

Annihilators of \mathcal{D} -modules in mixed characteristic

RANKEYA DATTA, NICHOLAS SWITALA, AND WENLIANG ZHANG

Let R be a polynomial or formal power series ring with coefficients in a DVR V of mixed characteristic with a uniformizer π . We prove that the R -module annihilator of any nonzero $\mathcal{D}(R, V)$ -module is either zero or is generated by a power of π . In contrast to the equicharacteristic case, nonzero annihilators can occur; we give an example of a top local cohomology module of the ring $\mathbb{Z}_2[[x_0, \dots, x_5]]$ that is annihilated by 2, thereby answering a question of Hochster in the negative. The same example also provides a counterexample to a conjecture of Lyubeznik and Yildirim.

1. Introduction

In [8], Huneke discussed 4 basic problems concerning local cohomology; these problems have guided the developments in the study of local cohomology modules for over two decades. As mentioned in the introduction of [8], “We will find all of these problems are connected with another question: what annihilates the local cohomology?” More concretely, Hochster’s [7, Question 6] asks the following:

Question 1.1. Is the top local cohomology module of a local Noetherian domain with support in a given ideal faithful? That is, if R is a local Noetherian domain and $I \subseteq R$ is an ideal of cohomological dimension δ , is $H_I^\delta(R)$ a faithful R -module?

Here $H_I^\delta(R)$ being faithful amounts to $\text{Ann}_R(H_I^\delta(R)) = (0)$. When R is a regular local ring containing a field, Question 1.1 has a positive answer; this was stated in [10, Theorem 1.1] and was attributed to Hochster and Huneke [9, Lemma 2.2] in characteristic $p > 0$ and to Lyubeznik in characteristic

The second author gratefully acknowledges NSF support through grant DMS-1604503. The third author is partially supported by the NSF through grant DMS-1606414 and CAREER grant DMS-1752081.

zero. As stated in [10, page 543], when R is a regular local ring of mixed characteristic, Question 1.1 remained open.

Question 1.1 also stems from a conjecture made by Lynch in [10]; Lynch conjectured that Question 1.1 has a positive answer for all Noetherian local rings even without assuming the ring is a domain. In [1, Example 3.2], a counterexample to Lynch's conjecture was found; note that the local ring in [1, Example 3.2] is not equidimensional.

The main purpose of this paper is to investigate Question 1.1 for regular local rings of mixed characteristic. One of our main results classifies annihilators of \mathcal{D} -modules as follows.

Theorem A (Theorem 3.1). *Let $R = V[[x_1, \dots, x_n]]$ or $R = V[x_1, \dots, x_n]$ where $(V, \pi V)$ is a DVR of mixed characteristic $(0, p)$. Let M be a nonzero $\mathcal{D}(R, V)$ -module. Then, either $\text{Ann}_R(M) = (0)$ or $\text{Ann}_R(M) = (\pi^\ell)$ for some $\ell \geq 1$.*

Note that local cohomology modules of R are primary examples of $\mathcal{D}(R, V)$ -modules (cf. §2), and that investigating local cohomology modules from the \mathcal{D} -module viewpoint has proven fruitful over the years (cf. [12]).

We also answer Question 1.1 in the negative in the case of regular local rings of mixed characteristic by considering a slight modification of Reisner's example.

Theorem B (Example 4.2 and Remark 4.3). *Let $A = \mathbb{Z}[x_0, \dots, x_5]$ and $\mathfrak{m} = (2, x_0, \dots, x_5)$. Let R be the \mathfrak{m} -adic completion of A , and let I be the ideal of R generated by the 10 monomials*

$$\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_4, x_0x_3x_5, x_0x_4x_5, \\ x_1x_2x_5, x_1x_3x_4, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5\}.$$

Then $\text{cd}(R, I) = \text{ara}(I) = 4$ and $\text{Ann}_R(H_I^4(R)) = (2) \neq (0)$, where $\text{ara}(I)$ denotes the arithmetic rank of I .

This turns out to be a counterexample to a conjecture of Lyubeznik and Yildirim as well (cf. Remark 4.7).

The paper is organized as follows. In §2, we review some basics of the theory of \mathcal{D} -modules; in §3, we classify the annihilators of $\mathcal{D}(R, V)$ -modules when R is a ring of polynomials or formal power series over a DVR $(V, \pi V)$ of mixed characteristic $(0, p)$; in §4, we answer Question 1.1 in the negative.

Acknowledgments

The authors thank Alberto Boix for comments on the paper. The second author thanks Tom Marley, who first interested him in these questions. The third author thanks Uli Walther for discussions. The authors also thank Benjamin Antieau and Kevin Tucker for helpful conversations. The authors are grateful to the referee for useful comments and suggestions.

2. Preliminaries on \mathcal{D} -modules

We begin by fixing some conventions. All rings are assumed to have a unit element 1. If R is a commutative ring and $r \in R$ is an element, we denote by (r) the principal ideal $rR \subseteq R$. All *local* commutative rings are assumed to be Noetherian. When we say that $(V, \pi V, k)$ is a *DVR of mixed characteristic* $(0, p)$, we mean that V is a rank-one discrete valuation ring of characteristic zero whose maximal ideal is the principal ideal $(\pi) = \pi V$ generated by π , and whose residue field $k = V/\pi V$ has characteristic $p > 0$. If $\omega \in V$, we denote by $\nu_\pi(\omega)$ the π -adic valuation of ω , that is, $\nu_\pi(\omega)$ is the exponent in the largest power of π dividing ω (so $\nu_\pi(\omega) = 0$ if and only if ω is a unit in V).

We now provide some necessary background material concerning \mathcal{D} -modules. If S is any commutative ring and $A \subseteq S$ is a commutative subring, then the ring $\mathcal{D}(S, A)$ of A -linear differential operators on S , a subring of $\text{End}_A(S)$, is defined recursively as follows [4, §16]. A differential operator $S \rightarrow S$ of order zero is multiplication by an element of S . Supposing that differential operators of order $\leq j - 1$ have been defined, $d \in \text{End}_A(S)$ is said to be a differential operator of order $\leq j$ if, for all $s \in S$, the commutator $[d, s] \in \text{End}_A(S)$ is a differential operator of order $\leq j - 1$. We write $\mathcal{D}^j(S)$ for the set of differential operators on S of order $\leq j$ and set $\mathcal{D}(S, A) = \cup_j \mathcal{D}^j(S)$. Every $\mathcal{D}^j(S)$ is naturally a left S -module. If $d \in \mathcal{D}^j(S)$ and $d' \in \mathcal{D}^l(S)$, it is easy to prove by induction on $j + l$ that $d' \circ d \in \mathcal{D}^{j+l}(S)$, so $\mathcal{D}(S, A)$ is a ring.

By a $\mathcal{D}(S, A)$ -module, we mean a *left* module over the ring $\mathcal{D}(S, A)$. We denote by $\text{Mod}_{\mathcal{D}(S, A)}$ the Abelian category of (left) $\mathcal{D}(S, A)$ -modules. The ring S itself has a $\mathcal{D}(S, A)$ -module structure; using the quotient rule, we can give a $\mathcal{D}(S, A)$ -module structure to the localization S_f for every $f \in S$ in such a way that the natural localization map $S \rightarrow S_f$ is a map of $\mathcal{D}(S, A)$ -modules. Using the Čech complex interpretation of local cohomology, it follows [12, Example 2.1] that the local cohomology modules $H_I^i(S)$

have $\mathcal{D}(S, A)$ -module structures for all finitely generated ideals $I \subseteq S$ and all $i \geq 0$.

We will be concerned with the special case in which $S = A[[x_1, \dots, x_n]]$ (resp. $S = A[x_1, \dots, x_n]$) is a formal power series (resp. polynomial) ring with coefficients in A . In this case, we can describe explicitly the structure of the ring $\mathcal{D}(S, A)$: we have

$$\mathcal{D}(S, A) = S\langle \partial_i^{[t]} \mid 1 \leq i \leq t, t \geq 1 \rangle$$

by [4, Thm. 16.11.2], where $\partial_i^{[t]}$ denotes the differential operator $\frac{1}{t!} \frac{\partial^t}{\partial x_i^t}$ (which is well-defined even if the natural number t is not invertible in S). If $a \in A$ is an element, then $a\mathcal{D}(S, A)$ is a two-sided ideal of $\mathcal{D}(S, A)$, and it follows from the displayed equality (with A replaced by the quotient $A/(a)$) that

$$\mathcal{D}(S, A)/a\mathcal{D}(S, A) \cong \mathcal{D}(S/(a), A/(a))$$

as rings. In particular, a $\mathcal{D}(S/(a), A/(a))$ -module is precisely a $\mathcal{D}(S, A)$ -module annihilated by a .

We will need the following proposition in the sequel.

Proposition 2.1. *Let $S = k[[x_1, \dots, x_n]]$ or $k[x_1, \dots, x_n]$ for some $n \geq 0$, where k is a field. If M is a nonzero $\mathcal{D}(S, k)$ -module, then $\text{Ann}_S(M) = (0)$.*

Proposition 2.1 is essentially contained in the proof of [2, Theorem 3.6] which only treats local cohomology modules; it follows from [3, Proposition 3.3]¹ and [2, Theorem 2.4]. Also note that Proposition 2.1 makes no assumption about the characteristic of k .

3. The annihilator of a $\mathcal{D}(R, V)$ -module

Throughout this section, $(V, \pi V, k)$ denotes a fixed DVR of mixed characteristic $(0, p)$, and R denotes either the ring $V[[x_1, \dots, x_n]]$ or $V[x_1, \dots, x_n]$ for some $n \geq 0$. In either case, we denote by \overline{R} the ring $R/(\pi)$, which is either a formal power series or polynomial ring over k . Our goal in this section is to classify the possible R -module annihilators of $\mathcal{D}(R, V)$ -modules, and our main result is the following:

Theorem 3.1. *Let M be a nonzero $\mathcal{D}(R, V)$ -module. Then, either $\text{Ann}_R(M) = (0)$ or $\text{Ann}_R(M) = \pi^\ell R$ for some $\ell \geq 1$.*

¹The proof of Proposition 3.3 in [2] is incomplete, and was corrected in [3].

In order to prove Theorem 3.1, we begin with a classification of $\mathcal{D}(R, V)$ -submodules of R .

Theorem 3.2. *Let $I \subseteq R$ be a nonzero $\mathcal{D}(R, V)$ -submodule of R . There exists a natural number $\ell \geq 0$ such that $I = \pi^\ell R$.*

Proof. We write elements $f \in R$ in multi-index notation as follows:

$$f = \sum_{\beta \in (\mathbb{Z}_{\geq 0})^n} \omega_\beta \mathbf{x}^\beta,$$

where all $\omega_\beta \in V$ and if $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{Z}_{\geq 0})^n$, then \mathbf{x}^β denotes the monomial $x_1^{\beta_1} \cdots x_n^{\beta_n}$. When R is a polynomial ring over V , we of course have $\omega_\beta = 0$ for almost all $\beta \in (\mathbb{Z}_{\geq 0})^n$.

We prove the formal power series case first. Let $R = V[[x_1, \dots, x_n]]$, let $I \subseteq R$ be a nonzero $\mathcal{D}(R, V)$ -submodule, and let $f \in I$ be given. If there exists $\beta \in (\mathbb{Z}_{\geq 0})^n$ such that ω_β is a unit in V (that is, $\nu_\pi(\omega_\beta) = 0$), then $I = R = (\pi^0)$, since

$$\partial_1^{[\beta_1]} \cdots \partial_n^{[\beta_n]}(\omega_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}) = \omega_\beta,$$

and so, since $\partial_1^{[\beta_1]} \cdots \partial_n^{[\beta_n]}(f)$ (which belongs to I by hypothesis) has a unit constant term, it is itself a unit in R . On the other hand, assume that for every $f = \sum_{\beta \in (\mathbb{Z}_{\geq 0})^n} \omega_\beta \mathbf{x}^\beta \in I$, we have $\nu_\pi(\omega_\beta) > 0$ for all $\beta \in (\mathbb{Z}_{\geq 0})^n$. Under this assumption, let ℓ be the minimal value of $\nu_\pi(\omega_\beta)$ among all ω_β occurring as coefficients in any $f \in I$. This ℓ is a well-defined, nonzero natural number; we claim that $I = (\pi^\ell)$. It is clear that $I \subseteq (\pi^\ell)$. For the converse inclusion, choose $f = \sum_{\beta \in (\mathbb{Z}_{\geq 0})^n} \omega_\beta \mathbf{x}^\beta \in I$ such that for some $\beta \in (\mathbb{Z}_{\geq 0})^n$ we have $\nu_\pi(\omega_\beta) = \ell$. Applying the differential operator $\partial_1^{[\beta_1]} \cdots \partial_n^{[\beta_n]}$, we obtain an element $g \in I$ whose *constant* term is of the form π^ℓ times a unit in V . But by the minimality ℓ , every other coefficient in g is divisible by π^ℓ ; factoring out π^ℓ , we can write g as π^ℓ times a unit h in R , from which it follows that $(\pi^\ell h)h^{-1} = \pi^\ell \in I$. Thus, $I = (\pi^\ell)$ as claimed.

On the other hand, suppose that $R = V[x_1, \dots, x_n]$. Again let $I \subseteq R$ be a nonzero $\mathcal{D}(R, V)$ -submodule, and let $f \in I$ be given. Let $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}_{\geq 0})^n$ be such that $\omega_\gamma \mathbf{x}^\gamma$ is the *leading term* of f with respect to the grlex term order. Then $\partial_1^{[\gamma_1]} \cdots \partial_n^{[\gamma_n]}(f) = \omega_\gamma \in I$. Scaling by a unit if needed, we conclude that $\pi^{\nu_\pi(\omega_\gamma)} \in I$. Now let ℓ be the minimal π -adic valuation of any of the (grlex) leading coefficients of elements of I . Since $I \neq (0)$, ℓ is a natural number. By the preceding argument, $\pi^\ell \in I$. We claim that, conversely, $I \subseteq$

(π^ℓ) and so the two are equal. Indeed, if $f \in I$ is not divisible by π^ℓ , then some nonzero term $\omega_\beta \mathbf{x}^\beta$ has $\nu_\pi(\omega_\beta) < \ell$. Let $\omega_\beta \mathbf{x}^\beta$ be the greatest term (under the glex order) whose coefficient has π -adic valuation less than ℓ . That is, f can be written as $f = f_1 + f_2$ where $\omega_\beta \mathbf{x}^\beta$ is the leading term of f_1 and π^ℓ divides f_2 . Since $\pi^\ell \in I$, this implies that $f_1 \in I$ and its leading coefficient ω_β is not divisible by π^ℓ , a contradiction to the hypothesis on ℓ . This completes the proof. \square

Before we proceed to a proof of Theorem 3.1, we need the following lemma. In this lemma, and the rest of the results in this section, the proofs for the formal power series and polynomial cases are identical.

Lemma 3.3. *Let M be a $\mathcal{D}(R, V)$ -module. Set $I = \text{Ann}_R(M)$ and*

$$J = (I : \pi^\infty) := \{a \in R \mid a\pi^m \in I \text{ for some integer } m \geq 0\}.$$

Then J is a $\mathcal{D}(R, V)$ -submodule of R .

Proof. Let $p = \pi^\ell u$, for some $u \in V^\times$. Then it is easy to see that

$$J = (I : \pi^\infty) = (I : (\pi^\ell)^\infty) = (I : p^\infty),$$

and so, it suffices for us to show that

$$J = (I : p^\infty) = \{a \in R \mid ap^m \in I \text{ for some integer } m \geq 0\}$$

is a $\mathcal{D}(R, V)$ -submodule of R . It further suffices to show that $\partial_i^{[t]}(r)$ remains in J for every $1 \leq i \leq n$, $t \geq 1$, and $r \in J$.

Since $r \in J$, there is an integer ℓ such that $(p^\ell r)M = 0$. First we consider the case when $t = 1$. For each $m \in M$, we have

$$0 = \partial_i(p^\ell r m) = p^\ell \partial_i(r m) = p^\ell (\partial_i(r)m + r \partial_i(m)) = p^\ell \partial_i(r)m,$$

which shows that $\partial_i(r) \in J$. Now an easy induction on t shows that $\partial_i^t(r) \in J$ for all $t \geq 1$. Since $\partial_i^t = t! \partial_i^{[t]}$, it follows that $t! \partial_i^{[t]}(r) \in J$ for all $t \geq 1$. Since every integer coprime to p is a unit in R , we have $p^{\nu_\pi(t!)} \partial_i^{[t]}(r) \in J$. By definition of $J = (I : p^\infty)$, there exists $m \geq 0$ such that $p^m (p^{\nu_\pi(t!)} \partial_i^{[t]}(r)) = p^{m+\nu_\pi(t!)} \partial_i^{[t]}(r) \in I$; again by definition of J , this means $\partial_i^{[t]}(r) \in J$. \square

We will also need the following consequence of Proposition 2.1.

Lemma 3.4. *Let M be a nonzero $\mathcal{D}(R, V)$ -module such that $\pi M = (0)$. Then $\text{Ann}_R(M) = \pi R$.*

Proof. By hypothesis, $(\pi) \subseteq \text{Ann}_R(M)$. For the converse inclusion, observe that since M is annihilated by π , it has a natural structure of $\mathcal{D}(R, V)/\pi\mathcal{D}(R, V) = \mathcal{D}(R/(\pi), V/(\pi)) = \mathcal{D}(\bar{R}, k)$ -module. Since $M \neq (0)$ and k is a field, we have $\text{Ann}_{\bar{R}}(M) = (0)$ by Proposition 2.1, so that $\text{Ann}_R(M) \subseteq (\pi)$. \square

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $I = \text{Ann}_R(M)$ and assume that $I \neq (0)$. By Lemma 3.3, $J = (I : \pi^\infty)$ is a nonzero $\mathcal{D}(R, V)$ -submodule of R , so by Theorem 3.2, we have $J = (\pi^e)$ for some natural number $e \geq 1$. In particular, $\pi^e \in J$, so for some $\ell \geq e$, $\pi^\ell \in I$ by definition of J . Assume that ℓ is the minimal integer such that $\pi^\ell \in I$. Since $M \neq (0)$, we must have $\ell \geq 1$.

We will use induction on ℓ to show that $I = (\pi^\ell)$. The base case, $\ell = 1$, is precisely Lemma 3.4. If $\ell \geq 2$, we consider πM . The definition of ℓ implies that $\pi M \neq (0)$. Since $\pi \in V$, the module πM is naturally a $\mathcal{D}(R, V)$ -submodule of M . Applying the induction hypotheses to πM (which is annihilated by $\pi^{\ell-1}$), by minimality of ℓ we have $\text{Ann}_R(\pi M) = (\pi^{\ell-1})$. It follows immediately that $\text{Ann}_R(M) = (\pi^\ell)$. \square

Theorem 3.1, in conjunction with Theorem 3.2, has the following interpretation: the R -module annihilators of $\mathcal{D}(R, V)$ -modules are precisely the $\mathcal{D}(R, V)$ -submodules of R .

4. Reisner's example in mixed characteristic

The main purpose of this section is to produce an example of a top local cohomology module of a regular local ring of mixed characteristic that has nonzero annihilator. Such an example provides a negative answer to Question 1.1 even in the special case of regular rings. We will begin with the following observation.

Proposition 4.1. *Let $A = \mathbb{Z}[x_1, \dots, x_n]$ for some $n \geq 0$ and let $I \subseteq A$ be a monomial ideal.*

- (a) *For all $\ell \geq 1$, let I_ℓ be the ideal generated by the ℓ -th powers of a given set of monomial generators for I . Suppose that $0 \neq \alpha \in \mathbb{Z}$ is such that α annihilates $\text{Ext}_A^j(A/I, A)$ for some $j \geq 0$. Then α annihilates $\text{Ext}_A^j(A/I_\ell, A)$ for all $\ell \geq 1$.*

(b) If α is as in part (a), then α annihilates $H_I^j(A)$; therefore, if $H_I^j(A) \neq (0)$ and such a nonzero α exists, then $\text{Ann}_A(H_I^j(A))$ is a nonzero proper ideal of A .

Proof. Part (b) follows immediately from part (a) since $H_I^j(A) \cong \varinjlim_{\ell} \text{Ext}_A^j(A/I_{\ell}, A)$. To prove part (a), let $\ell \geq 1$ be given and consider the ring $A_{\ell} = \mathbb{Z}[x_1^{\ell}, \dots, x_n^{\ell}]$. Since α is an integer, I (and therefore I_{ℓ}) is generated by monomials, and $\alpha \text{Ext}_A^j(A/I, A) = (0)$, it follows that $\alpha \text{Ext}_{A_{\ell}}^j(A_{\ell}/I_{\ell}, A_{\ell}) = (0)$. It is clear that A is a free A_{ℓ} -module, and hence

$$\text{Ext}_A^j(A/I_{\ell}, A) \cong \text{Ext}_{A_{\ell}}^j(A_{\ell}/I_{\ell}, A_{\ell}) \otimes_{A_{\ell}} A$$

as A -modules. It follows immediately that $\alpha \text{Ext}_A^j(A/I_{\ell}, A) = (0)$, completing the proof. \square

For the rest of this section, we consider the following example, considered by Reisner in [14, Remark 3] and associated with a minimal triangulation of the real projective plane. Let $A = \mathbb{Z}[x_0, \dots, x_5]$ and let I be the ideal of A generated by the 10 monomials

$$\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_4, x_0x_3x_5, x_0x_4x_5, \\ x_1x_2x_5, x_1x_3x_4, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5\}.$$

It is well-known that $\text{cd}(I, A) \leq 4$ (cf. [15]), i.e. $H_I^j(A) = (0)$ for $j \geq 5$. As we will see in the next example $H_I^4(A) \neq 4$ and hence $H_I^4(A)$ is a top local cohomology module. This allows us to give a negative answer to Question 1.1 as follows.

Example 4.2. Let A and I be as in the previous paragraph, and let $\mathfrak{m} = (2, x_0, \dots, x_5)$. It is straightforward to check² that $\text{Ext}_A^4(A/I, A) \cong A/\mathfrak{m}$, which is annihilated by 2. As in Proposition 4.1, for all $\ell \geq 1$, let I_{ℓ} be the ideal generated by the ℓ -th powers of the displayed monomial generators of I . Since $\text{Ext}_A^4(A/I, A) \cong A/\mathfrak{m}$, it follows from the proof of Proposition 4.1

²This can be seen directly from Hochster's formula; alternatively, as indicated by Lyubeznik in [11], it can be computed using the Taylor resolution of A/I .

that

$$\mathrm{Ext}_A^4(A/I_\ell, A) \cong A/(2, x_0^\ell, \dots, x_5^\ell)$$

for all $\ell \geq 1$. It follows from [11, Theorem 1]³ that the transition map $\mathrm{Ext}_A^4(A/I_\ell, A) \rightarrow \mathrm{Ext}_A^4(A/I_{\ell+1}, A)$ is injective for each $\ell \geq 1$. Therefore $H_I^4(A) \neq 0$ and is supported only in the maximal ideal \mathfrak{m} . It follows from Proposition 4.1 that $2 \cdot \mathrm{Ext}_A^4(A/I_\ell, A) = (0)$ for all $\ell \geq 1$; and hence $2 \cdot H_I^4(A) = (0)$. Now consider the \mathfrak{m} -adic completion $\widehat{A}^{\mathfrak{m}}$ of A , a complete unramified regular local ring. We again write I for the ideal $I\widehat{A}^{\mathfrak{m}} \subseteq \widehat{A}^{\mathfrak{m}}$. (Concretely, the ring $\widehat{A}^{\mathfrak{m}}$ is isomorphic to $\mathbb{Z}_2[[x_0, \dots, x_5]]$, where \mathbb{Z}_2 denotes the ring of 2-adic integers, and $I\widehat{A}^{\mathfrak{m}}$ is generated by the same ten monomials.) Local cohomology commutes with the flat base change $A \rightarrow \widehat{A}^{\mathfrak{m}}$, so $H_I^4(\widehat{A}^{\mathfrak{m}}) \neq (0)$ and $H_I^j(\widehat{A}^{\mathfrak{m}}) = (0)$ for all $j > 4$. Consequently $H_I^4(\widehat{A}^{\mathfrak{m}})$ is a top local cohomology module that is annihilated by 2. In particular, $\mathrm{Ann}_{\widehat{A}^{\mathfrak{m}}}(H_I^4(\widehat{A}^{\mathfrak{m}})) = (2)$ by Theorem 3.1.

Remark 4.3 (Arithmetic rank of the ideal I). It follows from [15, Example 5)] that the ideal I can be defined up to radical by 4 elements

$$\{x_0x_3x_5, x_0x_1x_3 + x_0x_4x_5 + x_2x_3x_5, \\ x_0x_2x_4 + x_1x_2x_5 + x_1x_3x_4, x_0x_1x_2 + x_1x_4x_5 + x_2x_3x_4\}$$

in both A and $\widehat{A}^{\mathfrak{m}}$. Since it is shown in Example 4.2 that $H_I^4(A) \neq 0$ and $H_I^4(\widehat{A}^{\mathfrak{m}}) \neq 0$, the arithmetic rank of I must be 4 in both A and $\widehat{A}^{\mathfrak{m}}$ and $\mathrm{cd}(A, I) = \mathrm{cd}(\widehat{A}^{\mathfrak{m}}, I) = 4$.

Remark 4.4. During the preparation of this paper, we learned that Hochster and Jeffries obtained the following result: *Let (R, \mathfrak{m}) be a Noetherian local domain of characteristic p . Assume that the arithmetic rank of an ideal I is the same as its cohomological dimension, which is denoted by δ . Then $H_I^\delta(R)$ is faithful.*

The combination of Example 4.2 and Remark 4.3 shows that the mixed-characteristic analogue of the aforementioned Hochster-Jeffries result does not hold.

Hernández, Núñez-Betancourt, Pérez, and Witt also studied the module $H_I^4(\widehat{A}^{\mathfrak{m}})$, concluding [6, Theorem 6.3] that this module has zero-dimensional

³In Lyubeznik's theorem, the ring was assumed to be a local ring containing a field. However, since each $\mathrm{Ext}_A^4(A/I_\ell, A)$ is annihilated by 2 and supported only in the maximal ideal \mathfrak{m} , it is naturally a module over the local ring \widehat{R} ; hence Lyubeznik's theorem is applicable.

support while its injective dimension as an $\widehat{A}^{\mathfrak{m}}$ -module is equal to 1. We finish this section with a finer analysis of the structure of $H_I^4(\widehat{A}^{\mathfrak{m}})$, that in particular recovers this result of [6].

Proposition 4.5. *Let $(R, \mathfrak{m}) = \widehat{A}^{\mathfrak{m}}$ denote the completion of A at the maximal ideal $\mathfrak{m} = (2, x_0, \dots, x_5)$ and let \bar{R} denote $R/(2)$. Then $H_I^4(R) \cong E_{\bar{R}}(\bar{R}/\mathfrak{m})$ as R -modules.*

Proof. As we have seen in Example 4.2 that the transition map

$$\text{Ext}_A^4(A/I_\ell, A) \rightarrow \text{Ext}_A^4(A/I_{\ell+1}, A)$$

is injective for each $\ell \geq 1$. Since

$$(x_0^{\ell+1}, \dots, x_5^{\ell+1}) : (x_0^\ell, \dots, x_5^\ell) = (x_0^{\ell+1}, \dots, x_5^{\ell+1}, x_0 \cdots x_5)$$

(which holds in both A and $A/(2)$), it is straightforward to check that this transition map is given by

$$\frac{A}{(2, x_0^\ell, \dots, x_5^\ell)} \xrightarrow{\cdot x_0 \cdots x_5} \frac{A}{(2, x_0^{\ell+1}, \dots, x_5^{\ell+1})}.$$

Therefore,

$$\begin{aligned} H_I^4(A) &\cong \varinjlim_{\ell} \text{Ext}_A^4(A/I_\ell, A) \\ &\cong \varinjlim_{\ell} (\cdots \rightarrow \frac{A}{(2, x_0^\ell, \dots, x_5^\ell)} \xrightarrow{\cdot x_0 \cdots x_5} \frac{A}{(2, x_0^{\ell+1}, \dots, x_5^{\ell+1})} \rightarrow \cdots) \\ &\cong H_{\mathfrak{m}}^6(A/(2)) \\ &\cong H_{\mathfrak{m}}^6(\bar{R}) \end{aligned}$$

Therefore $H_I^4(R) \cong H_I^4(A) \otimes_A R \cong H_{\mathfrak{m}}^6(\bar{R}) \cong E_{\bar{R}}(\bar{R}/\mathfrak{m})$ as R -modules. This completes the proof. □

Inspired by a question of Hellus ([5]), Lyubeznik and Yildirim conjectured in [13, Conjecture 1] that

Conjecture 4.6. Let (R, \mathfrak{m}) be a regular local ring and I be a non-zero ideal of R . If $H_I^j(R) \neq 0$, then $0 \in \text{Ass}_R(D(H_I^j(R)))$.

Remark 4.7. It is observed in [16] that our Proposition 4.5 provides a counterexample to Conjecture 4.6: let R, I be the same as in Proposition 4.5, then $D(H_I^4(R)) \cong R/(2)$ and hence $0 \notin \text{Ass}_R(D(H_I^4(R)))$.

References

- [1] K. Bahmanpour, *A note on Lynch's conjecture*, *Comm. Algebra* **45** (2017), no. 6, 2738–2745.
- [2] A. Boix and M. Eghbali, *Annihilators of local cohomology modules and simplicity of rings of differential operators*, *Beitr. Algebra Geom.* **59** (2018), no. 4, 665–684.
- [3] ———, *Correction to: Annihilators of local cohomology modules and simplicity of rings of differential operators*, *Beitr. Alg. Geom.* **59** (2018) 685–688.
- [4] A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas, quatrième partie*, *Publ. Math. IHÉS* **32** (1967) 5–361.
- [5] M. Hellus, *Local cohomology and Matlis duality*, habilitation thesis, University of Leipzig (2007).
- [6] D. Hernández, L. Núñez-Betancourt, F. Pérez, and E. Witt, *Lyubeznik numbers and injective dimension in mixed characteristic*, *Trans. Amer. Math. Soc.* **371** (2019), no. 11, 7533–7557.
- [7] M. Hochster, *Finiteness properties and numerical behavior of local cohomology*, *Comm. Algebra* **47** (2019), no. 6, 1–11.
- [8] C. Huneke, *Problems on local cohomology*, in *Free resolutions in commutative algebra and algebraic geometry* (Sundance, UT, 1990), Vol. 2 of *Res. Notes Math.*, 93–108, Jones and Bartlett, Boston, MA (1992).
- [9] C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), no. 3, 421–429.
- [10] L. R. Lynch, *Annihilators of top local cohomology*, *Comm. Algebra* **40** (2012), no. 2, 542–551.
- [11] G. Lyubeznik, *On the local cohomology modules $H_{\mathfrak{a}}^i(R)$ for ideals \mathfrak{a} generated by monomials in an R -sequence*, in *Complete intersections* (Acireale, 1983), Vol. 1092 of *Lecture Notes in Math.*, 214–220, Springer, Berlin (1984).
- [12] ———, *Finiteness properties of local cohomology modules (an application of D -modules to commutative algebra)*, *Invent. Math.* **113** (1993) 41–55.

- [13] G. Lyubeznik and T. Yildirim, *On the Matlis duals of local cohomology modules*, Proc. Amer. Math. Soc. **146** (2018), no. 9, 3715–3720.
- [14] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), no. 1, 30–49.
- [15] T. Schmitt and W. Vogel, *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. **245** (1979), no. 3, 247–253.
- [16] U. Walther and W. Zhang, *Local cohomology — an invitation*, in: Commutative Algebra, pp. 773–858, Springer, Cham, 2021.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI
COLUMBIA, MO 65201, USA
E-mail address: `rankeya.datta@missouri.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, IL 60607, USA
E-mail address: `nswitala@uic.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, IL 60607, USA
E-mail address: `wlzhang@uic.edu`

RECEIVED MAY 3, 2021

ACCEPTED OCTOBER 19, 2021