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Convexity of the weighted Mabuchi functional and the uniqueness of weighted extremal metrics

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We prove the uniqueness, up to a pull-back by an element of a suitable subgroup of complex automorphisms, of the weighted extremal Kähler metrics on a compact Kähler manifold introduced in our previous work [31]. This extends a result by Berman–Berndtsson [7] and Chen–Paun–Zeng [17] in the extremal Kähler case. Furthermore, we show that a weighted extremal Kähler metric is a global minimum of a suitable weighted version of the modified Mabuchi energy, thus extending our results from [31] from the polarized to the Kähler case. This implies a suitable notion of weighted K-semistability of a Kähler manifold admitting a weighted extremal Kähler metric.

1. Introduction

In a previous work [31], we introduced the notion of a *weighted extremal Kähler metric* on a Kähler manifold X, endowed with a Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$, a fixed compact real torus $\mathbb{T}$ inside the connected Lie group $\mathrm{Aut}_{\mathrm{red}}(X)$ of reduced automorphisms of X, and two arbitrary smooth positive functions (called weights) v, w defined on the fixed momentum image $\mathrm{P}_\alpha \subset \mathrm{Lie}(\mathbb{T})^*$ for the action of $\mathbb{T}$ with respect to any Kähler representative of α. More precisely, given these data, for any $\mathbb{T}$-invariant Kähler metric $\omega \in \alpha$ with normalized $\mathbb{T}$-momentum map $m_\omega : X \to \mathrm{P}_\alpha$, we define the v-scalar curvature by

$$\mathrm{Scal}_{\mathrm{v}}(\omega) := \mathrm{v}(m_\omega)\mathrm{Scal}(\omega) + 2\Delta_\omega(\mathrm{v}(m_\omega)) + \mathrm{Tr}(\mathrm{G}_\omega \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)), \tag{1}$$

where $\mathrm{Scal}(\omega)$ is the scalar curvature of ω, $m_\omega : X \to \mathfrak{t}^*$ is the momentum map of the $\mathbb{T}$-action normalized by $m_\omega(X) = \mathrm{P}_\alpha$, Δ_ω is the Riemannian Laplacian of the Kähler metric ω and $\mathrm{Hess}(\mathrm{v})$ is the hessian of v, viewed as bilinear form on $\mathfrak{t}^*$ whereas G_ω is the bilinear form with smooth coefficients on $\mathfrak{t}$, given by the restriction of the Kähler metric ω on fundamental vector fields. In a basis $\boldsymbol{\xi} = (\xi_i)_{i=1,\cdots,\ell}$ of $\mathfrak{t}$, we have

$$\mathrm{Tr}(\mathrm{G}_\omega \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\omega)) = \sum_{1 \le i,j \le \ell} \mathrm{v}_{,ij}(m_\omega)(\xi_i, \xi_j)_\omega,$$

where $\mathrm{v}_{,ij}$ stands for the partial derivatives of v with respect to the dual basis of $\boldsymbol{\xi}$.

Let $\mathrm{w} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$ be another smooth positive function on P_α. Similarly to the approach pioneered by Calabi [12], we are interested in the problem of finding a $\mathbb{T}$-invariant Kähler representative ω of α for which $\mathrm{Scal}_{\mathrm{v}}(\omega)/\mathrm{w}(m_\omega)$ is the momentum potential of a holomorphic vector field inside the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$. We have shown in [31] that the problem reduces to solve

$$\frac{\mathrm{Scal}_{\mathrm{v}}(\omega)}{\mathrm{w}(m_\omega)} = \ell_{\mathrm{ext}}(m_\omega), \tag{2}$$

where ℓ_{ext} is the (v, w)-extremal affine-linear function on $\mathfrak{t}^*$, determined from the data $(\alpha, \mathbb{T}, \mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$, and we shall refer to a Kähler metric satisfying the above condition as a (v, w)-*extremal Kähler metric* on $(X, \alpha, \mathbb{T}, \mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$.

Notice that if we take $\mathbb{T} = \{1\}$ and $\mathrm{v} = \mathrm{w} \equiv 1$, we obtain the much studied problem of the existence of cscK metric in α whereas taking $\mathbb{T}$ to be a *maximal* torus in $\mathrm{Aut}_{\mathrm{red}}(X)$ and $\mathrm{v} = \mathrm{w} \equiv 1$, our problem reduces to the famous Calabi problem of the existence of an extremal Kähler metric on (X, α). As we have noticed in [31], there is a number of other natural problems in Kähler geometry which can be reduced to the search of (v, w)-extremal Kähler metrics for special choices of $\mathbb{T}$ and the weight functions v and w, including the existence of conformally Kähler, Einstein–Maxwell metrics [3], the existence of extremal Sasaki metrics [1], the existence of Kähler–Ricci solitons [8, 29, 30], prescribing the scalar curvature on compact toric manifolds [25] and on semi-simple, rigid toric fibre bundles [2] as well as the recently introduced μ-cscK metrics in [30].

For a fixed $\mathbb{T}$-invariant Kähler metric $\omega \in \alpha$ let $\mathcal{K}(X, \omega)^{\mathbb{T}}$ denote the space of smooth $\mathbb{T}$-invariant Kähler potentials with respect to ω, i.e.

$$\mathcal{K}(X, \omega)^{\mathbb{T}} = \{\phi \in C^{\infty}(X, \mathbb{R})^{\mathbb{T}} | \omega_\phi = \omega + dd^c\phi > 0\}.$$

For $\phi \in \mathcal{K}(X, \omega)^{\mathbb{T}}$ we denote by $m_\phi : X \to \mathfrak{t}^*$ the corresponding ω_ϕ-momentum map, normalized by the condition $m_\phi(X) = m_\omega(X) =: P_\alpha$ or equivalently by $m_\phi = m_\omega + d^c\phi$ and by $\mathrm{Scal}_{\mathrm{v}}(\phi)$ the weighted scalar curvature of ω_ϕ introduced by (1). Also, we use the usual convention to denote by $\omega_\phi^{[n]} := \frac{\omega_\phi^n}{n!}$ the associated volume form. Following [31], for two weight functions $\mathrm{v}, \mathrm{w} \in C^{\infty}(\mathrm{P}_\alpha, \mathbb{R})$ such that $\mathrm{v} > 0$ and w is arbitrary, a Kähler potential $\phi \in \mathcal{K}(X, \omega)^{\mathbb{T}}$ defines a (v, w)-weighted cscK metric ω_ϕ if it satisfies

$$\mathrm{Scal}_{\mathrm{v}}(\phi) = c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\phi) \tag{3}$$

where $c_{\mathrm{v},\mathrm{w}}(\alpha)$ is a constant depending only on $(\mathrm{v}, \mathrm{w}, \alpha)$, given by

$$c_{\mathrm{v},\mathrm{w}}(\alpha) := \begin{cases} \frac{\int_X \mathrm{Scal}_{\mathrm{v}}(\omega)\omega^{[n]}}{\int_X \mathrm{w}(m_\omega)\omega^{[n]}}, & \text{if } \int_X \mathrm{w}(m_\omega)\omega^{[n]} \neq 0 \\ 1, & \text{if } \int_X \mathrm{w}(m_\omega)\omega^{[n]} = 0, \end{cases} \tag{4}$$

The (v, w)-weighted cscK metrics are critical points of the (v, w)-Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}} : \mathcal{K}(X, \omega)^{\mathbb{T}} \to \mathbb{R}$ defined on the Fréchet space $\mathcal{K}(X, \omega)^{\mathbb{T}}$ by its first variation

$$\begin{cases} (d\mathcal{M}_{\mathrm{v},\mathrm{w}})_\phi(\dot\phi) = -\displaystyle\int_X \dot\phi\big(\mathrm{Scal}_{\mathrm{v}}(\phi) - c_{\mathrm{v},\mathrm{w}}(\alpha)\mathrm{w}(m_\phi)\big)\omega_\phi^{[n]}, \\ \mathcal{M}_{\mathrm{v},\mathrm{w}}(0) = 0, \end{cases} \tag{5}$$

for all $\dot\phi \in T_\phi\mathcal{K}(X, \omega)^{\mathbb{T}} \cong C^{\infty}(X, \mathbb{R})^{\mathbb{T}}$, where $C^{\infty}(X, \mathbb{R})^{\mathbb{T}}$ stands for the space of $\mathbb{T}$-invariant smooth functions. As observed in [31, Section 3.2], when v, w are both positive, a Kähler metric ω_ϕ is (v,w)-extremal if and only if it is $(\mathrm{v},\ell_{\mathrm{ext}}\mathrm{w})$-cscK and the relative (v,w)-Mabuchi energy is defined in this case by

$$\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} = \mathcal{M}_{\mathrm{v},\ell_{\mathrm{ext}}\mathrm{w}}. \tag{6}$$

where ℓ_{ext} is the (v,w)-extremal affine linear function introduced in [31] via the orthogonal projection of $\mathrm{Scal}_{\mathrm{v}}(\phi)$ to the space of (pull-backs by m_ϕ) affine-linear functions on $\mathfrak{t}^*$ with respect to the weighted L^2-global product $\langle\varphi_1,\varphi_2\rangle_{\mathrm{w},\phi} := \int_X \varphi_1\varphi_2\mathrm{w}(m_\phi)\omega_\phi^{[n]}$. The critical points of the relative (v,w)-Mabuchi energy are precisely the (v,w)-extremal Kähler metrics in α.

The space $\mathcal{K}(X,\omega)^{\mathbb{T}}$ is an infinite dimentional Riemannian manifold with a natural Riemanniann metric, called the *Mabuchi metric* [33], defined by

$$\langle\dot\phi_1,\dot\phi_2\rangle_\phi := \int_X \dot\phi_1\dot\phi_2\omega_\phi^{[n]},$$

for any $\phi\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ and $\dot\phi_1,\dot\phi_2\in C^\infty(X,\mathbb{R})^{\mathbb{T}}$. The equation of a geodesic $(\phi_t)_{t\in[0,1]}\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ connecting two points $\phi_0,\phi_1\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ is given by [33, 34]

$$\ddot\phi_t = |d\dot\phi_t|^2_{\phi_t}. \tag{7}$$

It was shown by Donaldson [24] and Semmes [36] that by letting $\tau := e^{-t+is}$, the geodesic $(\phi_t)_{t\in[0,1]}\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ can be viewed as a smooth function $\Phi(x,\tau)$ on $\hat{X} := X\times\mathbb{A}$, where $\mathbb{A} := \{e^{-1}\leq|\tau|\leq 1\}$ is the corresponding annulus in $\mathbb{C}$, defined by

$$\Phi(x,\tau) := \phi_t(x), \tag{8}$$

which is invariant under the natural action of $\mathbb{G} := \mathbb{T}\times\mathbb{S}^1$ on $\hat{X}$, and satisfies the following degenerate Monge-Ampère equation on $\hat{X}$,

$$\left(\pi_X^*\omega + dd^c\Phi\right)^{n+1} = 0$$

where $\pi_X : \hat{X}\to X$ is the projection on the first factor. Hence, the problem of connecting two potentials $\phi_0,\phi_1\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ by a geodesic $(\phi_t)_{t\in[0,1]}\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ is equivalent to finding a solution $\Phi\in C^\infty(\hat{X},\mathbb{R})^{\mathbb{G}}$ to the following boundary value problem

$$
(9) \qquad \begin{cases} \left(\pi_X^*\omega + dd^c\Phi\right)^{n+1} = 0, \\ \omega + dd^c\Phi_{|X_\tau} > 0, \ \text{for } \tau \in \mathbb{A}, \\ \Phi(\cdot, e^{-1}) = \phi_1 \text{ and } \Phi(\cdot, 1) = \phi_0. \end{cases}
$$

where $X_\tau := \pi_{\mathbb{A}}^{-1}(\tau)$ is a fiber of the projection $\pi_{\mathbb{A}} : \hat{X} \to \mathbb{A}$.

In general, the space $\mathcal{K}(X, \omega)^{\mathbb{T}}$ is not geodesically convex by smooth geodesics (see [20, Theorem 1.2]). However, the boundary value problem (9) makes sense for $\mathbb{G}$-invariant bounded plurisubharmonic functions $\Phi \in \mathrm{PSH}(\hat{X}, \pi_X^\star\omega)^{\mathbb{G}} \cap L^\infty$, using the Bedford–Taylor interpretation of $\left(\pi_X^*\omega + dd^c\Phi\right)^{n+1}$ as a Borel measure on $\hat{X}$.

By a result of Chen [13], with complements of Blocki [10] and Chu–Tossati–Weinkove [19], the boundary value problem (9) admits a unique $\mathbb{G}$-invariant solution $\Phi \in C^{1,1}(\hat{X}, \mathbb{R})$ such that $\pi_X^*\omega + dd^c\Phi$ is a positive current with bounded coefficients, up to the boundary, corresponding to a family of functions $(\phi_t)_{t\in[0,1]}$ in the space $\mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$ of all $\mathbb{T}$-invariant functions $\phi \in C^{1,1}(X, \mathbb{R})$ such that ω_ϕ is a positive current with bounded coefficients. The curve $(\phi_t)_{t\in[0,1]} \subset \mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$ is called the *weak geodesic segment* joining $\phi_0, \phi_1 \in \mathcal{K}(X, \omega)^{\mathbb{T}}$. Consequently, the space $\mathcal{K}(X, \omega)^{\mathbb{T}}$ is geodesically convex by geodesics in the space $\mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$.

Building on the approach by finite dimensional approximations [26, 32, 35] in the extremal Kähler case, we proved in [31, Corollary 1] that when α is a polarization, (v, w)-extremal Kähler metrics are global minima of $\mathcal{M}_{\mathrm{v},\mathrm{w}}^{\mathrm{rel}}$. In this paper, we extend this result by removing the integrality condition on the Kähler class α.

To this end, we now follow the approach of Berman–Berndtsson [7] (see also Chen–Li–Paun [16]) who proved that $\mathcal{M}_{1,1}$ naturally extends to the space $\mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$ and is convex along the weak geodesics. Our main result of this paper is the following.

Theorem 1. *Let X be a compact Kähler manifold with Kähler class α, $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ a real torus with momentum polytope $\mathrm{P}_\alpha \subset \mathfrak{t}^*$ and $\mathrm{v} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R}_{>0})$, $\mathrm{w} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$. For any $\mathbb{T}$-invariant Kähler metric $\omega \in \alpha$, the (v, w)-Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ admits a natural extension as functional on the space $\mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$ which is convex in the pointwise sense along weak geodesics in $\mathcal{K}^{1,1}(X, \omega)^{\mathbb{T}}$ connecting smooth $\mathbb{T}$-invariant ω-Kähler potentials $\phi_0, \phi_1 \in \mathcal{K}(X, \omega)^{\mathbb{T}}$.*

Similarly to the case of cscK metrics, using the sub-slope property of convex functions, we obtain the following corollary giving an obstruction to the existence of (v,w)-cscK metrics in a Kähler class α, in terms of the boundedness of the corresponding (v,w)-Mabuchi energy.

Corollary 1. *Let $\phi_0, \phi_1 \in \mathcal{K}(X,\omega)^{\mathbb{T}}$. We have the following inequality*

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_1) - \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_0) \geq -\frac{d(\phi_1,\phi_0)}{\int_X \mathrm{v}(m_\omega)\omega^{[n]}} \parallel \mathrm{Scal}_{\mathrm{v}}(\phi_0) - \mathrm{w}(m_{\phi_0}) \parallel_{L^2(X,\mu_{\phi_0})}$$

where d is the distance corresponding to the Mabuchi metric and $\parallel \cdot \parallel_{L^2(X,\mu_{\phi_0})}$ is the usual L^2-norm on (X,μ_{ϕ_0}) with $\mu_{\phi_0} := \frac{\omega_{\phi_0}^{[n]}}{\mathrm{vol}(X,\alpha)}$. In particular, (v,w)-cscK metrics in a Kähler class α minimizes the corresponding (v,w)-Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}}$, and any (v,w)-extremal Kähler metric in α minimizes the relative weighted Mabuchi energy $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}}$.

By [31, Theorem 2], we obtain that the weighted K-semistability is a necessary condition for the existence of a (v,w)-extremal Kähler metric.

Corollary 2. *Let X be as in Theorem 1. If X admits a $\mathbb{T}$-invariant (v,w)-cscK metric in the Kähler class α, then for any smooth $\mathbb{T}$-equivariant Kähler test configuration $(\mathcal{X},\mathcal{A})$ of (X,α), which has reduced central fibre, the weighted Futaki invariant $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\mathcal{X},\mathcal{A})$ defined in [31] is non-negative.*

Our approach to prove Theorem 1 closely follows the scheme of Berman–Berndtsson's proof [7] in the cscK case (i.e. when $\mathrm{v} = \mathrm{w} \equiv 1$). A key point is proving the existence of a natural extension of the (v,w)-Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ as a continuous convex functional defined on the space $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$. To see that a similar extension of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ exists for arbitrary weights v,w we use the weighted Chen-Tian decomposition of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ found in [31, Theorem 5],

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi) = \mathrm{Ent}_{\mu_\omega}(\mu_{\mathrm{v}}(\phi)) + \mathcal{E}_{\mathrm{v},\mathrm{w}}(\phi), \tag{10}$$

where the first term is an entropy term of the probability measure

$$\mu_{\mathrm{v}}(\phi) := \frac{\mathrm{v}(m_\phi)\omega_\phi^{[n]}}{\mathrm{vol}(X,\mathrm{v}(m_\omega)\omega^{[n]})} \tag{11}$$

relatively to the reference smooth measure $\mu_\omega := \frac{\omega^{[n]}}{\mathrm{vol}(X,\alpha)}$. The second term $\mathcal{E}_{\mathrm{v},\mathrm{w}}$ is an energy type expression given by the integral over X of terms of the form $\phi\mathrm{u}(\phi)\omega_\phi^j \wedge \theta^{n-j}$ where θ are smooth two forms depending on ω, and

$\mathrm{u}(\phi)$ is a continuous function on X depending on v, w and ϕ. The presence of weights introduces an additional difficulty related to the definition and convexity of the momentum map with respect to weak geodesics in the space $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$. We solve this in Lemma 1 below using an approximation argument of Demailly [22]. For a weak geodesic $(\phi_t)_{t\in[0,1]} \subset \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$, with Φ being the corresponding solution of the boundary value problem (9) and $\phi_\tau := \Phi(\cdot, \tau)$ for $\tau \in \mathbb{A} = \{e^{-1} \leq |\tau| \leq 1\}$, Berman–Berndtsson showed in [7] that the function $\tau \mapsto \mathcal{M}_{1,1}(\phi_\tau)$ is weakly subharmonic on $\mathbb{A}$ and

$$dd^c\mathcal{M}_{1,1}(\phi_\tau) = \int_X T.$$

where T is a positive Radon measure on $\hat{X} = X \times \mathbb{A}$ and $\int_X$ denotes the fiber-wise integral on $\pi_{\mathbb{A}} : \hat{X} \to \mathbb{A}$. In the case when $\mathrm{v} > 0$ and w is an arbitrary function on the momentum polytope P_α, weak subharmonicity of $\tau \mapsto \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau)$ on $\mathbb{A}$ will follow from a similar expression

$$dd^c\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau) = \int_X \mathrm{v}(m_{\phi_\tau})T,$$

and the fact that $\mathrm{v}(m_{\phi_\tau})T$ is a positive Radon-measure. In particular, $\tau \mapsto \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau)$ is weakly convex. To get point-wise convexity, we will show that $\tau \mapsto \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau)$ is continuous on $\mathbb{A}$.

An important application of the approach in [7] is establishing the uniqueness of the cscK and extremal Kähler metrics in α, up to the natural action (by pull-backs) of the connected Lie group of reduced automorphisms $\mathrm{Aut}_{\mathrm{red}}(X)$. Similarly, we adapt the proof of the uniqueness of extremal Kähler metrics obtained by Chen–Paun–Zeng [17] to our weighted setting and obtain the following result.

Theorem 2. *Let X be as in Theorem 1 and let $G := \mathrm{Aut}^{\mathbb{T}}_{\mathrm{red}}(X)^\circ$ denote the connected component of identity of the commutator of $\mathbb{T}$ inside $\mathrm{Aut}_{\mathrm{red}}(X)$. Then, for any two $\mathbb{T}$-invariant (v,w)-extremal Kähler metrics ω_1 and ω_2 in α, there exits an element $f \in G$ such that $\omega_1 = f^*(\omega_2)$.*

Notice that if we take $\mathbb{T} = \{1\}$ and $\mathrm{v} = \mathrm{w} \equiv 1$ we get the uniqueness of cscK metrics modulo $\mathrm{Aut}_{\mathrm{red}}(X)$ obtained in [7, 17], whereas if we take $\mathbb{T}$ to be a maximal torus inside $\mathrm{Aut}_{\mathrm{red}}(X)$ and $\mathrm{v} = \mathrm{w} \equiv 1$, the above results yield the uniqueness of the $\mathbb{T}$-invariant extremal Kähler metrics modulo the complexification $\mathbb{T}^c$ of $\mathbb{T}$.

2. The weighted Mabuchi energy on $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$

Let X be a compact Kähler manifold of complex dimension $n \geq 2$. We denote by $\mathrm{Aut}_{\mathrm{red}}(X)$ the connected Lie group of automorphisms of X whose Lie algebra $\mathfrak{h}_{\mathrm{red}}$ is given by real holomorphic vector fields with zeros (see [27]). Let $\mathbb{T}$ be an ℓ-dimentional real torus in $\mathrm{Aut}_{\mathrm{red}}(X)$ with Lie algebra $\mathfrak{t}$, and ω a fixed $\mathbb{T}$-invariant Kähler form on X. The $\mathbb{T}$-action on X is ω-Hamiltonian (see [27]) and we choose $m_\omega : X \to \mathfrak{t}^*$ to be a ω-momentum map of $\mathbb{T}$. It is well known [4, 28] that $\mathrm{P}_\omega = m_\omega(X)$ is a convex polytope in $\mathfrak{t}^*$. For any smooth $\mathbb{T}$-invariant ω-Kähler potential $\phi \in \mathcal{K}(X,\omega)^{\mathbb{T}}$, let $\mathrm{P}_\phi := m_\phi(X)$ be the ω_ϕ-momentum image of X. By [4, 28], the following two facts are equivalent:

(i) $\mathrm{P}_\phi = \mathrm{P}_\omega$.

(ii) $\langle m_\phi, \xi\rangle = \langle m_\omega, \xi\rangle + (d^c\phi)(\xi)$ for any $\xi \in \mathfrak{t}$.

It follows that we can normalize m_ϕ such that $\mathrm{P}_\phi = \mathrm{P}_\omega$ is a ϕ-independent polytope $\mathrm{P}_\alpha \subset \mathfrak{t}^*$. For $\phi \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ the space of all $\mathbb{T}$-invariant functions $\phi \in C^{1,1}(X,\mathbb{R})$ such that ω_ϕ is a positive current with bounded coefficients, we define $m_\phi : X \to \mathfrak{t}^*$ by

$$\langle m_\phi, \xi\rangle = \langle m_\omega, \xi\rangle + (d^c\phi)(\xi),$$

for any $\xi \in \mathfrak{t}$.

Lemma 1. *For any $\phi \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$, we have $\mathrm{P}_\phi = \mathrm{P}_\alpha$.*

Proof. For any $k > 0$ we have $\omega_k = k\omega + \omega_\phi > 0$. By [22, Theorem 5.21] we can find a decreasing sequence $\phi_\epsilon \in C^\infty(X,\mathbb{R})^{\mathbb{T}}$ such that $\omega_{k,\epsilon} := (k+1)\omega + dd^c\phi_\epsilon$ is Kähler and $\phi_\epsilon \to \phi$ in C^1 topology as $\epsilon \to 0$. For any $\epsilon > 0$, we have $\mathrm{P}_{\omega_{k,\epsilon}} = (k+1)\mathrm{P}_\alpha$, and for any $\xi \in \mathfrak{t}$ we have

$$\langle m_{\omega_{k,\epsilon}}, \xi\rangle = (k+1)\langle m_\omega, \xi\rangle + (d^c\phi_\epsilon)(\xi). \tag{12}$$

Since ϕ_ϵ converge to ϕ in C^1 topology, passing to the limit when $\epsilon \to 0$ in (12), we obtain

$$\langle m_{\omega_{k,\epsilon}}, \xi\rangle \to (k+1)\langle m_\omega, \xi\rangle + (d^c\phi)(\xi) = \langle k m_\omega + m_\phi, \xi\rangle,$$

as $\epsilon \to 0$, for $\xi \in \mathfrak{t}$ fixed. It follows that

$$(k+1)\mathrm{P}_\alpha = \mathrm{P}_{\omega_{k,\epsilon}} = (\lim_{\epsilon\to 0} m_{\omega_{k,\epsilon}})(X) = (k m_\omega + m_\phi)(X) = k\mathrm{P}_\alpha + \mathrm{P}_\phi.$$

The result follows by taking the limit when $k \to 0$. $\square$

Remark 1. In [13], Chen considered the following family of elliptic boundary value problems with parameter $\epsilon > 0$,

$$\begin{cases} \left(\pi_X^*\omega + dd^c\Phi^\epsilon\right)^{n+1} = \epsilon\left(\pi_X^*\omega + \frac{\sqrt{-1}d\tau\wedge d\bar{\tau}}{2|\tau|^2}\right)^{n+1}, \\ \Phi^\epsilon(\cdot, e^{-1}) = \phi_1 \text{ and } \Phi^\epsilon(\cdot, 1) = \phi_0. \end{cases} \tag{13}$$

Solutions $\Phi^\epsilon \in \mathcal{K}(\hat{X}, \pi_X^*\omega)^{\mathbb{G}}$ of (13), are always smooth and approximate uniformly the weak solution Φ of (9). More precisely, Φ^ϵ is decreasing in ϵ and converges to the solution Φ of (9) in the weak $C^{1,1}$ topology as $\epsilon \to 0$ (see [13, Lemma 7]). The family of Kähler potentials $(\phi_t^\epsilon)_{t\in[0,1]} \subset \mathcal{K}(X,\omega)^{\mathbb{T}}$ is called an *ϵ-geodesic.*

If $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ is a weak geodesic segment, one can show that $\mathrm{P}_{\phi_t} = \mathrm{P}_\omega$ for any $t \in [0,1]$ using the fact that the ϵ-geodesic $(\phi_t^\epsilon)_{t\in[0,1]} \in \mathcal{K}(X,\omega)^{\mathbb{T}}$ converges to ϕ in the weak $C^{1,1}$-topology as $\epsilon \to 0$, together with the relation

$$m_{\phi_t^\epsilon} = m_\omega + d^c\phi_t^\epsilon.$$

Let $\mathrm{v} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R}_{>0})$ and $\mathrm{w} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$ two smooth functions. Now, we give the energy functionals allowing to define the (v,w)-Mabuchi energy (5) on weak geodesic segments.

Lemma 2. *The functional $\mathcal{E}_{\mathrm{w}} : \mathcal{K}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$ given by*

$$\begin{cases} \left(d\mathcal{E}_{\mathrm{w}}\right)_\phi(\dot{\phi}) = \displaystyle\int_X \dot{\phi}\,\mathrm{w}(m_\phi)\omega_\phi^{[n]}, \\ \mathcal{E}_{\mathrm{w}}(0) = 0, \end{cases} \tag{14}$$

for any $\dot{\phi} \in T_\phi\mathcal{K}(X,\omega)^{\mathbb{T}} \cong C^\infty(X,\mathbb{R})^{\mathbb{T}}$ is well-defined and has a natural extension to $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$.

Proof. The first claim in the Lemma is well known (see, for example, [8, Proposition 2.16]). Now we will extend $\mathcal{E}_{\mathrm{w}} : \mathcal{K}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$ to $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$.

Integrating the derivative of $\mathcal{E}_{\mathrm{w}}$ along the path $\epsilon \mapsto \epsilon\phi \in \mathcal{K}(X,\omega)^{\mathbb{T}}$ gives

$$\begin{aligned}\mathcal{E}_{\mathrm{w}}(\phi) &= \int_0^1 \left(\int_X \phi \mathrm{w}(m_{\epsilon\phi})\omega_{\epsilon\phi}^{[n]}\right) d\epsilon \\ &= \int_0^1 \left(\int_X \sum_{j=0}^{n} \phi\epsilon^{n-j}(1-\epsilon)^j \mathrm{w}(\epsilon m_\phi + (1-\epsilon)m_\omega)\omega_\phi^{[n-j]} \wedge \omega^{[j]}\right) d\epsilon \\ &= \int_X \phi \sum_{j=0}^{n} \mathrm{w}_{j,n}(m_\phi)\omega_\phi^{[n-j]} \wedge \omega^{[j]},\end{aligned}$$

where $\mathrm{w}_{j,n} : \mathrm{P}_\alpha \to \mathbb{R}$ is defined by

$$\mathrm{w}_{j,n}(p) := \int_0^1 \epsilon^{n-j}(1-\epsilon)^j \mathrm{w}(\epsilon p + (1-\epsilon)m_\omega) d\epsilon. \tag{15}$$

Using the expression

$$\mathcal{E}_{\mathrm{w}}(\phi) = \int_X \phi \sum_{j=0}^{n} \mathrm{w}_{j,n}(m_\phi)\omega_\phi^{[n-j]} \wedge \omega^{[j]}, \tag{16}$$

we can define the extension $\mathcal{E}_{\mathrm{w}} : \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$, since by Lemma 1 we have $\epsilon m_\phi + (1-\epsilon)m_\omega \in \mathrm{P}_\alpha$ by convexity. □

Lemma 3. *[8, Proposition 17] Let $(\phi_t)_{t\in[0,1]} \subset \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a geodesic segment connecting $\phi_0, \phi_1 \in \mathcal{K}(X,\omega)^{\mathbb{T}}$ and $\Phi \in \mathcal{K}^{1,1}(\hat{X}, \pi_X^*\omega)^{\mathbb{G}}$ the corresponding solution of the boundary value problem* (9)*. For any $\tau \in \mathbb{A}$, we have*

$$dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau) = 0,$$

where $\phi_\tau := \Phi(\cdot, \tau)$.

Definition 1. Let θ be a $\mathbb{T}$-invariant closed $(1,1)$-form on X. A θ-momentum map for the action of $\mathbb{T}$ on X is a smooth $\mathbb{T}$-invariant function $m_\theta : X \to \mathfrak{t}^*$ with the property $\theta(\xi,\cdot) = -d\langle m_\theta, \xi\rangle$ for all $\xi \in \mathfrak{t}$.

For example, if $\mathrm{Ric}(\omega)$ is the Ricci form of ω, then the $\mathrm{Ric}(\omega)$-momentum map for the action of $\mathbb{T}$ on X is given by (see e.g. [31, Lemma 5])

$$m_{\mathrm{Ric}(\omega)} := \frac{1}{2}\Delta_\omega(m_\omega).$$

Lemma 4. *[31, Lemma 4] Let θ be a fixed $\mathbb{T}$-invariant closed $(1,1)$-form and $m_\theta : X \to \mathfrak{t}^*$ a momentum map with respect to θ, see Definition 1. Then the functional $\mathcal{E}^\theta_{\mathrm{v}} : \mathcal{K}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$ given by*

$$\begin{cases} (d\mathcal{E}^\theta_{\mathrm{v}})_\phi(\dot\phi) = \int_X \dot\phi \left[\mathrm{v}(m_\phi)\theta \wedge \omega_\phi^{[n-1]} + \langle (d\mathrm{v})(m_\phi), m_\theta\rangle \omega_\phi^{[n]}\right], \\ \mathcal{E}^\theta_{\mathrm{v}}(0) = 0, \end{cases} \tag{17}$$

for any $\dot\phi \in C^\infty(X,\mathbb{R})^{\mathbb{T}}$ is well-defined and has a natural extension to $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$.

Proof. Similarly to $\mathcal{E}_{\mathrm{w}}$, we can define the extension $\mathcal{E}^\theta_{\mathrm{v}} : \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$, by using the following expression

$$\begin{aligned} \mathcal{E}^\theta_{\mathrm{v}}(\phi) &= \int_0^1 \left(\int_X \phi \left[\mathrm{v}(m_{\epsilon\phi})\theta \wedge \omega_{\epsilon\phi}^{[n-1]} + \langle (d\mathrm{v})(m_{\epsilon\phi}), m_\theta\rangle \omega_{\epsilon\phi}^{[n]}\right]\right) d\epsilon \\ &= \int_X \phi \Big[\sum_{j=0}^{n-1} \mathrm{v}_{j,n-1}(m_\phi)\omega_\phi^{[n-1-j]} \wedge \omega^{[j]} \wedge \theta \\ &\qquad + \sum_{j=0}^{n} \langle (d\mathrm{v}_{j,n})(m_\phi), m_\theta\rangle \omega_\phi^{[n-j]} \wedge \omega^{[j]}\Big], \end{aligned} \tag{18}$$

where $\mathrm{v}_{j,n} : \mathrm{P}_\alpha \to \mathbb{R}$ is given by (15). □

Now we give the Chen–Tian formula allowing to extend the (v,w)-Mabuchi energy to $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ to $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$.

Theorem 3. *[31, Theorem 5] We have the following expression for the (v,w)-Mabuchi energy,*

$$\mathcal{M}_{\mathrm{v},\mathrm{w}} = \mathcal{H}_{\mathrm{v}} - 2\mathcal{E}^{\mathrm{Ric}(\omega)}_{\mathrm{v}} + c_{(\mathrm{v},\mathrm{w})}(\alpha)\mathcal{E}_{\mathrm{w}}, \tag{19}$$

on $\mathcal{K}(X,\omega)^{\mathbb{T}}$ where $\mathcal{H}_{\mathrm{v}} : \mathcal{K}(X,\omega)^{\mathbb{T}} \to \mathbb{R}$ is given

$$\mathcal{H}_{\mathrm{v}}(\phi) := \int_X \log\left(\frac{\omega_\phi^n}{\omega^n}\right) \mathrm{v}(m_\phi)\omega_\phi^{[n]} = \mathrm{Ent}_{\mu_\omega}(\mu_{\mathrm{v}}(\phi)) + c(\alpha,\mathrm{v}). \tag{20}$$

where

$$\mathrm{Ent}_{\mu_\omega}(\mu_{\mathrm{v}}(\phi)) := \int_X \log\left(\frac{d\mu_{\mathrm{v}}(\phi)}{d\mu_\omega}\right) \frac{d\mu_{\mathrm{v}}(\phi)}{d\mu_\omega} d\mu_\omega$$

is the entropy of the probability measure $\mu_{\rm v}(\phi) := \frac{{\rm v}(m_\phi)\omega_\phi^{[n]}}{{\rm vol}(X,{\rm v}(m_\omega)\omega^{[n]})}$ *relatively to the reference smooth measure* $\mu_\omega := \frac{\omega^{[n]}}{{\rm vol}(X,\alpha)}$ *with , and* $c(\alpha, {\rm v})$ *is a constant depending on* α *and* ${\rm v}$*.*

For $\phi \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$, $\mu_{\rm v}(\phi)$ is a measure with bounded coefficient which is absolutely continuous with respect to μ_ω, thus ${\rm Ent}_{\mu_\omega}(\mu_{\rm v}(\phi))$ is well defined on $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$. Combining this with Lemmas 2 and 4 yields the following.

Corollary 3. *The equation* (19)*, extends the* $({\rm v},{\rm w})$*-Mabuchi energy* $\mathcal{M}_{\rm v,w}$ *to a functional on the space* $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$*.*

3. Convexity of the $({\rm v},{\rm w})$-Mabuchi energy along weak geodesics

The following formulas allow us to compute the second variations of the energy functionals $\mathcal{E}_{\rm v}^\theta$ and $\mathcal{E}_{\rm w}$ along weak geodesics.

Lemma 5. *Let* Φ *be a* $\mathbb{G}$*-invariant smooth function on* $\hat{X}$ *related to a family of* $\mathbb{T}$*-invariant functions* $(\phi_t)_{t\in[0,1]}$ *on* X *by* (8)*, and* θ *a* 2*-form on* X*. We have*

$$
\begin{aligned}
(\pi_X^*\omega + dd^c\Phi)^{[n+1]} &= -\big(\ddot{\phi}_t - |d\dot{\phi}_t|^2_{\phi_t}\big)\omega_{\phi_t}^{[n]} \wedge dt \wedge ds,\\
\pi_X^*\theta \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} &= -\Big(\big(\ddot{\phi}_t - |d\dot{\phi}_t|^2_{\phi_t}\big)\theta \wedge \omega_{\phi_t}^{[n-1]}\\
&\qquad + (\theta, d\dot{\phi}_t \wedge d^c\dot{\phi}_t)_\phi\, \omega_{\phi_t}^{[n]}\Big) \wedge dt \wedge ds.
\end{aligned}
$$

where $\dot{\phi}_t$ *and* $\ddot{\phi}_t$ *are the* t*-derivatives of* ϕ_t*.*

Proof. We have $dd^c\Phi = dd^c\phi + \gamma_\phi$ such that

$$
\gamma_\phi := -d^c\dot{\phi} \wedge dt - d\dot{\phi} \wedge ds - \ddot{\phi} dt \wedge ds.
$$

By a straightforward calculation we get $\gamma_\phi^2 = 2d\dot{\phi} \wedge d^c\dot{\phi} \wedge dt \wedge ds$ and $\gamma_\phi^3 = 0$. We calculate

$$
\begin{aligned}
(\omega + dd^c\Phi)^{[n+1]} &= (\omega_\phi + \gamma_\phi)^{[n+1]}\\
&= \omega_\phi^{[n+1]} + \omega_\phi^{[n]} \wedge \gamma_\phi + \frac{1}{2}\omega_\phi^{[m-1]} \wedge \gamma_\phi^2\\
&= -\big(\ddot{\phi} - |d\dot{\phi}|^2_\phi\big)\omega_\phi^{[n]} \wedge dt \wedge ds.
\end{aligned}
$$

For the second identity

$$\theta\wedge(\omega+dd^c\Phi)^{[n]}=\theta\wedge\left[\omega_\phi^{[n]}+\omega_\phi^{[n-1]}\wedge\gamma_\phi+\frac{1}{2}\omega_\phi^{[n-2]}\wedge\gamma_\phi^2\right]$$
$$(21)\qquad =\theta\wedge\omega_\phi^{[n-1]}\wedge\gamma_\phi+\frac{1}{2}\theta\wedge\omega_\phi^{[n-2]}\wedge\gamma_\phi^2$$
$$=-\big((\ddot\phi-|d\dot\phi|^2)\theta\wedge\omega_\phi^{[n-1]}+(\theta,d\dot\phi\wedge d^c\dot\phi)\omega_\phi^{[n]}\big)\wedge dt\wedge ds.$$

□

We start by computing the second variation of $\mathcal{E}_{\mathrm{v}}^\theta$ and $\mathcal{E}_{\mathrm{w}}$ on smooth families of smooth $\mathbb{T}$-invariant Kähler potentials.

Lemma 6. *Let $(\phi_t)_{t\in[0,1]}\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ be a smooth family of Kähler potentials and Φ the $\mathbb{G}$-invariant function on $\hat X$, corresponding to $(\phi_t)_{t\in[0,1]}$ given by* (8). *Let $\phi_\tau:=\Phi(\cdot,\tau)$.*

(i) The second variation of the function $\tau\mapsto\mathcal{E}_{\mathrm{w}}(\phi_\tau)$ on $\mathbb{A}$ is given by

$$(22)\qquad dd^c\mathcal{E}_{\mathrm{w}}(\tau)=\int_X \mathrm{w}(m_\Phi)\,(\pi_X^*\omega+dd^c\Phi)^{[n+1]}\,,$$

where $m_\Phi(x,\tau):=m_{\phi_\tau}$, and $\int_X$ is the push forward map on $\pi_{\mathbb{A}}:\hat X\to\mathbb{A}$.

(ii) The second variation of the function $\tau\mapsto\mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau)$ on $\mathbb{A}$ is given by

$$(23)\qquad dd^c\mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau)=\int_X \mathrm{v}(m_\Phi)\pi_X^*\theta\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]}+\langle d\mathrm{v}(m_\Phi),m_\theta\rangle(\pi_X^*\omega+dd^c\Phi)^{[n+1]}.$$

Proof. The proof of (i) is given in [9, Proposition 10.d]. For (ii), by the $\mathbb{S}^1$-invariance of Φ, we have

$$(24)\qquad dd^c\mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau)=\left[\frac{\partial^2}{\partial\tau\partial\bar\tau}\mathcal{E}_{\mathrm{v}}^\theta(\tau)\right]d\tau\wedge d\bar\tau=-\left[\frac{d^2}{dt^2}\mathcal{E}_{\mathrm{v}}^\theta(\phi_t)\right]dt\wedge ds.$$

Let $\mathcal{B}_{\mathrm{v}}$ be the 1-form on $\mathcal{K}(X,\omega)^{\mathbb{T}}$ defined by

$$(\mathcal{B}_{\mathrm{v}})_\phi(\dot\phi):=\int_X\dot\phi\left[\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]}+\langle(d\mathrm{v})(m_\phi),m_\theta\rangle\omega_\phi^{[n]}\right].$$

Using (17) we compute

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{E}^{\theta}_{\mathrm{v}}(\phi_t) &= (\boldsymbol{\delta}\mathcal{B}_{\mathrm{v}}(\dot{\phi}))_{\phi_t}(\dot{\phi}) \\
&\quad + \int_X \ddot{\phi}\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]} + \langle d\mathrm{v}(m_\phi), m_\theta\rangle\ddot{\phi}\omega_\phi^{[n]} \\
&= \int_X (\ddot{\phi} - |d\dot{\phi}|^2_\phi)\mathrm{v}(m_\phi)\theta\wedge\omega_\phi^{[n-1]} \\
&\quad + \int_X (\ddot{\phi} - |d\dot{\phi}|^2_\phi)\langle d\mathrm{v}(m_\phi), m_\theta\rangle\omega_\phi^{[n]} \\
&\quad + \int_X (\theta, d\dot{\phi}\wedge d^c\dot{\phi})_\phi \mathrm{v}(m_\phi)\omega_\phi^{[n]},
\end{aligned}
\tag{25}
$$

in the second equality we use the computation of $(\boldsymbol{\delta}\mathcal{B}_{\mathrm{v}}(\dot{\phi}))_{\phi_t}(\dot{\phi})$ given by [31, equation (18)] in the proof of [31, Lemma 4]. The identity (23) follows from Lemma 5 and the equations (24) and (25). □

Now we consider the second variations along a weak geodesic segment.

Lemma 7. *Let $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a weak geodesic segment and Φ the $\mathbb{G}$-invariant corresponding solution of the boundary value problem* (9) *on $\hat{X}$. The following identities holds in the weak sense of currents*

$$
dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau) = 0, \tag{26}
$$

$$
dd^c\mathcal{E}^{\theta}_{\mathrm{v}}(\phi_\tau) = \int_X \mathrm{v}(m_\Phi)\pi_X^*\theta\wedge\left(\pi_X^*\omega + dd^c\Phi\right)^{[n]}. \tag{27}
$$

Proof. The equation (26) is already established in [7, Proposition 2.16] and [9, Proposition 10.4]. The same argument works for (27). We give the proof for convenience of the reader. Let $(\phi^\epsilon_t)_{t\in[0,1]}$ be the ϵ-geodesic approximating $(\phi_t)_{t\in[0,1]}$ and Φ^ϵ the corresponding solution of the elliptic Dirichlet problem (13). By Lemma 6, we have

$$
\begin{aligned}
dd^c\mathcal{E}_{\mathrm{w}}(\phi^\epsilon_\tau) &= \int_X \epsilon\mathrm{w}(m_{\Phi^\epsilon})\Big(\pi_X^*\omega + \frac{\sqrt{-1}d\tau\wedge d\bar{\tau}}{2|\tau|^2}\Big)^{[n+1]}, \\
dd^c\mathcal{E}^{\theta}_{\mathrm{v}}(\phi^\epsilon_\tau) &= \int_X \mathrm{v}(m_{\Phi^\epsilon})\pi_X^*\theta\wedge(\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n]} \\
&\quad + \epsilon\langle d\mathrm{v}(m_{\Phi^\epsilon}), m_\theta\rangle\Big(\omega + \frac{\sqrt{-1}d\tau\wedge d\bar{\tau}}{2|\tau|^2}\Big)^{[n+1]}.
\end{aligned}
\tag{28}
$$

We have Φ^ϵ is decreasing in $\epsilon > 0$ and $\Phi^\epsilon \to \Phi$ in $(C^{1,1}, \|\cdot\|_{C^1} + \| dd^c \cdot \|_{L^\infty})$ when $\epsilon \to 0$. Using the identity

$$m_{\phi_\tau^\epsilon} = m_\omega + d^c \phi_\tau^\epsilon,$$

and the fact that v is smooth on P_α, we obtain

$$\mathcal{E}_{\mathrm{w}}(\phi_\tau^\epsilon) \to \mathcal{E}_{\mathrm{w}}(\phi_\tau) \text{ and } \mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau^\epsilon) \to \mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau),$$

since the Monge-Ampère measures converges weakly under decreasing limits. It follows that

$$dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau^\epsilon) \to dd^c\mathcal{E}_{\mathrm{w}}(\phi_\tau) \text{ and } dd^c\mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau^\epsilon) \to dd^c\mathcal{E}_{\mathrm{v}}^\theta(\phi_\tau),$$

in the weak sense of distributions. Passing to the limit when $\epsilon \to 0$ in the rhs of the equations of (28), and using the fact that $\pi_X^*\theta \wedge (\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n]} \to \pi_X^*\theta \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}$ in the sense of measures (since $\Phi^\epsilon \searrow \Phi$), we obtain (26) and (27). □

Corollary 4. *Let θ be a $\mathbb{T}$-invariant Kähler form. The functional $\mathcal{E}_{\mathrm{v}}^\theta$ is strictly convex on weak geodesic segments. In particular $\mathcal{E}_{\mathrm{v}}^\theta$ has at most one critical point in $\mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$.*

Proof. Using (25), we see that the following formula holds on any weak geodesic segment

$$\frac{d^2}{dt^2}\mathcal{E}_{\mathrm{v}}^\theta(\phi_t) = \int_X (\theta, d\dot{\phi} \wedge d^c\dot{\phi})_{\phi_t} \mathrm{v}(m_{\phi_t})\omega_{\phi_t}^{[n]} > 0,$$

since θ is a Kähler form. Thus, $t \mapsto \mathcal{E}_{\mathrm{v}}^\theta(\phi_t)$ is strictly convex. □

Now we consider the entropy part of the (v,w)-Mabuchi energy. For a family of $\mathbb{G}$-invariant volume forms Θ_τ on X we associate a function $\Psi := \log(\Theta_\tau)$ on $\hat{X}$, given locally on a holomorphic coordinate patch $(U, (z_j)_{j=1,n})$ on X by

$$\Psi_U = \log\left(\frac{\Theta_\tau}{\mathrm{vol_U}}\right), \tag{29}$$

where $\mathrm{vol_U}$ is the volume form of the flat Kähler metric $\frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ on U. For $(\phi_\tau)_{\tau\in\mathbb{A}} \subset \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$, we define

$$\begin{aligned}\mathcal{H}^{\Psi}_{\mathrm{v}}(\phi_\tau) &:= \int_X \log\big(\frac{\Theta_\tau}{\omega^n}\big)\mathrm{v}(m_{\phi_\tau})\omega_{\phi_\tau}^{[n]}\\ &= \int_X \log\big(\frac{e^{\psi_\tau}}{\omega^n}\big)\mathrm{v}(m_{\phi_\tau})\omega_{\phi_\tau}^{[n]},\end{aligned} \tag{30}$$

where $\psi_\tau := \Psi_{|X_\tau}$.

Lemma 8. *Let $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a weak geodesic segment and denote by Φ the associated $\mathbb{G}$-invariant function on $\hat{X}$. If Ψ (given by* (29)*) is smooth, then we have*

$$dd^c\big(\mathcal{H}^{\Psi}_{\mathrm{v}}(\phi_\tau) - 2\mathcal{E}^{\mathrm{Ric}(\omega)}_{\mathrm{v}}(\phi_\tau)\big) = \int_X \mathrm{v}(m_\Phi)dd^c\Psi \wedge \big(\pi_X^*\omega + dd^c\Phi\big)^{[n]}, \tag{31}$$

in the weak sense of currents.

Proof. Let $f(\tau)$ be a test function with support in $\mathbb{A}$ and $\hat{f} := \pi_{\mathbb{A}}^{\star} f$. We have

$$\begin{aligned}\langle dd^c\mathcal{H}^{\Psi}(\phi_\tau), f\rangle &= \int_{\mathbb{A}} dd^c f \int_X \log\big(\frac{e^{\psi_\tau}}{\omega^n}\big)\mathrm{v}(m_{\phi_\tau})\omega_{\phi_\tau}^{[n]}\\ &= \int_{\mathbb{A}} dd^c f \int_{X_\tau} \Big(\log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big)\mathrm{v}(m_\Phi)(\pi_X^*\omega + dd^c\Phi)^{[n]}\Big)_{|X_\tau}\\ &= \int_{\hat{X}} \log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big)\mathrm{v}(m_\Phi)dd^c\hat{f} \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}\\ &= \int_{\hat{X}} \log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big)d(\mathrm{v}(m_\Phi)) \wedge d^c\hat{f} \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}\\ &\quad - \int_{\hat{X}} \mathrm{v}(m_\Phi)d\log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big) \wedge d^c\hat{f} \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}\end{aligned}$$

Notice that $d(\mathrm{v}(m_\Phi)) \wedge d^c\hat{f} \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} = 0$, using approximation by an ϵ-geodesic Φ^ϵ and the fact that $d\hat{f}$ is zero on fundamental vector fields of the $\mathbb{T}$-action: Indeed

$$\begin{aligned}&d(\mathrm{v}(m_{\Phi^\epsilon})) \wedge d^c\hat{f} \wedge (\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n]}\\ &\qquad = \langle(d\mathrm{v})(m_{\Phi^\epsilon}), (d\hat{f})_{\mathfrak{t}}\rangle(\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n+1]} = 0,\end{aligned}$$

since $(d\hat{f})_{\mathfrak{t}}$ the restriction of $d\hat{f}$ on the fundamental vector fields of $\mathfrak{t}$ is zero ($(d\hat{f})_{\mathfrak{t}} = (df \circ \pi_*)_{\mathfrak{t}} = 0$). Using that $\Phi^\epsilon \searrow \Phi$ in $C^{1,1}$ topology, passing to the

limit as $\epsilon \to 0$, yields $d(\mathrm{v}(m_\Phi)) \wedge d^c \hat{f} \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} = 0$.
Integration by parts gives

$$\begin{aligned}\langle dd^c\mathcal{H}_{\mathrm{v}}^{\Psi}, f\rangle = &\int_{\hat{X}} \hat{f} d\log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big) \wedge d^c\mathrm{v}(m_\Phi) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} \\ &+ \int_{\hat{X}} \hat{f}\mathrm{v}(m_\Phi) dd^c \log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}.\end{aligned}$$

Notice that the first integral in the first equality is zero: Indeed, if $\Phi = \Phi^\epsilon$ is an ϵ-geodesic, then

$$\begin{aligned}&\int_{\hat{X}} \hat{f} d\log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big) \wedge d^c\mathrm{v}(m_{\Phi^\epsilon}) \wedge (\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n]} \\ &\quad = \int_{\hat{X}} \hat{f}\langle (d\mathrm{v})(m_{\Phi^\epsilon}), (d^c\Psi)_{\mathfrak{t}} - 2m_{\mathrm{Ric}(\omega)}\rangle (\pi_X^*\omega + dd^c\Phi^\epsilon)^{[n+1]} \\ &\quad = \epsilon \int_{\hat{X}} \hat{f}\langle (d\mathrm{v})(m_{\Phi^\epsilon}), (d^c\Psi)_{\mathfrak{t}} - 2m_{\mathrm{Ric}(\omega)}\rangle \Big(\pi_X^*\omega + \frac{\sqrt{-1}d\tau \wedge d\bar{\tau}}{2|\tau|^2}\Big)^{[n+1]},\end{aligned}$$

since $\Phi^\epsilon \searrow \Phi$ in $C^{1,1}$ topology, passing to the limit as $\epsilon \to 0$, yields

$$\int_{\hat{X}} \hat{f} d\log\big(\frac{e^{\Psi}}{\pi_X^*\omega^n}\big) \wedge d^c\mathrm{v}(m_\Phi) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} = 0.$$

It follows that,

$$\begin{aligned}\langle dd^c\mathcal{H}_{\mathrm{v}}^{\Psi}, f\rangle = &\int_{\hat{X}} \hat{f}\mathrm{v}(m_\Phi) dd^c\Psi \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} \\ &+ 2\int_{\hat{X}} \hat{f}\mathrm{v}(m_\Phi)\pi_X^*\mathrm{Ric}(\omega) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]}\end{aligned}$$

Combining the above equality with (27) completes the proof. □

Following [7], we consider the following modified version of the (v, w)-Mabuchi functional

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}^{\Psi} := \mathcal{H}_{\mathrm{v}}^{\Psi} - 2\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)} + \mathcal{E}_{\mathrm{w}}. \tag{32}$$

Notice that for $\Psi := \log(\omega + dd^c\phi_\tau)^n$ we have $\mathcal{M}_{\mathrm{v},\mathrm{w}}^{\Psi} = \mathcal{M}_{\mathrm{v},\mathrm{w}}$.

Corollary 5. *Under the hypothesis of Lemma 8, if Ψ is only locally bounded and $dd^c\Psi \geq 0$ as a current, then*

$$dd^c\mathcal{M}_{\mathrm{v}}^{\Psi}(\phi_\tau) = \int_X \mathrm{v}(m_\Phi) dd^c\Psi \wedge \left(\pi_X^*\omega + dd^c\Phi\right)^{[n]},$$

in the weak sense of currents.

Proof. Let Ψ_j be a sequence of uniformly bounded, $\mathbb{G}$-invariant smooth functions on $\hat{X}$ such that $\Psi_j \to \Psi$ almost everywhere on X and everywhere on $\mathbb{A}$. Using Lemma 8 we have

$$dd^c\mathcal{M}_{\mathrm{v}}^{\Psi_j}(\phi_\tau) = \int_X \mathrm{v}(m_\Phi) dd^c\Psi_j \wedge \left(\pi_X^*\omega + dd^c\Phi\right)^{[n]}.$$

By the dominated convergence theorem (notice that $\mathrm{v}(m_\Phi)$ is uniformly bounded), we can pass to the limit when $j \to \infty$ (see e.g. [22, Proposition 3.2]). $\square$

Now, we can use the arguments of Berman–Berndtsson in [7] to deduce the weak convexity of the (v, w)-Mabuchi energy along weak geodesic segments. We will need the following regularization result which is the main ingredient in the proof of Berman–Berndtsson for the weak convexity of the Mabuchi energy [7, Theorem 3.3].

Proposition 1 ([7]). *Let $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a weak geodesic segment, and Φ the corresponding weak solution of (9). Let $\Psi := \log(\pi_X^*\omega + dd^c\Phi)^n$.*

(i) There exist a family of locally bounded $\mathbb{G}$-invariant functions $(\Psi_A)_{A>0}$ on $\hat{X}$, such that $dd^c\Psi_A \geq 0$ in the weak sense of currents, and $\Psi_A \searrow \Psi$ as $A \to \infty$.

(ii) For fixed $A > 0$, there exist a family of $\mathbb{G}$-invariant functions $(\Psi_{k,A})_{A>0}$ on $\hat{X}$ with continuous dependence on $\tau \in \mathbb{A}$, such that the currents $T_{A,k} := dd^c\Psi_{k,A} \wedge (\pi_X^*\omega + dd^c\Phi)^n$ are positive and $\Psi_{k,A} \to \Psi_A$ pointwise almost everywhere on X and everywhere on τ as $k \to \infty$.

Using the above proposition together with Corollary 5, we get the following

Theorem 4. *Let $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a weak geodesic segment. The function $\tau \mapsto \mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_\tau)$ is weakly subharmonic on $\mathbb{A}$. In particular, $\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)$ is weakly convex along the weak geodesic (ϕ_t).*

Proof. By Corollary 5, since the function Ψ_A from (i) in Proposition 1 is locally bounded, we obtain

$$dd^c\mathcal{M}^{\Psi_A}_{\mathrm{v,w}}(\phi_\tau) = \int_X \mathrm{v}(m_\Phi)T_A. \tag{33}$$

where $T_A := dd^c\Psi_A \wedge (\pi_X^*\omega + dd^c\Phi)^n$. Now using the fact that $\mathrm{v}(m_\Phi)T_{A,k} \geq 0$ are positive Radon measures which converge weakly to $\mathrm{v}(m_\Phi)T_A$ as $k \to \infty$. It follows that $dd^c\mathcal{M}^{\Psi_A}_{\mathrm{v,w}}(\phi_\tau) \geq 0$. On the other hand we have $\mathcal{M}^{\Psi_A}_{\mathrm{v,w}}(\phi_\tau) \to \mathcal{M}_{\mathrm{v,w}}(\phi_\tau)$ as $A \to \infty$. Thus, $dd^c\mathcal{M}_{\mathrm{v,w}}(\phi_\tau) \geq 0$ in the weak sense of currents. □

To get the pointwise convexity of $t \mapsto \mathcal{M}_{\mathrm{v,w}}(\phi_t)$, we have to show that it is continuous. For the energy part $t \mapsto -2\mathcal{E}^{\mathrm{Ric}(\omega)}_{\mathrm{v}}(\phi_t) + \mathcal{E}_{\mathrm{w}}(\phi_t)$, it is clear from (16) and (18) that it is a continuous function, since $t \to \phi_t$ is a continuous family. As in the case when $\mathrm{v} \equiv 1$ on P_α (see [7]), it is not a priori clear that the entropy part $t \mapsto \mathcal{H}_{\mathrm{v}}(\phi_t)$ is continuous.

Theorem 5. *The* (v,w)*-Mabuchi energy* $\mathcal{M}_{\mathrm{v,w}}$ *is continuous along weak geodesics and therefore convex in the pointwise sense.*

Proof. The argument is very similar to the one of Berman–Berndtsson in [7, Theorem 3.4], the only difference is in the calculation of the second variation of the entropy term involving the weighted measure $\mathrm{v}(m_{\phi_\tau})\omega^{[n]}_{\phi_\tau}$.

Let $\kappa_\epsilon(s)$ be a sequence of strictly convex functions such that $\kappa'_\epsilon(s) \geq 1$ and $\kappa_\epsilon(s) \to s$ as $\epsilon \to 0$. Let ζ_j be a partition of unity subordinate to an open cover of X. We consider the following modification of the entropy term

$$\mathcal{H}^{\Psi_{A,k}}_{\mathrm{v},j,\epsilon}(\phi_\tau) = \int_X \zeta_j\kappa_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}(\cdot,\tau)}}{\omega^n}\big)\big)\mathrm{v}(m_{\phi_\tau})\omega^{[n]}_{\phi_\tau},$$

where $\Psi_{A,k}$ is given in Proposition 1 (ii) (see also [7, Theorem 3.3] for more details). From the calculations in the proof of Lemma 8 we have

$$\begin{aligned}
dd^c\mathcal{H}^{\Psi_{A,k}}_{\mathrm{v},j,\epsilon}(\phi_\tau) &= \int_X \zeta_j\mathrm{v}(m_{\phi_\tau})dd^c\kappa_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} \\
&= \int_X \zeta_j\mathrm{v}(m_{\phi_\tau})d\Big(\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)d^c\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\Big) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} \\
&= \int_X \zeta_j\mathrm{v}(m_{\phi_\tau})\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)dd^c\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]} \\
&\quad + \int_X \zeta_j\mathrm{v}(m_{\phi_\tau})\kappa''_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)d\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big) \\
&\qquad\qquad \wedge d^c\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big) \wedge (\pi_X^*\omega + dd^c\Phi)^{[n]},
\end{aligned}$$

It follows that,

$$
\begin{aligned}
dd^c\mathcal{H}^{\Psi_{A,k}}_{\mathrm{v},j,\epsilon}(\phi_\tau) &= \int_X \zeta_j \mathrm{v}(m_{\phi_\tau})\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)T_{A,k}\\
&+2\int_X \zeta_j \mathrm{v}(m_{\phi_\tau})\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)\pi_X^*\mathrm{Ric}(\omega)\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]}\\
&+\int_X \zeta_j \mathrm{v}(m_{\phi_\tau})\kappa''_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)d\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\\
&\qquad\wedge d^c\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]},
\end{aligned}
\tag{34}
$$

where $T_{A,k} := dd^c\Psi_{A,k}\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]}$. Now we introduce the following modified version of the (v,w)-Mabuchi energy:

$$
\mathcal{M}^{\Psi_{A,k}}_{\mathrm{v},\mathrm{w},j,\epsilon} := \mathcal{H}^{\Psi_{A,k}}_{\mathrm{v},j,\epsilon} - 2\mathcal{E}^{\theta_j}_{\mathrm{v},j},
$$

where $\theta_j := \zeta_j\mathrm{Ric}(\omega)$. Combining (34) with (26) and (27), we obtain

$$
\begin{aligned}
dd^c\mathcal{M}^{\Psi_{A,k}}_{\mathrm{v},\mathrm{w},j,\epsilon}(\phi_\tau) &= \int_X \zeta_j \mathrm{v}(m_{\phi_\tau})\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)T_{A,k}\\
&+2\int_X \Big[1-\kappa'_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)\Big]\zeta_j \mathrm{v}(m_{\phi_\tau})\pi_X^*\mathrm{Ric}(\omega)\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]}\\
&+\int_X \zeta_j \mathrm{v}(m_{\phi_\tau})\kappa''_\epsilon\big(\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\big)d\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\\
&\qquad\wedge d^c\log\big(\frac{e^{\Psi_{A,k}}}{\omega^n}\big)\wedge(\pi_X^*\omega+dd^c\Phi)^{[n]}.
\end{aligned}
$$

Since κ_ϵ is strictly convex, the integral in the last line is positive, and using that $\kappa'_\epsilon(s)\geq 1$, together with $T_{A,k}\geq 0$, it is also clear that the integral in the first line is positive. For the remaining integral we can bound it from below by $-C_{\epsilon,j}\frac{\sqrt{-1}d\tau\wedge d\bar\tau}{2|\tau|^2}$ for some $C_{\epsilon,j}\geq 0$. Thus,

$$
dd^c\mathcal{M}^{\Psi_{A,k}}_{\mathrm{v},\mathrm{w},j,\epsilon}(\phi_\tau) \geq -C_{\epsilon,j}\frac{\sqrt{-1}d\tau\wedge d\bar\tau}{2|\tau|^2}.
$$

It follows that the function $t\mapsto \mathcal{M}^{\Psi_{A,k}}_{\mathrm{v},\mathrm{w},j,\epsilon}(\phi_\tau)+C_{\epsilon,j}t^2$ (where $\tau=e^{-t+is}$) is weakly convex. On the other hand $\tau\mapsto\mathcal{M}^{\Psi_{A,k}}_{\mathrm{v},\mathrm{w},j,\epsilon}(\phi_\tau)$ is continuous since $\Psi_{A,k}$ is continuous in $\tau\in\mathbb{A}$, by Proposition 1 (ii). It follows that $t\mapsto$

$\mathcal{M}^{\Psi_{A,k}}_{\mathrm{v,w},j,\epsilon}(\phi_t) + C_{\epsilon,j}t^2$ is convex in the pointwise sense. Using the equation,

$$\mathcal{M}_{\mathrm{v,w}}(\phi_t) - \mathcal{E}_{\mathrm{w}}(\phi_t) = \lim_{\epsilon\to 0}\sum_j \mathcal{M}^{\Psi_{A,k}}_{\mathrm{v,w},j,\epsilon}(\phi_\tau) + C_\epsilon t^2,$$

where $C_\epsilon = \sum_j C_{j,\epsilon}$, we infer that $t \mapsto \mathcal{M}_{\mathrm{v,w}}(\phi_t) - \mathcal{E}_{\mathrm{w}}(\phi_t)$ is convex in the pointwise sense, thus continuous. By (20), the function $t \to \mathcal{H}_{\mathrm{v}}(\phi_t)$ is lower semicontinuous, then it is continuous on $[0,1]$. This, completes the proof. □

4. Proof of Corollary 1

Lemma 9. *Given a weak geodesic segment $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ connecting $\phi_0, \phi_1 \in \mathcal{K}(X,\omega)^{\mathbb{T}}$, we have the following inequalities*

$$\begin{aligned}\lim_{t\to 0^+}\frac{\mathcal{H}_{\mathrm{v}}(\phi_t)-\mathcal{H}_{\mathrm{v}}(\phi_0)}{t} \geq &-\int_X \tilde{\mathrm{v}}(m_{\phi_0})\dot{\phi}\,(\mathrm{Ric}(\omega_{\phi_0}) - \mathrm{Ric}(\omega))\wedge\omega_{\phi_0}^{[n-1]}\\ &-\int_X \langle (d\tilde{\mathrm{v}})(m_{\phi_0}), m_{\mathrm{Ric}(\omega_{\phi_0})} - m_{\mathrm{Ric}(\omega)}\rangle\dot{\phi}\omega_{\phi_0}^{[n]}\\ &-\int_X \dot{\phi}\Delta_{\phi_0}(\tilde{\mathrm{v}}(m_{\phi_0}))\omega_{\phi_0}^{[n]}\end{aligned}$$

where $\dot{\phi} = \frac{d\phi_t}{dt}\Big|_{t=0^+}$ *and* $\tilde{\mathrm{v}} := \frac{\mathrm{v}}{\mathrm{vol}(X,\mathrm{v}(m_\omega)\omega^{[n]})}$.

Proof. By convexity of the entropy with respect to the affine structure on the space of probability measures (see e.g. [7, 23]) and using (20), we get

$$\begin{aligned}\frac{\mathcal{H}_{\mathrm{v}}(\phi_t)-\mathcal{H}_{\mathrm{v}}(\phi_0)}{t} &= \frac{\mathrm{Ent}_{\mu_\omega}(\mu_{\mathrm{v}}(\phi_t)) - \mathrm{Ent}_{\mu_\omega}(\mu_{\mathrm{v}}(\phi_0))}{t}\\ &\geq \int_X \log\left(\frac{\mu_{\mathrm{v}}(\phi_0)}{\mu_\omega}\right)\frac{\mu_{\mathrm{v}}(\phi_t)-\mu_{\mathrm{v}}(\phi_0)}{t}\\ &= \int_X \log\left(\frac{\omega^n_{\phi_0}}{\mu_\omega}\right)\frac{\mu_{\mathrm{v}}(\phi_t)-\mu_{\mathrm{v}}(\phi_0)}{t}\\ &\quad+\int_X \log(\tilde{\mathrm{v}}(m_{\phi_0}))\frac{\mu_{\mathrm{v}}(\phi_t)-\mu_{\mathrm{v}}(\phi_0)}{t}\\ &= \int_X \log\left(\frac{\omega^n_{\phi_0}}{\mu_\omega}\right)\frac{\mu_{\mathrm{v}}(\phi_t)-\mu_{\mathrm{v}}(\phi_0)}{t}\\ &\quad+\int_X \frac{1}{t}\left(\log(\tilde{\mathrm{v}}(m_{\phi_0}))\tilde{\mathrm{v}}(m_{\phi_t})\omega_{\phi_t}^{[n]} - \log(\tilde{\mathrm{v}}(m_{\phi_0}))\tilde{\mathrm{v}}(m_{\phi_0})\omega_{\phi_0}^{[n]}\right)\\ &= \int_X \log\left(\frac{\omega^n_{\phi_0}}{\mu_\omega}\right)\frac{\mu_{\mathrm{v}}(\phi_t)-\mu_{\mathrm{v}}(\phi_0)}{t}\\ &\quad+\int_X \frac{1}{t}\left(\log(\tilde{\mathrm{v}}(m_{\phi_0})) - \log(\tilde{\mathrm{v}}(m_{\phi_t}))\right)\tilde{\mathrm{v}}(m_{\phi_t})\omega_{\phi_t}^{[n]}\end{aligned}$$

where $\mu_{\mathrm{v}}(\phi_t)$ is the probability measure (11) and we have used the fact that

$$\int_X \log(\tilde{\mathrm{v}}(m_{\phi_t}))\tilde{\mathrm{v}}(m_{\phi_t})\omega_{\phi_t}^{[n]} = \int_X \log(\tilde{\mathrm{v}}(m_{\phi_0}))\tilde{\mathrm{v}}(m_{\phi_0})\omega_{\phi_0}^{[n]} = \mathrm{const}$$

is a constant independent of t. We thus compute

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{\mathcal{H}_{\mathrm{v}}(\phi_t) - \mathcal{H}_{\mathrm{v}}(\phi_0)}{t} \\
&\quad \geq -\int_X \tilde{\mathrm{v}}(m_{\phi_0}) d\dot{\phi} \wedge d^c \log\left(\frac{\omega_{\phi_0}^n}{\omega^n}\right) \wedge \omega_{\phi_0}^{[n-1]} + \int_X \dot{\phi} dd^c(\tilde{\mathrm{v}}(m_{\phi_0})) \wedge \omega_{\phi_0}^{[n-1]} \\
&\quad = \int_X \dot{\phi}\Big(d(\tilde{\mathrm{v}}(m_{\phi_0})), d\log\left(\frac{\omega_{\phi_0}^{[n]}}{\omega^n}\right)\Big)_{\phi_0} \omega_{\phi_0}^{[n]} \\
&\qquad + \int_X \tilde{\mathrm{v}}(m_{\phi_0})\dot{\phi} dd^c \log\left(\frac{\omega_{\phi_0}^n}{\omega^n}\right) \wedge \omega_{\phi_0}^{[n-1]} + \int_X \dot{\phi} dd^c(\tilde{\mathrm{v}}(m_{\phi_0})) \wedge \omega_{\phi_0}^{[n-1]} \\
&\quad = -\int_X \tilde{\mathrm{v}}(m_{\phi_0})\dot{\phi}\left(\mathrm{Ric}(\omega_{\phi_0}) - \mathrm{Ric}(\omega)\right) \wedge \omega_{\phi_0}^{[n-1]} \\
&\qquad - \int_X \langle (d\tilde{\mathrm{v}})(m_{\phi_0}), m_{\mathrm{Ric}(\omega_{\phi_0})} - m_{\mathrm{Ric}(\omega)}\rangle \dot{\phi}\omega_{\phi_0}^{[n]} \\
&\qquad - \int_X \dot{\phi}\Delta_{\phi_0}(\tilde{\mathrm{v}}(m_{\phi_0}))\omega_{\phi_0}^{[n]}.
\end{aligned}$$

□

Now we are in position to give the proof of Corollary 1.

Proof of Corollary 1. Let $(\phi_t)_{t\in[0,1]} \in \mathcal{K}^{1,1}(X,\omega)^{\mathbb{T}}$ be a weak geodesic segment connecting $\phi_0, \phi_1 \in \mathcal{K}(X,\omega)^{\mathbb{T}}$. We suppose that $\tilde{\mathrm{v}} := \frac{\mathrm{v}}{\mathrm{vol}(X, \mathrm{v}(m_\omega)\omega^{[n]})} = \mathrm{v}$. We have

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{\mathcal{E}_{\mathrm{w}}(\phi_t) - \mathcal{E}_{\mathrm{w}}(\phi_0)}{t} = \int_X \dot{\phi}\mathrm{w}(m_{\phi_0})\omega_{\phi_0}^{[n]} \\
&\lim_{t\to 0^+} \frac{\mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_t) - \mathcal{E}_{\mathrm{v}}^{\mathrm{Ric}(\omega)}(\phi_0)}{t} \\
&\qquad = \int_X \dot{\phi}\big(\mathrm{v}(m_{\phi_0})\mathrm{Ric}(\omega) \wedge \omega_{\phi_0}^{[n-1]} + \langle (d\mathrm{v})(m_{\phi_0}), m_{\mathrm{Ric}(\omega)}\rangle \omega_{\phi_0}^{[n]}\big)
\end{aligned}$$

By Lemma 9 and Theorem 3 we get

$$\lim_{t\to 0^+} \frac{\mathcal{M}_{\mathrm{v,w}}(\phi_t) - \mathcal{M}_{\mathrm{v,w}}(\phi_0)}{t} \geq \int_X (-\mathrm{Scal}_{\mathrm{v}}(\phi_0) + \mathrm{w}(m_{\phi_0}))\dot{\phi}\omega_{\phi_0}^{[n]}.$$

Using the sub-slop inequality for the convex function $\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)$ we get

$$\begin{aligned}\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_1)-\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_0) &\geq \lim_{t\to 0^+}\frac{\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_t)-\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_0)}{t}\\ &\geq \int_X(-\mathrm{Scal}_{\mathrm{v}}(\phi_0)+\mathrm{w}(m_{\phi_0}))\dot{\phi}\omega_{\phi_0}^{[n]}.\end{aligned}$$

By Cauchy-Schwartz inequality we obtain

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_1)-\mathcal{M}_{\mathrm{v},\mathrm{w}}(\phi_0)\geq -d(\phi_1,\phi_0)\parallel \mathrm{Scal}_{\mathrm{v}}(\phi_0)-\mathrm{w}(m_{\phi_0})\parallel_{L^2(X,\mu_{\phi_0})}.$$

For the general case where $\tilde{\mathrm{v}}\neq\mathrm{v}$, we apply the above formula to the $(\tilde{\mathrm{v}},\frac{\mathrm{w}}{\mathrm{vol}(X,\mathrm{v}(m_\omega)\omega^{[n]})})$-Mabuchi energy. □

5. Uniqueness of weighted cscK metrics

This section is devoted to establish Theorem 2 from the introduction. We will generalise the approach of [7, 17] to the weighted setting. Our proof is closer to the method used by Chen–Paun–Zeng [17], based on a generalisation of the bifurcation technique of Bando–Mabuchi [5].

Proposition 2. *Let X be a compact Kähler manifold with Kähler class α, $\mathbb{T}\subset\mathrm{Aut}_{\mathrm{red}}(X)$ a real torus with momentum polytope $\mathrm{P}_\alpha\subset\mathfrak{t}^*$ and $\mathrm{v}\in C^\infty(\mathrm{P}_\alpha,\mathbb{R}_{>0})$, and $\mathrm{w}\in C^\infty(\mathrm{P}_\alpha,\mathbb{R}_{>0})$ a non vanishing function on P_α. If $\omega\in\alpha$ is a $\mathbb{T}$-invariant Kähler metric, and $\varphi_0\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ such that $\omega_{\varphi_0}\in\alpha$ a (v,w)-extremal metric. Then, there exist ω_{ϕ_0} in the orbit of ω_{φ_0} under the action of the group $G:=\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^\circ$, and a smooth function $\phi:[0,\epsilon)\times X\to\mathbb{R}$, such that $\phi_t:=\phi(t,\cdot)\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ satisfies the equation*

$$\mathrm{Scal}_{\mathrm{v}}(\phi_t)-t\big(\mathrm{v}(m_{\phi_t})\Lambda_{\phi_t}\omega+\big\langle(d\mathrm{v})(m_{\phi_t}),m_\omega\big\rangle\big)=\ell_{\mathrm{ext}}(m_{\phi_t})\mathrm{w}(m_{\phi_t}),\tag{35}$$

where $\Lambda_\phi\omega$ is the trace of ω with respect to ω_ϕ and ℓ_{ext} is the (v,w)-extremal affine linear function of $(\alpha,\mathbb{T},\mathrm{v},\mathrm{w})$.

The proof follows from an application of the inverse function theorem as in [17]. To this end we need to find the Kähler metric ω_{ϕ_0} in the G-orbit of the $(\mathrm{v},\ell_{\mathrm{ext}}\cdot\mathrm{w})$ metric ω_{φ_0}, as stated in the theorem.

Let $\hat{\mathcal{K}}(X,\omega)^{\mathbb{T}}$ denote the space of $\mathbb{T}$-invariant Kähler potentials $\phi\in\mathcal{K}(X,\omega)^{\mathbb{T}}$ normalized by $\int_X\phi\mathrm{w}(m_\omega)\omega^{[n]}=0$, and $K^\circ:=\mathrm{Isom}^{\mathbb{T}}(X,\omega_{\varphi_0})^\circ\cap G$ the connected component of identity of the group of Hamiltonian isometries of (X,ω_{φ_0}) commuting with $\mathbb{T}$. As we suppose by definition that

$\mathrm{Scal}_{\mathrm{v}}(\varphi_0)/\mathrm{w}(m_{\varphi_0}) = \ell_{\mathrm{ext}}(m_{\varphi_0})$ is the Killing potential of a vector field in $\mathfrak{t}$, by [31, Corollary B.1] K° is a maximal connected compact subgroup of G. Following [17], we consider the map

$$\Psi^{\omega} : \mathcal{O} \to \hat{\mathcal{K}}(X, \omega)^{\mathbb{T}},$$

defined on the homogeneous manifold $\mathcal{O} := G/K^{\circ}$ by $\Psi^{\omega}(\sigma) := \phi_{\sigma}$, where $\phi_{\sigma} \in \hat{\mathcal{K}}(X, \omega)^{\mathbb{T}}$ is the unique potential such that

$$\sigma^* \omega = \omega + dd^c \phi_{\sigma}. \tag{36}$$

In the case when ω is (v, w)-extremal metric, [31, Theorem B.1] yields the following result, which is a straightforward generalization of [17, Proposition 4.3] describing $(T_{\sigma}\Psi^{\omega})(T_{\sigma}\mathcal{O})$ the image of the differential of Ψ^{ω} in $\sigma \in \mathcal{O}$.

Lemma 10. *[17, Proposition 4.3] If ω is a (v, w)-extremal metric, then the image $(T_{\sigma}\Psi^{\omega})(T_{\sigma}\mathcal{O})$ is given by real holomorphic vector fields*

$$\xi = J\mathrm{grad}_{\phi_{\sigma}}(f) \in \mathfrak{k},$$

where $\mathfrak{k} := \mathrm{Lie}(K^{\circ})$, $\phi_{\sigma} = \Psi^{\omega}(\sigma)$ and $f \in C^{\infty}(X, \mathbb{R})$.

By a result due to Mabuchi [34], any real holomorphic vector field $\xi \in (T_{\sigma}\Psi^{\omega})(T_{\sigma}\mathcal{O})$, gives rise to a smooth geodesic ray $(\phi_t)_{t\in\mathbb{R}} \in \mathcal{K}(X, \omega)^{\mathbb{T}}$, defined by $\phi_t := \Psi^{\omega}(\exp(t\xi))$. Using, strict convexity of the functional $\mathcal{E}^{\theta}_{\mathrm{v}}$ along weak geodesics (see Corollary 4) and the fact that $\exp : T\mathcal{O} \to \mathcal{O}$ is onto, we obtain

Lemma 11. *[17, Lemma 2] If ω is a (v, w)-extremal metric, then for any $\mathbb{T}$-invariant Kähler form θ on X, the functional $\mathcal{E}^{\theta}_{\mathrm{v}} \circ \Psi^{\omega} : \mathcal{O} \to \mathbb{R}$ is proper. In particular $\mathcal{E}^{\theta}_{\mathrm{v}}$ admits a unique minimum point on the orbit $\Psi^{\omega}(\mathcal{O})$.*

Now we are in position to give a sketch for the proof of Proposition 2, which is not materially different than [17, Theorem 1.2].

Proof of Proposition 2. Since ω_{φ_0} is a (v, w)-extremal metric, we can take $\phi_0 \in \Psi^{\omega_{\varphi_0}}(\mathcal{O})$ be the unique minimiser of $\mathcal{E}^{\omega}_{\mathrm{v}}$ (we take $\theta = \omega$ in Lemma 11).

Using Lemma 10 and (17), we have

$$\left\langle \mathrm{w}(m_{\phi_0})^{-1}(\mathrm{v}(m_{\phi_0})\Lambda_{\phi_0}\omega + \langle (d\mathrm{v})(m_{\phi_0}), m_\omega\rangle), f\right\rangle_{\mathrm{w},\phi_0} = 0, \tag{37}$$

for any $f \in \mathfrak{k}_{\phi_0}$ in the space of ω_{ϕ_0}-Killing potentials of elements of $\mathrm{Lie}(K) := \mathfrak{k}$, where $\langle\cdot,\cdot\rangle_{\mathrm{w},\phi_0}$ is the weighted inner product

$$\langle f, h\rangle_{\mathrm{w},\phi_0} = \int_X fh\mathrm{w}(m_{\phi_0})\omega_{\phi_0}^{[n]}. \tag{38}$$

Let $\mathcal{K}^{2,k+4}(X,\omega)^{\mathbb{T}}$ be the open set of $\mathbb{T}$-invariant ω-Kähler potentials with $\mathrm{L}^{2,k+4}$ regularity. We consider the map:

$$\mathcal{F}_{\mathrm{v},\mathrm{w}} : \mathcal{K}^{2,k+4}(X,\omega)^{\mathbb{T}} \times [0,1] \to \mathrm{L}^{2,k}(X,\mathbb{R})^{\mathbb{T}} \times [0,1],$$

defined by

$$\begin{aligned}
&\mathcal{F}_{\mathrm{v},\mathrm{w}}(\phi,t) := (F_{\mathrm{v},\mathrm{w}}(\phi,t),t),\\
&F_{\mathrm{v},\mathrm{w}}(\phi,t) := \frac{\mathrm{Scal}_{\mathrm{v}}(\phi) - t\big(\mathrm{v}(m_\phi)\Lambda_\phi(\omega) + \big\langle (d\mathrm{v})(m_\phi), m_\omega\big\rangle\big)}{\mathrm{w}(m_\phi)}\\
&\qquad\qquad - \ell_{\mathrm{ext}}(m_\phi).
\end{aligned} \tag{39}$$

We have $\mathcal{F}_{\mathrm{v},\mathrm{w}}(\phi_0,0) = 0$. Using [31, Lemma B.1], we can calculate the differential at $(\phi_0,0)$ of $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ is given by

$$\begin{aligned}
T_{(\phi_0,0)}\mathcal{F}_{\mathrm{v},\mathrm{w}} &: \mathrm{L}^{2,k+4}(X,\mathbb{R})^{\mathbb{T}} \times \mathbb{R} \to \mathrm{L}^{2,k}(X,\mathbb{R})^{\mathbb{T}} \times \mathbb{R},\\
(T_{(\phi_0,0)}\mathcal{F}_{\mathrm{v},\mathrm{w}})(\dot\phi,\zeta) &= \Big((T_{(\phi_0,0)}F_{\mathrm{v},\mathrm{w}})(\dot\phi,\zeta),\zeta\Big),\\
(T_{(\phi_0,0)}F_{\mathrm{v},\mathrm{w}})(\dot\phi,\zeta) &= -\frac{\mathcal{D}^*_{\phi_0}\mathrm{v}(m_{\phi_0})\mathcal{D}_{\phi_0}\dot\phi}{\mathrm{w}(m_{\phi_0})}\\
&\qquad - \zeta\left[\frac{\mathrm{v}(m_{\phi_0})\Lambda_{\phi_0}\omega + \big\langle (d\mathrm{v})(m_{\phi_0}), m_\omega\big\rangle}{\mathrm{w}(m_{\phi_0})}\right],
\end{aligned}$$

where $\mathcal{D}_{\phi_0}\dot\phi := \sqrt{2}(\nabla^{\phi_0}d\dot\phi)^-$ is the J-anti-invariant part of the tensor $\nabla^{\phi_0}d\dot\phi$, with ∇^{ϕ_0} the g_{ϕ_0}-Levi-Civita connection, and $\mathcal{D}^*_{\phi_0}$ is the formal adjoint of $\mathcal{D}_{\phi_0}$.

Notice that $\mathbb{L}_{\mathrm{v},\mathrm{w}} := (\mathrm{w}(m_{\phi_0}))^{-1}\mathcal{D}^*_{\phi_0}\mathrm{v}(m_{\phi_0})\mathcal{D}_{\phi_0}$ is a fourth order $\langle\cdot,\cdot\rangle_{\mathrm{w},\phi_0}$-self adjoint $\mathbb{T}$-invariant elliptic linear operator. By standard elliptic theory we have the following $\langle\cdot,\cdot\rangle_{\mathrm{w},\phi_0}$-orthogonal decomposition

$$\mathrm{L}^{2,k}(X,\mathbb{R})^{\mathbb{T}} = \mathrm{Ker}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) \oplus \mathrm{Im}(\mathbb{L}_{\mathrm{v},\mathrm{w}}). \tag{40}$$

We have $\mathrm{Ker}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) = \mathfrak{k}_{\phi_0}$ since K is a maximal compact subgroup of G, and $\mathrm{Im}(\mathbb{L}_{\mathrm{v},\mathrm{w}}) = \mathrm{L}^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$. Using (40), it's clear that the linearization is neither injective nor surjective. Let Π_{w,ϕ_0} the $\langle\cdot,\cdot\rangle_{\mathrm{w},\phi_0}$-orthogonal projection on $\mathfrak{k}_{\phi_0}$.

We consider the following modification of the map $\mathcal{F}_{\mathrm{v},\mathrm{w}}$

$$\tilde{\mathcal{F}}_{\mathrm{v},\mathrm{w}} : \mathcal{K}^{2,k+4}(X,\omega)^{\mathbb{T}} \times [0,1] \to \mathfrak{k}_{\phi_0} \times \mathrm{L}^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}} \times [0,1]$$

defined by

$$\tilde{\mathcal{F}}_{\mathrm{v},\mathrm{w}}(f,\psi,t) := (f, (I - \Pi_{\mathrm{w},\phi_0}) \circ F_{\mathrm{v},\mathrm{w}}(f+\psi,t), t).$$

where $f \in \mathfrak{k}_{\phi_0}$ and $\psi \in \mathrm{L}^{2,k}_{\perp}(X,\mathbb{R})^{\mathbb{T}}$ such that $\phi := f+\psi \in \mathcal{K}^{2,k+4}(X,\omega)^{\mathbb{T}}$. Let $\phi_0 := f_0 + \psi_0$ be the orthogonal decomposition of ϕ_0 in (40). The derivative of $\tilde{\mathcal{F}}_{\mathrm{v},\mathrm{w}}$ in $(f_0,\psi_0,0)$, is given by

$$\begin{aligned}&(T_{(f_0,\psi_0,0)}\tilde{\mathcal{F}}_{\mathrm{v},\mathrm{w}})(f,\dot{\psi},\zeta)\\ &\quad = \Big(f, -\mathbb{L}_{\mathrm{v},\mathrm{w}}(\dot{\psi}) - \zeta \mathrm{w}(m_{\phi_0})^{-1}\big(\mathrm{v}(m_{\phi_0})\Lambda_{\phi_0}\omega + \big\langle (d\mathrm{v})(m_{\phi_0}), m_\omega\big\rangle\big), \zeta\Big).\end{aligned}$$

The decomposition (40) and the equation (37) show that $T_{(f_0,\psi_0,0)}\tilde{\mathcal{F}}_{\mathrm{v},\mathrm{w}}$ is bijective. By the inverse function theorem we obtain a path

$$\phi(f,t) := f + \psi(f,t) \in \mathcal{K}^{2,k+4}(X,\omega)^{\mathbb{T}}, \tag{41}$$

for $0 < t < \epsilon$ and $f \in \mathfrak{k}_{\phi_0}$, such that

$$(I - \Pi_{\mathrm{w},\phi_0}) \circ F_{\mathrm{v},\mathrm{w}}(\phi(f,t),t) = 0 \tag{42}$$

for $\parallel f - f_0 \parallel_{\mathrm{L}^{2,k+4}} < \epsilon$.

Now we introduce the functional $\mathcal{G}_{\mathrm{v},\mathrm{w}} : \mathfrak{k}_{\phi_0} \times (0,\epsilon) \to \mathfrak{k}_{\phi_0}$, defined by

$$\mathcal{G}_{\mathrm{v},\mathrm{w}}(f,t) := \Pi_{\mathrm{w},\phi_0} \circ F_{\mathrm{v},\mathrm{w}}(\phi(f,t),t),$$

where $\phi(f,t)$ is given by (41). To complete the proof we need to solve the equation

$$\mathcal{G}_{\mathrm{v},\mathrm{w}}(f(t),t) = 0,$$

for $t \in (0,\epsilon)$ and $f(t) \in \mathfrak{k}_{\phi_0}$. However, its not possible to apply the implicit function theorem. Indeed,

$$\left.\frac{\partial \mathcal{G}_{\mathrm{v,w}}}{\partial f}\right|_{(f_0,0)}(\dot{f}) = \Pi_{\mathrm{w},\phi_0} \circ \left.\frac{\partial F_{\mathrm{v,w}}}{\partial f}\right|_{(f_0,0)}\left(\dot{f} + \left.\frac{\partial \psi}{\partial f}\right|_{(f_0,0)}(\dot{f})\right) = 0$$

since, by differentiating (42) with respect to f, we get

$$\left.\frac{\partial \psi}{\partial f}\right|_{(f_0,0)}(\dot{f}) = 0. \tag{43}$$

To solve this problem, one can consider the map [17]:

$$\tilde{\mathcal{G}}_{\mathrm{v,w}}(f,t) := \begin{cases} \frac{\mathcal{G}_{\mathrm{v,w}}(f,t)}{t} & \text{if } t \neq 0, \\ \left.\frac{\partial \mathcal{G}_{\mathrm{v,w}}}{\partial t}\right|_{(f,0)} & \text{if } t = 0. \end{cases}$$

which is continuous on $\mathfrak{k}_{\phi_0} \times [0,1]$. We want to apply the implicit function theorem to solve the equation

$$\tilde{\mathcal{G}}_{\mathrm{v,w}}(f(t),t) = 0.$$

So we have to check that the derivative

$$Q_{\mathrm{v,w}} := \left.\frac{\partial \tilde{\mathcal{G}}_{\mathrm{v,w}}}{\partial f}\right|_{(f_0,0)}, \tag{44}$$

is invertible. To simplify notations we denote the derivative with respect to t of (41) by

$$\dot{\phi}(f) := \left.\frac{\partial \phi}{\partial t}\right|_{(f,0)}.$$

By differentiating (42) with respect to t, we get

$$(\mathcal{D}^*_{\phi_0}\mathrm{v}(m_{\phi_0})\mathcal{D}_{\phi_0})(\dot{\phi}(f_0)) + \mathrm{v}(m_{\phi_0})\Lambda_{\phi_0}\omega + \big\langle (d\mathrm{v})(m_{\phi_0}), m_\omega \big\rangle = 0. \tag{45}$$

A straightforward calculation yields

$$\begin{aligned} \tilde{\mathcal{G}}_{\mathrm{v,w}}(f,0) &= -\,\Pi_{\mathrm{w},\phi_0} \circ G_{\mathrm{v,w}}(f), \\ G_{\mathrm{v,w}}(f) &:= \frac{\mathcal{D}^*_{\phi}\mathrm{v}(m_\phi)\mathcal{D}_\phi(\dot{\phi}(f)) + \mathrm{v}(m_\phi)\Lambda_\phi\omega + \big\langle (d\mathrm{v})(m_\phi), m_\omega \big\rangle}{\mathrm{w}(m_\phi)}, \end{aligned}$$

where $\phi := \phi(f,0)$. For $\dot f \in \mathfrak{k}_{\phi_0}$ we denote $f_\varepsilon := f_0 + \varepsilon \dot f$ and $\phi_\varepsilon = f_\epsilon + \psi(f_\varepsilon, t)$, we then have

$$\begin{aligned}
\langle Q_{\mathrm{v},\mathrm{w}}(\dot f), \dot f\rangle_{\mathrm{w},\phi_0} &:= \int_X \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \tilde{\mathcal{G}}_{\mathrm{v},\mathrm{w}}(f_\varepsilon, 0)\right) \dot f \mathrm{w}(m_{\phi_0})\omega_{\phi_0}^{[n]} \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_X \Pi_{\mathrm{w},\phi_0}[G_{\mathrm{v},\mathrm{w}}(f_\varepsilon)]\dot f \mathrm{w}(m_{\phi_0})\omega_{\phi_0}^{[n]} \\
&= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_X G_{\mathrm{v},\mathrm{w}}(f_\varepsilon)\dot f \mathrm{w}(m_{\phi_\varepsilon})\omega_{\phi_\varepsilon}^{[n]} \\
&\quad - \int_X G_{\mathrm{v},\mathrm{w}}(f_0)\dot f \, \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\mathrm{w}(m_{\phi_\varepsilon})\omega_{\phi_\varepsilon}^{[n]}) \\
&= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_X G_{\mathrm{v},\mathrm{w}}(f_\varepsilon)\dot f \mathrm{w}(m_{\phi_\varepsilon})\omega_{\phi_\varepsilon}^{[n]} \quad \text{(using (45) } G_{\mathrm{v},\mathrm{w}}(f_0) = 0) \\
&= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_X \big(\mathrm{v}(m_{\phi_\varepsilon})\big(\mathcal{D}_{\phi_\varepsilon}(\dot\phi(f_\varepsilon)), \mathcal{D}_{\phi_\varepsilon}\dot f\big)_{\phi_\varepsilon} \\
&\qquad\qquad + [\mathrm{v}(m_{\phi_\varepsilon})\Lambda_{\phi_\varepsilon}\omega + \big\langle (d\mathrm{v})(m_{\phi_\varepsilon}), m_\omega\big\rangle]\dot f\big)\omega_{\phi_\varepsilon}^{[n]}.
\end{aligned} \tag{46}$$

Using the following variational formulas,

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \omega_{\phi_\varepsilon}^{[n]} &= -\Delta_{\phi_0}(\dot f)\omega_{\phi_0}^{[n]} \\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{v}(m_{\phi_\varepsilon}) &= \sum_{i=1}^{\ell} \mathrm{v}_{,i}(m_{\phi_0})(d^c\dot f)(\xi_i), \\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \langle (d\mathrm{v})(m_{\phi_\varepsilon}), m_\omega\rangle &= \sum_{i,j=1}^{\ell} \mathrm{v}_{,ij}(m_{\phi_0})(d^c\dot f)(\xi_j) m_\omega^{\xi_i}, \\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Lambda_{\phi_\varepsilon}\omega &= -(dd^c\dot f, \omega)_{\phi_0}, \\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{D}_{\phi_\varepsilon}\dot f &= -\mathcal{D}_{\phi_0}|d\dot f|^2_{\phi_0} \quad \text{(see Lemma 12 below)},
\end{aligned}$$

and the following calculation from the proof of [31, Lemma 4]

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_X [\mathrm{v}(m_{\phi_\varepsilon})\Lambda_{\phi_\varepsilon}\omega + \big\langle (d\mathrm{v})(m_{\phi_\varepsilon}), m_\omega\big\rangle]\dot f \omega_{\phi_\varepsilon}^{[n]} \\
&\quad = \int_X \mathrm{v}(m_{\phi_0})(\omega, d\dot f \wedge d^c\dot f)_{\phi_0}\omega_{\phi_0}^{[n]} \\
&\qquad - \int_X \big(\Lambda_{\phi_0}\omega + \big\langle (d\mathrm{v})(m_{\phi_0}), m_\omega\big\rangle\big) \, |d\dot f|^2_{\phi_0}\omega_{\phi_0}^{[n]},
\end{aligned}$$

we compute from (46):

$$
\begin{aligned}
\langle Q_{\mathrm{v},\mathrm{w}}(\dot f), \dot f\rangle_{\mathrm{w},\phi_0} &= \int_X (\mathcal{D}^*_{\phi_0}\mathrm{v}(m_{\phi_0})\mathcal{D}_{\phi_0})(\dot\phi(f_0))|d\dot f|^2_{\phi_0}\omega_{\phi_0}^{[n]}\\
&\quad - \int_X \mathrm{v}(m_{\phi_0})(\omega, d\dot f\wedge d^c\dot f)_{\phi_0}\omega_{\phi_0}^{[n]}\\
&\quad + \int_X \left(\Lambda_{\phi_0}\omega + \left\langle (d\mathrm{v})(m_{\phi_0}), m_\omega\right\rangle\right)|d\dot f|^2_{\phi_0}\omega_{\phi_0}^{[n]}\\
&= - \int_X \mathrm{v}(m_{\phi_0})(\omega, d\dot f\wedge d^c\dot f)_{\phi_0}\omega_{\phi_0}^{[n]},
\end{aligned}
$$

where we used (45) for the second equality. It follows that $Q_{\mathrm{v},\mathrm{w}}$ is bijective on $\mathfrak{k}_{\phi_0}$. Therefore, by the implicit function theorem, there exist a path $(f(t))_{t\in(0,\epsilon)}\in\mathfrak{k}_{\phi_0}$, $f(0)=f_0$ such that $\mathcal{G}_{\mathrm{v},\mathrm{w}}(f(t),t)=0$. From (42), we obtain

$$F_{\mathrm{v},\mathrm{w}}(\phi(f(t),t),t)=0,$$

for any $t\in(0,\epsilon)$, which completes the proof. □

Lemma 12. *We have*

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathcal{D}_{\phi_\varepsilon}\dot f = -\mathcal{D}_{\phi_0}|d\dot f|^2_{\phi_0}. \tag{47}
$$

Proof. Using [27, Lemma 1.23.2], and the fact that $\dot f$ is a Killing potential we obtain,

$$
\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathcal{D}_{\phi_\varepsilon}\dot f &= -\frac{\sqrt2}{2}\omega_{\phi_0}((\mathcal{L}_V J)\cdot,\cdot) \text{ where } V := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathrm{grad}_{\phi_\varepsilon}(\dot f)\\
&= -\frac{\sqrt2}{2}(\nabla^{\phi_0}V^\flat)^-,
\end{aligned}
\tag{48}
$$

where the musical isomorphism used in $V^\flat$ is with respect to the metric ω_{ϕ_0}. Using the equation $\omega_{\phi_\varepsilon}(\mathrm{grad}_{\phi_\varepsilon}(\dot f),\cdot)=Jd\dot f$, we obtain

$$
\begin{aligned}
V &= (dd^c\dot f)(J\mathrm{grad}_{\phi_0}(\dot f),\cdot)\\
&= \mathcal{L}_{J\mathrm{grad}_{\phi_0}(\dot f)}d^c\dot f - d((d^c\dot f)(J\mathrm{grad}_{\phi_0}(\dot f))) \text{ by Cartan formula,}\\
&= 0 - d|d\dot f|^2_{\phi_0} \text{ since } J\mathrm{grad}_{\phi_0}(\dot f) \text{ is real holomorphic.}
\end{aligned}
$$

Substituting the above expression of V back into (48), the expression (47) follows. □

Now we are in position to prove Theorem 2

Proof of Theorem 2. Using Proposition 2, the proof of Theorem 2 is very similar to [17, Corollary 1.3]. We give the argument for the sake of clarity. Suppose that $\varphi_0, \tilde{\varphi}_0 \in \mathcal{K}(X,\omega)^{\mathbb{T}}$, such that $\omega_{\varphi_0}, \omega_{\tilde{\varphi}_0}$ are two $\mathbb{T}$-invariant (v,w)-extremal metrics in the Kähler class α. Using Proposition 2, we get two paths $\phi : [0,\epsilon)\times X \to \mathbb{R}$ and $\tilde{\phi} : [0,\epsilon)\times X \to \mathbb{R}$ in $\mathcal{K}(X,\omega)^{\mathbb{T}}$ such that ϕ_0 (resp. $\tilde{\phi}_0$) is in the G-orbit of φ_0 (resp. $\tilde{\varphi}_0$) and ϕ_t (resp. $\tilde{\phi}_t$) solves (35). Notice that ϕ_t and $\tilde{\phi}_t$ are critical points of the functional $\mathcal{M}^{t\omega}_{(\mathrm{v},\ell_{\mathrm{ext}}\cdot\mathrm{w})} := \mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} + t\mathcal{E}^{\omega}_{\mathrm{v}}$. Indeed, by (17) and (5), we have

$$\big(d\mathcal{M}^{t\omega}_{(\mathrm{v},\ell_{\mathrm{ext}}\cdot\mathrm{w})}\big)_{\phi}(\dot{\phi}) =$$
$$-\int_X \Big(\frac{\mathrm{Scal}_{\mathrm{v}}(\phi) - t\big(\mathrm{v}(m_\phi)\Lambda_{\omega_\phi}\omega + \big\langle (d\mathrm{v})(m_\phi), m_\omega\big\rangle\big)}{\mathrm{w}(m_\phi)} - \ell_{\mathrm{ext}}(m_\phi)\Big)\dot{\phi}\mathrm{w}(m_\phi)\omega_\phi^{[n]}.$$

By convexity of $\mathcal{M}^{\mathrm{rel}}_{\mathrm{v},\mathrm{w}} = \mathcal{M}_{(\mathrm{v},\ell_{\mathrm{ext}}\mathrm{w})}$ along weak geodesics Theorem 4, and strict convexity of $\mathcal{E}^{\omega}_{\mathrm{v}}$ along weak geodesics Corollary 4, it follows that the functional $\mathcal{M}^{t\omega}_{(\mathrm{v},\ell_{\mathrm{ext}}\cdot\mathrm{w})}$ is strictly convex on weak geodesics. Thus, $\phi = \tilde{\phi}$ on $(0,\epsilon)\times X$. As $\epsilon \to 0$ we obtain $\varphi_0 = f^*\tilde{\varphi}_0$, for some $f \in G$. □

Remark 2. By [31, Corollary B.1], a (v,w)-extremal metric ω is always invariant under the action of a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(X)$. We can thus take in Theorem 2 $\mathbb{T}$ to be a maximal torus. In this case $G = \mathrm{Aut}^{\mathbb{T}}_{\mathrm{red}}(X)^\circ = \mathbb{T}^c$ the complixified torus. Indeed [1], by [31, Theorem B1] the group G is a reductive Lie group (at the level of Lie algebras we have $\mathrm{Lie}(G) = \mathfrak{k}\oplus J\mathfrak{k}$ where $\mathfrak{k}$ is the Lie algebra of the compact group $K := \mathrm{Isom}^{\mathbb{T}}(X,\omega)\cap \mathrm{Aut}_{\mathrm{red}}(X) = \mathrm{Isom}^{\mathbb{T}}(X,\omega)\cap G$). As $\mathbb{T}\subset K$ is simultaneously central and maximal, it follows that $\mathfrak{k} = \mathfrak{t}$, and thus $K^\circ = \mathbb{T}$ and $G = \mathbb{T}^c$. Thus, when $\mathbb{T}$ is maximal, any two (v,w)-extremal metrics $\omega_1,\omega_2\in\alpha$, there exist $f\in\mathbb{T}^c$ such that $\omega_2 = f^*\omega_1$.

6. Action of the Weyl group

As before, let (X,α) be a compact Kähler manifold and $\mathbb{T}$ a maximal real torus inside the connected Lie group $\mathrm{Aut}_{\mathrm{red}}(X)$ of reduced automorphisms of X, see Remark 2. We denote by $\mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ the normalizer of $\mathbb{T}$ inside $\mathrm{Aut}_{\mathrm{red}}(X)$. By [27, Lemma 4.14.2] we have the equality $\mathrm{Aut}_{\mathrm{red}}(X)^\circ_{\mathbb{T}} =$

[1] Thanks to V. Apostolov for this argument.

$\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^{\circ}$ between the corresponding connected components of identity, hence the following group

$$W := \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}/\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^{\circ} \tag{49}$$

is discrete. We refere to W as the *Weyl group* of $\mathrm{Aut}_{\mathrm{red}}(X)$, see Remark 3 below. The group $\mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ naturally acts on the space of $\mathbb{T}$-invariant Kähler metrics in α. The material in this section was suggested to us by the anonymous referee and aims to study the induced action of a certain subgroup of $\mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ on the space of (v,w)-extremal Kähler metrics. We shall show that for suitable weight functions, the group W may induce finite order elements inside the group $K = \mathrm{Isom}^{\mathbb{T}}(X,\omega) \cap \mathrm{Aut}_{\mathrm{red}}(X)$ of isometries of any $\mathbb{T}$-invariant (v,w)-extremal Kähler metric in α. Such a phenomenon can also lead to a stronger coercivity property of the weighted Mabuchi energy, for instance through computation of α and δ invariants on $\mathbb{T}$-invariant functions preserved by these further symmetries, as demonstrated in the work of Batyrev and Selivanova in the toric Fano case [6].

For every element $\sigma \in \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ the restriction of the differential of σ to $\mathfrak{t}$ defines a linear transformation $T_\sigma : \mathfrak{t} \to \mathfrak{t}$ by

$$T_\sigma(\xi) := \frac{d}{dt}_{|t=0} \sigma \circ \exp(t\xi) \circ \sigma^{-1}.$$

Notice that T_σ preserves the lattice $\Lambda \subset \mathfrak{t}$ of circle subgroups of $\mathbb{T}$, since a circle generator $\xi \in \Lambda$ defines a circle subgroup $\sigma \circ \exp(t\xi) \circ \sigma^{-1}$ whose generator is $T_\sigma(\xi)$, so $T_\sigma(\xi) \in \Lambda$.

Lemma 13. *Let P_α be a fixed momentum polytope for the $\mathbb{T}$ action on (X,α) and let $\sigma \in \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ with $T_\sigma^* : \mathfrak{t}^* \to \mathfrak{t}^*$ the dual map of the linear transformation $T_\sigma : \mathfrak{t} \to \mathfrak{t}$. There exist $a_\sigma \in \mathfrak{t}^*$ such that the affine transformation $A_\sigma := T_\sigma^* + a_\sigma$ of $\mathfrak{t}^*$ preserves the momentum polytope i.e. $\mathrm{P}_\alpha = A_\sigma(\mathrm{P}_\alpha)$.*

Proof. As $\sigma \in \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ and $\mathrm{Aut}_{\mathrm{red}}(X)$ acts trivially on $H^2_{\mathrm{dR}}(X)$, σ preserves the class α. Furthermore, since $\mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ preseves $\mathbb{T}$-invariance, if $\omega \in \alpha$ is a $\mathbb{T}$-invartiant Kähler form with normalized momentum map $m_\omega : X \to \mathrm{P}_\alpha$, then $\sigma^*\omega \in \alpha$ is a $\mathbb{T}$-invariant Kähler form with normalized momentum map $m_{\sigma^*\omega} : X \to \mathrm{P}_\alpha$. By the momentum map property for all $\xi \in \mathfrak{t}$ we have

$$\begin{aligned}-d\langle m_{\sigma^*\omega}, \xi\rangle &= \xi\lrcorner(\sigma^*\omega) = \sigma^*\left((\sigma_*\xi)\lrcorner\omega\right)\\ &= -d\left(\sigma^*\langle m_\omega, \sigma_*\xi\rangle\right) = -d\left(\langle\sigma^* m_\omega, T_\sigma(\xi)\rangle\right),\end{aligned}$$

hence $d\left(\langle m_{\sigma^*\omega} - T_\sigma^*(\sigma^* m_\omega), \xi\rangle\right) = 0$, and thus there exists an $a_\sigma \in \mathfrak{t}^*$ such that

$$m_{\sigma^*\omega} = T_\sigma^*(\sigma^* m_\omega) + a_\sigma = A_\sigma(\sigma^* m_\omega) \tag{50}$$

It follows that $A_\sigma(\mathrm{P}_\alpha) = \mathrm{P}_\alpha$ for the affine transformation $A_\sigma = T_\sigma^* + a_\sigma$ of $\mathfrak{t}^*$. □

Given two weight functions $\mathrm{v}, \mathrm{w} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R})$, we define the subgroup $N_{\mathrm{v},\mathrm{w}} \subset \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ of elements $\sigma \in \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}$ such that the corresponding affine map A_σ of Lemma 13 fixes the weights (v, w),

$$N_{\mathrm{v},\mathrm{w}} := \{\sigma \in \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}} \mid A_\sigma^*(\mathrm{v}, \mathrm{w}) = (\mathrm{v}, \mathrm{w})\}.$$

We have $\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^\circ \subset \mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X) \subset N_{\mathrm{v},\mathrm{w}}$, so we can introduce the group $W_{\mathrm{v},\mathrm{w}}$ as follows

$$W_{\mathrm{v},\mathrm{w}} := N_{\mathrm{v},\mathrm{w}}/\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^\circ \subset W.$$

where W is the Weyl group (49).

Lemma 14. *Let* $\mathrm{v}, \mathrm{w} \in C^\infty(\mathrm{P}_\alpha, \mathbb{R}_{>0})$ *be two weight functions and* $\omega \in \alpha$ *be a* $\mathbb{T}$*-invariant* (v, w)*-extremal Kähler metric. We denote* $K = \mathrm{Isom}^{\mathbb{T}}(X, \omega) \cap \mathrm{Aut}_{\mathrm{red}}(X)$*. Then there exists an injective morphism* $\rho : W_{\mathrm{v},\mathrm{w}} \to K/\mathbb{T}$*. In particular,* $W_{\mathrm{v},\mathrm{w}}$ *is finite and* K *contains a subgroup which is an extension of* $W_{\mathrm{v},\mathrm{w}}$ *by* $\mathbb{T}$*.*

Proof. For any $\sigma \in N_{\mathrm{v},\mathrm{w}}$, using (50) we get

$$\mathrm{v}(m_{\sigma^*\omega}) = \mathrm{v}(A_\sigma(\sigma^* m_\omega)) = \sigma^*((A_\sigma^*\mathrm{v})(m_\omega)) = \sigma^*(\mathrm{v}(m_\omega)).$$

Hence, $\mathrm{Scal}_{\mathrm{v}}(\sigma^*\omega) = \sigma^*\left(\mathrm{Scal}_{\mathrm{v}}(\omega)\right)$ and the affine (v, w)-extremal function ℓ_{ext} is also preserved by A_σ since A_σ preserves $(\mathrm{P}_\alpha, \mathrm{v}, \mathrm{w})$ and ℓ_{ext} is determined by the latter. It follows that $\sigma^*\omega$ is also (v, w)-extremal. By Remark 2, $\mathrm{Aut}_{\mathrm{red}}^{\mathbb{T}}(X)^\circ = \mathbb{T}^c$, and using Theorem 2, there exist $f \in \mathbb{T}^c$ such that $f^*\omega = \sigma^*\omega$. If we have a pair of elements $f_1, f_2 \in \mathbb{T}^c$ such that $f_1^*\omega = f_2^*\omega = \sigma^*\omega$, then $f_2 \circ f_1^{-1} \in \mathrm{Isom}^{\mathbb{T}}(X, \omega) \cap \mathbb{T}^c$. We write $f_2 \circ f_1^{-1} =$

$\exp(\xi + J\zeta) \in \mathbb{T}^c$ for $\xi, \zeta \in \mathfrak{t}$ and consider the segment of normalized potentials $(\phi_t)_{t\in[0,1]} \subset \hat{\mathcal{K}}(X,\omega)^{\mathbb{T}}$ induced from the family of Kähler metrics $\omega_t := \exp(t(\xi + J\zeta))^*\omega = \exp(tJ\zeta)^*\omega$ by

$$\omega_t - \omega = dd^c\phi_t.$$

By [27, Proposition 4.6.3]) $(\phi_t)_{t\in[0,1]}$ is a smooth geodesic. On the other hand since $f_2 \circ f_1^{-1} \in \mathrm{Isom}^{\mathbb{T}}(X,\omega)$ then $\phi_0 = \phi_1 = 0$. By uniqueness of the geodesic segments [13] it follows that $\zeta = 0$ and thus $f_2 \circ f_1^{-1} = \exp(\xi) \in \mathbb{T}$. Hence, we get a group morphism $\rho : W_{\mathrm{v,w}} \to K/\mathbb{T}$ defined by $\rho([\sigma]) = [\sigma \circ f^{-1}]$. If $\sigma \circ f^{-1} = u \in \mathbb{T}$ then $\sigma = u \circ f \in \mathbb{T}^c$ showing that ρ is injective.

The group $W_{\mathrm{v,w}}$ is finite since ρ is injective, K is compact and $K^\circ = \mathbb{T}$ by Remark 2. The subgroup $H := \pi^{-1}(\rho(W_{\mathrm{v,w}})) \subset K$, where $\pi : K \to K/\mathbb{T}$ is the canonical projection, fits into the following exact sequence

$$1 \to \mathbb{T} \hookrightarrow H \xrightarrow{\pi} \rho(W_{\mathrm{v,w}}) \simeq W_{\mathrm{v,w}} \to 1.$$

Hence H defines an extension of $W_{\mathrm{v,w}}$ by $\mathbb{T}$. $\square$

Remark 3. In the case of a polarized variety (X,L), the group $\mathrm{Aut}_{\mathrm{red}}(X)$ is identified with the group $\mathrm{Aut}(X,L)^\circ$ of automorphisms preserving the polarization (see [27, Proposition 8.1.2]). The latter is a linear algebraic group (see e.g. [11, Theorem 2.16]). In this case, for any maximal torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ it is well-known that $\mathrm{Aut}^{\mathbb{T}}_{\mathrm{red}}(X) = \mathrm{Aut}^{\mathbb{T}}_{\mathrm{red}}(X)^\circ$ is connected, and $W = \mathrm{Aut}_{\mathrm{red}}(X)_{\mathbb{T}}/\mathrm{Aut}^{\mathbb{T}}_{\mathrm{red}}(X)$ is the Weyl group (which is also known to be a finite group). Taking $\mathrm{v} = \mathrm{w} = 1$, we thus obtain the invariance of a $\mathbb{T}$-invariant extremal Kähler metric in $2\pi c_1(L)$ under the extension of the Weyl group by $\mathbb{T}$.

Note also that in the case of a Fano variety, it is now known (see e.g. [18, 21]) that a Kähler-Einstein metric must be, in fact, invariant under a maximal compact subgroup of $\mathrm{Aut}_{\mathrm{red}}(X)$.

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References

[1] V. Apostolov and D. M. J. Calderbank, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, Math. Ann, **379** (2021), 1047–10.

[2] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Extremal Kähler metrics on projective bundles over a curve.* Adv. Math. **227** (2011), 2385–2424.

[3] V. Apostolov, G. Maschler, *Conformally Kähler, Einstein-Maxwell geometry*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 5, 1319–1360.

[4] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.

[5] S. Bando, T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 11–40.

[6] V. V. Batyrev, E. N. Selivanova, *Einstein-Kähler metrics on symmetric toric Fano manifolds*, J. Reine Angew. Math. **512** (1999), 225–236.

[7] R. J. Berman and B. Berndtsson, *Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics*, J. Amer. Math. Soc. **30** (2017), no. 4, 1165—1196.

[8] R. J. Berman and D. Witt-Nyström, *Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons*, `arXiv:1401.8264`.

[9] B. Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry*, Invent. Math. **200** (2015), 149–200.

[10] Z. Blocki, The complex Monge–Ampère equation in Kähler geometry, Pluripotential theory, 95–141, Lecture Notes in Math., 2075, Fond. CIME/CIME Found. Subser., Springer, Heidelberg, 2013.

[11] M. Brion, Notes on automorphism groups of projective varieties, available at `http://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf` (2019).

[12] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, N.J., 1982.

[13] X. Chen, *The space of Kähler metrics*, J. Diff. Geom. **56** (2000), 189–234.

[14] X. Chen, *On the lower bound of the Mabuchi energy and its application*, Int Math Res Notices (2000) **2000** (12): 607–623.

[15] X. X. Chen and G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs*, Publ. Math. Inst. Hautes Etudes Sci. **107** (2008), 1–107.

[16] X. X. Chen, L. Li and M. Paun, Approximation of weak geodesics and subharmonicity of Mabuchi energy, Ann. Fac. Sci. Toulouse Math. (6) **25** (2016), no. 5, 935–957.

[17] X. X. Chen, M. Paun, Y. Zeng *On deformation of extremal metrics*, `arXiv:1506.01290v2`.

[18] C. Li *G-uniform stability and Kähler-Einstein metrics on singular Fano varieties*, Invent. Math. **227** (2022), no. 2, 661–744.

[19] J. Chu, V. Tosatti, B. Weinkove, *On the $C^{1,1}$ regularity of geodesics in the space of Kähler metrics*, Ann. PDE **3** (2017), no. 2, 3:15.

[20] T. Darvas *Morse theory and geodesics in the space of Kähler metrics*, Proc. Amer. Math. Soc. **142** (2014), no. 8, 2775–2782.

[21] T. Darvas and Y. Rubinstein, *Tian's properness conjectures and Finsler geometry of the space of Kähler metrics*, J. Amer. Math. Soc. **3**0 (2017), 347–387.

[22] J. P. Demailly, *Complex analytic and differential geometry*, book available online.

[23] A. Dembo, O. Zeitouni, Large deviations techniques and applications, Jones and Bartlett Publishers, Boston, MA, 1993.

[24] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, in: Northern California Symplectic Geometry Seminar, volume **196** of Amer. Math. Soc. Transl. Ser. 2, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.

[25] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.

[26] S. K. Donaldson, *Scalar curvature and projective embeddings* II., Q. J. Math. **56** (2005), no. 3, 345–356.

[27] P. Gauduchon, Calabi's extremal metrics: An elementary introduction, Lecture Notes.

[28] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[29] E. Inoue, *The moduli space of Fano manifolds with Kähler-Ricci solitons*, Adv. Math., **357** (2019), 106841, 65pp.

[30] E. Inoue, *Constant μ-scalar curvature ähler metric - formulation and foundational results*, J. Geom. Anal. **32** (2022), Article number 145.

[31] A. Lahdili, *Kähler metrics with constant weighted scalar curvature and weighted K-stability*, Proc. London Math. Soc. (3) **119** (2019), 1065–1114.

[32] C. Li, *Constant scalar curvature Kähler metrics obtains minimum of K-energy*, Int. Math. Res. Not., Vol. **2011** (2011), no. 9, 2161–2175.

[33] T. Mabuchi, *K-energy maps integrating Futaki invariants.* Tohoku Math. J. (2) **38** (1986), no. 4, 575–593

[34] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds.* I. Osaka J. Math. (2) **24** (1987), 227—252.

[35] Y. Sano and C. Tipler, *Extremal Kähler metrics and lower bound of the modified K-energy*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 9, 2289–2310.

[36] S. Semmes, *Complex Monge–Ampère and symplectic manifolds*, Amer. J. Math. **114** (1992), 495–550.

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