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Stein-fillable open books of genus one that do not admit positive factorisations

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We construct an infinite family of genus one open book decompositions supporting Stein-fillable contact structures and show that their monodromies do not admit positive factorisations. This extends a line of counterexamples in higher genera and establishes that a correspondence between Stein fillings and positive factorisations only exists for planar open book decompositions.

1. Introduction

In the foundational work [7], Giroux has established a one-to-one correspondence between isotopy classes of contact structures on a 3-manifold Y and positive stabilisation classes of open book decompositions of Y, enabling one to consider questions of contact and symplectic geometry through a powerful lens of surface mapping class groups. In particular, a natural question when studying contact manifolds is that of fillability, i.e., determining when a contact manifold can be the boundary of a symplectic manifold in some compatible way; in this paper, we are concerned with Stein fillability. Results of Giroux coupled with the work of Loi and Piergallini [11], Akbulut and Özbağcı [1] and Plamenevskaya [15] drew a further connection between the worlds of symplectic geometry and surface diffeomorphisms, establishing that a contact manifold is Stein-fillable if and only if the monodromy of *some* open book supporting it admits a positive factorisation into Dehn twists. The picture, however, is still complicated: for example, proving that a contact manifold is not Stein-fillable this way entails the usually intractable task of obstructing positive factorisability of *all* monodromies of supporting open books.

A tempting but untrue strengthening of this result would be the claim that the monodromy of *every* open book (Σ, φ) supporting a Stein-fillable contact manifold (Y, ξ) factorises positively. Indeed, a result of Wendl [20] implies that if the genus $g(\Sigma) = 0$, then Stein fillings of (Y, ξ), up to symplectic deformation, are in one-to-one correspondence with positive factorisations of φ, up to conjugation. However, if $g(\Sigma) \geqslant 2$, it follows from the work of the second author [18] and Baker, Etnyre and Van Horn-Morris [2] that φ need not admit any positive factorisation. The case of $g(\Sigma) = 1$ has been previously studied by Lisca [10] who has shown that if Σ has one boundary component and Y is a Heegaard Floer L-space, then (Y, ξ) is Stein-fillable if and only if φ admits a positive factorisation. The purpose of this paper is to exhibit for the first time a family of Stein-fillable contact manifolds supported by open books with $g(\Sigma) = 1$ whose monodromies do not factorise positively.

Theorem 1.1. Let $n \geqslant 0$. Then $(\Sigma_{1,2}, \varphi_n)$ with $\varphi_n = \tau_\alpha \tau_\beta \tau_\gamma^{-1} \tau_{\delta_1} \tau_{\delta_2}^{4+n}$, as illustrated in Figure 1, is an open book decomposition supporting a Stein-fillable contact manifold, but φ_n does not admit a positive factorisation.

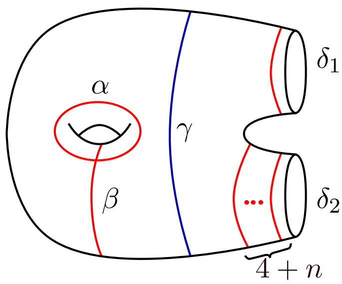

Figure 1. An open book decomposition $(\Sigma_{1,2}, \tau_\alpha \tau_\beta \tau_\gamma^{-1} \tau_{\delta_1} \tau_{\delta_2}^{4+n})$ with $n \geqslant 0$.

Remark 1.2. The open books in our examples have pages $\Sigma_{1,2}$ with two boundary components, and we note that one can add 1-handles to them to obtain any surface $\Sigma_{g,n}$ with $g, n \geqslant 1$ other than $\Sigma_{1,1}$. Since adding a 1-handle to the page of an open book amounts to, on the level of 3-manifolds, taking a contact connected sum with $S^1 \times S^2$ endowed with its unique Stein-fillable contact structure, it also preserves Stein fillability. Moreover, if one attaches a 1-handle while extending the monodromy by the identity on the co-core of the 1-handle, it does not change the property of not being positively factorisable (see [10, Remark 5.3]). Hence, this leaves open only the case of open books whose pages are one-holed tori.

The structure of the paper is as follows. In Section 2 we recall some pertinent facts about open book decompositions and their interplay with contact structures, as well as the notion of Stein fillability. In Section 3 we construct, via transverse contact surgery, an infinite family of Stein-fillable contact manifolds with explicitly given genus one open books. Finally, in Section 4 we show that the monodromies of those open books do not admit positive factorisations.

2. Basic definitions

Recall that an *open book decomposition* (or just *open book*) of a closed 3-manifold Y is a pair (L, π) where $L \subset Y$ is an oriented link, called the *binding*, and $\pi : Y \setminus L \to S^1$ is a fibration such that $\pi^{-1}(s)$ for any $s \in S^1$ is the interior of a compact orientable surface Σ_π, called the *page*, with $\partial\Sigma_\pi = L$. Now, given a compact oriented surface Σ with boundary, denote by Γ_Σ the *mapping class group* of Σ consisting of isotopy classes of orientation-preserving self-diffeomorphisms of Σ that restrict to the identity on $\partial\Sigma$; we shall confuse classes in Γ_Σ with their representatives. Any locally trivial bundle over oriented S^1 with the fibre Σ is canonically diffeomorphic to the fibration $M_\varphi \to S^1$ for $M_\varphi = [0,1] \times \Sigma/(0, \varphi(x)) \sim (1, x)$ and φ an orientation-preserving self-diffeomorphism of Σ, taken up to conjugation. Hence, an open book decomposition (L, π) of a 3-manifold $Y_{(L,\pi)}$ determines a map $\varphi_\pi \in \Gamma_{\Sigma_\pi}$, called the *monodromy*. On the other hand, given a pair (Σ, φ) with $\varphi \in \Gamma_\Sigma$ and $\partial\Sigma \neq \varnothing$, we may construct a closed 3-manifold $Y_{(\Sigma,\varphi)}$ in the following way: take the mapping torus M_φ, identify all its boundary components with $\bigsqcup_n S^1 \times S^1$ for some $n > 0$, where in each $S^1 \times S^1$ the first factor comes from the quotient of the unit interval and the second from $\partial\Sigma$, and glue in solid tori $\bigsqcup_n D^2 \times S^1$ via the identity map $\bigsqcup_n \partial D^2 \times S^1 \to \bigsqcup_n S^1 \times S^1$. The resultant $Y_{(\Sigma,\varphi)}$ admits an open book decomposition with the binding given by the cores $\bigsqcup_n \{0\} \times S^1$ of the solid tori, the page Σ and monodromy φ. Hence, we can pass between (L, π) and (Σ, φ) to determine an open book decomposition of a closed 3-manifold up to diffeomorphism.

Given $\varphi \in \Gamma_\Sigma$, we say that φ admits a *positive factorisation* if it can be written as a product of positive Dehn twists about essential simple closed curves in Σ. We will denote by Γ_Σ^+ the sub-monoid of Γ_Σ consisting of isotopy classes of positively factorisable maps.

Recall also that a *(positive) contact structure* on Y is an oriented plane field $\xi \subset TY$ given by $\ker\alpha$ for some 1-form $\alpha \in \Omega^1(Y)$ satisfying $\alpha \wedge d\alpha > 0$. We say that ξ is *supported* by an open book decomposition of Y if $\alpha > 0$ on the binding and $d\alpha > 0$ on the interior of the pages. In fact, every open book decomposition of Y supports some contact structure [17]. Moreover, as recounted in Section 1, Giroux has shown [7] that there exists a one-to-one correspondence between contact structures on Y up to contact isotopy and open book decompositions of Y up to *positive stabilisation*, i.e., up to adding a 1-handle to the page and pre-composing the monodromy with a positive Dehn twist about some closed curve in the page intersecting the co-core of the 1-handle once. If a positive stabilisation of (Σ, φ) yields (Σ', φ'), then the contact manifolds (Y, ξ) and (Y', ξ') supported by those respective open books are *contactomorphic*, i.e., there exists a diffeomorphism $Y \to Y'$ that induces a map carrying ξ to ξ'.

A *Stein surface* is a complex surface W endowed with a Morse function $f : W \to \mathbb{R}$ such that for any non-critical point c of f, the level set $f^{-1}(c)$ inherits a contact structure ξ_c, induced by the complex tangencies, that orients $f^{-1}(c)$ as when $f^{-1}(c)$ is viewed as the boundary of the complex manifold $f^{-1}((-\infty, c])$. We say that a contact manifold (Y, ξ) is *Stein-fillable* if Y is orientation-preserving diffeomorphic to such $f^{-1}(c)$ and ξ is isotopic to ξ_c. If the 3-manifold is understood, we might simply say that ξ is Stein-fillable. A necessary condition for (Y, ξ) to be Stein-fillable is for ξ to be *tight*, i.e., there being no embedded disc $D^2 \subset Y$ tangent to ξ everywhere along ∂D^2; if there is such a disc, ξ is *overtwisted.* As noted in the introduction, (Y, ξ) is Stein-fillable if and only if the monodromy of some open book decomposition of Y supporting ξ admits a positive factorisation.

Having refreshed our memory, we are now ready to tackle the task of constructing Stein-fillable contact manifolds supported by genus one open books whose monodromies do not positively factorise.

3. A family of Stein-fillable contact manifolds supported by genus one open books

The goal of this section is to use methods of Conway [3] to construct a family of Stein-fillable contact manifolds by surgery techniques, and to determine supporting genus one open book decompositions. Recall that an oriented knot $K \subset (Y, \xi)$ is *transverse* if its oriented tangent vector is always positively transverse to ξ. By *transverse surgery* on K we mean an analogue of the usual surgery operation in the contact category, defined by Gay [5], in which we first cut out, then re-glue a contact neighbourhood of K to obtain a new contact manifold; the adjective *'inadmissible'* characterises 'adding the twisting' near the knot, while *admissible* transverse surgery 'removes the twisting'.

3.1. An algorithm for describing open books supporting transverse surgered manifolds

First, we collect necessary ingredients from [3] to describe open books supporting the result of inadmissible transverse surgery on a knot that is already a component of the binding of some open book of the original manifold.

Suppose (Σ, φ) is an open book, and Σ has a boundary component K, forming a part of the binding. In the following, by 'stabilising K' we mean adding a 1-handle across K and pre-composing the monodromy with a positive Dehn twist about a curve that is boundary-parallel to one of the two new boundary components. After that, we continue denoting by K the other boundary component, without a twist about it. Recall that given a rational number $r < 0$, we can write it as a negative continued fraction $[a_1 + 1, a_2, \ldots, a_n]^-$, where

$$r = a_1 + 1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}$$

and $a_i \leqslant -2$ for all i. The following two propositions give the desired procedure.

Proposition 3.1 ([3, Proposition 3.9]). Let $r \in \mathbb{Q}$ with $r < 0$ and $r = [a_1 + 1, a_2, \ldots, a_n]^-$. The open book supporting admissible transverse r-surgery with respect to the page slope on the binding component K is obtained by, for each $i = 1, \ldots, n$ in order, stabilising K positively $|a_i + 2|$ times and adding a positive Dehn twist about K.

Proposition 3.2 ([3, Proposition 3.12]). Let $r = p/q \in \mathbb{Q}$ with $r > 0$, n a positive integer such that $1/n < r$, and $r' = p/(q - np)$. The open book supporting inadmissible transverse r-surgery with respect to the page slope on the binding component K is obtained by first adding n negative Dehn twists about K, and then performing admissible transverse r'-surgery on K.

3.2. Transverse (+5)-surgery on a right-handed trefoil

Given a simple closed curve σ in a surface Σ, denote the positive Dehn twist about σ by τ_σ. Consider a transverse right-handed trefoil knot T in $(S^3, \xi_{\mathrm{std}})$, where ξ_{std} is the standard tight contact structure on S^3. By stabilising the standard open book for $(S^3, \xi_{\mathrm{std}})$ given by the positive Hopf band, we can take T to be the binding of an open book $(\Sigma_{1,1}, \tau_\alpha\tau_\beta)$ with one-holed torus pages supporting $(S^3, \xi_{\mathrm{std}})$; this open book is shown on the left of Figure 2.

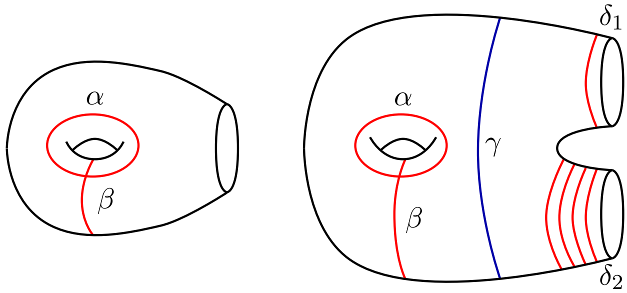

Figure 2. On the left: an open book $(\Sigma_{1,1}, \tau_\alpha\tau_\beta)$, obtained as a plumbing of two positive Hopf bands, supporting $(S^3, \xi_{\mathrm{std}})$. On the right: an open book $(\Sigma_{1,2}, \tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2}^4)$ supporting $(L(5,1), \xi)$, the result of transverse $(+5)$-surgery on a right-handed trefoil in $(S^3, \xi_{\mathrm{std}})$.

In the notation of Proposition 3.2, we have $r = p/q = 5/1$, and, choosing $n = 1$, we get that $r' = -5/4 = [-3+1, -2, -2, -2]^-$. Hence, an open book supporting (Y_0, ξ_0), the product of inadmissible transverse $(+5)$-surgery on T, is obtained by adding a negative boundary twist τ_γ^{-1} to the monodromy, stabilising once, adding a twist about K, then adding three more twists about K. Renaming K to δ_2 and the other boundary component to δ_1, we conclude that (Y_0, ξ_0) is supported by the open book $(\Sigma_{1,2}, \varphi_0)$ shown in Figure 2 with $\varphi_0 = \tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2}^4$. Since $r > 2g(\Sigma_{1,1}) = 2$ and $(\Sigma_{1,1}, \tau_\alpha\tau_\beta)$ supports the contact manifold $(S^3, \xi_{\mathrm{std}})$ whose Ozsváth–Szabó contact invariant $c(\xi_{\mathrm{std}})$ is non-zero [14], by [8, Theorem 3] we have that $c(\xi_0)$ is also non-zero and hence (Y_0, ξ_0) is tight. Figure 3 shows that Y_0 can be obtained by -5-surgery on the unknot in S^3 and thus is diffeomorphic to the lens space $L(5,1)$. By work of McDuff [13] and Plamenevskaya and Van Horn-Morris [16], every tight contact structure on $L(p,1)$ with $p \neq 4$ has a unique Stein filling, hence so does (Y_0, ξ_0).

Finally, we observe that (Y_n, ξ_n), the product of n-fold Legendrian surgery on the δ_2 component of $(\Sigma_{1,2}, \varphi_0)$, is supported by the open book $(\Sigma_{1,2}, \varphi_n)$, where $\varphi_n = \tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2}^{4+n}$. Since Legendrian surgery preserves Stein fillability [4, 19], this yields an infinite family of Stein-fillable contact manifolds supported by $(\Sigma_{1,2}, \varphi_n)$ for $n \geqslant 0$.

Remark 3.3. Note that every tight contact structure on a lens space $L(p,1)$ is supported by a positive planar open book [16]; hence, (Y_0, ξ_0)

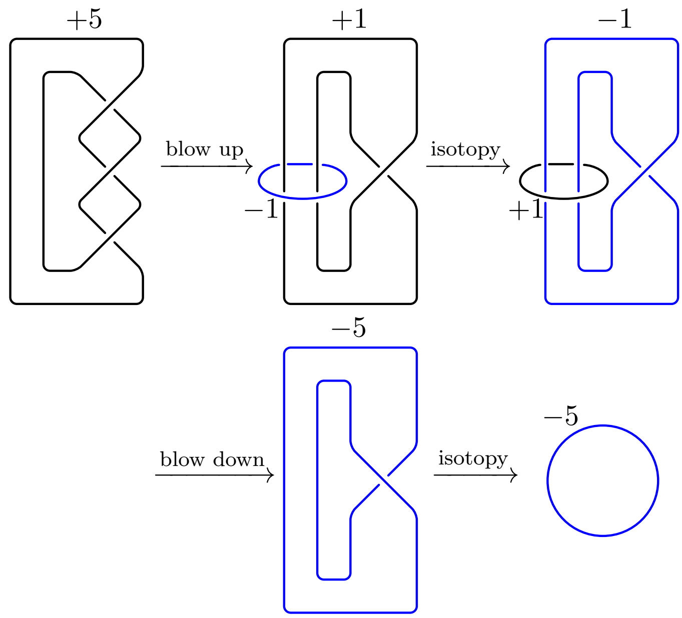

Figure 3. A surgery diagram showing that, topologically, the +5-surgery on a trefoil and the −5-surgery on the unknot give diffeomorphic 3-manifolds.

is also supported by a genus one positive open book obtained by stabilising the corresponding planar open book. It would be of interest to know whether the same is true of (Y_n, ξ_n) for $n > 0$, and to explicitly determine the sequence of stabilisations and destabilisations of $(\Sigma_{1,2}, \varphi_0)$ that yield a positive planar open book for (Y_0, ξ_0).

4. Non-positivity of φ_n

The purpose of this section is to show that the mapping classes $\varphi_n \in \Gamma_{\Sigma_{1,2}}$ do not admit positive factorisations into Dehn twists for all $n \geqslant 0$.

Recall that Luo [12], building on work of Gervais [6], showed that the mapping class group of a compact oriented surface admits a presentation in which generators are Dehn twists, and all relations are supported in subsurfaces homeomorphic to either $\Sigma_{1,1}$ or $\Sigma_{0,4}$. The latter case corresponds to the well-known *lantern relation*, which equates the composition of Dehn twists along curves isotopic to the four boundary components of the subsurface with a composition of three other twists, illustrated in Figure 4. Note that if one or more of the boundary curves are homotopically trivial, the relation reduces to the identity. In what follows, given a surface Σ, we will accordingly refer to any sub-surface homeomorphic to $\Sigma_{0,4}$, none of whose boundary components bound discs in Σ, as a *lantern*.

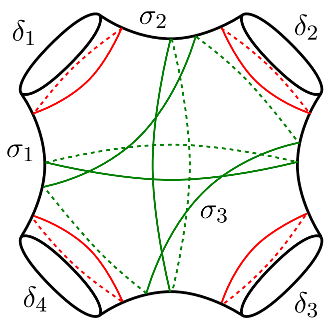

Figure 4. The lantern relation on $\Sigma_{0,4}$ is $\tau_{\delta_1}\tau_{\delta_2}\tau_{\delta_3}\tau_{\delta_4} = \tau_{\sigma_1}\tau_{\sigma_2}\tau_{\sigma_3}$, up to cyclic permutation of τ_{σ_i} and reordering of τ_{δ_i}.

We begin with a simple observation. Letting $|\epsilon|_w$ denote the total exponent of τ_ϵ in a word w of Dehn twists, we have:

Lemma 4.1. Let δ_1 and δ_2 denote curves isotopic to the boundary components of $\Sigma_{1,2}$ and let w be a word in Dehn twists about curves on $\Sigma_{1,2}$. Then the number $|\delta_2|_w - |\delta_1|_w$ depends only on the mapping class of w.

Proof. Using the presentation of Luo in [12], we see that any non-trivial relation which contains either τ_{δ_i} must be a lantern relation; the claim follows immediately by showing that every lantern in $\Sigma_{1,2}$ has boundary components isotopic to each δ_i. To see this, let $\Lambda \subset \Sigma_{1,2}$ be any lantern, and ϵ a curve isotopic to a boundary component of Λ but not isotopic to either δ_i. Now, if ϵ is non-separating in $\Sigma_{1,2}$, then $\overline{\Sigma_{1,2} \setminus \epsilon}$ is a lantern, so Λ is as claimed. On the other hand, if ϵ is separating, then as it is not boundary-parallel in $\Sigma_{1,2}$ it must cut the surface into $\Sigma_{0,3} \sqcup \Sigma_{1,1}$, neither of which contains a lantern, giving a contradiction. □

We are now ready to prove that φ_n cannot be written as a product of positive twists for any $n \geqslant 0$.

Theorem 4.2. The monodromy $\varphi_n \in \Gamma_{\Sigma_{1,2}}$ represented by the word $\tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2}^{4+n}$ does not admit a positive factorisation for any $n \geqslant 0$.

Proof. Suppose otherwise, and let w be be a positive factorisation of φ_n. Then $|\delta_1|_w \geqslant 0$ and, by Lemma 4.1, we have $|\delta_2|_w \geqslant 3+n$. Since boundary-parallel Dehn twists commute with all other twists, we can write $w = w'\tau_{\delta_2}^{3+n}$ for w' a positive factorisation of $\varphi' = \tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2}$.

Now, following the procedure used in Section 3, we recover that $(\Sigma_{1,2}, \varphi')$, shown in Figure 5, supports (Y', ξ'), the result of inadmissible transverse $(+2)$-surgery on a right-handed trefoil.

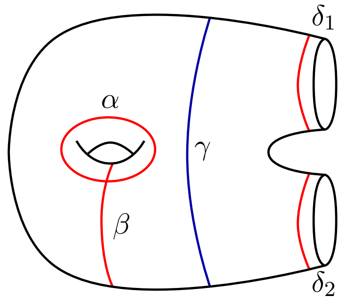

Figure 5. An open book decomposition $(\Sigma_{1,2}, \tau_\alpha\tau_\beta\tau_\gamma^{-1}\tau_{\delta_1}\tau_{\delta_2})$ supporting the result of inadmissible transverse $(+2)$-surgery on a right-handed trefoil in $(S^3, \xi_{\mathrm{std}})$.

Denote by $M(e_0; r_1, r_2, r_3)$ the Seifert fibred space given by the surgery description in Figure 6. Consider $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{4})$. One can verify by sequentially blowing down -1-framed components that it is orientation-preserving diffeomorphic to Y', the $(+2)$-surgery on a right-handed trefoil in S^3. However, $(M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{4}), \xi)$ is not Stein-fillable for any contact structure ξ by [9, Theorem 1.4]. By Giroux [7], it follows that no monodromy of an open book decomposition supporting (Y', ξ') admits a positive factorisation. Hence no positive factorisation of φ' exists, supplying a contradiction.

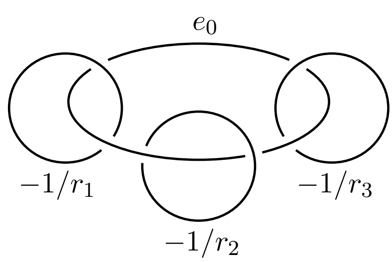

Figure 6. The Seifert fibred 3-manifold $M(e_0; r_1, r_2, r_3)$.

□

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