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A Riemannian metric on hyperbolic components

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We introduce a Riemannian metric on certain hyperbolic components in the moduli space of degree at least 2 rational maps in one complex variable. Our metric is constructed by considering the measure-theoretic entropy of a rational map with respect to some equilibrium state.

1. Introduction

For each $d \geq 2$, let Rat_d (resp. Poly_d) be the space of degree d rational maps (resp. polynomials) in one complex variable. Denote by $\mathrm{rat}_d := \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ (resp. $\mathrm{poly}_d := \mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})$) the moduli space of degree d rational maps (resp. polynomials), modulo the action by conjugation of the group of Möbius transformations (resp. affine maps). Then rat_d is a complex $(2d-2)$-dimensional orbifold and poly_d is a complex $(d-1)$-dimensional subspace of rat_d. The Hausdorff dimension function $\delta : \mathrm{Rat}_d \to [0, 2]$, sending $f \in \mathrm{Rat}_d$ to the Hausdorff dimension of the Julia set $J(f)$ of f is invariant under conjugacy. Hence it descends to a well-defined function on rat_d, which we also call the Hausdorff dimension function δ:

$$\delta : \mathrm{rat}_d \to [0, 2],$$

sending the conjugacy class $[f] \in \mathrm{rat}_d$ to the Hausdorff dimension of $J(f)$.

In [7], McMullen proposed an analogue of the Weil-Petersson metric on the moduli space of degree d Blaschke products. Via Bers embedding, this metric induces a metric on the central hyperbolic component $\mathcal{H}_0$ in poly_d, that is $\mathcal{H}_0 \subset \mathrm{poly}_d$ is the hyperbolic component containing $[z^d]$. The goal of this paper is to introduce a natural metric on other hyperbolic components in poly_d (or in rat_d).

For a hyperbolic rational map $f \in \mathrm{Rat}_d$, there exists a unique invariant probability measure ν on $J(f)$ such that the pressure of the potential $-\delta(f) \log |f'| : J(f) \to \mathbb{R}$ is zero. Then the measure-theoretic entropy $h_\nu(f) = \delta(f)\mathrm{Ly}_\nu(f)$, where $\mathrm{Ly}_\nu(f)$ is the Lyapunov exponent of f with respect to ν. Note that there is a small neighborhood $U(f) \subset \mathrm{Rat}_d$ of f such that the natural holomorphic motion on $U(f) \times J(f)$ with identity on $\{f\} \times J(f)$ induces a homeomorphism $\phi_g : J(f) \to J(g)$, for any $g \in U(f)$. Then the pushforward measure $(\phi_g)_*\nu$ is a g-invariant probability measure on $J(g)$. Consider the map

$$G_f : U(f) \to \mathbb{R}$$

sending $g \in U(f)$ to $\delta(g)\mathrm{Ly}_{(\phi_g)_*\nu}(g)$. It is well-known that the Hausdorff dimension function δ is real-analytic on $U(f)$, see [13]. Moreover, it turns out that the map $g \mapsto \mathrm{Ly}_{(\phi_g)_*\nu}(g)$ is also real-analytic on $U(f)$ (Proposition 2.10), and that G_f has a local minimum at f (Proposition 4.1). Hence the Hessian of G_f is well-defined at f. It then induces a symmetric bilinear form $||\cdot||_G$ on the tangent space T_f of Rat_d at f as follows: given $\vec{w} \in T_f \cong \mathbb{R}^{4d+2}$, for a smooth real 1-dimensional path $\gamma : (-1, 1) \to U(f)$ with $\gamma(0) = f$ and $\gamma'(0) = \vec{w} \in T_f$, define

$$||\vec{w}||_G^2 := \frac{\partial^2 G_f}{\partial \vec{w} \partial \vec{w}} = \frac{d^2}{dt^2}\bigg|_{t=0} G_f(\gamma(t)),$$

which in fact only depends on the initial conditions of γ. The form $||\cdot||_G$ descends to a 2-form on the corresponding hyperbolic component in rat_d (see Section 4.2) and we again denote this form by $||\cdot||_G$.

Our main result asserts that $||\cdot||_G$ is nondegenerate on certain hyperbolic components.

Theorem 1.1. *Let $\mathcal{H}$ be a hyperbolic component in* rat_d*. Suppose that every $[f] \in \mathcal{H}$ has a repelling multiplier which is not a real number. Then on $\mathcal{H}$, the form $||\cdot||_G$ is a Riemannian metric and is conformal equivalent to the standard pressure metric.*

A key ingredient to prove Theorem 1.1 is a distribution result on repelling multipliers for elements in $\mathcal{H}$ (see Section 2.2). With this distribution result, for a point $[f] \in \mathcal{H}$, we consider an analytic path $[f_t], t \in \mathbb{D}$, in $\mathcal{H}$ with $[f_0] = [f]$ and show that under the assumption in Theorem 1.1, the quantities $\frac{d}{dt}|_{t=0} \log|\lambda_t|/\log|\lambda_0|$ cannot be constant for all repelling multipliers λ_t of $[f_t]$ (Proposition 5.1). However, from the thermodynamic formalism, if $||v||_G$ is degenerate for some tangent vector $v \in T_{[f]}\mathcal{H}$, there exists a path $[f_t]$ with a constant $\frac{d}{dt}|_{t=0} \log|\lambda_t|/\log|\lambda_0|$ for all repelling multipliers λ_t. This gives a contradiction and proves the positive-definiteness of $||\cdot||_G$.

By Sullivan's dictionary between rational maps and Kleinian groups, our work is inspired by the work of Bridgeman [2] and Bridgeman-Taylor [3]. They established an extension of the Weil-Peterson metric to the quasi-Fuchsian space $QF(S)$ of a closed surface S of genus at least 2. In particular, Bridgeman [2, Main Theorem] proved that the extension is degenerate on the so-called pure bending vectors. Moreover, using our metric and applying an analog of the proof of [2, Theorem 1.2], we can obtain that the function δ has no local maximum on the hyperbolic components mentioned in Theorem 1.1, although it has already proven by Ransford [12] with a distinct method.

A result [4, Theorem 1] of Eremenko and van Strien states that if all the repelling multipliers of a rational map are real, then its Julia set is contained in a circle. Thus Theorem 1 implies immediately the following.

Corollary 1.2. *Let $\mathcal{H}$ be a hyperbolic component in* rat_d *such that $\delta(\mathcal{H}) \subset (1, 2)$. Then on $\mathcal{H}$, the form $||\cdot||_G$ is a Riemannian metric and is conformal equivalent to the standard pressure metric.*

Now we restrict our attention to hyperbolic components in poly_d. Recall that the shift locus is the hyperbolic component in poly_d such that for each element, all its critical points are in the basin of ∞. By a characterization of the hyperbolic polynomials $f \in \mathrm{Poly}_d$ with $\delta(f) > 1$ (Proposition 2.2), from Corollary 1.2, we have the following.

Corollary 1.3. *Let $\mathcal{H}$ be a hyperbolic component in* poly_d *that is neither the central hyperbolic component nor the shift locus. Then on $\mathcal{H}$, the form $||\cdot||_G$ is a Riemannian metric and is conformal equivalent to the standard pressure metric.*

The central hyperbolic component and the shift locus are the only hyperbolic components in poly_d containing elements with all real multipliers. In the central hyperbolic component, the point $[z^d]$ is the only element whose Hausdorff dimension of Julia set is not in $(1, 2)$. Computer experiments show that it is possible to extend the metric $||\cdot||_G$ to the central component. We give a direct computation to show it is the case for $d = 2$. But it is unclear to us for a rigorous proof in the general case (see Section 5.2).

Organization of the paper

The paper is organized as follows. In Section 2, we prove some preparatory results regarding the Hausdorff dimension of Julia sets (Proposition 2.2) and some properties of multipliers (Proposition 2.6) and Lyapunov exponents (Proposition 2.10). In Section 3, we review thermodynamic formalism and the pressure metric on the moduli space. We introduce the non-negative form $||\cdot||_G$ in Section 4 and prove the main result Theorem 1.1 in Section 5.

2. Complex dynamics background

In this section, we give an expository account for the basics in complex dynamics. It covers Hausdorff dimension of Julia sets, distribution of multipliers and Lyapunov exponents.

2.1. Hausdorff dimension of Julia sets for hyperbolic maps

Let $f \in \mathbb{C}(z)$ be a rational map of degree at least 2. Denote by $F(f)$ and $J(f)$ the Fatou set and Julia set of f, respectively. Recall that f is *hyperbolic* if all the critical points under iterations converge to attracting cycles. In this subsection, we state some results about the Hausdorff dimension of $J(f)$.

The following result, due to Przytycki [10], concerns the dimensions of boundaries of immediate attracting basins. It is a continuation of Zdunik's work [16]. Recall that for an f-invariant set K, its hyperbolic Hausdorff dimension is the supremum of the Hausdorff dimensions of f-invariant subsets X of K such that $f|_X$ is expending.

Proposition 2.1. *[10, Theorem A] Let $f \in \mathbb{C}(z)$ be a rational map of degree at least 2. Suppose f is not a finite Blaschke product in some holomorphic coordinates or a quotient of a Blaschke product by a rational function of degree 2. Assume f has an attracting cycle and denote by B its immediate basin. If each component of B is simply connected, then the hyperbolic Hausdorff dimension of ∂B is larger than 1.*

If $P \in \mathbb{C}[z]$ is a polynomial of degree at least 2, the Julia set $J(P)$ is the boundary of the basin $B_\infty(P)$ of ∞. If $J(P)$ is connected, equivalently all the critical points of P are away from $B_\infty(P)$, Zdunik's result [15] implies that the Hausdorff dimension of $J(P)$ is larger than 1 unless P is conjugate to a monomial or a $\pm$ Chebyshev polynomial. Indeed, in this case the Hausdorff dimension of the measure of maximal entropy for P is 1, see [6].

For a hyperbolic polynomial P, if $J(P)$ is not a Cantor set, equivalently not all the critical points are contained in $B_\infty(P)$, then the Hausdorff dimension of $J(P)$ is larger than 1 unless P is conjugate to a monomial:

Proposition 2.2. *Let P be a hyperbolic polynomial of degree at least 2. Suppose P is not conjugate to a monomial. If $J(P)$ is not a Cantor set, then the Hausdorff dimension of $J(P)$ is larger than 1.*

Proof. Since P is hyperbolic and has a critical point not in the basin of ∞, it follows that P has an attracting cycle in $\mathbb{C}$. Then the immediate basin of the attracting cycle is a union of finitely many simply connected components. Note that the boundary of this immediate basin is contained in $J(P)$. Thus by Proposition 2.1, we only need to deal with the case that P is a quotient of a Blaschke product by a rational function of degree 2. In this case, the map P is a $\pm$ Chebyshev polynomial in some holomorphic coordinates. It is impossible since P is hyperbolic. Then the conclusion follows. □

Proposition 2.2 concerns with hyperbolic polynomials. In a recent paper [11, Theorem 2] Przytycki and Zdunik proved that a more general result asserting that for any polynomial P of degree $d \geq 2$ whose Julia set $J(P)$ is not a Cantor set, the Hausdroff dimension of $J(P)$ is 1 if and only if P is affine conjugate to the monomial z^d or a $\pm$ Chebyshev polynomial.

Recall that poly_d is the moduli space of degree d polynomials. For the descended map $\delta : \text{poly}_d \to [0, 2]$, sending $[P]$ to the Hausdorff dimension of $J(P)$, the above proposition immediately implies the following.

Corollary 2.3. *For $d \geq 2$, let $\mathcal{H} \subset \text{poly}_d$ be a hyperbolic component. Suppose $\mathcal{H}$ is not the shift locus. Then $\delta(\mathcal{H} - \{[z^d]\}) \subset (1, 2)$.*

2.2. Distribution of multipliers

Let $f \in \mathbb{C}(z)$ be a hyperbolic rational map of degree at least 2. Given any primitive periodic orbit $\hat{z} = \{z, f(z), \cdots, f^{k-1}(z)\}$ on the Julia set $J(f)$, we consider its *multiplier* $\lambda(\hat{z})$ given by, in the local coordinates,

$$\lambda(\hat{z}) = (f^k)'(z).$$

Let $\mathcal{O}$ denote the set of all primitive periodic orbits of f in $J(f)$. For $T > 0$, consider the counting function

$$N_T(\mathcal{O}) := \#\{\hat{z} \in \mathcal{O} \mid |\lambda(\hat{z})| < T\}.$$

Since f is hyperbolic, $N_T(\mathcal{O})$ is finite for any $T > 0$. In [8], Oh and Winter proved the following asymptotics for $N_T(\mathcal{O})$.

Theorem 2.4. *[8, Theorem 1.1 (1)] Let $f \in \mathbb{C}(z)$ be a hyperbolic rational map of degree at least 2. Suppose that f is not conjugate to a monomial. Then there exists $\epsilon := \epsilon(f) > 0$ such that*

$$N_T(\mathcal{O}) = Li(T^\beta) + O(T^{\beta-\epsilon})$$

where $Li(t) = \int_2^t \frac{dt}{\log t}$ is the offset logarithmic integral and $\beta := \delta(f)$ is the Hausdorff dimension of $J(f)$.

Remark 2.5. Under the assumptions in Theorem 2.4, we set

$$\epsilon^* = \epsilon^*(f) := \sup_{\{T_i\}_{i\geq 1}} \sup \left\{\epsilon > 0 : N_{T_i}(\mathcal{O}) = Li({T_i}^\beta) + O({T_i}^{\beta-\epsilon})\right\},$$

where the first supremum is taken over all $\epsilon > 0$ satisfying the indicated equality for a given strictly increasing sequence $\{T_i\}_{i\geq 1}$ tending to ∞, and the second supremum is taken over all strictly increasing sequences $\{T_i\}_{i\geq 1}$ tending to ∞. Then $0 < \epsilon^* \leq \beta$ by Theorem 2.4. In our later argument, we can pick $\epsilon = \epsilon(f) = \epsilon^*/2$ in Theorem 2.4. Moreover, if $\epsilon^* < \beta$, for any $0 < \epsilon' < \beta - \epsilon^*$,

$$\lim_{T\to+\infty} \frac{|N_T(\mathcal{O}) - Li(T^\beta)|}{T^{\beta-(\epsilon^*+\epsilon')}} = +\infty,$$

It follows that there exists $C(T) := C_{\epsilon'}(T)$ depending on ϵ' with $C(T) \to \infty$ as $T \to \infty$ such that

$$N_T(\mathcal{O}) - Li(T^\beta) = C(T)T^{\beta-(\epsilon^*+\epsilon')} + o(T^{\beta-(\epsilon^*+\epsilon')}).$$

If $\epsilon^* = \beta$, we set $\epsilon' = 0$ and $C(T) = N_T(\mathcal{O}) - Li(T^\beta)$. We will use the term $C(T)$ for a specific $\epsilon'_0 > 0$ in the proof of Theorem A.1. We mention here that an analog of Riemann hypothesis conjectures that $\epsilon^* < \beta$.

Using the above theorem, we prove the following result regarding the existence of multipliers within an annulus which we will use in Section 5.

Proposition 2.6. *Let $f \in \mathbb{C}(z)$ be a hyperbolic rational map of degree at least 2. Suppose that f is not conjugate to a monomial. Let $\epsilon > 0$ be as in Remark 2.5. Suppose $\{T_n\} \subset \mathbb{R}_{>0}$ is a sequence with $T_n \to +\infty$, as $n \to +\infty$. Then for $S_n \geq T_n^\alpha$ with $\alpha > 1 - \epsilon/2$,*

$$N_{T_n+S_n}(\mathcal{O}) - N_{T_n}(\mathcal{O}) \to \infty \text{ as } n \to \infty.$$

Proof. It suffices to prove the conclusion for $1 - \epsilon/2 < \alpha < 1$. In this case, $T_n^\alpha \leq S_n = o(T_n)$. Note that $Li(x) = x/\ln(x) + o(x/\ln(x))$ for sufficiently large x. Denote $\beta := \delta(f) > \epsilon$. Then we have

$$Li((T_n + S_n)^\beta) - Li(T_n^\beta) = \beta\frac{T_n^{\beta-1}S_n}{\ln T_n} + o\left(\frac{T_n^{\beta-1}S_n}{\ln T_n}\right).$$

On the other hand, there exists $\epsilon' \geq \epsilon$ such that

$$\begin{aligned}(N_{T_n+S_n}(\mathcal{O}) &- Li((T_n + S_n)^\beta)) - (N_{T_n}(\mathcal{O}) - Li(T_n^\beta))\\ &= O((T_n - S_n)^{\beta-\epsilon}) - O(T_n^{\beta-\epsilon})\\ &= O(T_n^{\beta-\epsilon'}).\end{aligned}$$

Then

$$\begin{aligned}(N_{T_n+S_n}(\mathcal{O}) &- Li((T_n + S_n)^\beta)) - (N_{T_n}(\mathcal{O}) - Li(T_n^\beta))\\ &= o\left(\frac{T_n^{\beta-\epsilon/2}}{\ln T_n}\right) = o(Li((T_n + S_n)^\beta) - Li(T_n^\beta)).\end{aligned}$$

The first equality holds since $\beta - \epsilon' < \beta - \epsilon/2$. The second equality holds since $\beta - \epsilon/2 < \beta - 1 + \alpha$.

Therefore, we have

$$N_{T_n+S_n}(\mathcal{O}) - N_{T_n}(\mathcal{O}) = \beta \frac{T_n^{\beta-1} S_n}{\ln T_n} + o\left(\frac{T_n^{\beta-1} S_n}{\ln T_n}\right).$$

Since $\beta > \epsilon/2$, we have, as $n \to \infty$,

$$\frac{T_n^{\beta-1} S_n}{\ln T_n} \to \infty.$$

The conclusion follows. □

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk. We say that a family $\{f_t\}_{t\in\mathbb{D}}$ is a *holomorphic family of degree $d \geq 2$ rational maps* if each f_t is a rational map of degree d and each coefficient of f_t is a holomorphic function in t. In this case, the periodic points and hence the corresponding multipliers of f_t are holomorphic in t, up to a base change. We will use the following notation throughout the paper. For a real number $s \in \mathbb{R}$ and a nonzero complex number $a \in \mathbb{C} \setminus \{0\}$, we write a^s for $\exp(s\mathrm{Log}(a))$, where $\mathrm{Log}(a)$ is the principle value of the logarithm of a. We will mainly consider the modulus of a^s.

Corollary 2.7. *Let $\{f_t\}_{t\in\mathbb{D}}$ be a holomorphic family of hyperbolic rational maps of degree at least 2, and let a_t be the multiplier of a repelling cycle of f_t. Suppose that f_0 is not conjugate to a monomial. Then there exists $0 < \kappa_0 < 1$ such that for any $\kappa \in (\kappa_0, 1)$, the following holds: for any arbitrary multiplier b_t of a repelling cycle of f_t, any sufficiently large positive integer $n \geq 1$ and any t close to 0, there exist $\theta_{t,n} := \theta(t, n, \kappa, b_t) \in [0, 2\pi)$ and multipliers $\lambda_{t,n}$ of f_t of the form*

$$\lambda_{t,n} = e^{i\theta_{t,n}}(a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)),$$

where the little-o term is taken with respect to n.

Proof. For each $t \in \mathbb{D}$, let $\epsilon_t := \epsilon(f_t)$ be as in Remark 2.5. By Theorem A.1 in Appendix A, there exist a small disk $D' \subset \mathbb{D}$ centered at 0 and $\epsilon' > 0$ such that $\epsilon_t > \epsilon'$ for all $t \in D'$. Let $\kappa_0 = 1 - \epsilon'/2$. Then $0 < \kappa_0 < 1$. For $t \in D'$, set

$$\epsilon_t' = \frac{1 - \epsilon_t/2}{1 - \epsilon'/2} \in (0, 1).$$

Then for $\kappa \in (\kappa_0, 1)$, we have $\kappa\epsilon_t' > 1 - \epsilon_t/2$.

For $t \in D'$ and $n \geq 1$, consider the annulus

$$A_{t,n} = \{z \in \mathbb{C} \mid |a_t^n + a_t^{n\kappa} b_t| \leq |z| \leq |a_t^n + a_t^{n\kappa} b_t| + |a_t^{n\kappa} b_t|^{\epsilon'_t}\}.$$

We first claim that for each t, there exists a multiplier of f_t in the annulus $A_{t,n}$ for sufficiently large n. Indeed, apply Proposition 2.6 to each f_t with $T_n = |a_t^n + a_t^{n\kappa} b_t|$, $S_n = |a_t^{n\kappa} b_t|^{\epsilon'_t}$ and $\alpha = \kappa\epsilon'_t$.

Let $\lambda_{t,n}$ be such a multiplier of f_t contained in $A_{t,n}$. Then, we must have $|\lambda_{t,n}| = |a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)|$ with $|o(a_t^{n\kappa} b_t)| \leq |a_t^{n\kappa} b_t|^{\epsilon'_t}$. Therefore $\lambda_{t,n} = e^{i\theta_{t,n}}(a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t))$ for some $\theta_{t,n} \in [0, 2\pi)$. □

Remark 2.8. Let us take the notations in Corollary 2.7 and in its proof. For each $n \gg 1$, since $\lambda_{t,n}$ is holomorphic, up to base change, in t and $|\lambda_{0,n}| > 1$, there exists a small disk $D_n \subset \mathbb{D}$ centered at 0 on which we can choose $\theta_n := \theta_{t,n}$ in Corollary 2.7 independent of $t \in D_n$. Indeed, pick $\theta_n = \mathrm{Arg}(\lambda_{0,n}) - \mathrm{Arg}(a_0^n + a_0^{n\kappa} b_0)$. On D_n, denote by $E_n(t) := \lambda_{t,n}/e^{i\theta_n} - (a_t^n + a_t^{n\kappa} b_t)$. Then up to base change $t \to s^\ell$ for some integer ℓ, the map $E_n(t)$ naturally extends to a holomorphic function in the small disk $D' \subset \mathbb{D}$ centered at 0, that is, for $t \in D'$,

$$E_n(t) = e^{i(\theta_{t,n} - \theta_n)}(a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)) - (a_t^n + a_t^{n\kappa} b_t).$$

To abuse the notation, we denote this extension by $E_n(t)$ also. Now we give a uniform upper bound of $|E_n(t)/a_t^n|$ in any small compact set. Shrinking ϵ' in the proof of Corollary 2.7 if necessary, we can assume that $\chi := \sup\{\epsilon'_t : t \in D'\} < 1$. For a compact subset $X \subset D'$, set $a_{\max} := \max\{|a_t| : t \in X\}$, $a_{\min} := \min\{|a_t| : t \in X\}$ and $b_{\max} := \max\{|b_t| : t \in X\}$. Then $a_{\max} \geq a_{\min} > 1$ and $b_{\max} > 1$. We consider compact subsets $X \subset D'$ small enough such that $a_{\max}^\chi < a_{\min}$. We claim that there exists $N \geq 1$ such that if $n \geq N$, then for any $t \in X$,

$$\begin{aligned} |E_n(t)| &= \left| e^{i(\theta_{t,n} - \theta_n)}(a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)) - (a_t^n + a_t^{n\kappa} b_t) \right| \\ &\leq |a_t^n| + |a_t^{n\kappa} b_t| + |o(a_t^{n\kappa} b_t)| + |a_t^n| + |a_t^{n\kappa} b_t| \\ &< 5\,|a_t^n|. \end{aligned}$$

Let us see the existence of N and the last inequality. Noting that $0 < \kappa < 1$, we conclude that there exists $N_1 \geq 1$ such that $b_{\max} < a_{\min}^{n(1-\kappa)}$ for all $n \geq N_1$, and hence for any $t \in X$,

$$|a_t^{n\kappa} b_t| \leq |a_t^{n\kappa}| \, b_{\max} < |a_t^{n\kappa}| \, a_{\min}^{n(1-\kappa)} \leq |a_t^{n\kappa}| \, |a_t|^{n(1-\kappa)} = |a_t^n| \, .$$

Moreover, since $o(a_t^{n\kappa} b_t) = \lambda_{t,n} - (a_t^n + a_t^{n\kappa} b_t)$ is holomorphic for each n, there exists $t_n \in X$ such that

$$|o(a_t^{n\kappa} b_t)| \leq \max\{|o(a_t^{n\kappa} b_t)| : t \in X\} = |o(a_{t_n}^{n\kappa} b_{t_n})|.$$

Noting that for each $t \in D'$, $|o(a_t^{n\kappa} b_t)| \leq |a_t^{n\kappa} b_t|^{\epsilon'_t}$, as $n \to \infty$, in the proof of Corollary 2.7, we conclude that there exists $N \geq N_1$ such that for any $n \geq N$ and any $t \in X$,

$$\begin{aligned} |o(a_t^{n\kappa} b_t)| &\leq |o(a_{t_n}^{n\kappa} b_{t_n})| \leq (a_{\max}^{n\kappa} b_{\max})^{\chi} \leq a_{\min}^{n\kappa} b_{\max}^{\chi} \leq a_{\min}^{n\kappa} b_{\max} \\ &< a_{\min}^{n\kappa} a_{\min}^{n(1-\kappa)} = a_{\min}^n \leq |a_t^n| \, . \end{aligned}$$

This gives the existence of N and the desired inequality. Furthermore, for sufficiently large n, shrinking D_n if necessary, we have the following better estimate for $E_n(t)$ on D_n. By the choice of D_n and the definition of θ_n, we have that for $t \in D_n$,

$$\begin{aligned} E_n(t) &= e^{i(\theta_{t,n} - \theta_n)} (a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)) - (a_t^n + a_t^{n\kappa} b_t) \\ &= (a_t^n + a_t^{n\kappa} b_t + o(a_t^{n\kappa} b_t)) - (a_t^n + a_t^{n\kappa} b_t) \\ &= o\left(a_t^{n\kappa} b_t\right) . \end{aligned}$$

Remark 2.9. Since f_t has only countably many multipliers, there are uncountably many $\kappa \in (\kappa_0, 1)$ giving rise to the same values of $\lambda_{t,n}$ although the expressions of $\lambda_{t,n}$ are different.

2.3. Lyapunov exponents

For a hyperbolic rational map f of degree at least 2, we denote by $\mathcal{M}_f$ the set of f-invariant probability measures on the Julia set $J(f)$. Recall that $U(f)$ is a small neighborhood of f in Rat_d such that the natural holomorphic motion induces a homeomorphism $\phi_g : J(f) \to J(g)$ for $g \in U(f)$. For $\mu \in \mathcal{M}_f$, the Lyapunov exponent $\mathrm{Ly}_\mu(f) = \int_{J(f)} \log |f'| d\mu$ induces a well-defined function

$\mathrm{Ly} = \mathrm{Ly}_f : \mathcal{M}_f \times U(f) \to \mathbb{R}$ by

$$\mathrm{Ly}(\mu, g) := \mathrm{Ly}_{(\phi_g)_*\mu}(g) = \int_{J(g)} \log |g'| d(\phi_g)_*\mu.$$

In this subsection, we discuss analytic properties of the function $\mathrm{Ly}(\cdot, \cdot)$.

Recall that if X is a smooth manifold and $C^\infty(X, \mathbb{R})$ is the set of smooth real-valued functions on X, the C^∞-topology on $C^\infty(X, \mathbb{R})$ is given by $\psi_n \to \psi$ if the derivatives of ψ_n converge uniformly on compact subsets of X to the derivatives of ψ.

Proposition 2.10. *The map* $\mathrm{Ly} : \mathcal{M}_f \times U(f) \to \mathbb{R}$ *satisfies the following properties:*

1) *for each* $\mu \in \mathcal{M}_f$*, the function* $\mathrm{Ly}(\mu, \cdot) : U(f) \to \mathbb{R}$ *is harmonic and hence is real analytic; and*
2) *the map from* $\mathcal{M}_f$ *to* $C^\infty(U(f), \mathbb{R})$*, sending* μ *to* $\mathrm{Ly}(\mu, \cdot)$*, is continuous.*

To prove the proposition, we first show that if $\mu_n \to \mu$ in $\mathcal{M}_f$, the functions $\mathrm{Ly}(\mu_n, \cdot)$ converge uniformly on compact subsets of $U(f)$.

Lemma 2.11. *If* $\mu_n \to \mu$ *in* $\mathcal{M}_f$ *with respect to the weak-* topology, then* $\mathrm{Ly}(\mu_n, \cdot) \to \mathrm{Ly}(\mu, \cdot)$ *uniformly on compact subsets of* $U(f)$*.*

Proof. Let $g \in U(f)$. For any $\epsilon > 0$, we first claim that there is a neighborhood W of g in $U(f)$ such that for any $\nu \in \mathcal{M}_f$ and any $h \in W$,

$$|\mathrm{Ly}(\nu, g) - \mathrm{Ly}(\nu, h)| < \epsilon.$$

Indeed,

$$\begin{aligned}
|\mathrm{Ly}(\nu, g) - \mathrm{Ly}(\nu, h)| &= \left| \int_{J(f)} \log |g' \circ \phi_g| d\nu - \int_{J(f)} \log |h' \circ \phi_h| d\nu \right| \\
&= \left| \int_{J(f)} (\log |g' \circ \phi_g| - \log |h' \circ \phi_h|) d\nu \right| \\
&= \left| \int_{J(f)} \log \left| \frac{g' \circ \phi_g}{h' \circ \phi_h} \right| d\nu \right| \\
&\le \max \left\{ \left| \log \left| \frac{g' \circ \phi_g(z)}{h' \circ \phi_h(z)} \right| \right| : z \in J(f) \right\}.
\end{aligned}$$

Consider

$$\alpha_g(h,z) := \frac{g' \circ \phi_g(z)}{h' \circ \phi_h(z)}.$$

Then α_g is well-defined on $U(f) \times J(f)$ since the Julia sets do not contain critical points. Moreover, α_g is continuous in both h and z. Note that $\alpha_g(g,z) = 1$ for all $z \in J(f)$. Since $J(f)$ is compact, we can choose a sufficiently small neighborhood W of g such that $|\alpha_g(h,z)| < e^{\epsilon}$ for all $h \in W$ and all $z \in J(f)$. Hence the claim holds.

Now consider the sequence $\{\mu_n\}$. For any $\epsilon > 0$, by the previous claim, we can choose a neighborhood V of g in $U(f)$ such that for all μ_n and all $h \in V$, we have $|\mathrm{Ly}(\mu_n, g) - \mathrm{Ly}(\mu_n, h)| < \epsilon$. It follows that the sequence $\{\mathrm{Ly}(\mu_n, \cdot)\}$ is equicontinuous on any compact subset of $U(f)$. Moreover, by definition of $\mathrm{Ly}(\mu_n, \cdot)$ and $\mathrm{Ly}(\mu, \cdot)$, we have that $\mathrm{Ly}(\mu_n, \cdot)$ converges pointwise to $\mathrm{Ly}(\mu, \cdot)$. It follows that $\mathrm{Ly}(\mu_n, \cdot)$ locally uniformly converges to $\mathrm{Ly}(\mu, \cdot)$. □

Proof of Proposition 2.10. For $\mu \in \mathcal{M}_f$, by definition of $\mathrm{Ly}(\mu, \cdot)$, we have

$$\mathrm{Ly}(\mu, g) = \int_{J(f)} \log |g' \circ \phi_g| d\mu.$$

Since ϕ_g is holomorphic in g, the map $g \mapsto g' \circ \phi_g$ is holomorphic in g. Then $\log |g' \circ \phi_g|$ is harmonic in g. Therefore $\mathrm{Ly}(\mu, \cdot)$ is harmonic. In particular, it is real-analytic. This completes the proof of statement (1).

For statement (2), if $\mu_n \to \mu$, again by Lemma 2.11, the sequence $\mathrm{Ly}(\mu_n, \cdot)$ converges to $\mathrm{Ly}(\mu, \cdot)$ uniformly on compact sets. As $\mathrm{Ly}(\mu_n, \cdot)$ are harmonic, uniform convergence on compact sets implies uniform convergence of derivatives on compact sets. □

3. Thermodynamic formalism and the pressure metric

In this section, we first review the thermodynamic formalism for conformal repellers. In particular, we discuss the topological pressure of a Hölder continuous function and the pressure metric on the space of cohomology classes of Hölder continuous functions with pressure zero. Standard references are [9, 14, 17]. Via the thermodynamic mapping, the pressure metric pulls back to a non-negative two-form on a hyperbolic component in the moduli space rat_d of degree $d \geq 2$ rational maps.

3.1. Conformal repellers

Let f be a holomorphic function from an open subset $V \subset \mathbb{C}$ into $\mathbb{C}$ and let J be a compact subset of V. The triple (J, V, f) is a *conformal repeller* if

1) (Expansiveness) there exist $C > 0$ and $\lambda > 1$ such that $|(f^n)'(z)| \geq C\lambda^n$ for every $z \in J$ and $n \geq 1$,
2) (Invariance) $f^{-1}(V) \subset V$ is relatively compact in V with $J = \cap_{n\geq 1} f^{-n}(V)$, and
3) (Topological exactness) for any open set U with $U \cap J \neq \emptyset$, there exists an $n > 0$ such that $J \subset f^n(U \cap J)$.

An important property of conformal repellers is the existence of a Markov partition. A *Markov partition* of J is a finite cover of J by sets R_j, $1 \leq j \leq N$ satisfying the following conditions:

1) each set R_j is the closure of its interior $\mathrm{Int}R_j$,
2) the interiors of the R_j are pariwise disjoint,
3) if $x \in \mathrm{Int}R_j$ and $f(x) \in \mathrm{Int}R_\ell$, then $R_\ell \subset f(R_j)$, and
4) each restriction $f|_{R_j}$ is injective.

Let (J, V, f) be a conformal repeller and let $(R_1, \cdots, R_m)$ be a Markov partition of J. Define a matrix A by

$$A_{j,\ell} = \begin{cases} 1, & \text{if } R_\ell \subset f(R_j), \\ 0, & \text{otherwise.} \end{cases}$$

Then every point $x \in J$ corresponds to an infinite sequence $\{\ell_k\}_{k\geq 0}$ where $\ell_k \in \{1, \cdots, m\}$ and $A_{\ell_k, \ell_{k+1}} = 1$. Let Σ be the set of all such sequences, i.e.

$$\Sigma = \{\{\ell_k\}_{k\geq 0} \mid \ell_k \in \{1, \cdots, m\}, A_{\ell_k, \ell_{k+1}} = 1\}$$

and $\sigma : \Sigma \to \Sigma$ be the shift map, i.e.

$$\sigma(\ell_0, \ell_1, \ell_2 \cdots) = (\ell_1, \ell_2, \ell_3, \cdots).$$

There is a standard metric on Σ defined as

$$d(x, y) = 2^{-N(x,y)}$$

where $N(x,y) = \min\{n : x_n \neq y_n\}$. Given $\alpha \in (0,1]$, a continuous function ϕ is *α-Hölder continuous* if there exists a constant $C > 0$ such that

$$|\phi(x) - \phi(y)| \leq Cd(x,y)^\alpha$$

for any $x, y \in \Sigma$. Denote by $C^\alpha(\Sigma)$ the space of α-Hölder continuous real-valued functions on Σ. We say that a continuous function ϕ is *Hölder continuous* if it is α-Hölder continuous for some $\alpha \in (0,1]$.

Given a conformal repeller (J, V, f), there is a projection map $\Psi_f : \Sigma \to J$ sending a sequence $\{\ell_k\}_{k\geq 0}$ to $z \in J$ such that $f^{\ell_k}(z) \in R_{\ell_k}$. Let $\phi_f = -\log|f' \circ \Psi_f|$. Then ϕ_f is a Hölder continuous function. Bowen's theorem states that the Hausdorff dimension of J is the unique solution to the equation $\mathcal{P}(t\phi_f) = 0$ (see [17]). Here $\mathcal{P}$ is the topological pressure which we introduce now.

3.2. The pressure function

In this subsection, we review definitions of topological pressure and equilibrium states. Then we summarize formulas for the derivatives of the pressure function. A standard reference is [9].

Given $\phi \in C^\alpha(\Sigma)$, the *transfer operator* $\mathcal{L}_\phi : C^\alpha(\Sigma) \to C^\alpha(\Sigma)$ is defined by

$$\mathcal{L}_\phi(g)(y) = \sum_{\sigma(x)=y} e^{\phi(x)} g(x).$$

By Ruelle-Perron-Frobenius theorem, there is a positive eigenfunction e^ψ, unique up to scale, such that

$$\mathcal{L}_\phi(e^\psi) = \rho(\mathcal{L}_\phi) e^\psi,$$

where $\rho(\mathcal{L}_\phi)$ is the isolated maximal eigenvalue of the transfer operator and the rest of the spectrum is contained in a disk of radius $r < \rho(\mathcal{L}_\phi)$.

The *pressure* of ϕ is defined by

$$\mathcal{P}(\phi) = \log \rho(\mathcal{L}_\phi).$$

Alternatively, the pressure $\mathcal{P}(\phi)$ can also be defined using variational methods. Let $\mathcal{M}_\sigma$ be the set of σ-invariant probability measures on Σ. Then

$$\mathcal{P}(\phi) = \sup_{m \in \mathcal{M}_\sigma} \left(h_m(\sigma) + \int_\Sigma \phi \, dm \right).$$

where $h_m(\sigma)$ is the measure-theoretic entropy of σ with respect to the measure m. A measure $m = m(\phi) \in \mathcal{M}_\sigma$ is called an *equilibrium state* of ϕ if $\mathcal{P}(\phi) = h_m(\sigma) + \int_\Sigma \phi dm$. It is well-known that ϕ has a unique equilibrium state, see [1].

The equilibrium state $m(\phi)$ is also related to the spectral data of transfer operators. If $\mathcal{P}(\phi) = 0$, then $\mathcal{L}_\phi(e^\psi) = e^\psi$. It follows that there is a unique positive measure μ on Σ such that

$$\int_\Sigma \mathcal{L}_\phi(\tilde{\phi}) d\mu = \int_\Sigma \tilde{\phi} d\mu$$

for all $\tilde{\phi} \in C^\alpha(\Sigma)$ and $\int_\Sigma e^\psi d\mu = 1$. We have

$$m(\phi) = e^\psi \mu.$$

Note that $m(\phi)$ is an ergodic, σ-invariant probability measure with positive entropy.

The *asymptotic variance* (cf. "variance" in [7]) of a Hölder continuous function $\psi : \Sigma \to \mathbb{R}$ with $\int \psi dm(\phi) = 0$ is given by

$$Var(\psi, m(\phi)) = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \left| \sum_{i=0}^{n-1} \psi \circ \sigma^i(x) \right|^2 dm(\phi).$$

It follows that, as $t \to 0$,

$$\mathcal{P}(\phi + t\psi) = \mathcal{P}(\phi) + (t^2/2) Var(\psi, m(\phi)) + O(t^3)$$

([7], p. 373 Equation (1.5)). We summarize the following formulas for the derivatives of the pressure $\mathcal{P}$.

Proposition 3.1. *[7, Theorem 2.2] Let ϕ_t be a smooth path in $C^\alpha(\Sigma)$, let $m = m(\phi_0)$ and let $\dot{\phi}_0 = d\phi_t/dt|_{t=0}$. We have*

$$\left.\frac{d\mathcal{P}(\phi_t)}{dt}\right|_{t=0} = \int_\Sigma \dot{\phi}_0 dm$$

and, if the first derivative of $\mathcal{P}(\phi_t)$ is zero, then

$$\left.\frac{d^2\mathcal{P}(\phi_t)}{dt^2}\right|_{t=0} = Var(\dot{\phi}_0, m) + \int \ddot{\phi}_0 dm.$$

3.3. The pressure metric

Recall that two continuous functions ϕ_1 and ϕ_2 are *cohomologous*, denoted by $\phi_1 \sim \phi_2$, if there exists a continuous function $h : \Sigma \to \mathbb{R}$ such that $\phi_1(x) - \phi_2(x) = h(\sigma(x)) - h(x)$. The pressure function $\mathcal{P} : C^\alpha(\Sigma) \to \mathbb{R}$ depends only on the cohomology classes [9]. We show in this subsection that it defines a metric in the thermodynamic setting.

Let $\mathcal{C}(\Sigma)$ be the set of cohomology classes of Hölder continuous functions with pressure zero, that is,

$$\mathcal{C}(\Sigma) = \{\phi : \phi \in C^\alpha(\Sigma) \text{ for some } \alpha, \mathcal{P}(\phi) = 0\}/\sim$$

where $\phi_1 \sim \phi_2$ if ϕ_1 and ϕ_2 are cohomologous.

If $[\phi] \in \mathcal{C}(\Sigma)$, let m be an equilibrium state for ϕ. Then by the formula for the derivative of the pressure $\mathcal{P}$, the tangent space of $\mathcal{C}(\Sigma)$ at $[\phi]$ can be identified with

$$T_{[\phi]}\mathcal{C}(\Sigma) = \left\{\psi \;\middle|\; \psi \text{ Hölder}, \int_\Sigma \psi dm = 0\right\} / \sim .$$

We define the *pressure metric* $||\cdot||_{pm}$ on $\mathcal{C}(\Sigma)$ as follows (see [7], p. 375). Given $[\psi] \in T_{[\phi]}\mathcal{C}(\Sigma)$, define

$$||[\psi]||^2_{pm} := \frac{Var(\psi, m)}{-\int_\Sigma \phi dm}.$$

We claim that $||\cdot||_{pm}$ is non-degenerate. Indeed, on one hand, by convexity of $\mathcal{P}$ and Proposition 3.1, the second derivative

$$\left.\frac{d^2\mathcal{P}(\phi + t\psi)}{dt^2}\right|_{t=0} = Var(\psi, m(\phi))$$

is non-negative on the tangent space $T_{[\phi]}\mathcal{C}(\Sigma)$. In fact, the variance is zero if and only if ψ is cohomologous to zero ([9], Proposition 4.12). On the other hand, we have $\int_\Sigma \phi dm < 0$ since $0 = \mathcal{P}(\phi) = \int_\Sigma \phi dm + h_m(\sigma)$ and $h_m(\sigma) > 0$.

3.4. Thermodynamic mapping on hyperbolic components

Let $\mathcal{H} \subset \mathrm{rat}_d$ be a hyperbolic component. For $[f] \in \mathcal{H}$, there exists a neighborhood V of $J(f)$ such that $(J(f), V, f)$ is a conformal repeller. Moreover,

$(J(f), f)$ admits a Markov partition. Recall that $\Psi_f : \Sigma \to J(f)$ is the projection map as in Section 3.1.

Recall that $\delta(f)$ is the Hausdorff dimension of the Julia set $J(f)$. The function $-\log|f' \circ \Psi_f| : \Sigma \to \mathbb{R}$ is Hölder continuous and by Bowen's theorem, we have

$$\mathcal{P}(-\delta(f)\log|f' \circ \Psi_f|) = 0.$$

Note that if $f_1 \in \mathrm{Rat}_d$ is Möbius conjugate to f, then $\delta(f_1) = \delta(f)$ and for all $k \geq 1$ and any k-periodic point x_k of σ,

$$\log|(f_1^k)'(\Psi_{f_1}(x_k))|) = \log|(f^k)'(\Psi_f(x_k))|).$$

It follows that

$$-\delta(f_1)\log|(f_1^n)'(\Psi_{f_1}(x_k))| = -\delta(f)\log|(f^n)'(\Psi_f(x_k))|.$$

By Livsic Theorem, we have $-\delta(f_1)\log|f_1' \circ \Psi_{f_1}|$ and $-\delta(f)\log|f' \circ \Psi_f|$ are cohomologous. Thus, there is a *thermodynamic mapping*

$$\mathscr{E} : \mathcal{H} \to \mathcal{C}(\Sigma),$$

given by

$$\mathscr{E}([f]) = [-\delta(f)\log|f' \circ \Psi_f|].$$

We remark that a result [5, Theorem 2] of Levin implies that the map $\mathscr{E}$ is a local embedding, although we will not use this fact in the paper.

We define a non-negative metric $||\cdot||_{\mathcal{P}}$ on $\mathcal{H}$ as the pullback of the pressure metric $||\cdot||_{pm}$ on $\mathcal{C}(\Sigma)$. Indeed, $||\cdot||_{\mathcal{P}}$ is non-negative since $||\cdot||_{pm}$ is positive-definite. Abusing notation, we also call $||\cdot||_{\mathcal{P}}$ the *pressure metric* on $\mathcal{H}$.

Now we derive a formula for $||\cdot||_{\mathcal{P}}$. Given $[f] \in \mathcal{H}$ and $\vec{v} \in T_{[f]}\mathcal{H} \cong \mathbb{R}^{4d-4}$, let $c(t) := [f_t], t \in (-1,1)$ be a smooth real 1-dimesional path in $\mathcal{H}$ such that $c(0) = [f]$ and $c'(0) = \vec{v}$. Moreover, we can assume that all f_t are in the same lift of $\mathcal{H}$ in Rat_d. Under the thermodynamic mapping, $c(t)$ corresponds to the following one-parameter family of pressure zero Hölder continuous functions on Σ:

$$g(t,x) = -\delta(f_t)\log|f_t'(\Psi_{f_t}(x))|, \quad t \in (-1,1).$$

Since the Julia sets $J(f_t)$ form a holomorphic motion as t varies, the dependence of the projection map Ψ_{f_t} on t is given by the holomorphic motion.

Denote by $\dot{g}(0,x) = \frac{d}{dt}\big|_{t=0} g(t,x)$. Then by definition of the pressure metric,

$$||\vec{v}||^2_{\mathcal{P}} = \frac{Var(\dot{g}(0,x),\nu)}{-\int_\Sigma g(0,x)d\nu(x)}$$

where ν is the equilibrium state for $g(0,x)$.

Since $\mathcal{P}(g(t,x)) = 0$ for all $t \in (-1,1)$, we have $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{P}(g(t,x)) = 0$. By Proposition 3.1, we have

$$Var(\dot{g}(0,x),\nu) + \int_\Sigma \ddot{g}(0,x)d\nu(x) = 0.$$

It follows that

$$||\vec{v}||^2_{\mathcal{P}} = \frac{Var(\dot{g}(0,x),\nu)}{-\int_\Sigma g(0,x)d\nu(x)} = \frac{\int_\Sigma \ddot{g}(0,x)d\nu(x)}{\int_\Sigma g(0,x)d\nu(x)}.$$

4. A symmetric bilinear form $||\cdot||_G$

Our main goal in this section is to define a non-negative 2-form $||\cdot||_G$ on hyperbolic components in Rat_d and then show it descends to a non-negative 2-form on a hyperbolic component in rat_d.

4.1. The 2-form on hyperbolic components in Rat_d

Suppose $\widetilde{\mathcal{H}}$ is a hyperbolic component in Rat_d and fix $f \in \widetilde{\mathcal{H}}$. Let $U(f)$ be as in Section 1 and let ν be the equilibrium state of the Hölder potential $-\delta(f)\log|f'| : J(f) \to \mathbb{R}$ which has pressure zero. Recall that the function $\mathrm{Ly}(\nu,\cdot) : U(f) \to \mathbb{R}$ is given by

$$\mathrm{Ly}(\nu,g) = \int_{J(g)} \log|g'|d\left((\phi_g)_*\nu\right) = \int_{J(f)} \log|g'\circ\phi_g|d\nu,$$

where $\phi_g : J(f) \to J(g)$ is the quasi-conformal conjugacy. Now consider the real analytic function $G_f : U(f) \to \mathbb{R}$ given by

$$G_f(g) = \delta(g)\mathrm{Ly}(\nu,g).$$

In what follows, we will show the Hessian of the function G_f is well-defined at f and gives us a non-negative 2-form on $\widetilde{\mathcal{H}}$. Note that the Hessian of a smooth real-valued function $G : X \to \mathbb{R}$ on a smooth manifold X is not well-defined at a point $x \in X$ unless $G'(x) = 0$ (see [3, Section 7]). We first show G_f has a minimum at f and hence $G'_f(f) = 0$ in the following result.

Proposition 4.1. *Fix $f \in \widetilde{\mathcal{H}}$ and ν as above. Then for all $g \in U(f)$, we have*

$$\frac{\delta(f)}{\delta(g)} \leq \frac{\mathrm{Ly}(\nu, g)}{\mathrm{Ly}(\nu, f)}.$$

Proof. To ease notation, set $m_g := (\phi_g)_* \nu$. Then

$$\mathrm{Ly}(\nu, g) = \int_{J(g)} \log |g'| dm_g.$$

Since $-\delta(f) \log |f'| : J(f) \to \mathbb{R}$ has pressure zero and $\nu = m_f$ is its equilibrium state, by the variational definition of pressure,

$$h_{m_f}(f) = \delta(f) \int_{J(f)} \log |f'| dm_f,$$

where $h_{m_f}(f)$ is the measure-theoretic entropy of f with respect to m_f.

Since entropy is invariant under topological conjugacy, it follows that $h_{m_f}(f) = h_{(\phi_g)_* m_f}(g)$. We have $h_{m_f}(f) = h_{m_g}(g)$. Since m_f is f-invariant, it follows that m_g is g-invariant. Again, by the variational definition of pressure, we have

$$h_{m_g}(g) \leq \delta(g) \int_{J(g)} \log |g'| dm_g.$$

Hence $\delta(f)\mathrm{Ly}(\nu, f) \leq \delta(g)\mathrm{Ly}(\nu, g)$ and the conclusion follows. □

Therefore, the Hessian of G_f at f is well-defined and it defines a symmetric bilinear form $|| \cdot ||_G$ on the tangent space $T_f\widetilde{\mathcal{H}}$ as follows. Let $\gamma(t), t \in (-1, 1)$ be a smooth real 1-dimensional path in $U(f)$ with $\gamma(0) = f$ and $\gamma'(0) = \vec{w} \in T_f\widetilde{\mathcal{H}} \cong \mathbb{R}^{4d+2}$. Define

$$||\vec{w}||_G^2 := \frac{\partial^2 G_f}{\partial \vec{w} \partial \vec{w}} = \frac{d^2}{dt^2}\Big|_{t=0} G_f(\gamma(t)).$$

Note that $||\vec{w}||_G^2$ only depends on f and $\vec{w}$. Indeed,

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} G_f(\gamma(t)) &= G_f''(\gamma(0)) \cdot (\gamma'(0))^2 + G_f'(\gamma(0))\gamma''(0) \\ &= G_f''(\gamma(0)) \cdot (\gamma'(0))^2 \end{aligned}$$

since by Proposition 4.1, we have $G_f'(\gamma(0)) = G_f'(f) = 0$. Moreover, again by Proposition 4.1, we have $||\vec{w}||_G^2 \geq 0$.

4.2. The 2-form on hyperbolic components in rat_d

Let $\mathcal{H} \subset \mathrm{rat}_d$ be a hyperbolic component. For $[f] \in \mathcal{H}$ and $\vec{v} \in T_{[f]}\mathcal{H} \cong \mathbb{R}^{4d-4}$, let $c(t)$ be a smooth real 1-dimensional curve in $\mathcal{H}$ defined on $(-1,1)$ with $c(0) = [f]$ and $c'(0) = \vec{v}$. Consider two distinct lifts $\tilde{c}(t)$ and $\tilde{c}_1(t)$ in Rat_d. Since our analysis is local, we may assume that $\tilde{c}(t) \subset U(\tilde{c}(0))$ and $\tilde{c}_1(t) \subset U(\tilde{c}_1(0))$ as in the previous section. By the definition of $||\cdot||_G$, we have $||\tilde{c}'(0)||_G = ||\tilde{c}_1'(0)||_G$. Indeed, since $\tilde{c}(t)$ and $\tilde{c}_1(t)$ are Möbius conjugate, $G_{\tilde{c}(0)}(\tilde{c}(t)) = G_{\tilde{c}_1(0)}(\tilde{c}_1(t))$ on $(-1,1)$. Thus the 2-form $||\cdot||_G$ descends to a 2-form on $\mathcal{H}$. Abusing notation, we also denote the 2-form on $\mathcal{H}$ by $||\cdot||_G$ and therefore

$$||\vec{v}||_G := ||\tilde{c}'(0)||_G.$$

Write $\tilde{c}(t) = f_t \in \mathrm{Rat}_d$. Recall notations from Section 3. For $x \in \Sigma$,

$$g(t,x) = -\delta(f_t)\log|f_t' \circ \Psi_{f_t}(x)|,$$

where $t \in (-1,1)$. Denote by $\dot{g}(0,x) = \frac{d}{dt}\big|_{t=0} g(t,x)$ and ν the equilibrium state for $g(0,x)$.

Proposition 4.2. *The form $||\cdot||_G$ is conformal equivalent to the pressure form $||\cdot||_{\mathcal{P}}$. More precisely, fixing the notations as above, we have*

$$||\vec{v}||^2_{\mathcal{P}} = \frac{||\vec{v}||^2_G}{-\int_\Sigma g(0,x)d\nu(x)}.$$

Proof. By straightforward calculation,

$$||\vec{v}||^2_{\mathcal{P}} = \frac{Var(\dot{g}(0,x),\nu)}{-\int_\Sigma g(0,x)d\nu(x)} = \frac{\int_\Sigma \ddot{g}(0,x)d\nu(x)}{\int_\Sigma g(0,x)d\nu(x)} = \frac{-||\vec{v}||^2_G}{\int_\Sigma g(0,x)d\nu(x)}.$$

The last equality holds by definition of $||\cdot||_G$. □

Recall that a continuous function $\phi : \Sigma \to \mathbb{R}$ is a coboundary if it is cohomologous to zero.

Corollary 4.3. *Fix the notations as above. The following are equivalent.*

1) $||\vec{v}||_G = 0$.

2) $||\vec{v}||_{\mathcal{P}} = 0$.

3) $\dot{g}(0,x)$ *is a coboundary.*

Proof. The equivalence follows immediately from Proposition 4.2 and a standard fact from Thermodynamic Formalism that $Var(\dot{g}(0,x),\nu)=0$ if and only if $\dot{g}(0,x)$ is a coboundary. □

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For quadratic polynomials, we also show that $||\cdot||_G$ is positive-definite on the central hyperbolic component.

5.1. Proof of Theorem 1.1

We first show the following result concerning the derivative of repelling multipliers for an analytic family of hyperbolic rational maps.

Proposition 5.1. *Suppose $f\in\mathbb{C}(z)$ is a hyperbolic rational map of degree at least 2. Let $\{f_t\}_{t\in\mathbb{D}}$ be a holomorphic family of hyperbolic rational maps such that $f_0=f$ and some repelling multiplier has nonzero derivative at $t=0$. Assume that there exists $K\in\mathbb{R}$ such that*

$$\frac{d}{dt}\Big|_{t=0}\log|\lambda_t| = K\log|\lambda_0|$$

for all repelling multipliers λ_t of f_t. Then all the repelling multipliers of f are real.

Proof. If f is conjugate to a monomial, then all the multipliers of f are real. Now we assume that f is not conjugate to a monomial. Let a_t be a repelling multiplier of f_t such that $a_t'|_{t=0}\neq 0$, and let b_t be a repelling multiplier of f_t. By Corollary 2.7, there exists $\kappa_0\in(0,1)$ such that for any $\kappa\in(\kappa_0,1)$ and t close to 0, the map f_t has a repelling cycle with multiplier

$$\lambda_{t,n}=e^{i\theta_{t,n}}(a_t^n+a_t^{n\kappa}b_t+o(a_t^{n\kappa}b_t)).$$

Then

$$\frac{d}{dt}\Big|_{t=0}\log|\lambda_{t,n}| = K\log|\lambda_{0,n}|.$$

Consider D_n and $E_n(t)$ as in Remark 2.8 for this $\lambda_{t,n}$. Then on D_n,

$$|\lambda_{t,n}|=|a_t^n+a_t^{n\kappa}b_t+E_n(t)|=|a_t|^n\cdot|1+a_t^{n(\kappa-1)}b_t+E_n(t)/a_t^n|.$$

To ease notation, set $\eta := \kappa - 1$. Then $\eta \in (\kappa_0 - 1, 0)$. It follows that

$$\begin{aligned}\log|\lambda_{t,n}| &= n\log|a_t| + \log|1 + a_t^{n\eta}b_t + E_n(t)/a_t^n| \\ &= n\log|a_t| + Re(\log(1 + a_t^{n\eta}b_t + E_n(t)/a_t^n)).\end{aligned}$$

Now we begin to compute $\frac{d}{dt}\Big|_{t=0}\log|\lambda_{t,n}|$. First note that

$$\frac{d}{dt}\Big|_{t=0}\log(1 + a_t^{n\eta}b_t + E_n(t)/a_t^n) = \frac{\frac{d}{dt}\Big|_{t=0}(a_t^{n\eta}b_t) + \frac{d}{dt}\Big|_{t=0}(E_n(t)/a_t^n)}{1 + a_0^{n\eta}b_0 + E_n(0)/a_0^n}.$$

We claim that

$$\frac{d}{dt}\Big|_{t=0}(E_n(t)/a_t^n) = o(na_0^{n\eta}).$$

Suppose on the contrary that there exist $\alpha > 0$ and a strictly increasing subsequence $\{n_i\}_{i\geq 1}$ such that $\left|\frac{d}{dt}\Big|_{t=0}(E_{n_i}(t)/a_t^{n_i})\right| > \alpha\,|n_i a_0^{n_i\eta}|$ for all sufficiently large $n_i \gg 1$. Considering the extension of $E_n(t)$ as in Remark 2.8, we have that $E_n(t)/a_t^n$ is holomorphic on $\mathbb{D}$. Adding a nonzero constant if necessary, which does not change the derivative, we can write $E_n(t)/a_t^n = (g_n(t))^n$ for some holomorphic function $g_n(t)$. Moreover, considering a small closed disk $\overline{D} \subset \mathbb{D}$ centered at 0, by Remark 2.8, we have $|E_n(t)/a_t^n|$ is uniformly bounded on $\overline{D}$, so is $|g_n(t)|$. It follows that $\{g_n(t)\}$ is normal on D, and hence passing to subsequence if necessary, g_{n_i} converges uniformly on D to a holomorphic map $h(t)$. We also have $g'_{n_i}(0)$ converges to $h'(0) \in \mathbb{C}$. Note that $g_{n_i}(0) = o(a_0^{\eta})$ for sufficiently large $n_i \gg 1$ since by Remark 2.8 we have $E_n(0) = o(a_0^{n\kappa}b_0)$. We conclude that

$$\frac{d}{dt}\Big|_{t=0}(E_{n_i}(t)/a_t^{n_i}) = n_i(g_{n_i}(0))^{n_i-1}g'_{n_i}(0) = o(n_i a_0^{n_i\eta}),$$

which contradicts the choice of $\{n_i\}_{i\geq 1}$. So the claim holds.

Thus we obtain

$$\frac{d}{dt}\Big|_{t=0}\log(1 + a_t^{n\eta}b_t + E_n(t)/a_t^n) = (1 + o(1))\left(\frac{d}{dt}\Big|_{t=0}(a_t^{n\eta}b_t) + o(na_0^{n\eta})\right).$$

It follows that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\log|\lambda_{t,n}| &= n\frac{d}{dt}\Big|_{t=0}\log|a_t| + Re\left(\frac{d}{dt}\Big|_{t=0}(\log(1 + a_t^{n\eta}b_t + E_n(t)/a_t^n)\right) \\ &= n\frac{d}{dt}\Big|_{t=0}\log|a_t| + (1 + o(1))\,Re\left(\frac{d}{dt}\Big|_{t=0}(a_t^{n\eta}b_t) + o(na_0^{n\eta})\right).\end{aligned}$$

Applying the equation $\frac{d}{dt}\Big|_{t=0} \log|a_t| = K\log|a_0|$, we obtain

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} \log|\lambda_{t,n}| - K\log|\lambda_{0,n}| \\
&= (1+o(1))\, Re\left(\frac{d}{dt}\Big|_{t=0}(a_t^{n\eta} b_t) + o(n a_0^{n\eta})\right) \\
&\quad - K Re(\log(1 + a_0^{n\eta} b_0 + E_n(0)/a_0^n)) \\
&= (1+o(1))\, Re\left(\frac{d}{dt}\Big|_{t=0}(a_t^{n\eta} b_t) + o(n a_0^{n\eta})\right) \\
&\quad - (K+o(1)) Re(a_0^{n\eta} b_0 + E_n(0)/a_0^n) \\
&= (1+o(1))\left(Re(n\eta a_0^{n\eta-1} a_t'|_{t=0} b_0 + a_0^{n\eta} b_t'|_{t=0}) + Re(o(n a_0^{n\eta}))\right) \\
&\quad - (K+o(1)) Re(a_0^{n\eta} b_0 + E_n(0)/a_0^n).
\end{aligned}$$

Dividing by $n|a_0|^{n\eta}$ in the above equality and taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} Re\left(\eta \frac{a_0^{n\eta}}{|a_0|^{n\eta}} \frac{a_t'|_{t=0}}{a_0} b_0\right) = 0.$$

Since η is real and nonzero, it follows that

$$\lim_{n\to\infty} Re\left(\frac{a_0^{n\eta}}{|a_0|^{n\eta}} \frac{a_t'|_{t=0}}{a_0} b_0\right) = 0.$$

Set $u_0 := a_0/|a_0| = e^{i\theta_0}$. Then $u_0^\eta = a_0^\eta/|a_0|^\eta = e^{i\eta\theta_0}$. Choose a subsequence n_j such that $u_0^{n_j\eta} \to 1$ as $j \to \infty$. Then we have

$$\lim_{j\to\infty} Re\left(u_0^{n_j\eta} \frac{a_t'|_{t=0}}{a_0} b_0\right) = Re\left(\frac{a_t'|_{t=0}}{a_0} b_0\right).$$

It follows that

$$Re\left(\frac{a_t'|_{t=0}}{a_0} b_0\right) = 0.$$

Now we claim that a_0 is real. Suppose otherwise. Then we have $\theta_0 \in (0, 2\pi)$ and $\theta_0 \neq \pi$. To obtain a contradiction, we discuss in the following two cases.

Case 1: $\theta/\pi \in (0,2)$ is irrational. Pick $\kappa \in (\kappa_0, 1)$ to be rational. Then $\eta \in (\kappa_0 - 1, 0)$ is rational. Choose a subsequence n_k such that $u_0^{n_k\eta} \to i$ as

$k \to \infty$. Then

$$\lim_{k\to\infty} Re\left(\frac{a_0^{n_k\eta}}{|a_0|^{n_k\eta}}\frac{a_t'|_{t=0}}{a_0}b_0\right) = Im\left(\frac{a_t'|_{t=0}}{a_0}b_0\right).$$

It follows that

$$Im\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) = 0.$$

Thus

$$\frac{a_t'|_{t=0}}{a_0}b_0 = 0.$$

Since $a_t'|_{t=0} \neq 0$, we have $b_0 = 0$. It is impossible since b_0 is the multiplier of a repelling cycle of f.

Case 2: $\theta/\pi \in (0,2)$ is rational and $\theta/\pi \neq 1$. Pick $\kappa \in (\kappa_0, 1)$ to be irrational. Then $\eta \in (\kappa_0 - 1, 0)$ is irrational. Write $\theta/\pi = p/q$ for two (not necessarily coprime) integers p and q such that $u_0^q = 1$. Set $n_\ell = \ell q + 1$. Then $u_0^{n_\ell} = u_0$. It follows that

$$\begin{aligned} 0 &= \lim_{\ell\to\infty} Re\left(u_0^{n_\ell\eta}\frac{a_t'|_{t=0}}{a_0}b_0\right) \\ &= Re\left(u_0^{\eta}\frac{a_t'|_{t=0}}{a_0}b_0\right) \\ &= Re(u_0^{\eta})Re\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) - Im(u_0^{\eta})Im\left(\frac{a_t'|_{t=0}}{a_0}b_0\right). \end{aligned}$$

Since $Re\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) = 0$, we have

$$Im(u_0^{\eta})Im\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) = 0.$$

Note that $Im(u_0^{\eta}) \neq 0$. It follows that

$$Im\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) = 0.$$

Then we obtain the same contradiction as in Case 1. This proves the claim.

Note that b_t is an arbitrary repelling multiplier of f_t in the above argument. If we set $b_t = a_t$, we have

$$Re\left(\frac{a_t'|_{t=0}}{a_0}a_0\right) = 0.$$

Hence $a_t'|_{t=0}$ is purely imaginary. Since a_0 is real, it follows that $a_t'|_{t=0}/a_0$ is purely imaginary.

We claim that b_0 is real. Indeed, it immediately follows from that $Re\left(\frac{a_t'|_{t=0}}{a_0}b_0\right) = 0$ and $Re\left(\frac{a_t'|_{t=0}}{a_0}\right) = 0$.

Since b_0 is arbitrary, all the repelling multipliers of f are real. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.2, we only need to show $||\cdot||_G$ is positive-definite on $\mathcal{H}$. We prove by contradiction. Suppose that there exist $[f] \in \mathcal{H}$ and a nonzero tangent vector $\vec{v} \in T_{[f]}\mathcal{H} \cong \mathbb{R}^{4d-4}$ such that $||\vec{v}||_G = 0$. Then consider a holomorphic path $c(t) = [f_t] \in \mathcal{H}$ with $t \in \mathbb{D}$ such that $\{f_t\}_{t\in\mathbb{D}}$ is a holomorphic family of hyperbolic rational maps with $f_0 = f$ and identifying $\mathbb{D}$ to a real 2-dimensional disk, the directional derivative of $c(t)$ along the positive real axis is $\vec{v}$. We show that there exists a constant $K \in \mathbb{R}$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \log|\lambda_t| = K \log|\lambda_0|$$

for all multipliers λ_t of repelling periodic orbits of f_t. Since $\log|\lambda_t|$ is real analytic in $t \in \mathbb{D}$, it is sufficient to treat the parameter $t \in (-1, 1)$, so in next paragraph, we assume $t \in (-1, 1)$.

By Corollary 4.3, the derivative $\dot{g}(0, x)$ of the map $g(t, x) = -\delta(f_t)\log|f_t' \circ \Psi_{f_t}(x)|$ is a coboundary. Recall that σ is the left shift map on Σ. By definition of coboundary, there exists a continuous function $h : \Sigma \to \mathbb{R}$ such that $\dot{g}(0, x) = h(x) - h(\sigma(x))$. Let $x \in \Sigma$ be a periodic point of σ, i.e. $\sigma^n(x) = x$ for some $n \geq 1$. Then

$$\begin{aligned}
0 &= h(x) - h(\sigma^n(x)) \\
&= \left.\frac{d}{dt}\right|_{t=0} g(t, x) + \left.\frac{d}{dt}\right|_{t=0} g(t, \sigma(x)) + \cdots + \left.\frac{d}{dt}\right|_{t=0} g(t, \sigma^{n-1}(x)) \\
&= -\left.\frac{d}{dt}\right|_{t=0} \delta(f_t)\log|(f_t^n)' \circ \Psi_{f_t}(x)|.
\end{aligned}$$

Applying the chain rule, we obtain

$$\frac{d}{dt}\Big|_{t=0} \log|(f_t^n)' \circ \Psi_{f_t}(x)| = -\frac{\frac{d}{dt}\big|_{t=0}\delta(f_t)}{\delta(f_0)} \cdot \log|(f_0^n)' \circ \Psi_{f_0}(x)|.$$

Then $K := \frac{d}{dt}\big|_{t=0}\delta(f_t)/\delta(f_0)$ is the desired real number.

Reparametrizing $c(t)$ if necessary, the family f_t has a multiplier of some repelling periodic orbit with nonzero derivative at $t = 0$ since $v \neq 0$. Then by Proposition 5.1, all the repelling multipliers of f are real. It contradicts to the assumption on f. □

5.2. The component $\mathcal{H}_0$ in poly$_2$

In this subsection, we show $||\cdot||_G$ is also positive-definite on the main cardioid of the Mandelbrot set.

Theorem 5.2. *If $\mathcal{H}_0$ is the central hyperbolic component in poly$_2$, then $||\cdot||_G$ is positive-definite on $\mathcal{H}_0$.*

Proof. Consider a curve $P_t(z) = z^2 + c(t)$ with $P_t \in \mathcal{H}_0$. We first claim that if $c(0) \neq 0$, then $||\vec{v}||_G \neq 0$ for any nonzero tangent vector $\vec{v} \in T_{P_0}\mathcal{H}$. Indeed, if $||\vec{v}||_G = 0$, then again, there exists a constant $K = \delta'(v)/\delta(0) \in \mathbb{R}$ such that

$$\frac{d}{dt}\Big|_{t=0} \log|\lambda_t| = K \log|\lambda_0|$$

for all multipliers λ_t of repelling cycles of P_t. Since $\delta(P_0) > 1$, Proposition 5.1 gives a contradiction.

Therefore, it suffices to check that $||\cdot||_G$ is nondegenerate on the tangent space $T_{[z^2]}\mathcal{H}$ at $[z^2]$. Suppose $||\vec{v}||_G = 0$ for some nonzero $\vec{v} \in T_{[z^2]}\mathcal{H}$. Then there exists a constant $K = \delta'(v)$ such that

$$\frac{d}{dt}\Big|_{t=0} \log|\lambda_t| = K \log|\lambda_0|$$

for all multipliers λ_t of repelling cycles of P_t. But here $K = 0$ since $P_0(z) = z^2$ is the local minimum for the Hausdorff dimension function. Also we note that all multipliers λ_0 of the repelling cycles of $P_0(z) = z^2$ are real numbers.

Therefore, $\frac{d}{dt}\Big|_{t=0} \log|\lambda_t| = 0$ implies that

$$Re\left(\frac{d}{dt}\Big|_{t=0}\lambda_t\right) = 0$$

for all multipliers λ_t of repelling cycles of f_t.

A contradiction follows from direct computations. The multiplier for the repelling 1-cycle is $1+\sqrt{1-4c(t)}$. Plugging into the above equation and using $c_0 = 0$,

$$Re\left(\frac{d}{dt}\Big|_{t=0}\lambda_t\right) = Re\left(\frac{-2c'(0)}{\sqrt{1-4c_0}(1+\sqrt{1-4c_0})}\right) = 0$$

implies the tangent vector $\vec{v} = c'(0)$ must be $\pm i$, namely the purely imaginary direction. On the other hand, there are two 3-cycles and their multipliers are
$-4\left(-c(t) - 2 \pm c(t)\sqrt{-4c(t)-7}\right)$, respectively. Therefore,

$$Re\left(\frac{d}{dt}\Big|_{t=0}\lambda_t\right) = Re\left(\frac{-c'(0) + c'(0)\sqrt{-4c_0-7} - \frac{2c_0c'(0)}{\sqrt{-4c_0-7}}}{-c_0 - 2 + c_0\sqrt{-4c_0-7}}\right).$$

But $c_0 = 0$ and $\vec{v} = \pm i$ do not give $Re\left(\frac{d}{dt}\big|_{t=0}\lambda_t\right) = 0$, which is a contradiction.

Hence, $||\cdot||_G$ is positive-definite on $\mathcal{H}_0$. □

If $\mathcal{H}_0$ is the central component in poly_d for $d \geq 3$, then by the same argument as in Theorem 5.2, the form $||\cdot||_G$ is positive-definite on the tangent space $T_{[P]}\mathcal{H}_0$ if $[P] \neq [z^d]$. Therefore the positive-definiteness of $||\cdot||_G$ on $\mathcal{H}_0$ is reduced to the positive-definiteness of $||\cdot||_G$ on the tangent space $T_{[z^d]}\mathcal{H}_0$. However, the proof of Theorem 5.2 is much difficult to reproduce for $T_{[z^d]}\mathcal{H}_0$ when $d \geq 3$. In fact, the positive-definiteness of $||\cdot||_G$ on $T_{[z^d]}\mathcal{H}_0$ is equivalent to a negative answer of the following question.

Question 5.3. For $d \geq 3$, let $\{P_t\}_{t\in\mathbb{D}}$ be a holomorphic family with $P_0(z) = z^d$. Are $\lambda'_t|_{t=0}$ purely imaginary for all repelling multipliers λ_t of P_t?

A positive answer for the above question seems improbable. When $d = 3$, we consider repelling cycles of period 3 and compute the multipliers numerically. There exist repelling multipliers whose derivatives are not purely imaginary. However, it is still unclear to us how to obtain a conceptual proof.

Appendix A: A local upper bound of the exponent of the error term

Let $f \in \mathbb{C}(z)$ be a hyperbolic rational map of degree $d \geq 2$ and let $\{f_t\}_{t\in\mathbb{D}}$ be a holomorphic family of hyperbolic rational maps of degree d such that $f_0 = f$. Denote by δ_t the Hausdorff dimension of the Julia set $J(f_t)$. Recall from Section 2.2 that for $T > 0$, $N_T(\mathcal{O}_t)$ is the number of primitive periodic orbits of f_t in $J(f_t)$ whose multipliers have absolute value less than T. Now suppose f is not conjugate to a monomial. Let $\epsilon_t^* := \epsilon^*(f_t)$ be as in Remark 2.5 for f_t and let $\epsilon_t := \epsilon(f_t)$ in Theorem 2.4 be $\epsilon_t := \epsilon_t^*/2$. In this appendix, we establish that $\delta_t - \epsilon_t$ has an upper bound on a small neighborhood of 0, which completes the proof of Corollary 2.7 and is of independent interest. By Remark 2.5, it suffices to show that $\epsilon^*(f_t)$ has a lower bound near $t = 0$.

Theorem A.1. *Fix the notations as above. There exist a small disk $D' \subset \mathbb{D}$ centered at 0 and $\epsilon > 0$ such that $\epsilon_t^* > \epsilon$ for $t \in \mathbb{D}$.*

Proof. Suppose on the contrary that there exists a sequence $\{t_n\}$ such that as $n \to \infty$, $t_n \to 0$ and $\epsilon_{t_n}^* \to 0$. We will obtain a contradiction to the estimate of $N_T(\mathcal{O}_t)$.

Fix $0 < \epsilon_t' < \min\{\delta_t - \epsilon_t^*, \epsilon_t^*\}$ if $\epsilon_t^* < \delta_t$; and set $\epsilon_t' = 0$ if $\epsilon_t^* = \delta_t$. Since $\epsilon_{t_n}^* \to 0$, then $\epsilon_{t_n}' \to 0$ as $n \to \infty$. When $t = 0$, for ϵ_0', let $C(T)$ be as in Remark 2.5 for f_0. Now fix $\tau \in (0, 1)$ and $\eta > 0$, choose $T_\eta > 0$ sufficiently large so that for any $T \geq T_\eta$, the following hold:

(A.1) $(C(T) - \eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')} < N_T(\mathcal{O}_0) - Li(T^{\delta_0}) < (C(T) + \eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')}$;

(A.2) $|Li((T \pm \tau)^{\delta_0}) - Li(T^{\delta_0})| < 2\tau \frac{T^{\delta_0 - 1}}{\ln T}$;

(A.3) $(C(T) - 2\eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')} < (C(T) \pm \eta)(T \pm \tau)^{\delta_0 - (\epsilon_0^* + \epsilon_0')} < (C(T) + 2\eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')}$;

(A.4) $-\eta T^{\delta_{t_n} - (\epsilon_{t_n}^* + \epsilon_{t_n}')} < (C(T) \pm 2\eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')} < \eta T^{\delta_{t_n} - (\epsilon_{t_n}^* + \epsilon_{t_n}')}$ for n sufficiently large; and

(A.5) $-\eta T^{\delta_{t_n} - (\epsilon_{t_n}^* + \epsilon_{t_n}')} < 2\tau \frac{T^{\delta_0 - 1}}{\ln T} < \eta T^{\delta_{t_n} - (\epsilon_{t_n}^* + \epsilon_{t_n}')}$ for n sufficiently large.

Inequality (A.1) is from the asymptotic expression of $N_T(\mathcal{O}_0)$, see Remark 2.5. Inequalities (A.2) and (A.3) are from Taylor expansions. Inequalities (A.4) and (A.5) are from the fact that δ_t is real-analytic and hence continuous and the assumption $\epsilon_{t_n} \to 0$.

Since the multipliers of f_t are continuous in t and $N_T(\mathcal{O}_0)$ is finite for each fixed above such T, choose $t' := t'(T) > 0$ such that for $|t| < t'$,

$$N_{T-\tau}(\mathcal{O}_0) < N_T(\mathcal{O}_t) < N_{T+\tau}(\mathcal{O}_0). \tag{A.6}$$

Then

$$N_{T-\tau}(\mathcal{O}_0) - Li(T^{\delta_0}) < N_T(\mathcal{O}_t) - Li(T^{\delta_0}) < N_{T+\tau}(\mathcal{O}_0) - Li(T^{\delta_0}). \tag{A.7}$$

Note that we can choose $t' \to 0$ as $T \to \infty$.

Substituting T by $T \pm \tau$ in inequality (A.1), we have

$$\begin{aligned}(C(T) - \eta)(T \pm \tau)^{\delta_0 - (\epsilon_0^* + \epsilon_0')} &< N_{T\pm\tau}(\mathcal{O}_0) - Li((T \pm \tau)^{\delta_0}) \\ &< (C(T) + \eta)(T \pm \tau)^{\delta_0 - (\epsilon_0^* + \epsilon_0')}.\end{aligned} \tag{A.8}$$

We obtain an estimate for $N_T(\mathcal{O}_t) - Li(T^{\delta_0})$ by first combining inequalities (A.7) with (A.2) and (A.8):

$$\begin{aligned}&(C(T) - \eta)(T - \tau)^{\delta_0 - (\epsilon_0^* + \epsilon_0')} - 2\tau \frac{T^{\delta_0 - 1}}{\ln T} \\ &< N_{T-\tau}(\mathcal{O}_0) - Li((T - \tau)^{\delta_0}) + Li((T - \tau)^{\delta_0}) - Li(T^{\delta_0}) \\ &< N_T(\mathcal{O}_t) - Li(T^{\delta_0}) \\ &< N_{T+\tau}(\mathcal{O}_0) - Li((T + \tau)^{\delta_0}) + Li((T + \tau)^{\delta_0}) - Li(T^{\delta_0}) \\ &< (C(T) + \eta)(T + \tau)^{\delta_0 - (\epsilon_0^* + \epsilon_0')} + 2\tau \frac{T^{\delta_0 - 1}}{\ln T}.\end{aligned} \tag{A.9}$$

Then we apply inequality (A.3) to the first and the last line in inequality (A.9):

$$\begin{aligned}(C(T) - 2\eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')} - 2\tau \frac{T^{\delta_0 - 1}}{\ln T} &< N_T(\mathcal{O}_t) - Li(T^{\delta_0}) \\ &< (C(T) + 2\eta)T^{\delta_0 - (\epsilon_0^* + \epsilon_0')} + 2\tau \frac{T^{\delta_0 - 1}}{\ln T}.\end{aligned} \tag{A.10}$$

Now we obtain an estimate for $N_T(\mathcal{O}_t) - Li(T^{\delta_t})$. Since δ_t is real analytic, from the expansion of $Li(T^{\delta_t})$, there exist small $t'' > 0$ and large $T'' > 0$ such that for any $|t| < t''$ and any $T' \geq T''$,

$$|Li((T')^{\delta_t}) - Li((T')^{\delta_0})| < (T')^{(\delta_0 - \epsilon_0)/2}.$$

We can assume that $T_\eta > T''$. Then

$$|Li(T^{\delta_t}) - Li(T^{\delta_0})| < T^{(\delta_0-\epsilon_0)/2}. \tag{A.11}$$

Now for $|t| < \min\{t', t''\}$, since

$$Li(T^{\delta_t}) - Li(T^{\delta_0}) = (Li(T^{\delta_t}) - N_T(\mathcal{O}_t)) + (N_T(\mathcal{O}_t) - Li(T^{\delta_0})),$$

using inequalities (A.11) and (A.10), we get the following bound for $N_T(\mathcal{O}_t) - Li(T^{\delta_t})$:

$$\begin{aligned} |N_T(\mathcal{O}_t) - Li(T^{\delta_t})| &\le |Li(T^{\delta_t}) - Li(T^{\delta_0})| + |N_T(\mathcal{O}_t) - Li(T^{\delta_0})| \\ &< T^{(\delta_0-\epsilon_0)/2} + |C(T) + 2\eta| T^{\delta_0-(\epsilon_0^*+\epsilon_0')} + 2\tau \frac{T^{\delta_0-1}}{\ln T}. \end{aligned} \tag{A.12}$$

Now consider the sequence $\{t_n\}$. Applying inequalities (A.4) and (A.5) to (A.12), we have

$$|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})| \le 3\eta T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}. \tag{A.13}$$

for each large n with $|t_n| < \min\{t', t''\}$. Since t' depends on T, we remark here that $n \to \infty$ as $T \to \infty$. Then passing to a subsequence of T if necessary, we have

$$\lim_{\substack{T\to\infty \\ |t_n|<\min\{t',t''\}}} \frac{T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}}{|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})|} \ge \frac{1}{3\eta}, \tag{A.14}$$

where the limit is possibly ∞.

However, note that by Remark 2.5

$$\lim_{T\to\infty} \frac{T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}}{|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})|} = 0. \tag{A.15}$$

Moreover, by the real analyticity of δ_t in t and the assumptions on $\epsilon_{t_n}^*$ and ϵ_{t_n}', we have

$$\lim_{n\to\infty} \frac{T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}}{|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})|} = \frac{T^{\delta_0}}{|N_T(\mathcal{O}_0) - Li(T^{\delta_0})|}. \tag{A.16}$$

Applying Moore-Osgood theorem, by the limits (A.15) and (A.16), we have

$$\lim_{\substack{T\to\infty \\ n\to\infty}} \frac{T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}}{|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})|} = \lim_{n\to\infty}\lim_{T\to\infty} \frac{T^{\delta_{t_n}-(\epsilon_{t_n}^*+\epsilon_{t_n}')}}{|N_T(\mathcal{O}_{t_n}) - Li(T^{\delta_{t_n}})|} = 0,$$

which contradicts the limit (A.14). This completes the proof. □

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