef_A(X)} = Psef_{BC}(X)^*$,*

6) *G is open and* $\overline{Nef_A(X)} = \overline{G}$.

7) *If* E_j *are the irreducible curves of negative self-intersection on* X*, then*

$$Psef_{BC}(X) = Nef_{BC}(X) + \sum_j \{[E_j]\}_{BC}$$

and

$$\overline{Psef_A(X)} = \overline{Nef_A(X)} + \sum_j \{[E_j]\}_A.$$

8) $\overline{Nef_A(X)} = \{\alpha \in H^{1,1}_A(X,\mathbb{R}) \mid \forall \epsilon > 0 \ \exists \eta_\epsilon \in \alpha \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X) \ \eta_\epsilon > -\epsilon\omega\}$.

Proof. 1) By arguing similarly to [HL83, Section 2] one gets the following facts: the operator $i\partial\bar\partial : \mathcal{E}^{1,1}_{\mathbb{R}}(X) \to \mathcal{E}^{2,2}_{\mathbb{R}}(X)$ has closed range since its cokernel is finite dimensional, [Ser55, Lemme 2], its dual $i\partial\bar\partial : {\mathcal{D}'}^{0}_{\mathbb{R}}(X) \to {\mathcal{D}'}^{1,1}_{\mathbb{R}}(X)$ has closed range by the closed range theorem, [Sch71, IV 7.7], and thus the quotient topology induced by the projection $\pi : \{T \in {\mathcal{D}'}^{1,1}_{\mathbb{R}}(X) \mid \mathrm{d}T = 0\} \to H^{1,1}_{BC}(X,\mathbb{R})$ on $H^{1,1}_{BC}(X,\mathbb{R})$ is separated. Now the cone of closed positive currents is generated by the compact set $K := \{T \in {\mathcal{D}'}^{1,1}_{\mathbb{R}}(X) \mid \mathrm{d}T = 0, \ \langle T, \omega\rangle = 1\}$, hence $\mathrm{Psef}_{BC}(X)$ is generated by its image $\pi(K)$ in $H^{1,1}_{BC}(X,\mathbb{R})$. This image is compact and does not contain 0. Thus $\mathrm{Psef}_{BC}(X)$ is closed in $H^{1,1}_{BC}(X,\mathbb{R})$. The same argument shows that $\mathrm{Nef}_{BC}(X)$ is closed as well.

2) Let us denote the cone $\{\alpha \in H^{1,1}_{BC}(X,\mathbb{R}) \mid \forall \epsilon > 0 \ \exists \eta_\epsilon \in \alpha \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X) \ \eta_\epsilon \geq -\epsilon\omega\}$ by $P_{nef}(X)$. The inclusion $\mathrm{Nef}_{BC}(X) \supset P_{nef}(X)$ is proved in [Lam99, Proposition 4.1]. We show the second assertion by duality. Let T be a nef pluriharmonic current on X and which is d-closed. By [Lam99, Théorème 1.2] the Bott-Chern cohomology class $\{T\}_{BC}$ is in $P_{nef}(X)$ if for all positive pluriharmonic currents T' one has $\langle \{T\}_{BC}, \{T'\}_A\rangle \geq 0$, where $\{T'\}_A$ is the Aeppli cohomology class of T'. By Proposition 2.2 the positive, $i\partial\bar\partial$-closed $(1,1)$-current T' has a decomposition $T' = \sum_j c_j[E_j] + T''$, where $c_j \geq 0$ are positive real numbers, E_j are irreducible compact curves on X and T'' is a nef pluriharmonic current. The inequality $\langle \{T\}_{BC}, \{T''\}_A\rangle \geq 0$, is a consequence of Lemma 5.3 below. Now if we write T as a limit of Gauduchon forms, $T = \lim_{n\to\infty} \omega_n$, and choose a smooth representative η in the class $\{E\}$ of the integration current along a curve E, we get $\langle \{T\}_{BC}, \{E\}_A\rangle = \langle \{T\}_{BC}, \{\eta\}_A\rangle = T(\eta) = \lim_{n\to\infty} \int_X \omega_n \wedge \eta = \lim_{n\to\infty} \int_E \omega_n \geq 0$.

3) It is proved in [Lam99, Théorème 7.1] that $P_{nef}(X) = \mathbb{R}_{\geq 0}\{\tau\}_{BC}$, so $\mathrm{Nef}_{BC}(X) = P_{nef}(X) = \mathbb{R}_{\geq 0}\{\tau\}_{BC}$.

4) Let T be a positive closed current with nef class $\{T\}_{BC}$. Then as before T has a decomposition $T = \sum_j c_j[E_j] + T'$, and this time T' is closed and nef pluriharmonic. Both T and T' are thus d-exact. This implies that $\sum_j c_j[E_j]$ is d-exact as well. If X is elliptic the sum $\sum_j c_j[E_j]$ may be infinite but it is in any case nef pluriharmonic. If X is not elliptic, the divisor $\sum_j c_jE_j$ on X is homologically trivial and the corresponding integration current is nef pluriharmonic.

5) This follows from [Lam99, Théorème 1.2].

6) As in (1) (see also [HL83, Lemma 6]) one can see that the operators $p_1 \circ \mathrm{d} : \mathcal{E}^1_{\mathbb{R}}(X) \to \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ and $p_2 \circ \mathrm{d} : \mathcal{D}'^1(X) \to \mathcal{D}'^{1,1}_{\mathbb{R}}(X)$ have closed range, where $p_1 : \mathcal{E}^2_{\mathbb{R}}(X) \to \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ and $p_2 : \mathcal{D}'^{1,1}_{\mathbb{R}}(X) \to \mathcal{D}'^{1,1}_{\mathbb{R}}(X)$ are the natural projections. Thus the quotient topologies induced on $H^{1,1}_A(X,\mathbb{R})$ both from the space of pluriharmonic forms and from the space of pluriharmonic currents are separated. It follows that G is open and that if $T = \lim_{n\to\infty} \omega_n$ is a weak limit of Gauduchon forms, then $\{T\}_A = \lim_{n\to\infty}\{\omega_n\}_A \in \overline{G}$ hence $\overline{\mathrm{Nef}_A(X)} = \overline{G}$.

7) This assertion is a consequence of [Lam99, Proposition 4.3]. Note however that in loc. cit. one needs to take the closure of $\mathrm{Psef}_A(X)$, see also Remark 5.2.

8) We denote the set $\{\alpha \in H^{1,1}_A(X,\mathbb{R}) \mid \forall \epsilon > 0\ \exists \eta_\epsilon \in \alpha \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X)\ \eta_\epsilon > -\epsilon\omega\}$ by $\Pi_{nef}(X)$. Let $\alpha \in \Pi_{nef}(X)$ and let $\eta_\epsilon \in \alpha \cap \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ be such that $\eta_\epsilon > -\epsilon\omega$. We set $\Omega_\epsilon := \eta_\epsilon + \epsilon\omega$. Then the classes $\{\Omega_\epsilon\}_A = \alpha + \epsilon\{\omega\}_A$ are in G and tend to α as ϵ tends to zero. Thus $\alpha \in \overline{G} = \overline{\mathrm{Nef}_A(X)}$ and $\Pi_{nef}(X) \subset \overline{\mathrm{Nef}_A(X)}$. Conversely, since we clearly have $G \subset \Pi_{nef}(X)$, we get $\overline{\mathrm{Nef}_A(X)} \subset \overline{\Pi_{nef}(X)}$ and the desired equality of cones follows since $\Pi_{nef}(X)$ is closed, [CRŞ19, Lemma 2.3]. □

Remark 5.2. *The above proof cannot be mimicked to show closedness for* $\mathit{Psef}_A(X)$ *and* $\mathit{Nef}_A(X)$ *since there exist non-trivial d-exact currents on* X*, hence the projection to* $H^{1,1}_A(X,\mathbb{R})$ *of a corresponding generating compact set of positive pluriharmonic currents will contain* 0*.*

In fact, if X *is an Enoki surface with just one irreducible curve* C*, one can renormalize* τ *so that* $\{\tau\}_{BC} = \{[C]\}_{BC}$ *and one gets* $\dim H^{1,1}_{BC}(X,\mathbb{R}) =$

$\dim H^{1,1}_A(X,\mathbb{R}) = 2$, $Psef_{BC}(X) = Nef_{BC}(X) = \mathbb{R}_{\geq 0}\{\tau\}_{BC}$. *By Proposition 2.2 it follows that any positive pluriharmonic current is nef pluriharmonic. Moreover, if such a current vanishes on* τ, *then it must be d-exact by Proposition 2.4. Hence we get*

$$Psef_A(X) = Nef_A(X) = \{\alpha \in H^{1,1}_A(X,\mathbb{R}) \mid \langle \alpha, \{\tau\}_{BC}\rangle > 0\} \cup \{0\}.$$

Lemma 5.3. *Let* T, T' *be nef pluriharmonic* $(1,1)$*-currents on* X *such that* T *is d-closed. Then for any sequences* $(\omega_n)_n$, $(\omega'_n)_n$ *of Gauduchon forms converging weakly to* T *and to* T' *respectively, we have:*

$$\lim_{n,m\to\infty} \langle \omega_n, \omega'_m\rangle = \langle \{T\}_{BC}, \{T'\}_A\rangle.$$

Proof. Let $\alpha_1, ..., \alpha_n$ be closed $(1,1)$-forms on X whose classes generate $H^{1,1}_{dR}(X,\mathbb{R})$ and such that $\int_X \alpha_i \wedge \alpha_j = -\delta_{ij}$. Then in Aeppli cohomology T' is cohomologous to some form $\delta\omega + A'$, where $A' = \sum_{j=1}^n a'_j \alpha_j$, $\delta, a_j \in \mathbb{R}$ and $\delta = \langle T', \tau\rangle \geq 0$. Similarly ω'_n are cohomologous to some $(\delta + \epsilon'_n)\omega + A'_n$, with $A'_n = \sum_{j=1}^n a'_{n,j}\alpha_j$. Evaluating on τ and on each α_j one obtains $\lim_{n\to\infty} \epsilon'_n = 0$ and $\lim_{n\to\infty} a'_{n,j} = a'_j$. Thus

$$\omega'_n = (\delta + \epsilon'_n)\omega + A'_n + \bar\partial\sigma'_n + \partial\bar\sigma'_n$$

for some $(1,0)$-forms σ'_n. We have

$$\begin{aligned}
0 \geq \int_X (A'_n)^2 &= \int_X (A'_n + \mathrm{d}(\sigma'_n + \bar\sigma'_n))^2 \\
&= \int_X (\omega'_n - (\delta + \epsilon'_n)\omega + \partial\sigma'_n + \bar\partial\bar\sigma'_n)^2 \\
&= \int_X (\omega'_n - (\delta + \epsilon'_n)\omega)^2 + 2\int_X \partial\sigma'_n \wedge \bar\partial\bar\sigma'_n \\
&= \int_X (\omega'_n)^2 - 2\int_X (\delta + \epsilon'_n)\omega \wedge \omega'_n \\
&\quad + \int_X (\delta + \epsilon'_n)^2\omega^2 + 2 \parallel \partial\sigma'_n \parallel^2_{L^2},
\end{aligned}$$

hence $\parallel \partial\sigma'_n \parallel^2_{L^2} \leq \int_X (\delta + \epsilon'_n)\omega \wedge \omega'_n$ and the right hand term tends to $\delta\langle T', \omega\rangle$ when n tends to infinity. Thus the sequence $(\parallel \partial\sigma'_n \parallel_{L^2})_n$ is bounded.

The same argument works for T and this time we get

$$\omega_n = \epsilon_n \omega + A_n + \bar{\partial}\sigma_n + \partial\bar{\sigma}_n,$$

with $A_n = \sum_{j=1}^{n} a_{n,j}\alpha_j$, $\lim_{n\to\infty} \epsilon_n = 0$, $\lim_{n\to\infty} a_{n,j} = 0$, and $\lim_{n\to\infty} \|\partial\sigma'_n\|_{L^2} = 0$.

Thus

$$\begin{aligned}
&\lim_{n,m\to\infty} \langle \omega_n, \omega'_m\rangle \\
&\quad = \lim_{n,m\to\infty} \langle \epsilon_n\omega + A_n + \bar{\partial}\sigma_n + \partial\bar{\sigma}_n, (\delta + \epsilon'_m)\omega + A'_m + \bar{\partial}\sigma'_m + \partial\bar{\sigma}'_m\rangle \\
&\quad = \lim_{n,m\to\infty} \langle \epsilon_n\omega + \bar{\partial}\sigma_n + \partial\bar{\sigma}_n, (\delta + \epsilon'_m)\omega + \bar{\partial}\sigma'_m + \partial\bar{\sigma}'_m\rangle \\
&\quad = \lim_{n,m\to\infty} \langle \epsilon_n\omega + \bar{\partial}\sigma_n + \partial\bar{\sigma}_n, (\delta + \epsilon'_m)\omega\rangle \\
&\qquad + \lim_{n,m\to\infty} \langle \epsilon_n\omega + \bar{\partial}\sigma_n + \partial\bar{\sigma}_n, \bar{\partial}\sigma'_m + \partial\bar{\sigma}'_m\rangle \\
&\quad = \langle T, \delta\omega\rangle + \lim_{n,m\to\infty} \langle \epsilon_n\omega, \bar{\partial}\sigma'_m + \partial\bar{\sigma}'_m\rangle \\
&\quad = \langle T, \delta\omega\rangle = \langle \{T\}_{BC}, \{T'\}_A\rangle.
\end{aligned}$$

□

Acknowledgements

We express our thanks to the referees whose careful reading and detailed comments helped us to improve the presentation. Our collaboration benefited from the support of the GDRI ECO-Math.

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Institute of Mathematics of the Romanian Academy
010702 Bucharest, Romania
E-mail address: Ionut.Chiose@imar.ro

Institut de Mathématiques Élie Cartan, Université de Lorraine
B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail address: Matei.Toma@univ-lorraine.fr

Received August 20, 2020
Accepted October 4, 2021