

Continuity of the gradient of the fractional maximal operator on $W^{1,1}(\mathbb{R}^d)$

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We establish that the map $f \mapsto |\nabla \mathcal{M}_\alpha f|$ is continuous from $W^{1,1}(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, where $\alpha \in (0, d)$, $q = \frac{d}{d-\alpha}$ and \mathcal{M}_α denotes either the centered or non-centered fractional Hardy–Littlewood maximal operator. In particular, we cover the cases $d > 1$ and $\alpha \in (0, 1)$ in full generality, for which results were only known for radial functions.

1. Introduction

Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $0 \leq \alpha < d$, the centered fractional Hardy–Littlewood maximal operator is defined by

$$M_\alpha f(x) := \sup_{r>0} \frac{r^\alpha}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy$$

for every $x \in \mathbb{R}^d$. The non-centered version of M_α , denoted by \widetilde{M}_α , is defined by taking the supremum over all balls $B(z, r)$ such that x is contained in the closure of $B(z, r)$. In what follows, we use \mathcal{M}_α to denote either the centered or non-centered version, in the sense that if we formulate a result or a proof for \mathcal{M}_α , we mean that it holds for both M_α and \widetilde{M}_α . The non-fractional case $\alpha = 0$ corresponds to the classical maximal function, which we denote by $M = M_0$, $\widetilde{M} = \widetilde{M}_0$ and $\mathcal{M} = \mathcal{M}_0$.

The study of regularity properties for \mathcal{M} and \mathcal{M}_α started with the influential works of Kinnunen [13] and Kinnunen and Saksman [14], where it was established that

$$(1.1) \quad |\nabla \mathcal{M}_\alpha f(x)| \leq \mathcal{M}_\alpha |\nabla f|(x)$$

a.e. in \mathbb{R}^d . The mapping properties of \mathcal{M}_α then imply that the map $f \mapsto \mathcal{M}_\alpha f$ is bounded from $W^{1,p}(\mathbb{R}^d)$ to $W^{1,q}(\mathbb{R}^d)$ when $1 < p \leq d/\alpha$ and $\frac{1}{q} =$

$\frac{1}{p} - \frac{\alpha}{d}$. At the endpoint $p = 1$ this boundedness fails since $\mathcal{M}_\alpha f \notin L^q(\mathbb{R}^d)$ unless $f = 0$ a.e.. However, one can still consider the following question:

(1.2) Is the map $f \mapsto |\nabla \mathcal{M}_\alpha f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)$?

By a dilation argument, this is equivalent to proving that there exists a constant $C > 0$ such that

$$(1.3) \quad \|\nabla \mathcal{M}_\alpha f\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

This question was first explored in the classical case $\alpha = 0$ and $d = 1$ [1, 15, 23] and, more recently, for $d > 1$, radial functions and non-centered \widetilde{M} [18]. For $\alpha > 0$, this boundedness was first considered in [6], where the case $d = 1$ was settled for \widetilde{M}_α . Moreover, they observed that the case $d > 1$, $1 \leq \alpha < d$ follows via Sobolev embedding and the smoothing property of \mathcal{M}_α obtained by Kinnunen and Saksman [14], which ensures that if $1 \leq \alpha < d$ and $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq d/\alpha$, then

$$(1.4) \quad |\nabla \mathcal{M}_\alpha f(x)| \leq (d - \alpha) \mathcal{M}_{\alpha-1} f(x)$$

a.e. in \mathbb{R}^d .

For $0 < \alpha < 1$, the first boundedness result in higher dimensions was established for \widetilde{M}_α in [19] for radial functions. Analogous results in both $d = 1$ and $d > 1$ were obtained for M_α in [3], where a pointwise relation between ∇M_α and $\nabla \widetilde{M}_\alpha$ was observed for the first time for $\alpha > 0$. That relation revealed that both operators behave quite similarly, unlike it was previously thought; note that without taking the gradient the two maximal functions are comparable. Very recently, the question (1.2) was established in full generality by the fourth author in [24] for $\alpha > 0$ for both the centered and the uncentered operator \widetilde{M}_α , completing the remaining open cases in the fractional setting (that is, $d > 1$, $0 < \alpha < 1$ and general f). And indeed, the proof in [24] is almost identical for both maximal operators. It is based on the corresponding bound for the dyadic maximal operator in the non-fractional case $\alpha = 0$ in [25]. Other interesting related results in the context of fractional maximal functions have recently been proven in [4, 9, 12, 21, 22].

In this manuscript we explore the continuity of the map $f \mapsto |\nabla \mathcal{M}_\alpha f|$ for $\alpha > 0$. Note that this map is not sublinear, and thus its boundedness from $W^{1,1}(\mathbb{R}^d)$ to $L^{d/(d-\alpha)}(\mathbb{R}^d)$ does not immediately imply its continuity as a map between those function spaces. For $p > 1$, the continuity can be established by the methods developed by Luiro [16], which rely on the Lebesgue space mapping properties of \mathcal{M}_α . Once again, the endpoint case $p = 1$ is

more intricate. For $d = 1$ the continuity was established by the third author [20] for the non-centered case and by the first and third authors [2] for the centered case. For $d > 1$, similarly to the boundedness, we shall distinguish between the ranges $1 \leq \alpha < d$ and $0 < \alpha < 1$. For the former range, the result can be obtained via the inequality (1.4) and dominated convergence theorem arguments. This was proven in [2]. The range $0 < \alpha < 1$ is harder as one can no longer appeal to (1.4). Positive results under a radial assumption on f were obtained by the first and third authors in both the non-centered [2] and centered case [3]. We refer to [5, 7, 10, 17] for complementary results regarding the continuity of \widetilde{M} .

Here we establish the following complete result for $\alpha > 0$, which in particular yields the continuity in the remaining open cases, that is, for $d > 1$, $0 < \alpha < 1$ and general functions $f \in W^{1,1}(\mathbb{R}^d)$.

Theorem 1.1. *Let $\mathcal{M}_\alpha \in \{\widetilde{M}_\alpha, M_\alpha\}$. If $0 < \alpha < d$, the operator $f \mapsto |\nabla \mathcal{M}_\alpha f|$ maps continuously $W^{1,1}(\mathbb{R}^d)$ into $L^{d/(d-\alpha)}(\mathbb{R}^d)$.*

As observed in [2], it suffices to establish the continuity for any compact set $K \subseteq \mathbb{R}^d$. For any given $\delta > 0$, we consider two types of points in K , depending on whether the ball with maximal average has *large* radius (larger than δ) or *small* radius (smaller than δ). The techniques from [2, 3] immediately apply to prove the continuity for the points whose maximal ball has *large* radius: the radially assumption was not used in that situation.

Thus, in order to establish continuity in Theorem 1.1, it suffices to bound contributions coming from points whose maximal ball has *small* radius, i.e. radius smaller than δ , and show that they go to zero for $\delta \rightarrow 0$. This is the main novelty of this paper. To obtain this bound for points with *small* radius, we first note that on any compact set K , $\mathcal{M}_\alpha f$ is bounded away from 0. Then we use the Poincaré–Sobolev inequality, which becomes stronger the smaller the radius is and the larger the average of the function is. Then we apply a refined version of (1.4) which allows us to invoke a local version of the boundedness (1.3) in [24] on the subset of points with *small* radius. This yields the desired result. In the passage, we also use a refined version of (1.1).

The proof of Theorem 1.1 is presented in Section 4. Auxiliary results which feature prominently in the proof are presented in Sections 2 and 3.

Notation. Given a measurable set $E \subseteq \mathbb{R}^d$, we denote by $E^c := \mathbb{R}^d \setminus E$ the complementary set of E in \mathbb{R}^d . For $c \in \mathbb{R}$, we denote by cE the concentric set to E dilated by c . The integral average of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ over E is denoted by $f_E \equiv \bar{f}_E f := |E|^{-1} \int_E f$. Given a ball $B \subseteq \mathbb{R}^d$, we denote its radius by

$r(B)$. The volume of the d -dimensional unit ball is denoted by ω_d . The weak derivative of f is denoted by ∇f .

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2. Families of good balls

In this section we develop some estimates and identities regarding the weak derivative of the maximal functions of interest. We shall only be concerned with $0 < \alpha < d$, although many of the arguments can also be extended to $\alpha = 0$.

2.1. The truncated fractional maximal function

An important object for our purposes are the truncated fractional maximal operators which, for a given $\delta > 0$, are defined as

$$M_{\alpha}^{\delta} f(x) := \sup_{r > \delta} r^{\alpha} \int_{B(x,r)} |f(y)| \, dy \quad \text{and}$$

$$\widetilde{M}_{\alpha}^{\delta} f(x) := \sup_{\substack{\bar{B}(z,r) \ni x \\ r > \delta}} r^{\alpha} \int_{B(z,r)} |f(y)| \, dy.$$

We use $\mathcal{M}_{\alpha}^{\delta}$ to denote either M_{α}^{δ} or $\widetilde{M}_{\alpha}^{\delta}$. Note that if $\delta = 0$, we recover the original operators $\mathcal{M}_{\alpha} = \mathcal{M}_{\alpha}^0$. The following is a well-known and elementary result; see for instance [3, Lemma 2.4] and [11, Lemma 8].

Proposition 2.1. *Let $0 < \alpha < d$ and $\delta > 0$. If $f \in L^1(\mathbb{R}^d)$, then $\mathcal{M}_{\alpha}^{\delta} f$ is Lipschitz continuous (in particular, a.e. differentiable).*

2.2. Weak derivative and approximate derivative

As mentioned in the introduction, the fourth author proved in [24], after partial contributions by many, the following result.

Theorem 2.2 ([24, Theorem 1.1]). *Let $0 < \alpha < d$ and $f \in W^{1,1}(\mathbb{R}^d)$. Then $\mathcal{M}_\alpha f$ is weakly differentiable and the bound*

$$\|\nabla \mathcal{M}_\alpha f\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)} \leq C_{d,\alpha} \|\nabla f\|_{L^1(\mathbb{R}^d)}$$

holds, where the constant $C_{d,\alpha} > 0$ only depends on α and the dimension.

It will be convenient in our arguments to also recall the concept of approximate derivative. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *approximately differentiable* at a point $x_0 \in \mathbb{R}^d$ if there exists a vector $Df(x_0) \in \mathbb{R}^d$ such that, for any $\varepsilon > 0$, the set

$$(2.1) \quad A_\varepsilon := \left\{ x \in \mathbb{R}^d : \frac{|f(x) - f(x_0) - \langle Df(x_0), x - x_0 \rangle|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. In this case, $Df(x_0)$ is called the *approximate derivative* of f at x_0 and it is uniquely determined. It is well-known that if f is weakly differentiable, then f is approximately differentiable a.e. and the weak and approximate derivatives coincide [8, Theorem 6.4].

The approximate derivative satisfies the following property, which will play a rôle in Propositions 2.4 and 2.6 below.

Lemma 2.3. *Let f be approximately differentiable at a point $x \in \mathbb{R}^d$. Then there exists a sequence $\{h_n\}_{n \in \mathbb{N}}$ with $|h_n| \rightarrow 0$ such that*

$$|Df(x)| = - \lim_{n \rightarrow \infty} \frac{f(x + h_n) - f(x)}{|h_n|},$$

where $Df(x)$ denotes the approximate derivative of f at x .

Proof. Let $0 < \varepsilon < \pi/2$. By the definition of the approximate derivative, there exists $0 < \rho < \varepsilon$ such that

$$(2.2) \quad |A_\varepsilon \cap B(x, \rho)| \geq \left(1 - \frac{\omega_{d-1}}{d\omega_d} (\sin \varepsilon)^{d-1} (\cos \varepsilon)^d\right) |B(x, \rho)|$$

where A_ε is as in (2.1).

If $Df(x) = 0$, the result simply follows by the definition of A_ε and taking $\varepsilon = 1/n$.

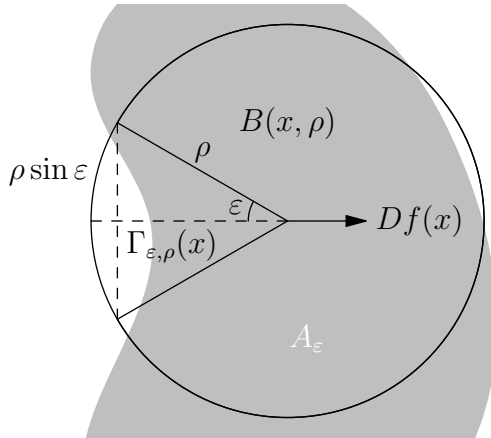


Figure 1. The sets $\Gamma_{\varepsilon, \rho}(x)$ and A_ε intersect.

Assume next $Df(x) \neq 0$. For each $h \in \mathbb{R}^d$, let β_h denote the angle formed by h and $-Df(x)$, so that

$$-\langle Df(x), h \rangle = |Df(x)| |h| \cos \beta_h.$$

The set

$$\Gamma_{\varepsilon, \rho}(x) := \{x + h : h \in B(0, \rho), \beta_h \leq \varepsilon\}$$

has measure

$$|\Gamma_{\varepsilon, \rho}(x)| > \int_0^{\rho \cos \varepsilon} \omega_{d-1} (r \sin \varepsilon)^{d-1} dr = \frac{\omega_{d-1}}{d} (\sin \varepsilon)^{d-1} (\cos \varepsilon)^d \rho^d.$$

Thus, it follows from (2.2) that $\Gamma_{\varepsilon, \rho}(x) \cap A_\varepsilon \neq \emptyset$, so by the definition of A_ε there is an $h \in \mathbb{R}^d$ such that

$$(2.3) \quad \frac{|f(x+h) - f(x) - \langle Df(x), h \rangle|}{|h|} < \varepsilon, \\ \beta_h \leq \varepsilon \quad \text{and} \quad 0 < |h| < \rho < \varepsilon.$$

By the triangle inequality, for h satisfying (2.3),

$$\left| |Df(x)| + \frac{f(x+h) - f(x)}{|h|} \right| \leq \left| |Df(x)| + \frac{\langle Df(x), h \rangle}{|h|} \right| + \left| \frac{f(x+h) - f(x)}{|h|} - \frac{\langle Df(x), h \rangle}{|h|} \right|$$

$$\begin{aligned} &< |Df(x)| |1 - \cos \beta_h| + \varepsilon \\ &\leq |Df(x)| |1 - \cos \varepsilon| + \varepsilon. \end{aligned}$$

As $|Df(x)| \neq 0$, the result now follows taking $\varepsilon = \min\{1/2n, 1/\sqrt{|Df(x)|n}\}$ and the corresponding $h_n = h$ from the previous display. \square

The approximate derivative of Mf for a.e. approximately differentiable functions $f \in L^1(\mathbb{R}^d)$ was studied by Hajlasz and Malý [11]. In particular, their arguments show that if $f \in L^1$ is a.e. approximately differentiable, then $\mathcal{M}_\alpha f$ is a.e. approximately differentiable.

2.3. The families of good balls

Let $0 < \alpha < d$ and $\delta \geq 0$. For the uncentered maximal operator, given a function $f \in W^{1,1}(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d$, define the family of *good balls* for f at x as

$$\begin{aligned} \mathcal{B}_{\alpha,x}^\delta \equiv \mathcal{B}_{\alpha,x}^\delta(f) &:= \left\{ B(z,r) : r \geq \delta, x \in \overline{B(z,r)}, \right. \\ &\left. \mathcal{M}_\alpha^\delta f(x) = r^\alpha \int_{B(z,r)} |f(y)| \, dy \right\}. \end{aligned}$$

For the centered maximal operator we use the same definition, except that $z = x$. Note that $\mathcal{B}_{\alpha,x}^\delta \neq \emptyset$ for all $x \in \mathbb{R}^d$ if $\delta > 0$. Moreover, by the Lebesgue differentiation theorem $\mathcal{B}_{\alpha,x} \equiv \mathcal{B}_{\alpha,x}^0 \neq \emptyset$ for a.e. $x \in \mathbb{R}^d$, and if $B(z,r) \in \mathcal{B}_{\alpha,x}^0$, then $r > 0$. This immediately implies that for a.e. x there exists $\delta_x > 0$ such that if $0 \leq \delta < \delta_x$, then

$$\mathcal{M}_\alpha^\delta f(x) = \mathcal{M}_\alpha f(x).$$

This type of observation will be used at the derivative level in the forthcoming Lemma 3.5.

2.4. Luiro’s Formula

An important tool for our purposes is the so called Luiro’s formula, which relates the derivative of the maximal function with the derivative of the original function. It corresponds to a refinement of Kinnunen’s inequality (1.1) and has its roots in [16, Theorem 3.1].

Proposition 2.4. *Let $0 < \alpha < d$, $\delta \geq 0$ and $f \in W^{1,1}(\mathbb{R}^d)$. Then, for a.e. $x \in \mathbb{R}^d$ and $B = B(z, r) \in \mathcal{B}_{\alpha,x}^\delta$, the weak derivative $\nabla \mathcal{M}_\alpha^\delta f$ satisfies*

$$(2.4) \quad \nabla \mathcal{M}_\alpha^\delta f(x) = r^\alpha \int_B \nabla |f|(y) \, dy.$$

Proof. This essentially follows from an argument of Hajlasz and Malý [11, Theorem 2], which we include for completeness. By §2.2 the weak gradient of $\mathcal{M}_\alpha^\delta f$ equals its approximate gradient almost everywhere, so it suffices to show (2.4) at a point x at which $\mathcal{M}_\alpha^\delta f$ is approximately differentiable and for which there exists $B = B(z_x, r_x) \in \mathcal{B}_{\alpha,x}^\delta$. Define the function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(y) &:= \mathcal{M}_\alpha^\delta f(y) - r^\alpha \int_{B(z_x+y-x, r_x)} |f(t)| \, dt \\ &= \mathcal{M}_\alpha^\delta f(y) - r^\alpha \int_{B(z_x-x, r_x)} |f(t-y)| \, dt, \end{aligned}$$

which satisfies $\varphi \geq 0$ and $\varphi(x) = 0$. Thus, φ has a minimum at x . Furthermore, φ is approximately differentiable at x (note that one can differentiate under the integral sign) and by Lemma 2.3 there exists a sequence $\{h_n\}_{n \in \mathbb{N}}$ with $|h_n| \rightarrow 0$ such that

$$|D\varphi(x)| = - \lim_{n \rightarrow \infty} \frac{\varphi(x+h_n) - \varphi(x)}{|h_n|}.$$

As φ has a minimum at x , the right-hand side is non-positive and thus $D\varphi(x) = 0$, which yields the desired result. \square

Remark 2.5. Proposition 2.4 continues to hold for $\alpha = 0$, replacing the weak derivative by the approximate derivative in the cases where the weak differentiability of \mathcal{M} is currently unknown.

2.5. Refined Kinnunen–Saksman Inequality

The Kinnunen–Saksman inequality (1.4) admits a refinement in terms of the good balls, in the same spirit as Luiro’s formula (2.4) improves over Kinnunen’s pointwise inequality (1.1). It is noted that further refinements involving boundary terms (that is, averages along spheres) have been obtained in [19] and [3] for \widetilde{M}_α and M_α respectively, although these are not required for the purposes of this paper.

Proposition 2.6. *Let $0 < \alpha < d$, $\delta \geq 0$ and $f \in W^{1,1}(\mathbb{R}^d)$. Then, for a.e. $x \in \mathbb{R}^d$ and $B = B(z, r) \in \mathcal{B}_{\alpha,x}^\delta$, the weak derivative $\nabla \mathcal{M}_\alpha^\delta f$ satisfies*

$$(2.5) \quad |\nabla \mathcal{M}_\alpha^\delta f(x)| \leq (d - \alpha)r^{\alpha-1} \int_B |f(y)| \, dy.$$

Proof. By §2.2 the weak gradient of $\mathcal{M}_\alpha^\delta f$ equals its approximate gradient almost everywhere, so it suffices to show (2.5) at a point x at which $\mathcal{M}_\alpha^\delta f$ is approximately differentiable and for which there exists $B = B(z, r) \in \mathcal{B}_{\alpha,x}^\delta$. By Lemma 2.3 there is a sequence $\{h_n\}_{n \in \mathbb{N}}$ with $|h_n| \rightarrow 0$ and

$$|\nabla \mathcal{M}_\alpha^\delta f(x)| = \lim_{n \rightarrow \infty} \frac{\mathcal{M}_\alpha^\delta f(x) - \mathcal{M}_\alpha^\delta f(x + h_n)}{|h_n|}.$$

Now the proof follows from the classical Kinnunen–Saksman [14] reasoning, which we include for completeness. Note that $x + h_n \in B(z + h_n, r + |h_n|)$, and that for the centered maximal operator we have $z = x$. This implies

$$\mathcal{M}_\alpha^\delta f(x + h_n) \geq (r + |h_n|)^\alpha \int_{B(z+h_n, r+|h_n|)} |f(y)| \, dy.$$

Therefore

$$\begin{aligned} & \frac{\mathcal{M}_\alpha^\delta f(x) - \mathcal{M}_\alpha^\delta f(x + h_n)}{|h_n|} \\ & \leq \frac{1}{\omega_d |h_n|} \left(r^{\alpha-d} \int_{B(z,r)} |f(y)| \, dy - (r + |h_n|)^{\alpha-d} \int_{B(z+h_n, r+|h_n|)} |f(y)| \, dy \right) \\ & \leq \frac{1}{\omega_d |h_n|} \left(r^{\alpha-d} \int_{B(z+h_n, r+|h_n|)} |f(y)| \, dy \right. \\ & \quad \left. - (r + |h_n|)^{\alpha-d} \int_{B(z+h_n, r+|h_n|)} |f(y)| \, dy \right) \\ & = \frac{r^{\alpha-d} - (r + |h_n|)^{\alpha-d}}{\omega_d |h_n|} \int_{B(z+h_n, r+|h_n|)} |f(y)| \, dy \\ & \rightarrow \frac{(d - \alpha)r^{\alpha-d-1}}{\omega_d} \int_{B(z,r)} |f(y)| \, dy \end{aligned}$$

for $n \rightarrow \infty$, which concludes the proof. □

Remark 2.7. Proposition 2.6 continues to hold for $\alpha = 0$, replacing the weak derivative by the approximate derivative in the cases where the weak differentiability of \mathcal{M} is currently unknown.

2.6. A refined fractional maximal function

In view of the Kinnunen–Saksman type inequality (2.5), it is instructive to define the operator

$$\mathcal{M}_{\alpha,-1}f(x) = \sup_{B \in \mathcal{B}_{\alpha,x}(f)} r(B)^{\alpha-1} \int_B |f(y)| \, dy,$$

so that for any $0 < \alpha < d$,

$$(2.6) \quad |\nabla \mathcal{M}_{\alpha}f(x)| \leq (d - \alpha)\mathcal{M}_{\alpha,-1}f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Furthermore, this extends to the case $\delta > 0$, that is,

$$(2.7) \quad |\nabla \mathcal{M}_{\alpha}^{\delta}f(x)| \leq (d - \alpha)\mathcal{M}_{\alpha,-1}f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Indeed, let $\delta > 0$ and $B \in \mathcal{B}_{\alpha,x}^{\delta}$. Then, there exists $C \in \mathcal{B}_{\alpha,x}$ such that $r(C) \leq r(B)$. This immediately yields

$$r(B)^{\alpha-1} \int_B |f| \leq r(C)^{\alpha-1} \int_C |f| \leq \mathcal{M}_{\alpha,-1}f(x),$$

which implies (2.7) via Proposition 2.6.

The proof of Theorem 2.2 in [24] is obtained through the analogous bound on $\mathcal{M}_{\alpha,-1}$. Indeed, such a bound is of local nature. The following is a local version of [24, Theorem 1.2]; the details are explained in [24, Remark 1.8].

Theorem 2.8. *Let $0 < \alpha < d$ and $E \subseteq \mathbb{R}^d$ be a measurable set. There exist constants $c > 1$ and $C_{d,\alpha} > 0$ such that the inequality*

$$\|\mathcal{M}_{\alpha,-1}f\|_{L^{d/(d-\alpha)}(E)} \leq C_{d,\alpha}\|\nabla f\|_{L^1(D)}$$

holds for all $f \in W^{1,1}(\mathbb{R}^d)$, where

$$D = \bigcup_{B \in \mathcal{I}_E} cB \quad \text{and} \quad \mathcal{I}_E := \{B \in \mathcal{B}_{\alpha,x}; \text{ for some } x \in E\}.$$

Remark 2.9. For $0 < \alpha < d$ one has, combining (2.6) and Theorem 2.8, that

$$\|\nabla \mathcal{M}_{\alpha}f\|_{L^{d/(d-\alpha)}(E)} \leq (d - \alpha)C_{d,\alpha}\|\nabla f\|_{L^1(D)},$$

where $C_{d,\alpha}$ is the constant in Theorem 2.8.

2.7. Poincaré–Sobolev Inequality

Another important tool for our purposes is the following.

Lemma 2.10. *Let $0 < \alpha < d$ and $c > 1$. Then there is a constant $C_{d,\alpha,c} > 0$ such that for all $f \in W^{1,1}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $B = B(z, r) \in \mathcal{B}_{\alpha,x}(f)$ the inequality*

$$\int_{cB} |f(y)| \, dy \leq C_{d,\alpha,c} r \int_{cB} |\nabla f(y)| \, dy$$

holds.

Proof. By the triangle inequality and the Poincaré–Sobolev inequality there is a C_d such that

$$\int_{cB} \left| |f(y)| - |f_{cB}| \right| \, dy \leq \int_{cB} |f(y) - f_{cB}| \, dy \leq C_d r \int_{cB} |\nabla f(y)| \, dy.$$

Since $B \in \mathcal{B}_{\alpha,x}$ we have $c^\alpha |f|_{cB} \leq |f|_B$. This, the triangle inequality and $|f_{cB}| \leq |f|_{cB}$ yield

$$\begin{aligned} c^d \int_{cB} \left| |f(y)| - |f_{cB}| \right| \, dy &\geq \int_B \left| |f(y)| - |f_{cB}| \right| \, dy \\ &\geq |f|_B - |f|_{cB} \geq (c^\alpha - 1) \int_{cB} |f(y)| \, dy. \end{aligned}$$

Then, combining the above, we obtain

$$\int_{cB} |f(y)| \, dy \leq \frac{c^d C_d}{c^\alpha - 1} r \int_{cB} |\nabla f(y)| \, dy,$$

as desired. □

3. Convergences

In this section we review some auxiliary convergence results established in the series of papers [2, 7] which reduce the proof of Theorem 1.1 to the convergence of the difference $\mathcal{M}_\alpha f_j - \mathcal{M}_\alpha^\delta f_j$ on a compact set.

3.1. A Sobolev space lemma

We start recalling an auxiliary result concerning the convergence of the modulus of a sequence in $W^{1,1}(\mathbb{R}^d)$. This is useful in view of the identity (2.4).

Lemma 3.1 ([2, Lemma 2.3]). *Let $f \in W^{1,1}(\mathbb{R}^d)$ and $\{f_j\}_{j \in \mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d)$ be such that $\|f_j - f\|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$. Then $\| |f_j| - |f| \|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$.*

3.2. Convergence outside a compact set

By Theorem 2.2 and the work of the first and third author in [2] we have that it suffices to study the convergence in a compact set.

Proposition 3.2 ([2, Proposition 4.10]). *Let $0 < \alpha < d$, $f \in W^{1,1}(\mathbb{R}^d)$ and $\{f_j\}_{j \in \mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d)$ such that $\|f_j - f\|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$. Then, for any $\varepsilon > 0$ there exists a compact set K and $j_\varepsilon > 0$ such that*

$$\|\nabla \mathcal{M}_\alpha f_j - \nabla \mathcal{M}_\alpha f\|_{L^{d/(d-\alpha)}((3K)^\varepsilon)} < \varepsilon$$

for all $j \geq j_\varepsilon$.

3.3. Continuity of $\mathcal{M}_\alpha^\delta$ in $W^{1,1}(\mathbb{R}^d)$, $\delta > 0$

A key observation is the a.e. convergence of the maximal function $\mathcal{M}_\alpha^\delta f_j$ at the derivative level.

Lemma 3.3. *Let $0 < \alpha < d$, $\delta \geq 0$, $f \in W^{1,1}(\mathbb{R}^d)$ and $\{f_j\}_{j \in \mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d)$ be such that $\|f_j - f\|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$. Then*

$$\nabla \mathcal{M}_\alpha^\delta f_j(x) \rightarrow \nabla \mathcal{M}_\alpha^\delta f(x) \quad \text{a.e. as } j \rightarrow \infty.$$

A version of this result for the full \mathcal{M}_α is given in [2, Lemma 2.4]. The proof for $\mathcal{M}_\alpha^\delta$ is identical (in fact, it slightly simplifies), and relies on Luiro’s formula for $\mathcal{M}_\alpha^\delta$, that is, Proposition 2.4. We omit further details. For $\delta > 0$, we have the following norm convergence.

Proposition 3.4. *Let $0 < \alpha < d$, $\delta > 0$, $f \in W^{1,1}(\mathbb{R}^d)$ and $\{f_j\}_{j \in \mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d)$ be such that $\|f_j - f\|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$. Let $K \subseteq \mathbb{R}^d$ be a compact set.*

$$\|\nabla \mathcal{M}_\alpha^\delta f - \nabla \mathcal{M}_\alpha^\delta f_j\|_{L^{d/(d-\alpha)}(K)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Proof. By Proposition 2.4 and Lemma 3.1 there exists $j_0 \in \mathbb{N}$ such that

$$|\nabla \mathcal{M}_\alpha^\delta f_j(x)| \leq \frac{1}{\omega_d \delta^{d-\alpha}} \|\nabla |f_j|\|_1 \leq \frac{1}{\omega_d \delta^{d-\alpha}} \|\nabla |f|\|_1 + 1$$

for all $j \geq j_0$, a.e. $x \in K$.

Furthermore, by Lemma 3.3

$$\nabla \mathcal{M}_\alpha^\delta f_j(x) \rightarrow \nabla \mathcal{M}_\alpha^\delta f(x) \quad \text{a.e. as } j \rightarrow \infty.$$

The convergence on $L^{d/(d-\alpha)}(K)$ then follows from the dominated convergence theorem. \square

3.4. δ -convergence of $\nabla \mathcal{M}_\alpha^\delta f$

Here we establish that $\nabla \mathcal{M}_\alpha^\delta f$ provides a good approximation for $\nabla \mathcal{M}_\alpha f$ in $L^{d/(d-\alpha)}(\mathbb{R}^d)$ when $\delta \rightarrow 0$; note, however, that for the proof of Theorem 1.1 it suffices to establish it in a compact set. This relies on the Theorem 2.2.

Lemma 3.5. *Let $0 < \alpha < d$ and $f \in W^{1,1}(\mathbb{R}^d)$. Then*

$$\|\nabla \mathcal{M}_\alpha f - \nabla \mathcal{M}_\alpha^\delta f\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Proof. Recall from §2.3 that for a.e. $x \in \mathbb{R}^d$ one has that if $B(z, r) \in \mathcal{B}_{\alpha, x}^\delta$, then $r > 0$. This and Luro’s formula (2.4) imply that for a.e. $x \in \mathbb{R}^d$ there exists $\delta_x > 0$ such that

$$\nabla \mathcal{M}_\alpha^\delta f(x) = \nabla \mathcal{M}_\alpha f(x) \quad \text{for all } 0 \leq \delta < \delta_x,$$

and thus $\nabla \mathcal{M}_\alpha^\delta f(x) \rightarrow \nabla \mathcal{M}_\alpha f(x)$ for a.e. $x \in \mathbb{R}^d$ as $\delta \rightarrow 0$. Furthermore, as proven in (2.7), for a.e. $x \in \mathbb{R}^d$ we have that

$$|\nabla \mathcal{M}_\alpha^\delta f(x)| \leq (d - \alpha) \mathcal{M}_{\alpha, -1} f(x) \quad \text{for all } \delta \geq 0.$$

Since $f \in W^{1,1}(\mathbb{R}^d)$, Theorem 2.8 ensures that $\mathcal{M}_{\alpha, -1} f \in L^{d/(d-\alpha)}(\mathbb{R}^d)$ and we can then conclude the result by the dominated convergence theorem. \square

4. Proof of Theorem 1.1

Let $f \in W^{1,1}(\mathbb{R}^d)$ and $\{f_j\}_{j \in \mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d)$ be a sequence of functions such that $\|f_j - f\|_{W^{1,1}(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$. If $f = 0$ then the result follows directly

from the boundedness, that is Theorem 2.2. From now on we assume that $f \neq 0$. Let $\varepsilon > 0$. Then by Proposition 3.2 it is sufficient to prove that for any compact set $K \subseteq \mathbb{R}^d$ there exists $j^* \in \mathbb{N}$ such that

$$(4.1) \quad \|\nabla \mathcal{M}_\alpha f - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(K)} < 3\varepsilon$$

for all $j \geq j^*$. To this end, for any $\delta > 0$, use the triangle inequality to bound

$$(4.2) \quad \begin{aligned} \|\nabla \mathcal{M}_\alpha f - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(K)} &\leq \|\nabla \mathcal{M}_\alpha f - \nabla \mathcal{M}_\alpha^\delta f\|_{L^{d/(d-\alpha)}(K)} \\ &\quad + \|\nabla \mathcal{M}_\alpha^\delta f - \nabla \mathcal{M}_\alpha^\delta f_j\|_{L^{d/(d-\alpha)}(K)} \\ &\quad + \|\nabla \mathcal{M}_\alpha^\delta f_j - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(K)}. \end{aligned}$$

To finish the proof, it suffices to show that for $\varepsilon > 0$ fixed, there exist a δ^* and a j^* such that for $\delta = \delta^*$ and all $j \geq j^*$, each of the summands on the right hand side of (4.2) is bounded by ε . We choose δ^* depending on ε, K and f , and j^* depending on $\delta^*, \varepsilon, K, f$ and the sequence $\{f_j\}_{j \in \mathbb{N}}$.

For the first term, we know by Lemma 3.5 that there exists a $\delta' > 0$ such that

$$\|\nabla \mathcal{M}_\alpha f - \nabla \mathcal{M}_\alpha^\delta f\|_{L^{d/(d-\alpha)}(K)} < \varepsilon$$

for all $0 \leq \delta \leq \delta'$. For the second term, we have by Proposition 3.4 that for every $\delta > 0$ there exists a $j(\delta) \in \mathbb{N}$ such that

$$\|\nabla \mathcal{M}_\alpha^\delta f - \nabla \mathcal{M}_\alpha^\delta f_j\|_{L^{d/(d-\alpha)}(K)} < \varepsilon$$

for all $j \geq j(\delta)$. The rest of the section is devoted to proving a favourable bound for the third term. More precisely, we will show that there are $\tilde{\delta} > 0$ and $\tilde{j} \in \mathbb{N}$ such that for all $0 \leq \delta \leq \tilde{\delta}$ and $j \geq \tilde{j}$,

$$(4.3) \quad \|\nabla \mathcal{M}_\alpha^\delta f_j - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(K)} < \varepsilon.$$

Temporarily assuming this, we can then conclude that for $\delta = \delta^* := \min\{\delta', \tilde{\delta}\}$ and $j \geq j^* := \max\{j(\delta^*), \tilde{j}\}$, the right-hand side of (4.2) is bounded by at most 3ε , as desired for (4.1).

We now turn to the proof of (4.3). We start by noting that there exists a $\lambda_0 > 0$ and a $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ and $x \in K$ we have $\mathcal{M}_\alpha f_j(x) > \lambda_0$. Indeed, as $f \in L^1(\mathbb{R}^d)$, there exists a ball B_0 that contains K with $\int_{B_0} |f| > \frac{1}{2} \int_{\mathbb{R}^d} |f|$. As $\|f_j - f\|_1 \rightarrow 0$ as $j \rightarrow 0$, by the triangle inequality, there exists $j_0 > 0$ such that for all $j \geq j_0$ we have $\int_{B_0} |f_j| > \frac{1}{2} \int_{B_0} |f| >$

$\frac{1}{4} \int_{\mathbb{R}^d} |f|$. Then, for every $j \geq j_0$ and $x \in K$ we have

$$\mathcal{M}_\alpha f_j(x) \geq 2^\alpha r(B_0)^\alpha \int_{B(x, 2r(B_0))} |f_j| > \frac{(2r(B_0))^{\alpha-d}}{4\omega_d} \int_{\mathbb{R}^d} |f|,$$

where in the last inequality we have used that $B(x, 2r(B_0)) \supset B_0$ for all $x \in K$. Thus, we can take λ_0 to be the right-hand side of the inequality above. Furthermore by Proposition 2.4, if there exists a $B \in \mathcal{B}_{\alpha,x}(f_j)$ such that $r(B) \geq \delta$ then $\nabla \mathcal{M}_\alpha f_j(x) = \nabla \mathcal{M}_\alpha^\delta f_j(x)$. Define

$$E_{\lambda_0, \delta, j} := \left\{ x \in K : \text{if } B \in \mathcal{B}_{\alpha,x}(f_j), \right. \\ \left. \text{then } r(B) < \delta \text{ and } r(B)^\alpha \int_B |f_j| > \lambda_0 \right\}.$$

By the previous two observations, Proposition 2.6 and a crude application of the triangle inequality, one has

$$\begin{aligned} \|\nabla \mathcal{M}_\alpha^\delta f_j - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(K)} &= \|\nabla \mathcal{M}_\alpha^\delta f_j - \nabla \mathcal{M}_\alpha f_j\|_{L^{d/(d-\alpha)}(E_{\lambda_0, \delta, j})} \\ &\leq 2(d - \alpha) \|\mathcal{M}_{\alpha, -1} f_j\|_{L^{d/(d-\alpha)}(E_{\lambda_0, \delta, j})}. \end{aligned}$$

for all $j \geq j_0$. Define the indexing set

$$\mathcal{I}_{\lambda_0, \delta, j} := \left\{ B \in \mathcal{B}_{\alpha,x}(f_j) : x \in K, r(B) < \delta \text{ and } r(B)^\alpha \int_B |f_j| > \lambda_0 \right\}$$

and consider the set

$$D_{\lambda_0, \delta, j} := \bigcup_{B \in \mathcal{I}_{\lambda_0, \delta, j}} cB,$$

where c is the constant from Theorem 2.8. Then, by Theorem 2.8, we have

$$\|\mathcal{M}_{\alpha, -1} f_j\|_{L^{d/(d-\alpha)}(E_{\lambda_0, \delta, j})} \leq C_{d, \alpha} \|\nabla f_j\|_{L^1(D_{\lambda_0, \delta, j})}$$

for any $\delta > 0$. Thus, the proof of (4.3) is reduced to showing that there exist a $\tilde{\delta} > 0$ and a $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$ and $0 \leq \delta \leq \tilde{\delta}$ we have

$$(4.4) \quad \|\nabla f_j\|_{L^1(D_{\lambda_0, \delta, j})} < \frac{\varepsilon}{2(d - \alpha)C_{d, \alpha}},$$

as one can then take $\tilde{j} := \max\{j_0, j_1\}$.

In order to prove (4.4), we first use the triangle inequality and that $\|\nabla f_j - \nabla f\|_{L^1(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$ to find a $j_2 \in \mathbb{N}$ such that

$$(4.5) \quad \|\nabla f_j\|_{L^1(D_{\lambda_0, \delta, j})} \leq \|\nabla f\|_{L^1(D_{\lambda_0, \delta, j})} + \frac{\varepsilon}{4(d - \alpha)C_{d, \alpha}}.$$

for any $\delta > 0$ and $j \geq j_2$.

Next, let $x \in D_{\lambda_0, \delta, j}$. Then there is a $B \in \mathcal{I}_{\lambda_0, \delta, j}$ with $x \in cB$. So, by Lemma 2.10, we have

$$\begin{aligned} \lambda_0 &\leq c^d r(B)^\alpha \int_{cB} |f_j| \leq C_{d, \alpha, c} c^{d+1} r(B)^{\alpha+1} \int_{cB} |\nabla f_j| \\ &\leq C_{d, \alpha, c} c^{d-\alpha+1} \delta \widetilde{M}_\alpha |\nabla f_j|(x), \end{aligned}$$

where \widetilde{M}_α in the above inequality denotes the uncentered fractional maximal operator. Hence, by the weak $(1, d/(d - \alpha))$ inequality for \widetilde{M}_α ,

$$(4.6) \quad \begin{aligned} |D_{\lambda_0, \delta, j}| &\leq \left| \left\{ x : \widetilde{M}_\alpha |\nabla f_j|(x) \geq \frac{\lambda_0}{C_{d, \alpha, c} c^{d-\alpha+1} \delta} \right\} \right| \\ &\leq C_{d, \alpha, c, \lambda_0} \delta^{d/(d-\alpha)} \|\nabla f_j\|_1^{d/(d-\alpha)} \\ &\leq C_{d, \alpha, c, \lambda_0} \delta^{d/(d-\alpha)} \left(1 + \|\nabla f\|_1^{d/(d-\alpha)} \right) \end{aligned}$$

if $j \geq j_3$ for some $j_3 \in \mathbb{N}$, using that $\|\nabla f_j - \nabla f\|_{L^1(\mathbb{R}^d)} \rightarrow 0$ as $j \rightarrow \infty$.

Finally, note that as $\nabla f \in L^1(\mathbb{R}^d)$, there exists $\rho > 0$ such that for all $A \subseteq \mathbb{R}^d$ satisfying $|A| < \rho$, one has

$$(4.7) \quad \|\nabla f\|_{L^1(A)} < \frac{\varepsilon}{4(d - \alpha)C_{d, \alpha}}.$$

As the right-hand side of (4.6) goes to zero for $\delta \rightarrow 0$ uniformly in j , there exists $\tilde{\delta} > 0$ such that $|D_{\lambda_0, \delta, j}| < \rho$ for all $j \geq j_3$ and $\delta < \tilde{\delta}$. Thus, taking $j_1 := \max\{j_2, j_3\}$, (4.4) follows from combining (4.5) and (4.7) with $A = D_{\lambda_0, \delta, j}$. This implies the claimed inequality (4.3) and therefore finishes the proof of Theorem 1.1. □

Remark. Note that in the above proof, instead of using Lemma 3.5 to bound the first term in (4.2), we could have also bounded it running the same scheme as for the third term.

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