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Zero–one laws for eventually always hitting points in rapidly mixing systems

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In this work we study the set of eventually always hitting points in shrinking target systems. These are points whose long orbit segments eventually hit the corresponding shrinking targets for all future times. We focus our attention on systems where translates of targets exhibit near perfect mutual independence, such as Bernoulli schemes and the Gauß map. For such systems, we present tight conditions on the shrinking rate of the targets so that the set of eventually always hitting points is a null set (or co-null set respectively).

1. Introduction

Let (X,μ,T) be a measure preserving system, and let $\mathcal{B}=\{B_n : n\in\mathbb{N}\}$ be a sequence of measurable subsets of X. The *hitting set* $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ is defined as the set of $x\in X$ such that

$$T^n x\in B_n \text{ for infinitely many } n\in\mathbb{N}. \tag{1.1}$$

If $\sum_n \mu(B_n)$ is finite, it follows from the Borel–Cantelli Lemma that $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ has measure zero. Conversely, if $\sum_n \mu(B_n)$ is infinite then in certain settings the hitting set $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ has full measure. Results pertaining to this dichotomy, where

$$\sum_n \mu(B_n)\begin{cases}<\infty\\ =\infty\end{cases} \iff \mathcal{H}_{\text{i.o.}}(\mathcal{B}) \text{ has } \begin{cases}\text{zero}\\ \text{full}\end{cases} \text{ measure,} \tag{1.2}$$

are referred to as dynamical Borel-Cantelli lemmas.

The earliest result of this type is due to Kurzweil [Ku]. He proved that for $X=[0,1]$ and T a rotation by α, (1.2) holds for any sequence of *nested* intervals ($B_1\supset B_2\supset\ldots$) if and only if α is badly approximable. Later, there was an important paper of Philipp [P] in which it is shown that (1.2) holds in the cases where $X=[0,1]$, $\mathcal{B}$ consists of (not necessarily nested) intervals, and T is either the map $x\mapsto\beta x \bmod 1$ or the Gauß map $x\mapsto 1/x \bmod 1$. See e.g. [S, KM, CK, HNPV, Ke, KY] for further results, and [A] for a survey.

Let us say that $(X,\mu,T,\mathcal{B})$ is a *shrinking target system* if the sets B_n are nested[1] and

$$\lim_{n\to\infty}\mu(B_n)=0. \tag{1.3}$$

For $m\in\mathbb{N}$, write $O_m(x) := \{Tx, T^2x,\ldots,T^mx\}$ for the m-th orbit segment of a point $x\in X$ under the transformation T. Certainly, if x belongs to $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ then $O_m(x)\cap B_m\neq\varnothing$ for infinitely many m. On the other hand, if $O_m(x)\cap B_m\neq\varnothing$ for infinitely many m then either $x\in\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ or $T^mx\in\bigcap_{n\in\mathbb{N}}B_n$ for some m. Thus, under the additional assumption (1.3), $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$

[1] We remark that shrinking target systems with a nested sequence of targets are sometimes also referred to as *monotone shrinking target systems* in the literature.

coincides almost everywhere with the set

$$\{x \in X : O_m(x) \cap B_m \neq \varnothing \text{ infinitely often}\}. \tag{1.4}$$

In this paper, we study a natural variation of the set defined in (1.4). Following the terminology introduced by Kelmer [Ke], we define the *eventually always hitting set* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ to be the set of $x \in X$ such that for all but finitely many $m \in \mathbb{N}$ there exists $n \in \{1, \dots, m\}$ such that $T^n x \in B_m$. Equivalently,

$$\mathcal{H}_{\text{e.a.}}(\mathcal{B}) := \{x \in X : O_m(x) \cap B_m \neq \varnothing \text{ eventually always}\}. \tag{1.5}$$

By comparing (1.4) and (1.5), we see that up to a set of measure zero the eventually always hitting property is a strengthening of (1.1). It is also not hard to show that in any ergodic shrinking target system, the set of eventually always hitting points obeys a zero–one law (see Proposition 2.1 and Corollary 2.2 below). It is therefore natural to ask:

> Under what conditions on the shirking rate of the size of the targets in $\mathcal{B}$ can one expect $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ to have zero or full measure respectively?

This question has already been addressed for certain special classes of shrinking target systems. Bugeaud and Liao [BL] looked at maps $x \mapsto \beta x \bmod 1$ on $X = [0, 1]$ and computed the Hausdorff dimension of sets $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ for families of rapidly shrinking targets $\mathcal{B}$. In the set-up of [Ke], X is the unit tangent bundle of a finite volume hyperbolic manifold of constant negative curvature, T is the time-one map of the geodesic flow on X, and $\mathcal{B}$ consists of rotation-invariant subsets of X. Under these conditions, it was shown that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure whenever the series $\sum_{j=1}^{\infty} \frac{1}{2^j \mu(B_{2^j})}$ diverges. This was later generalized by Kelmer and Yu [KY] to higher rank homogeneous spaces, and by Kelmer and Oh [KO] to the set-up of actions on geometrically finite hyperbolic manifolds of infinite volume. More recently, several results in this direction were obtained by Kirsebom, Kunde and Persson [KKP] for some classes of interval maps, including the doubling map, some quadratic maps, the Gauß map, and the Manneville-Pomeau map. See comments after Corollaries 1.5 and 1.7 below for a comparison of some results from [KKP] with our results.

1.1. The main technical result

Our main technical result concerns systems whose targets satisfy a long-term independence property that arises in connection with rapid mixing. In such cases, we give sufficient conditions for the set of eventually always hitting points to either have zero or full measure. The class of systems to which this applies contains several relevant examples, such as product systems, Bernoulli schemes and the Gauß map.

The long-term independence property that we impose in our theorem asserts, roughly speaking, that any target $B_m \in \mathcal{B}$ becomes "evenly spread out" under the transformation T in the sense that $\mu(B_n \cap T^{-k} B_m) \approx \mu(B_n)\mu(B_m)$ for all $k \geqslant k(n,m)$, where $k(n,m)$ depends on n and m. The precise formulation is more technical and involves

$$\begin{gathered}\Xi_{n,m} := \text{ the algebra of subsets of } X \\ \text{generated by } \{T^{-j}B_i : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}.\end{gathered} \tag{1.6}$$

It states the following:

$$\begin{aligned}&\text{For all } m,n \in \mathbb{N} \text{ with } n \leqslant m, \text{ all } A \in \Xi_{n,m}, \text{ and all } B \in \Xi_{m,m} \\ &\text{one has } \big|\mu\big(A \cap T^{-(n+F(m))}B\big) - \mu(A)\mu(B)\big| \leqslant \eta(m)\mu(A)\mu(B),\end{aligned} \tag{1.7}$$

where $\eta\colon \mathbb{N} \to [0,1]$ is some function satisfying $\lim_{m\to\infty} \eta(m) = 0$, and $F\colon \mathbb{N} \to \mathbb{N}$ is another function satisfying

$$F(m) \leqslant \frac{1}{(\log m)^{1+\delta}\mu(B_m)} \text{ for some } \delta > 0 \text{ and all large enough } m \in \mathbb{N}. \tag{1.8}$$

We also define the set

$$E_m \coloneqq \{x \in X : O_m(x) \cap B_m = \varnothing\}, \tag{1.9}$$

which describes the collection of all points in X for which none of the first m iterates under the transformation T visits the target B_m. Note that

$$X \smallsetminus \mathcal{H}_{\text{e.a.}}(\mathcal{B}) = \limsup E_n \coloneqq \textstyle\bigcap_{n\in\mathbb{N}} \bigcup_{m\geqslant n} E_m. \tag{1.10}$$

Theorem 1.1. *Let $(X, \mu, T, \mathcal{B})$ be a shrinking target system satisfying (1.7). If*

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} < \infty$$

for some $\varepsilon > 0$*, then* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has full measure. On the other hand, if*

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \infty,$$

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has zero measure.*

1.2. Product systems

For our first application of Theorem 1.1, fix an arbitrary probability space (Y, ν), and let $A_1 \supset A_2 \supset \ldots$ be a sequence of measurable subsets of Y with $\nu(A_n) \to 0$ as $n \to \infty$. Consider the shrinking target system $(X, \mu, T, \mathcal{B})$, where $X := Y^{\mathbb{N}\cup\{0\}}$, $\mu := \nu^{\otimes \mathbb{N}\cup\{0\}}$, $T\colon X \to X$ denotes the left shift, and the shrinking targets $\mathcal{B} := \{B_1 \supset B_2 \supset \ldots\}$ are defined as $B_n := \{x \in X : x[0] \in A_n\}$. The elements in $\mathcal{B}$ have the convenient property that

$$\mu(B_n \cap T^{-k} B_m) = \mu(B_n)\mu(B_m), \quad \forall\, k, n, m \in \mathbb{N}, \tag{1.11}$$

which immediately implies that the shrinking target system $(X, \mu, T, \mathcal{B})$ satisfies condition (1.7) with $\eta(m) = 0$ and $F(m) = 0$ for all $m \in \mathbb{N}$.

Theorem 1.2. *Let* $(X, \mu, T, \mathcal{B})$ *be the shrinking target system described above. If*

$$\sum_{n=1}^{\infty} \frac{\big(1 - \mu(B_n)\big)^{n(1-\varepsilon)}}{n} < \infty$$

for some $\varepsilon > 0$*, then* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has full measure. On the other hand, if*

$$\sum_{n=1}^{\infty} \frac{\big(1 - \mu(B_n)\big)^{n}}{n} = \infty,$$

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has zero measure.*

From Theorem 1.2 one can derive the following corollary.

Corollary 1.3. *Let* $(X, \mu, T, \mathcal{B})$ *be as in Theorem 1.2. Suppose that there exists* $C > 1$ *such that for all but finitely many* m *one has*

$$\mu(B_m) \geqslant \frac{C \log\log m}{m};$$

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has full measure. If, on the other hand,*

$$\mu(B_m) \leqslant \frac{\log\log m}{m}$$

for all but finitely many m*, then* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has zero measure.*

1.3. Bernoulli schemes

Another class of systems that satisfy (1.7) for a natural choice of shrinking targets are Bernoulli schemes. Let (X, T) denote the full symbolic shift in 2 letters[2], that is, $X := \{0,1\}^{\mathbb{N}\cup\{0\}}$. Let $T\colon X \to X$ be the left shift on X, and denote by μ the $(1/2, 1/2)$-Bernoulli measure on X. Given a non-decreasing unbounded sequence of indices $(r_m)_{m\in\mathbb{N}}$, consider the corresponding sequence of shrinking targets $\mathcal{B} = \{B_1 \supset B_2 \supset \cdots\}$ defined as

$$B_m := \big\{x \in X : x[0] = x[1] = \cdots = x[r_m - 1] = 0\big\}, \qquad \forall\, m \in \mathbb{N}. \tag{1.12}$$

Note that $\mu(B_m) = 2^{-r_m}$. It is then straightforward to verify that the resulting shrinking target system $(X, \mu, T, \mathcal{B})$ satisfies condition (1.7) with $\eta(m) = 0$ and $F(m) = r_m$ for all $m \in \mathbb{N}$.

Theorem 1.4. *Let* $(X, \mu, T, \mathcal{B})$ *be as above, and assume that either one of the following two conditions is satisfied:*

$$\exists\, D > 2 \text{ such that } \mu(B_m) \geqslant \tfrac{D\log\log m}{m} \text{ for all but finitely many } m \in \mathbb{N}; \tag{1.13}$$

$$\exists\, \tau > 1 \text{ such that } \mu(B_m) \leqslant \tfrac{1}{(\log m)^{\tau}} \text{ for all but finitely many } m \in \mathbb{N}. \tag{1.14}$$

If

$$\sum_{n=1}^{\infty} \frac{\big(1 - \mu(B_n)\big)^{\frac{n(1-\varepsilon)}{2}}}{n} < \infty \tag{1.15}$$

for some $\varepsilon > 0$*, then* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has full measure. On the other hand, if*

$$\sum_{n=1}^{\infty} \frac{\big(1 - \mu(B_n)\big)^{\frac{n}{2}}}{n} = \infty, \tag{1.16}$$

[2] The same results hold for shifts on $\{0, \ldots, b-1\}^{\mathbb{N}\cup\{0\}}$ for any integer $b > 2$; we chose to restrict ourselves to the case $b = 2$ to simplify the presentation.

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has zero measure.*

Note that (1.13) implies $\sum_{n=1}^{\infty} \frac{1}{n}\big(1-\mu(B_n)\big)^{\frac{n(1-\varepsilon)}{2}} < \infty$, which means that if we are in case (1.13), then automatically $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure. This observation is part of a dichotomy that is described by the following analogue of Corollary 1.3.

Corollary 1.5. *Let* $(X, \mu, T, \mathcal{B})$ *be as in Theorem 1.4. If* (1.13) *holds, then* $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has full measure. If, on the other hand,*

$$\mu(B_m) \leqslant \frac{2\log\log m}{m} \quad \text{for all but finitely many } m \in \mathbb{N}, \tag{1.17}$$

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ *has zero measure.*

We remark that it is proved in [KKP, Theorem 1] that if (X, μ, T) is as in Theorem 1.4 and $\mathcal{B} = \{B_m\}$ is a family of nested intervals, then

$$\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = \begin{cases} 1 & \text{if } \mu(B_m) \geq \frac{c(\log m)^2}{m} \quad \text{for some } c > 0 \text{ and large enough } m; \\ 0 & \text{if } \mu(B_m) \leq \ c/m \quad\quad \text{for some } c > 0 \text{ and large enough } m. \end{cases}$$

Note that the above inequalities are significantly stronger than (1.13) and (1.17) respectively; on the other hand, our method is applicable only to specific families of targets given by (1.12).

1.4. The Gauß map

The *Gauß map* is the map T on the interval $X := [0,1]$ defined as

$$T(x) := \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

There is an explicit T-invariant Borel probability measure on $[0,1]$ called the *Gauß measure* (cf. [EW, Lemma 3.5]):

$$\mu(B) := \frac{1}{\log 2} \int_B \frac{\mathsf{d}x}{1+x}, \qquad \text{for all measurable } B \subset [0,1].$$

The Gauß map and the Gauß measure are tightly connected to the theory of continued fractions. Any irrational number $x \in [0,1]$ has a unique *simple* *continued fraction expansion*

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \qquad a_1, a_2, \ldots \in \mathbb{N},$$

which we write as $[a_1, a_2, \ldots]$. Note that if $x = [a_1, a_2, \ldots]$, then $T(x) = [a_2, a_3, \ldots]$. Thus T acts as the left shift on the continued fraction representation of a number. This identification leads us to a natural shrinking target problem where the targets are determined by digit restrictions in the continued fraction expansion. Let $(k_m)_{m\in\mathbb{N}}$ be a non-decreasing sequence of natural numbers, and consider the sequence of shrinking targets $\mathcal{B} = \{B_1 \supset B_2 \supset \ldots\}$ given by

$$B_m := \{[a_1, a_2, \ldots] : a_1 \geqslant k_m\} = [0, 1/k_m] \tag{1.18}$$

for all $m \in \mathbb{N}$;[3] note that

$$\mu(B_m) = \frac{\log(1 + 1/k_m)}{\log 2}. \tag{1.19}$$

We show in Section 6 that the shrinking target system $(X, \mu, T, \mathcal{B})$ satisfies condition (1.7) for any $F(m)$ that satisfies (1.8) and $\eta(m) = \mathrm{O}\Big(\exp\big(-C\sqrt{F(m)}\,\big)\Big)$ for some universal constant $C > 0$. Combining this with Theorem 1.1 allows us to derive the following result.

Theorem 1.6. *Let $(X, \mu, T, \mathcal{B})$ be as described above, and assume that either there exists $\sigma < 1$ such that $k_m \leqslant \frac{\sigma m}{\log\log m}$ for all but finitely many $m \in \mathbb{N}$, or there exists $\tau > 0$ such that $k_m \geqslant (\log m)^\tau$ for all but finitely*

[3]Technically speaking, the equality in (1.18) is incorrect as written and should instead be $\{[a_1, a_2, \ldots] : a_1 \geqslant k_m\} = [0, 1/k_m] \setminus \mathbb{Q}$, because the set in the left-hand side consists of infinite continued fraction expansions which only yield irrational numbers. However, since all shrinking target problems that we consider are insensitive to adding or removing a zero-measure set from the targets or the underlying space, we brush aside this issue for the sake of cleaner notation and simpler expressions.

many $m \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} \frac{\big(1-\mu(B_n)\big)^{(\log 2)(1-\varepsilon)n}}{n} < \infty$$

for some $\varepsilon > 0$, then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure. On the other hand, if

$$\sum_{n=1}^{\infty} \frac{\big(1-\mu(B_n)\big)^{2(\log 2)(1+\varepsilon)n}}{n} = \infty,$$

for some $\varepsilon > 0$, then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has zero measure.

Corollary 1.7. *Let $(X, \mu, T, \mathcal{B})$ be as described above. If there exists $C_1 > 1$ such that for all but finitely many m one has*

$$\log(1 + 1/k_m) \geqslant \frac{C_1 \log\log m}{m},$$

then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure. On the other hand, if there exists $C_2 < 1/2$ such that

$$\log(1 + 1/k_m) \leqslant \frac{C_2 \log\log m}{m}$$

for all but finitely many m, then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has zero measure.

We remark that [KKP, Theorem 3], which asserts that

$$\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = \begin{cases} 1 & \text{if } k_m = \frac{cm}{(\log m)^2} \text{ where } c > 0 \text{ is sufficiently small,} \\ 0 & \text{if } k_m = cm \quad\quad\;\; \text{for any } c > 0, \end{cases}$$

is a consequence of the above corollary.

Remark 1.8. During the period of revision for this paper, Theorems 1.4 and 1.6 have already been improved upon by other authors. In the recent preprint [HKKP] it is shown that if (X, μ, T) is as in Theorem 1.4 and the targets $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ are as in (1.12) with the additional assumption that $n \mapsto n\mu(B_n)$ is eventually non-decreasing, then

(1.20)

$$\sum_n \mu(B_n) e^{-\frac{n}{2}\mu(B_n)} \begin{cases} < \infty \\ = \infty \end{cases} \iff \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \text{ has } \begin{cases} \text{full} \\ \text{zero} \end{cases} \text{measure.}$$

The comparison between the convergence conditions in (1.20) and Theorem 1.4 provides an interesting insight. To see that the conditions given in

(1.20) conform to and, in fact, improve upon the conditions in Theorem 1.4, we must show that

$$\text{(1.21)} \qquad \sum_n \frac{\big(1-\mu(B_n)\big)^{\frac{n}{2}}}{n} = \infty \quad \Longrightarrow \quad \sum_n \mu(B_n) e^{-\frac{n}{2}\mu(B_n)} = \infty,$$

as well as

$$\text{(1.22)} \qquad \sum_n \frac{\big(1-\mu(B_n)\big)^{\frac{n(1-\varepsilon)}{2}}}{n} < \infty \quad \Longrightarrow \quad \sum_n \mu(B_n) e^{-\frac{n}{2}\mu(B_n)} < \infty.$$

We will do so under the simplifying assumption that $\frac{c}{n} \leqslant \mu(B_n)$ for some $c \in (0,1)$ and all large enough n. This assumption is not overly restrictive, because if, on the contrary, $\mu(B_n) < \frac{c}{n}$ for some $c \in (0,1)$ and infinitely many n, then it is well known that we are in the $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 0$ case (see [Ke, Proposition 12]).

To prove (1.21), note that from the basic inequality $(1-x) \leqslant e^{-x}$, which holds for all $x \in \mathbb{R}$, we immediately get $\big(1-\mu(B_n)\big)^{\frac{n}{2}} \leqslant e^{-\frac{n}{2}\mu(B_n)}$. In view of $\frac{c}{n} \leqslant \mu(B_n)$, this implies $\frac{1}{n}\big(1-\mu(B_n)\big)^{\frac{n}{2}} \leqslant c^{-1}\mu(B_n)e^{-\frac{n}{2}\mu(B_n)}$ and the claim follows.

For the proof of (1.22), we utilize the comparison test to show that convergence of the left series implies convergence of the right series. It suffices to show

$$\limsup_{n\to\infty} \frac{\mu(B_n)e^{-\frac{n}{2}\mu(B_n)}}{\frac{1}{n}\big(1-\mu(B_n)\big)^{\frac{n(1-\varepsilon)}{2}}} \leqslant 1.$$

Since $n^{2/n} \to 1$ and $\mu(B_n) \leqslant 1$, the above is implied by

$$\limsup_{n\to\infty} \frac{e^{-\mu(B_n)}}{\big(1-\mu(B_n)\big)^{1-\varepsilon}} \leqslant 1.$$

This follows from the inequality $e^{-x} \leqslant (1-x)^{1-\varepsilon}$, which holds for all sufficiently small positive x.

Results similar to (1.20) are also established in [HKKP] for more general interval maps T of $X = [0,1]$ with a Gibbs measure μ on X and some additional regularity conditions (see [HKKP, Theorems 3.2 and 4.1]). In parallel, results about the Gauß map were also obtained (see [HKKP, paragraph after Corollary 2.5]): If $X = [0,1]$, T is the Gauß map, μ the Gauß measure, the targets $\mathcal{B}$ are as in (1.18), and $n \mapsto n\mu(B_n)$ is eventually non-decreasing, then

$$\sum_n \mu(B_n)e^{-n\mu(B_n)} \begin{cases} < \infty \\ = \infty \end{cases} \iff \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \text{ has } \begin{cases} \text{full} \\ \text{zero} \end{cases} \text{ measure.}$$

It seems plausible that the methods of [HKKP] are applicable to the set-up of §1.2, and thus Theorem 1.2 can also be upgraded to a necessary and sufficient condition.

2. General properties of $\mathcal{H}_{\text{e.a.}}$ sets

Before embarking on the proofs of Theorems 1.1, 1.2, 1.4, and 1.6, we gather in this section some general results regarding $\mathcal{H}_{\text{e.a.}}$ sets that apply to all ergodic shrinking target systems. In Subsection 2.1 below, we show that all $\mathcal{H}_{\text{e.a.}}$ sets obey a zero–one law. Thereafter, in Subsections 2.2 and 2.3 we present general necessary and sufficient conditions for $\mathcal{H}_{\text{e.a.}}$ sets to have full measure.

2.1. The zero–one law for eventually always hitting points

We begin with showing that $\mathcal{H}_{\text{e.a.}}$ sets are essentially invariant, a result that was obtained independently in [KKP].

Proposition 2.1. *Let $(X, \mu, T, \mathcal{B})$ be a (not necessarily ergodic) shrinking target system. Then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ is essentially invariant under T, that is,*

$$\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B}) \triangle T^{-1}\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 0.$$

Proof. Let $Y := \{x \in X : Tx \in B_n \text{ for infinitely many } n\}$. Since B_n are nested, we have that $Y = \bigcap_{n\in\mathbb{N}} T^{-1}B_n$ and, using $\mu(B_n) \to 0$ as $n \to \infty$, it follows from the monotone convergence theorem that Y has zero measure.

We now claim that if $x \in \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \setminus Y$, then $Tx \in \mathcal{H}_{\text{e.a.}}(\mathcal{B})$. To verify this claim recall that

$$x \in \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \iff O_n(x) \cap B_n \neq \varnothing \text{ for all but finitely many } n.$$

Note also that $O_n(x) = \{Tx\} \cup O_{n-1}(Tx)$. Therefore, if $x \notin Y$, then $O_n(x) \cap B_n$ is non-empty for cofinitely many n if and only if $O_{n-1}(Tx) \cap B_n \neq \varnothing$ for cofinitely many n. Hence

$$x \in \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \setminus Y \implies O_n(x) \cap B_n \neq \varnothing \text{ for all but finitely many } n$$

$$\begin{aligned} &\implies O_{n-1}(Tx)\cap B_n \neq \varnothing \text{ for all but finitely many } n\\ &\implies O_{n-1}(Tx)\cap B_{n-1} \neq \varnothing \text{ for all but finitely many } n\\ &\implies O_{n}(Tx)\cap B_n \neq \varnothing \text{ for all but finitely many } n, \end{aligned}$$

where in the second to last implication we have used that $B_n \subset B_{n-1}$. This proves that if $x \in \mathcal{H}_{\text{e.a.}}(\mathcal{B}) \setminus Y$ then $Tx \in \mathcal{H}_{\text{e.a.}}(\mathcal{B})$. Therefore $\mathcal{H}_{\text{e.a.}}(\mathcal{B}) \setminus Y \subset T^{-1}\mathcal{H}_{\text{e.a.}}(\mathcal{B})$. Since $\mu(Y) = 0$ and T is measure preserving, we conclude that

$$\mu\Big(\mathcal{H}_{\text{e.a.}}(\mathcal{B}) \triangle T^{-1}\mathcal{H}_{\text{e.a.}}(\mathcal{B})\Big) = 0.$$

This finishes the proof. □

In the presence of ergodicity, all essentially invariant sets are trivial. Therefore Proposition 2.1 implies the following corollary.

Corollary 2.2. *If $(X, \mu, T, \mathcal{B})$ is an ergodic shrinking target system, then $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ is either a null set or a co-null set.*

2.2. General sufficient condition for $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 1$

For $m, n \in \mathbb{N}$ define

$$E_{n,m} := \{x : O_n(x) \cap B_m = \varnothing\}, \tag{2.1}$$

which can also be written as

$$E_{n,m} = \bigcap_{i=1}^{n} T^{-i} B_m^c. \tag{2.2}$$

Note that $E_{n,m} \in \Xi_{n,m}$ for all $m, n \in \mathbb{N}$, and the sets E_m defined in (1.9) coincide with $E_{m,m}$. The following result is taken from [Ke] and plays an important role in our proof of Theorem 1.1:

Lemma 2.3 ([Ke, Lemma 13]). *Suppose there exists a non-decreasing sequence m_j such that $\sum_{j=0}^{\infty} \mu(E_{m_{j-1},m_j}) < \infty$. Then $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 1$.*

2.3. General necessary condition for $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 1$

The next result establishes a necessary condition for $\mathcal{H}_{\text{e.a.}}$ sets to have full measure, conditional under the assumption that the sets E_m are asymptotically independent.

Theorem 2.4. *Let $(m_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence and $(X,\mu,T,\mathcal{B})$ a shrinking target system with the property that*

$$\mu(E_{m_s}\cap E_{m_t}) \leqslant \big(1+\mathrm{o}_{t\to\infty}(1)\big)\mu(E_{m_s})\mu(E_{m_t})^{1-2^{s-t+2}} + \mathrm{O}\big(\mu(E_{m_s})v_t\big) \tag{2.3}$$

where $(v_t)_{t\in\mathbb{N}}$ is a sequence of non-negative numbers satisfying $\sum_{t\in\mathbb{N}} v_t < \infty$. If $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big)=1$, then necessarily $\sum_{j=1}^{\infty}\mu(E_{m_j})<\infty$.

For the proof of Theorem 2.4 we need the following lemma.

Lemma 2.5. *Let $0<q_i<1$ for $i=1,\dots,N$. Then*

$$\sum_{m>n}^{N}\left(q_n q_m^{1-2^{n-m+2}} - q_n q_m\right) = \mathrm{O}\left(\sum_{n=1}^{N} q_n\right).$$

Proof. Recall Bernoulli's inequality, which asserts that $(1+y)^r-1\leqslant ry$ for all $r\in(0,1)$ and $y>-1$. If we apply this inequality with $y=\frac{1}{q_m}-1$ and $r=2^{n-m+2}$ we obtain

$$\left(\tfrac{1}{q_m}\right)^{2^{n-m+2}}-1 \;\leqslant\; 2^{n-m+2}\left(\tfrac{1}{q_m}-1\right) \;\leqslant\; \frac{1}{2^{m-n-2}q_m}.$$

This gives

$$\begin{aligned}\sum_{m>n}^{N}\left(q_n q_m^{1-2^{n-m+2}} - q_n q_m\right) &= \sum_{m>n}^{N} q_n q_m\left(\left(\tfrac{1}{q_m}\right)^{2^{n-m+2}}-1\right)\\ &\leqslant \sum_{m>n}^{N}\frac{q_n}{2^{m-n+2}} = \mathrm{O}\left(\sum_{n=1}^{N} q_n\right).\end{aligned}$$

□

Proof of Theorem 2.4. By way of contradiction, assume that $\sum_{j=1}^{\infty}\mu(E_{m_j})=\infty$. Let $1_j=1_{E_{m_j}}$ denote the indicator function of E_{m_j}, and define $q_j\coloneqq\mu(E_{m_j})$. Consider the normalized deviation

$$D_N=\frac{\sum_{j=1}^{N}1_j}{\sum_{j=1}^{N}q_j}-1.$$

Its L^2-norm is

$$\begin{aligned}
\|D_N\|_2^2 &= \frac{2\sum_{t>s}^{N}\langle 1_s - q_s, 1_t - q_t\rangle}{\left(\sum_{j=1}^{N} q_j\right)^2} + \frac{\sum_{j=1}^{N}\langle 1_j - q_j, 1_j - q_j\rangle}{\left(\sum_{j=1}^{N} q_j\right)^2}\\
&= \frac{2\sum_{t>s}^{N}\langle 1_s - q_s, 1_t - q_t\rangle}{\left(\sum_{j=1}^{N} q_j\right)^2} + \mathrm{o}_{N\to\infty}(1)\\
&= \frac{2\sum_{t>s}^{N}\left(\mu(E_{m_s}\cap E_{m_t}) - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2} + \mathrm{o}_{N\to\infty}(1).
\end{aligned}$$

Fix $\varepsilon > 0$. As guaranteed by (2.3), there exists $M \in \mathbb{N}$ such that for all $s, t \in \mathbb{N}$ with $t \geqslant M$ one has

$$\mu(E_{m_s}\cap E_{m_t}) \leqslant (1+\varepsilon) q_s q_t^{1-2^{s-t+2}} + \mathrm{O}(q_s v_t).$$

Hence

$$\begin{aligned}
\|D_N\|_2^2 &\leqslant \frac{2\sum_{M<s<t<N}\left(q_s q_t^{1-2^{s-t+2}}(1+\varepsilon) - q_s q_t + \mathrm{O}(q_s v_t)\right)}{\left(\sum_{j=1}^{N} q_j\right)^2} + \mathrm{o}_{N\to\infty}(1)\\
&= \frac{2(1+\varepsilon)\sum_{M<s<t<N}\left(q_s q_t^{1-2^{s-t+2}} - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2} +\\
&\quad + \frac{2\varepsilon\sum_{M<s<t<N} q_s q_t}{\left(\sum_{j=1}^{N} q_j\right)^2} + \mathrm{O}\left(\frac{\sum_{M<s<t<N} q_s v_t}{\left(\sum_{j=1}^{N} q_j\right)^2}\right) + \mathrm{o}_{N\to\infty}(1)\\
&\leqslant \frac{2(1+\varepsilon)\sum_{M<s<t<N}\left(q_s q_t^{1-2^{s-t+2}} - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2} + \varepsilon + \mathrm{O}\left(\frac{\sum_{t=1}^{N} v_t}{\sum_{j=1}^{N} q_j}\right) + \mathrm{o}_{N\to\infty}(1).
\end{aligned}$$

Since by assumption $\sum_{j=1}^{\infty} q_j = \infty$ and $\sum_{t=1}^{\infty} v_t < \infty$, the term $\mathrm{O}\left(\frac{\sum_{t=1}^{N} v_t}{\sum_{j=1}^{N} q_j}\right)$ goes to 0 as $N \to \infty$. Also, using Lemma 2.5, we can control the term

$$\frac{2(1+\varepsilon)\sum_{M<s<t<N}\left(q_s q_t^{1-2^{s-t+2}} - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2}.$$

Indeed,

$$\begin{aligned}
\frac{2(1+\varepsilon)\sum_{M<s<t<N}\left(q_s q_t^{1-2^{s-t+2}} - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2} &\leqslant \frac{2(1+\varepsilon)\sum_{1<s<t<N}\left(q_s q_t^{1-2^{s-t+2}} - q_s q_t\right)}{\left(\sum_{j=1}^{N} q_j\right)^2}\\
= \mathrm{O}\left(\frac{2(1+\varepsilon)\left(\sum_{j=1}^{N} q_j\right)}{\left(\sum_{j=1}^{N} q_j\right)^2}\right) &= \mathrm{O}\left(\frac{1}{\sum_{j=1}^{N} q_j}\right) = \mathrm{o}_{N\to\infty}(1).
\end{aligned}$$

This proves that $\|D_N\|_2^2 \leqslant \varepsilon + \mathrm{o}_{N\to\infty}(1)$. Since ε was chosen arbitrarily, we obtain $\|D_N\|_2^2 = \mathrm{o}_{N\to\infty}(1)$. The decay of the L^2-norm of D_N implies that $\limsup E_{m_j}$ has full measure. Therefore $\mu(\limsup E_n) = 1$, which, in view of (1.10), contradicts $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 1$. □

3. Proof of the main technical result

This section is dedicated to the proof of Theorem 1.1 and is divided into three subsections. In Subsection 3.1 we study the asymptotic behavior of $\mu(E_{m_{j-1},m_j})$ for certain lacunary sequences (m_j), which is needed for the proof of Theorem 1.1 in combination with Lemma 2.3. In Subsection 3.2 we proceed to study the asymptotic independence of E_{m_j} along dyadic sequences (m_j), which we need for the application of Theorem 2.4. Finally, in Subsection 3.3, we combine all these results to finish the proof of Theorem 1.1.

3.1. Estimates for the measure of E_{m_{j-1},m_j}

Proposition 3.1. *Let $(X,\mu,T,\mathcal{B})$ be a shrinking target system satisfying (1.7). Let $m,n,k \in \mathbb{N}$ with $kn \leqslant m$. Then*

$$\mu(E_{kn,m}) = \big(1 + \mathrm{o}_{m\to\infty}(1)\big)\,\mu(E_{n,m})^k + \mathrm{O}\big(kF(m)\mu(B_m)\big). \tag{3.1}$$

For the proof of Proposition 3.1 it will be convenient to write $E^*_{n,m}$ for the set

$$E^*_{n,m} := \begin{cases} T^{-F(m)}E_{n-F(m),m}, & \text{if } n > F(m),\\ X, & \text{otherwise.}\end{cases} \tag{3.2}$$

Note that $E^*_{n,m}$ always contains $E_{n,m}$ as a subset. This inclusion follows quickly from the definition of $E_{n,m}$ (see (2.1) and (2.2)), because

$$E_{n,m} = \bigcap_{j=1}^{n} T^{-j}B_m^c \subset \bigcap_{j=F(m)+1}^{n} T^{-j}B_m^c = T^{-F(m)}E_{n-F(m),m} = E^*_{n,m}.$$

In general, this inclusion is proper, and the sets $E_{n,m}$ and $E^*_{n,m}$ are not identical. However they are approximately the same. Indeed, since we are only interested in the case where the quantity $F(m)$ is much smaller than m, the difference in measure between $E_{n,m}$ and $E^*_{n,m}$ becomes negligible (as we will see in the proofs of Proposition 3.1 and Lemma 3.4 below). For that reason, we suggest to think of $E^*_{n,m}$ as an approximation of $E_{n,m}$.

The advantage of using $E^*_{n,m}$ over $E_{n,m}$ is that for any $\ell \in \mathbb{N}$ with $\ell \geqslant n - F(m)$ and any set $C \in \Xi_{\ell,m}$ one has

$$\left|\mu(C \cap T^{-\ell} E^*_{n,m}) - \mu(C)\mu(E^*_{n,m})\right| \leqslant \eta(m)\mu(C)\mu(E^*_{n,m}), \tag{3.3}$$

which follows directly from (1.7) by choosing $A = C$ and $B = E_{n-F(m),m}$.

Proof of Proposition 3.1. Recall that $E_{n,m} = \bigcap_{i=1}^{n} T^{-i} B^c_m$. We can split off the first $F(m)$ terms in this intersection and thus write $E_{n,m}$ as the intersection of two sets,

$$E_{n,m} = \underbrace{\bigcap_{i=1}^{\min\{n,F(m)\}} T^{-i} B^c_m}_{R} \cap E^*_{n,m}, \tag{3.4}$$

where $E^*_{n,m}$ is as defined in (3.2). We can think of $E^*_{n,m}$ as the “main part” of $E_{n,m}$ and of R as the “remainder”. Since

$$E_{kn,m} = \bigcap_{j=1}^{kn} T^{-j} B^c_m = \bigcap_{j=0}^{k-1} T^{-jn} \left(\bigcap_{i=1}^{n} T^{-i} B^c_m \right) = \bigcap_{j=0}^{k-1} T^{-jn} E_{n,m},$$

we can now write

$$\mu(E_{kn,m}) = \mu\left(\bigcap_{j=0}^{k-1} T^{-jn} E_{n,m} \right) = \mu\left(R' \cap \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right),$$

where $R' := \bigcap_{j=0}^{k-1} T^{-jn} R$. From this it follows that

$$\mu(E_{kn,m}) \leqslant \mu\left(\bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right), \tag{3.5}$$

which provides us with a suitable upper bound on $\mu(E_{kn,m})$. We also want to find a good lower bound for $\mu(E_{kn,m})$. Observe that the measure of R can trivially be bounded from below by $1 - F(m)\mu(B_m)$. Therefore, we can bound the measure of R' from below by $1 - k\mu(R^c) \geqslant 1 - kF(m)\mu(B_m)$.

This gives the estimate

$$
\begin{aligned}
\mu(E_{kn,m}) &= \mu\left(R' \cap \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m}\right) \\
&\geqslant \mu\left(\bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m}\right) - kF(m)\mu(B_m).
\end{aligned} \tag{3.6}
$$

To finish the proof, we only have to apply (3.3) $(k-1)$ times to find that

$$
\mu\left(\bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m}\right) \geqslant \big(1-\eta(m)\big)^{k-1} \mu(E^*_{n,m})^k \tag{3.7}
$$

and

$$
\mu\left(\bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m}\right) \leqslant \big(1+\eta(m)\big)^{k-1} \mu(E^*_{n,m})^k. \tag{3.8}
$$

Finally, since $\mu(E^*_{n,m}) = \mu(E_{n,m}) + \mathrm{O}\big(F(m)\mu(B_m)\big)$, we obtain

$$
\mu(E^*_{n,m})^k = \mu(E_{n,m})^k + \mathrm{O}\big(kF(m)\mu(B_m)\big). \tag{3.9}
$$

Putting together (3.5), (3.6), (3.7), (3.8), and (3.9) proves (3.1). □

From Proposition 3.1 we can now derive the following corollary.

Corollary 3.2. *For any shrinking target system $(X, \mu, T, \mathcal{B})$ that satisfies (1.7) and any $m, n, k \in \mathbb{N}$ with $kn \leqslant m$,*

$$
\mu\left(E_{kn,m}\right) = \big(1 + \mathrm{o}_{m\to\infty}(1)\big)\mu(E_{(k+1)n,m})^{\frac{k}{k+1}} + \mathrm{O}\left(\big(kF(m)\mu(B_m)\big)^{\frac{k}{k+1}}\right).
$$

Proof. From Proposition 3.1,

$$
\begin{aligned}
\mu(E_{kn,m}) &= \big(1+\mathrm{o}(1)\big)\mu(E_{n,m})^k + \mathrm{O}\big(kF(m)\mu(B_m)\big) \\
&= \left(\big(1+\mathrm{o}(1)\big)\mu(E_{n,m})^{k+1}\right)^{\frac{k}{k+1}} + \mathrm{O}\big(kF(m)\mu(B_m)\big) \\
&= \Big(\big(1+\mathrm{o}(1)\big)\mu(E_{(k+1)n,m}) + \mathrm{O}\big(kF(m)\mu(B_m)\big)\Big)^{\frac{k}{k+1}} \\
&\quad + \mathrm{O}\big(kF(m)\mu(B_m)\big) \\
&= \big(1+\mathrm{o}(1)\big)\mu(E_{(k+1)n,m})^{\frac{k}{k+1}} + \mathrm{O}\left(\big(kF(m)\mu(B_m)\big)^{\frac{k}{k+1}}\right).
\end{aligned}
$$

This finishes the proof. □

Proposition 3.3. *Let $(X,\mu,T,\mathcal{B})$ be a shrinking target system satisfying (1.7). Let $k \geqslant 2$, and define*

$$m_j \coloneqq k \left\lfloor \frac{(k+1)^{\frac{j}{2}}}{k^{\frac{j}{2}}} \right\rfloor .$$

Then

$$\mu(E_{m_{j-1},m_j}) = \big(1+\mathrm{o}_{j\to\infty}(1)\big)\mu(E_{m_{j+1},m_j})^{\frac{k}{k+1}} + \mathrm{O}\left(\big(kF(m_j)\mu(B_{m_j})\big)^{\frac{k}{k+1}}\right).$$

Proof. Set $n_j \coloneqq \left\lfloor \frac{(k+1)^{\frac{j}{2}}}{k^{\frac{j}{2}}} \right\rfloor$. Then $m_{j-1} = kn_{j-1}$. Observe that

$$\begin{aligned} n_{j+1} - \tfrac{k+1}{k} n_{j-1} &= \left\lfloor \frac{(k+1)^{\frac{j+1}{2}}}{k^{\frac{j+1}{2}}} \right\rfloor - \tfrac{k+1}{k} \left\lfloor \frac{(k+1)^{\frac{j-1}{2}}}{k^{\frac{j-1}{2}}} \right\rfloor \\ &= -\left\{ \frac{(k+1)^{\frac{j+1}{2}}}{k^{\frac{j+1}{2}}} \right\} + \tfrac{k+1}{k} \left\{ \frac{(k+1)^{\frac{j-1}{2}}}{k^{\frac{j-1}{2}}} \right\} = \mathrm{O}(1), \end{aligned}$$

and therefore $kn_{j+1} - (k+1)n_{j-1} = \mathrm{O}(k)$.

Observe also that $(k+2)n_{j-1} = \frac{k+2}{k+1}\,\frac{k+1}{k}\, m_{j-1}$, and hence $|m_{j+1} - (k+2)n_{j-1}|$ is bounded from above by $2k$. It follows that $|m_{j+1} - (k+1)n_{j-1}| = \mathrm{O}(k)$, and hence

$$\mu(E_{m_{j+1},m_j}) = \mu(E_{(k+1)n_{j-1},m_j}) + \mathrm{O}\big(k\mu(B_{m_j})\big).$$

In view of Corollary 3.2 we obtain

$$\begin{aligned} \mu(E_{m_{j-1},m_j}) &= \mu(E_{kn_{j-1},m_j}) \\ &= \big(1+\mathrm{o}(1)\big)\mu(E_{(k+1)n_{j-1},m_j})^{\frac{k}{k+1}} + \mathrm{O}\left(\big(kF(m_j)\mu(B_{m_j})\big)^{\frac{k}{k+1}}\right) \\ &= \big(1+\mathrm{o}(1)\big)\left(\mu(E_{m_{j+1},m_j}) + \mathrm{O}\big(k\mu(B_{m_j})\big)\right)^{\frac{k}{k+1}} \\ &\quad + \mathrm{O}\left(\big(kF(m_j)\mu(B_{m_j})\big)^{\frac{k}{k+1}}\right) \\ &= \big(1+\mathrm{o}(1)\big)\mu(E_{m_{j+1},m_j})^{\frac{k}{k+1}} + \mathrm{O}\left(\big(kF(m_j)\mu(B_{m_j})\big)^{\frac{k}{k+1}}\right). \end{aligned}$$

This completes the proof. □

3.2. Independence of dyadic samples

Lemma 3.4. *Let $(X,\mu,T,\mathcal{B})$ be a shrinking target system satisfying* (1.7)*. For every $s\in\mathbb{N}$ let m_s be a number in $[2^s,2^{s+1})$. Then, for all $t>s$,*

$$\begin{aligned}(3.10)\qquad \mu(E_{m_s}\cap E_{m_t}) &\leqslant \big(1+\mathrm{o}_{t\to\infty}(1)\big)\mu(E_{m_s})\mu(E_{m_t})^{1-\frac{2^{s+2}}{2^t}}\\ &\quad+\mathrm{O}\Big(\mu(E_{m_s})\big(F(m_t)\mu(B_{m_t})\big)^{1-\frac{2^{s+2}}{2^t}}\Big).\end{aligned}$$

Proof. It follows from the definition of E_{m_s} and E_{m_t} (see (2.1) and (2.2)) and the fact that $B_{m_t}\subset B_{m_s}$ that

$$E_{m_s}\cap E_{m_t}=E_{m_s}\cap T^{-m_s}E_{m_t-m_s,m_t}.$$

Since $E_{m_t-m_s,m_t}$ is a subset of $E^*_{m_t-m_s,m_t}$, we trivially have

$$\begin{aligned}(3.11)\qquad \mu(E_{m_s}\cap E_{m_t}) &= \mu(E_{m_s}\cap T^{-m_s}E_{m_t-m_s,m_t})\\ &\leqslant \mu(E_{m_s}\cap T^{-m_s}E^*_{m_t-m_s,m_t}).\end{aligned}$$

It follows from (1.7) that

$$(3.12)\qquad \mu(E_{m_s}\cap T^{-m_s}E^*_{m_t-m_s,m_t})\leqslant\big(1+\eta(m_t)\big)\mu(E_{m_s})\mu(E^*_{m_t-m_s,m_t}).$$

Putting together (3.11) and (3.12) we obtain

$$(3.13)\qquad \mu(E_{m_s}\cap E_{m_t})\leqslant\big(1+\eta(m_t)\big)\mu(E_{m_s})\mu(E^*_{m_t-m_s,m_t}).$$

Let $k:=\lfloor m_t/m_s\rfloor-1$. Since (3.10) trivially holds for $t=s+1$, we can assume that $t>s+1$ and hence $k\neq0$. In light of (3.13) we see that for the proof of Lemma 3.4 it is beneficial to find a good upper bound on the measure of the set $E^*_{m_t-m_s,m_t}$, preferably in terms of the measure of E_{m_t}. In order to find such an upper bound, we will first prove the following inequality:

$$(3.14)\qquad \frac{\mu(E^*_{m_t-m_s,m_t})^{\frac{1}{k}}}{1+\eta(m_t)}\leqslant\mu(E^*_{m_s,m_t}).$$

Since $km_j\leqslant m_t-m_s$, the set $E^*_{m_t-m_s,m_t}$ is a subset of $E^*_{km_s,m_t}$ and hence $\mu(E^*_{m_t-m_s,m_t})\leqslant\mu(E^*_{km_s,m_t})$. Therefore, (3.14) is implied by

$$(3.15)\qquad \frac{\mu(E^*_{km_s,m_t})^{\frac{1}{k}}}{1+\eta(m_t)}\leqslant\mu(E^*_{m_s,m_t}).$$

Note that

$$
\begin{aligned}
E^*_{km_s,m_t} &= \bigcap_{i=F(m_t)+1}^{km_s} T^{-i}B_{m_t}\\
&= \bigcap_{i=F(m_t)+1}^{m_s} T^{-i}B_{m_t} \cap \bigcap_{i=m_s+1}^{2m_s} T^{-i}B_{m_t} \cap \cdots \cap \bigcap_{i=(k-1)m_s+1}^{km_s} T^{-i}B_{m_t}.
\end{aligned}
$$

Also observe that for any $\ell \in \{1, \ldots, k-1\}$,

$$
\bigcap_{i=\ell m_s+1}^{(\ell+1)m_s} T^{-i}B_{m_t} \ \subset\ T^{-\ell m_s}\left(\bigcap_{i=F(m_t)+1}^{m_s} T^{-i}B_{m_t}\right) = T^{-\ell m_s}E^*_{m_s,m_t}.
$$

This proves that

$$
E^*_{km_s,m_t} \ \subset\ \bigcap_{\ell=0}^{k-1} T^{-\ell m_s}E^*_{m_s,m_t}.
$$

If we now apply property (1.7) to $\mu\left(\bigcap_{\ell=0}^{k-1} T^{-\ell m_s}E^*_{m_s,m_t}\right)$ $(k-1)$ times, then we see that

$$
\mu(E^*_{km_s,m_t}) \leqslant \mu\left(\bigcap_{\ell=0}^{k-1} T^{-\ell m_s}E^*_{m_s,m_t}\right) \leqslant \big(1+\eta(m_t)\big)^{k-1}\mu(E^*_{m_s,m_t})^k.
$$

This completes the proof of (3.15), and hence also of (3.14).

Next, consider the trivial identity

$$
\mu\left(E^*_{m_t-m_s,m_t}\right) = \frac{\mu\big(E^*_{m_s,m_t}\big)\mu\big(T^{-m_s}E^*_{m_t-m_s,m_t}\big)}{\mu\big(E^*_{m_s,m_t}\big)}. \tag{3.16}
$$

Using (3.14) we get

$$
\begin{aligned}
&\frac{\mu\big(E^*_{m_s,m_t}\big)\mu\big(T^{-m_s}E^*_{m_t-m_s,m_t}\big)}{\mu\big(E^*_{m_s,m_t}\big)}\\
&\qquad \leqslant \big(1+\eta(m_t)\big)\mu(E^*_{m_s,m_t})\mu(T^{-m_s}E^*_{m_t-m_s,m_t})^{1-\frac{1}{k}}\\
&\qquad \leqslant \big(1+\eta(m_t)\big)\Big(\mu(E^*_{m_s,m_t})\mu(T^{-m_s}E^*_{m_t-m_s,m_t})\Big)^{1-\frac{1}{k}}.
\end{aligned} \tag{3.17}
$$

Using property (1.7) once more, we conclude

$$
\begin{aligned}
&\big(\mu(E^*_{m_s,m_t})\mu(T^{-m_s}E^*_{m_t-m_s,m_t})\big)^{1-\frac{1}{k}}\\
&\qquad \leqslant \big(1+\eta(m_t)\big)\mu\left(E^*_{m_s,m_t}\cap T^{-m_s}E^*_{m_t-m_s,m_t}\right)^{1-\frac{1}{k}}.
\end{aligned} \tag{3.18}
$$

As in the proof of Proposition 3.1 (cf. equation (3.4)), we can use the relation

$$E_{m_t-m_s,m_t} = \bigcap_{i=1}^{\min\{m_t-m_s,F(m_t)\}} T^{-i}B^c_{m_t} \ \cap \ E^*_{m_t-m_s,m_t}$$

to obtain the estimate

$$\mu(E^*_{m_t-m_s,m_t}) - \mu\left(\bigcup_{i=1}^{\min\{m_t-m_s,F(m_t)\}} T^{-i}B_{m_t}\right) \leqslant \mu(E_{m_t-m_s,m_t}).$$

Since $\mu\left(\bigcup_{i=1}^{\min\{m_t-m_s,F(m_t)\}} T^{-i}B_{m_t}\right) \leqslant F(m_t)\mu(B_{m_t})$ and $E_{m_t-m_s,m_t} \subset E^*_{m_t-m_s,m_t}$, we have

$$\mu(E^*_{m_t-m_s,m_t} \triangle E_{m_t-m_s,m_t}) = \mathrm{O}\big(F(m_t)\mu(B_{m_t})\big).$$

In a similar fashion, one can derive

$$\mu(E^*_{m_s,m_t} \triangle E_{m_s,m_t}) = \mathrm{O}\big(F(m_t)\mu(B_{m_t})\big).$$

This gives

$$\begin{aligned}\mu(E^*_{m_s,m_t} &\cap T^{-m_s}E^*_{m_t-m_s,m_t}) \\ &= \mu(E_{m_s,m_t} \cap T^{-m_s}E_{m_t-m_s,m_t}) + \mathrm{O}\big(F(m_t)\mu(B_{m_t})\big) \\ &= \mu(E_{m_t}) + \mathrm{O}\big(F(m_t)\mu(B_{m_t})\big).\end{aligned}$$

It follows that

$$\begin{aligned}\mu(E^*_{m_s,m_t} &\cap T^{-m_s}E^*_{m_t-m_s,m_t})^{1-\frac{1}{k}} \\ &= \mu(E_{m_t})^{1-\frac{1}{k}} + \mathrm{O}\Big(\big(F(m_t)\mu(B_{m_t})\big)^{1-\frac{1}{k}}\Big).\end{aligned} \tag{3.19}$$

Combining (3.13), (3.16), (3.17), (3.18) and (3.19) yields

$$\begin{aligned}\mu(E_{m_s} \cap E_{m_t}) \leqslant &\big(1 + \mathrm{o}_{t\to\infty}(1)\big)\mu(E_{m_s})\mu(E_{m_t})^{1-\frac{1}{k}} \\ &+ \mathrm{O}\Big(\big(\mu(E_{m_s})F(m_t)\mu(B_{m_t})\big)^{1-\frac{1}{k}}\Big).\end{aligned}$$

Finally, since $k = \lfloor m_t/m_s \rfloor - 1 \geqslant 2^t/2^{s+1} - 1 \geqslant 2^t/2^{s+2}$, one has

$$1 - \frac{1}{k} \ \geqslant \ 1 - \frac{2^{s+2}}{2^t}.$$

This finishes the proof. □

3.3. Proof of Theorem 1.1

We need one more lemma before proving Theorem 1.1.

Lemma 3.5. *Suppose that* (1.8) *holds for some* $\delta > 0$ *and all but finitely many* $m \in \mathbb{N}$. *Let* $\sigma > 1$ *and let* $(m_j)_{j\in\mathbb{N}}$ *be a sequence of natural numbers such that*

$$m_{j+1}/m_j \geqslant \sigma \text{ for all large enough } j \in \mathbb{N}. \tag{3.20}$$

Define $w_j := F(m_j)\mu(B_{m_j})$. *Then*

$$\sum_{j\in\mathbb{N}} w_j^{\frac{k}{k+1}} < \infty$$

for all $k \geqslant 2\delta^{-1}$.

Proof. In view of (3.20), there exists some $c > 0$ such that $m_j \geqslant c\sigma^j$ for all large enough $j \in \mathbb{N}$. Hence from (1.8) we can conclude that

$$w_j \leqslant \frac{1}{(\log m_j)^{1+\delta}} \ll \frac{1}{j^{1+\delta}} \qquad \text{for all but finitely many } j.$$

Since $\sum_{j\in\mathbb{N}} \left(\frac{1}{j^{1+\delta}}\right)^{\frac{k}{k+1}} < \infty$ for all k with $k \geqslant 2\delta^{-1}$, the claim follows. □

Proof of Theorem 1.1. First assume there exists $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} < \infty.$$

By assumption, there exists $\delta > 0$ such that $F(m) \leqslant \big(\log^{1+\delta}(m)\mu(B_m)\big)^{-1}$ for all but finitely many $m \in \mathbb{N}$. Fix such a δ. Pick now $k \in \mathbb{N}$ with $1/k < \min\{\varepsilon, \delta/2\}$.

Next let $(m_j)_{j\in\mathbb{N}}$ be defined as in Proposition 3.3, that is, $m_j := k\left\lfloor \frac{(k+1)^{\frac{j}{2}}}{k^{\frac{j}{2}}} \right\rfloor$. It is easy to check that (3.20) holds, and, by combining Proposition 3.3 with Lemma 3.5 we see that the series $\sum_{j\in\mathbb{N}} \mu(E_{m_{j-1},m_j})$ converges if and only if so does the series $\sum_{j\in\mathbb{N}} \mu(E_{m_{j+1},m_j})^{\frac{k}{k+1}}$. We now have

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} = \sum_{j=1}^{\infty} \sum_{n\in[m_j,m_{j+1})} \frac{\mu(E_n)^{1-\varepsilon}}{n} \geqslant \sum_{j=1}^{\infty} \frac{1}{m_{j+1}} \sum_{n\in[m_j,m_{j+1})} \mu(E_n)^{1-\varepsilon}.$$

For any n with $m_j \leqslant n < m_{j+1}$ we have $\mu(E_n) \geqslant \mu(E_{m_{j+1},m_j})$. Therefore

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} \geqslant \sum_{j=1}^{\infty} \frac{1}{m_{j+1}} \sum_{n\in[m_j,m_{j+1})} \mu(E_{m_{j+1},m_j})^{1-\varepsilon}$$
$$\geqslant \sum_{j=1}^{\infty} \frac{m_{j+1}-m_j}{m_{j+1}} \mu(E_{m_{j+1},m_j})^{1-\varepsilon} \geqslant \frac{1}{2k} \sum_{j=1}^{\infty} \mu(E_{m_{j+1},m_j})^{\frac{k}{k+1}},$$

where the last inequality follows because $1-\varepsilon \leqslant \frac{k}{k+1}$ and $\frac{m_{j+1}-m_j}{m_{j+1}} \geqslant \frac{1}{2k}$. Since $\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} < \infty$, we conclude that $\sum_{j=1}^{\infty} \mu(E_{m_{j+1},m_j})^{\frac{k}{k+1}} < \infty$, and therefore also $\sum_{j\in\mathbb{N}} \mu(E_{m_{j-1},m_j}) < \infty$. In view of Lemma 2.3, this implies that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure, which completes the proof of the first part of Theorem 1.1.

For the second part, assume $\sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \infty$. For every $j \in \mathbb{N}$ let m_j be a number in $[2^j, 2^{j+1})$ that satisfies

$$\mu(E_{m_j}) = \max_{n\in[2^j,2^{j+1})} \mu(E_n).$$

Then

$$\sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \sum_{j=0}^{\infty} \sum_{n\in[2^j,2^{j+1})} \frac{\mu(E_n)}{n} \leqslant \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{n\in[2^j,2^{j+1})} \mu(E_n) \leqslant \sum_{j=0}^{\infty} \mu(E_{m_j}).$$

It follows that $\sum_{j=1}^{\infty} \mu(E_{m_j}) = \infty$.

According to Lemma 3.4, for all pairs $s, t \in \mathbb{N}$ with $t > s$ we have

$$\mu(E_{m_s} \cap E_{m_t}) \leqslant \big(1 + \mathrm{o}_{t\to\infty}(1)\big)\mu(E_{m_s})\mu(E_{m_t})^{1-\frac{2^{s+2}}{2^t}} + \mathrm{O}\Big(\mu(E_{m_s})\big(F(m_t)\mu(B_{m_t})\big)^{1-\frac{2^{s+2}}{2^t}}\Big). \tag{3.21}$$

Define $v_t \coloneqq (F(m_t)\mu(B_{m_t}))^{1-2^{s+2-t}}$, and set

$$w_{0,j} \coloneqq F(m_{2j})\mu(B_{m_{2j}}) \text{ and } w_{1,j} \coloneqq F(m_{2j-1})\mu(B_{m_{2j-1}}).$$

Then we have

$$\sum_{t\in\mathbb{N}} v_t = \sum_{j\in\mathbb{N}} v_{2j} + \sum_{j\in\mathbb{N}} v_{2j-1} = \sum_{j\in\mathbb{N}} w_{0,j}^{1-2^{s+2-2j}} + \sum_{j\in\mathbb{N}} w_{1,j}^{1-2^{s+3-2j}}.$$

Since $w_{0,j} = F(m_{2j})\mu(B_{m_{2j}})$ and $m_{2(j+1)}/m_{2j} \geqslant 2$ for all $j \in \mathbb{N}$, it follows from Lemma 3.5 that

$$\sum_{j\in\mathbb{N}} w_{0,j}^{1-2^{s+2-2j}} < \infty.$$

In an analogous way, one can show that

$$\sum_{j\in\mathbb{N}} w_{1,j}^{1-2^{s+3-2j}} < \infty.$$

Therefore, we have $\sum_{t\in\mathbb{N}} v_t < \infty$, which, in combination with (3.21), proves that

$$\mu(E_{m_s} \cap E_{m_t}) \leqslant \big(1 + \mathrm{o}_{t\to\infty}(1)\big)\mu(E_{m_s})\mu(E_{m_t})^{1-2^{s-t+2}} + \mathrm{O}\big(\mu(E_{m_s})v_t\big)$$

where $(v_t)_{t\in\mathbb{N}}$ satisfies $\sum_{t\in\mathbb{N}} v_t < \infty$. Thus, by Theorem 2.4, we conclude that $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big)$ is not equal to 1. Since $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big)$ is essentially invariant (see Proposition 2.1), we must have $\mu\big(\mathcal{H}_{\text{e.a.}}(\mathcal{B})\big) = 0$, which finishes the proof. □

4. Shrinking target systems with independent targets

Let us now show how Theorem 1.2 and the corresponding Corollary 1.3 follow from the results we have obtained so far.

Proof of Theorem 1.2. If follows immediately from property (1.11) and the definition of E_m (see (1.9)) that

$$\mu(E_m) = \big(1 - \mu(B_m)\big)^m.$$

Hence Theorem 1.2 follows from Theorem 1.1. □

Proof of Corollary 1.3. First assume for all but finitely many $m \in \mathbb{N}$ that

$$\mu(B_m) \geqslant \frac{C \log\log m}{m}.$$

Choose any $b \in (1, C)$. Using the inequality $(1 + x) \leqslant e^x$, which holds for all real numbers x, we obtain (with $x = -(C\log(k\log b/2))/\lfloor b^k\rfloor$) that for all sufficiently large k,

$$\left(1 - \frac{C\log\log\lfloor b^k\rfloor}{\lfloor b^k\rfloor}\right)^{\lfloor b^k\rfloor} \leqslant \left(1 - \frac{C\log\log(b^{k/2})}{\lfloor b^k\rfloor}\right)^{\lfloor b^k\rfloor}$$

$$= \left(1 - \frac{C \log\left(\frac{k \log b}{2}\right)}{\lfloor b^k \rfloor}\right)^{\lfloor b^k \rfloor}$$
$$\leqslant e^{-C \log\left(\frac{k \log b}{2}\right)} = \frac{e^{-C \log\left(\frac{\log b}{2}\right)}}{k^C}.$$

Then

$$\sum_{m=1}^{\infty} \frac{(1-\mu(B_m))^{m(1-\varepsilon)}}{m} \leqslant \sum_{m=1}^{\infty} \frac{\left(1 - \frac{C \log\log m}{m}\right)^{m(1-\varepsilon)}}{m}$$
$$\leqslant \sum_{k=1}^{\infty} \left(1 - \frac{C \log\log \lfloor b^k \rfloor}{\lfloor b^k \rfloor}\right)^{\lfloor b^k \rfloor (1-\varepsilon)}$$
$$= \mathrm{O}\left(\sum_{k=1}^{\infty} \frac{1}{k^{C(1-\varepsilon)}}\right).$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{C(1-\varepsilon)}} < \infty$ for sufficiently small ε, it follows from Theorem 1.2 that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure.

The second part follows from an analogous calculation where instead of the inequality $(1+x) \leqslant e^x$ one uses the inequality $(1+x) \geqslant e^{x-x^2}$, which holds for all $x \in (-1/2, 0]$. Indeed,

$$\sum_{m=1}^{\infty} \frac{\mu(E_m)}{m} = \sum_{m=1}^{\infty} \frac{(1-\mu(B_m))^m}{m} \geqslant \sum_{m=1}^{\infty} \frac{\left(1 - \frac{\log\log m}{m}\right)^m}{m}$$
$$\geqslant \frac{1}{2} \sum_{k=1}^{\infty} \left(1 - \frac{\log k}{2^k}\right)^{2^k} \geqslant \frac{1}{2} \sum_{k=1}^{\infty} \left(e^{-\frac{\log k}{2^k} - \frac{\log^2 k}{2^{2k}}}\right)^{2^k}$$
$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\frac{\log^2 k}{2^k}} = \infty.$$

Therefore, by Theorem 1.2, $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has zero measure. □

5. Bernoulli schemes and a proof of Theorem 1.4

In this section we give a proof of Theorem 1.4. Let $(r_n)_{n \in \mathbb{N}}$ and $(X, \mu, T, \mathcal{B})$ be as in Subsection 1.3. Given a point $x \in X = \{0,1\}^{\mathbb{N} \cup \{0\}}$ we denote by $x[1, \dots, n]$ the word $x[1]x[2]\dots x[n]$.

In order to derive Theorem 1.4 from Theorem 1.1 we first need to understand the measure of the set $E_n = \bigcap_{j=1}^{n} T^{-j} B_n^c$. Note that E_n consists exactly of all the points $x \in \{0,1\}^{\mathbb{N}\cup\{0\}}$ with the property that the word $x[1, n+r_n]$ does not contain r_n consecutive zeros. To estimate $\mu(E_n)$, it will therefore be convenient to beforehand estimate the average number of zeros in $x[1, n+r_n]$. For each $n \geqslant 1$ and $x \in X$, let $V_n(x) := \max\{\text{number of consecutive zeros in } x[1,\ldots,n]\}$. Let $\log_2 x := \frac{\log x}{\log 2}$. Our main tool is the following estimate from [FS].

Proposition 5.1 ([FS, Proposition V.1.]). *Let $a(n) := 2^{\{\log_2 n\}}$, where we use $\{.\}$ to denote the fractional part of a real number. One has*

$$\mu\big(V_n < \lfloor \log_2 n \rfloor + h\big) = \exp\big(-a(n)2^{-h-1}\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right).$$

Note that

$$E_{n,m} = \{x \in X : V_{n+r_m}(x) < r_m\}. \tag{5.1}$$

Using this, we can get the first order asymptotics for $\mu(E_{n,m})$.

Theorem 5.2. *One has*

$$\mu(E_{n,m}) = \exp\big(-\tfrac{n}{2}\mu(B_m)\big)(1 + \mathrm{o}_{m\to\infty}(1)) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right).$$

Proof. We will write $h_{n,m} := r_m - \lfloor \log_2(n+r_m) \rfloor$. In view of (5.1),

$$\mu(E_{n,m}) = \exp\left(-a(n+r_m)2^{-h_{n,m}-1}\right) + \mathrm{O}\left(\frac{\log(n+r_m)}{\sqrt{n+r_m}}\right).$$

We can replace $\mathrm{O}\left(\frac{\log(n+r_m)}{\sqrt{n+r_m}}\right)$ with $\mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)$. Thus,

$$\begin{aligned}
\mu(E_{n,m}) &= \exp\big(-a(n+r_m)2^{-h_{n,m}-1}\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\\
&= \exp\big(-2^{\{\log_2(n+r_m)\}}2^{-(r_m - \lfloor \log_2(n+r_m)\rfloor)-1}\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\\
&= \exp\big(-2^{\log_2(n+r_m)-r_m-1}\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)\\
&= \exp\big(-(n+r_m)2^{-r_m-1}\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)
\end{aligned}$$

$$= \exp\big(-n2^{-r_m-1}\big)\big(1+\mathrm{o}_{m\to\infty}(1)\big) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right)$$
$$= \exp\left(-\frac{n}{2}\mu(B_m)\right)\left(1+\mathrm{o}_{m\to\infty}(1)\right) + \mathrm{O}\left(\frac{\log n}{\sqrt{n}}\right).$$

From this the claim follows. □

Choosing $n = m$ in Theorem 5.2 yields the following corollary.

Corollary 5.3. *One has*

$$\mu(E_m) = \exp\big(-\tfrac{m}{2}\mu(B_m)\big)\big(1+\mathrm{o}_{m\to\infty}(1)\big) + \mathrm{O}\left(\frac{\log m}{\sqrt{m}}\right).$$

Remark 5.4. Theorem 5.2 can also be useful to estimate the measure of sets of the from E_{m_{j-1},m_j}, which are of interest because of Lemma 2.3. For the proof of Theorem 1.4, which we will present at the end of this section, we are particularly interested in the case where

$$m_j = \lfloor b^j \rfloor$$

for some $b > 1$. In this case, it follows from Theorem 5.2 that

$$\mu(E_{m_{j-1},m_j}) = \exp\big(-\tfrac{m_{j-1}}{2}\mu(B_{m_j})\big)\big(1+\mathrm{o}_{m\to\infty}(1)\big) + \mathrm{O}\left(\frac{j}{b^{\frac{j}{2}}}\right). \tag{5.2}$$

Since $m_{j-1} \geqslant \frac{m_j}{c}$ for all but finitely many j as long as $c > b$, we deduce from (5.2) that

$$\mu(E_{m_{j-1},m_j}) \leqslant \exp\big(-\tfrac{m_j}{2c}\mu(B_{m_j})\big)\big(1+\mathrm{o}_{m\to\infty}(1)\big) + \mathrm{O}\left(\frac{j}{b^{\frac{j}{2}}}\right) \tag{5.3}$$

for all $c > b$.

Theorem 5.5. *Let $(X, \mu, T, \mathcal{B})$ be a shrinking target system. If there exists $\varepsilon > 0$ such that*

$$\sum_{m=1}^{\infty} \frac{\big(1-\mu(B_m)\big)^{\frac{m(1-\varepsilon)}{2}}}{m} < \infty,$$

then there exists $\varepsilon' > 0$ such that

$$\sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)^{1-\varepsilon'}}{m} < \infty.$$

Also, if

$$\sum_{m=1}^{\infty} \frac{\big(1-\mu(B_m)\big)^{\frac{m}{2}}}{m} = \infty$$

then

$$\sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)}{m} = \infty.$$

Proof. Using the basic inequality $(1+x)^r \leqslant \exp(rx)$ (which holds for all $x \geqslant -1$ and $r > 0$ and follows readily from $\ln(1+x) \leqslant x$) it is straightforward to show that

$$\big(1-\mu(B_m)\big)^{\frac{m}{2}} \leqslant \exp\big(-\tfrac{m}{2}\mu(B_m)\big).$$

From this the implication

$$\sum_{m=1}^{\infty} \frac{\big(1-\mu(B_m)\big)^{\frac{m}{2}}}{m} = \infty \implies \sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)}{m} = \infty$$

follows.

For the other implication, we can use the inequality

$$(1+x)^{\delta r} \geqslant \exp(rx), \tag{5.4}$$

which holds for every $\delta < 1$, all $r > 0$, and all non-positive x that are sufficiently close to 0, where the closeness to 0 depends only on δ but not on r. The validity of (5.4) follows from $\ln(1+x) \geqslant \delta^{-1}x$ by exponentiating both sides and then raising to the power of r. Hence, for $\delta = \frac{1-\varepsilon}{1-\varepsilon'}$ we get that for all but finitely many m

$$\big(1-\mu(B_m)\big)^{\frac{m(1-\varepsilon)}{2}} \geqslant \exp\big(-\tfrac{m}{2}\mu(B_m)\big)^{1-\varepsilon'},$$

where $\varepsilon' > 0$ can be any number that is strictly smaller than ε. This implies

$$\sum_{m=1}^{\infty} \frac{\big(1-\mu(B_m)\big)^{\frac{m(1-\varepsilon)}{2}}}{m} < \infty \implies \sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)^{1-\varepsilon'}}{m} < \infty.$$

□

Next, we will present a proof of Theorem 1.4. For the reader's benefit, let us briefly outline the main ideas behind the proof first. Recall that by assumption either (1.13) or (1.14) are satisfied. We will show below that (1.14) forces conditions (1.7) and (1.8) to be satisfied for an appropriate choice of F and η, which will allow us to derive the conclusion of Theorem 1.4 from Theorem 1.1. On the other hand, under the assumption (1.13) we cannot guarantee that (1.8) is satisfied, because the measure of the targets B_m might not shrink sufficiently fast. In this case, instead of using Theorem 1.1, our argument will build on Remark 5.4 together with Lemma 2.3.

Proof of Theorem 1.4. Let us first deal with (1.13). Let c be such that $1 < c < D/2$, and define $\eta := \frac{D}{4} - \frac{c}{2}$. Since $\mu(B_m) \geqslant \frac{2(c+2\eta)\log\log m}{m}$ for all but finitely many m, it follows that $\frac{m}{2}\mu(B_m) \geqslant (c+2\eta)\log\log m \geqslant (c+\eta)\log\left(\frac{\log m}{\log c}\right)$. Applying the map $x \mapsto \exp(-x/c)$ to both sides of this inequality yields

$$\exp\left(-\frac{m}{2c}\mu(B_m)\right) \leqslant \left(\frac{\log c}{\log m}\right)^{1+\frac{\eta}{c}} \tag{5.5}$$

for all but finitely many m. Let $b \in (1, c)$ be arbitrary. After substituting $m_j = \lfloor b^j \rfloor$ for m in (5.5), we are left with

$$\exp\left(-\frac{m_j}{2}\mu(B_{m_j})\right) \leqslant \frac{1}{j^{1+\frac{\eta}{c}}}.$$

Combining this with (5.3) shows that $\sum_{j\in\mathbb{N}} \mu(E_{m_{j-1},m_j}) < \infty$. In light of Lemma 2.3, this proves that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure.

Next, we deal with (1.14). Pick $F(m) = r_m$ and $\eta = 0$. Choose $\delta > 0$ sufficiently small such that $\tau \geqslant \frac{1+\delta}{1-\delta}$. Since

$$\mu(B_m)\log_2\left(\frac{1}{\mu(B_m)}\right) \leqslant \mu(B_m)^{1-\delta}$$

for all but finitely many $m \in \mathbb{N}$ (because $\lim_{m\to\infty}\mu(B_m) = 0$), we deduce that

$$\begin{aligned} F(m)\mu(B_m) &= \mu(B_m)\log_2\left(\tfrac{1}{\mu(B_m)}\right) \\ &\leqslant \mu(B_m)^{1-\delta} \leqslant \left(\frac{1}{(\log m)^\tau}\right)^{\frac{1}{1-\delta}} \leqslant \frac{1}{(\log m)^{1+\delta}}. \end{aligned}$$

Hence F satisfies (1.8). By construction, the shrinking target system also satisfies (1.7). In light of Corollary 5.3 there exists a constant $C > 1$ such that

$$C^{-1}\exp\big(-\tfrac{m}{2}\mu(B_m)\big) - C\,\frac{\log m}{\sqrt{m}} \leqslant \mu(E_m) \leqslant C\exp\big(-\tfrac{m}{2}\mu(B_m)\big) + C\,\frac{\log m}{\sqrt{m}} \tag{5.6}$$

holds for all but finitely many m. Since

$$\sum_{m\in\mathbb{N}} \frac{\log m}{m\sqrt{m}} < \infty,$$

we conclude that

$$\sum_{m=1}^{\infty} \frac{\mu(E_m)}{m} = \infty \iff \sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)}{m} = \infty. \tag{5.7}$$

Then, using the inequalities $x^{1-\varepsilon} - y^{1-\varepsilon} \leqslant (x-y)^{1-\varepsilon}$ (which holds for all $x \geqslant y \geqslant 0$) and $(x+y)^{1-\varepsilon} \leqslant x^{1-\varepsilon} + y^{1-\varepsilon}$ (which holds for all $x, y \geqslant 0$), it follows from (5.6) that

$$C^{-1}\exp\big(-\tfrac{m}{2}\mu(B_m)\big)^{1-\varepsilon} - C\left(\frac{\log m}{\sqrt{m}}\right)^{1-\varepsilon} \leqslant \mu(E_m)^{1-\varepsilon} \leqslant C\exp\big(-\tfrac{m}{2}\mu(B_m)\big)^{1-\varepsilon} + C\left(\frac{\log m}{\sqrt{m}}\right)^{1-\varepsilon}.$$

Combining this with

$$\sum_{m\in\mathbb{N}} \frac{\left(\frac{\log m}{\sqrt{m}}\right)^{1-\varepsilon}}{m} < \infty$$

shows that we also have

$$\sum_{m=1}^{\infty} \frac{\mu(E_m)^{1-\varepsilon}}{m} < \infty \iff \sum_{m=1}^{\infty} \frac{\exp\big(-\frac{m}{2}\mu(B_m)\big)^{1-\varepsilon}}{m} < \infty. \tag{5.8}$$

Hence, in light of (5.7) and (5.8), Theorem 1.4 follows directly from Theorem 1.1 together with Theorem 5.5. □

Corollary 1.5 can be derived from Theorem 1.4 the same way that Corollary 1.3 was derived from Theorem 1.2. Therefore we omit its proof.

6. The Gauß map and the Gauß measure

In this section let $(X, \mu, T, \mathcal{B})$ denote the shrinking target system considered in Subsection 1.4, where X is the interval $[0,1]$, $T\colon [0,1] \to [0,1]$ is the Gauß map, μ is the Gauß measure, and $\mathcal{B} = \{B_1 \supset B_2 \supset \ldots\}$ is defined by (1.18).

We begin by showing that for this shrinking target system condition (1.7) holds for any $F(m)$ that satisfies (1.8) and $\eta(m) = \mathrm{O}\Big(\big(-C\sqrt{F(m)}\,\big)\Big)$ for some universal constant $C > 0$. The following result of Phillipp will be crucial for making this deduction.

Lemma 6.1 ([P]). *There exists a constant $\lambda \in (0,1)$ such that for all $k, n \in \mathbb{N}$, all sets of the form*

$$A = \big\{[a_1, a_2, \ldots] : a_1 = r_1, \ldots, a_n = r_n\big\}, \quad \text{where } r_1, \ldots, r_n \in \mathbb{N} \text{ are arbitrary,} \tag{6.1}$$

and all measurable sets $B \subset [0,1]$ one has

$$\mu(A \cap T^{-n-k}B) = \mu(A)\mu(B)\Big(1 + \mathrm{O}\big(\lambda^{\sqrt{k}}\big)\Big).$$

From Lemma 6.1 we can derive the following corollary.

Corollary 6.2. *Let Θ_n be the σ-algebra on $[0,1]$ generated by all sets of the form (6.1). There exists a constant $\lambda \in (0,1)$ such that for all $k, n \in \mathbb{N}$, all $A \in \Theta_n$, and all measurable sets $B \subset [0,1]$ one has*

$$\mu(A \cap T^{-n-k}B) = \mu(A)\mu(B)\Big(1 + \mathrm{O}\big(\lambda^{\sqrt{k}}\big)\Big).$$

Proof. Fix $k, n \in \mathbb{N}$. Any set $A \in \Theta_n$ can be written (up to null sets) as a disjoint union

$$A = \bigcup_{j \in J} A_j$$

where J is either finite or countably infinite, and for each $j \in J$ the set A_j is of the form

$$A_j = \big\{[a_1, a_2, \ldots] : a_1 = r_1^{(j)}, \ldots, a_n = r_n^{(j)}\big\}$$

for some $r_1^{(j)}, \ldots, r_n^{(j)} \in \mathbb{N}$. In light of Lemma 6.1 we have

$$\mu(A_j \cap T^{-n-k}B) = \mu(A_j)\mu(B)\Big(1 + \mathrm{O}\big(\lambda^{\sqrt{k}}\big)\Big)$$

for every $j \in J$. Moreover, seeing that $(A_j)_{j\in J}$ are pairwise disjoint and J is countable, we have

$$\mu(A) = \sum_{j\in J} \mu(A_j) \quad \text{and} \quad \mu(A \cap T^{-n-k}B) = \sum_{j\in J} \mu(A_j \cap T^{-n-k}B).$$

The claim now follows. □

Recall that given $n \leqslant m$ and a shrinking target $\mathcal{B}$ we have defined the algebra $\Xi_{n,m}$ by (1.6). Since the σ-algebra Θ_m appearing in Corollary 6.2 contains $\Xi_{n,m}$ as a sub-algebra, it follows from the conclusion of Corollary 6.2 that (1.7) holds for any $F(m)$ that satisfies (1.8) and $\eta(m) = \mathrm{O}\left(\left(- C\sqrt{F(m)}\,\right)\right)$ for some universal constant $C > 0$.

Proposition 6.3. *For all $n \leqslant m \in \mathbb{N}$ we have*

$$\left(1 - \frac{2}{k_m}\right)^n \leqslant \mu(E_{n,m}) \leqslant \left(1 - \frac{1}{k_m + 1}\right)^n. \tag{6.2}$$

In what follows, for any $t \in [0,1]$ let $M_t : [0,1] \to [0,1]$ denote the map $M_t(x) = tx$. For the proof of Proposition 6.3 we will need the following lemma.

Lemma 6.4. *For all $k \in \mathbb{N}$ and all measurable $A \subset [0,1]$ we have*

$$\mu\Big(\big[0, \tfrac{1}{k}\big] \cap T^{-1}A\Big) = \mu\big(M_{1/k}A\big). \tag{6.3}$$

Proof. Note that both $A \mapsto \mu\big([0,\frac{1}{k}] \cap T^{-1}A\big)$ and $A \mapsto \mu(M_{1/k}A)$ are Borel measures on $[0,1]$. To show that two Borel measures coincide it suffices to verify equality for a family of sets that generate the Borel σ-algebra. In particular, instead of proving (6.3) for all measurable sets $A \subset [0,1]$, it suffices to show

$$\mu\Big(\big[0, \tfrac{1}{k}\big] \cap T^{-1}[0,s]\Big) = \mu\big(\big[0, \tfrac{s}{k}\big]\big) \tag{6.4}$$

for all $0 \leqslant s \leqslant 1$.

Note that $T^{-1}[0,s] = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{n+s}, \frac{1}{n}\right]$ and hence $\left[0, \frac{1}{k}\right] \cap T^{-1}[0,s] = \bigcup_{n\geqslant k} \left[\frac{1}{n+s}, \frac{1}{n}\right]$. We conclude that

$$\mu\Big(\left[0, \tfrac{1}{k}\right] \cap T^{-1}[0,s]\Big) = \sum_{n=k}^{\infty} \mu\big(\left[\tfrac{1}{n+s}, \tfrac{1}{n}\right]\big).$$

Therefore we have

$$\mu\big(\left[\tfrac{1}{n+s}, \tfrac{1}{n}\right]\big) = \int_{\frac{1}{n+s}}^{\frac{1}{n}} \frac{\mathrm{d}x}{1+x} = \log\left(1+\tfrac{1}{n}\right) - \log\left(1+\tfrac{1}{n+s}\right) = \log\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+s}}\right).$$

An analogous calculation yields

$$\mu\big(\left[\tfrac{s}{n+1}, \tfrac{s}{n}\right]\big) = \int_{\frac{s}{n+1}}^{\frac{s}{n}} \frac{\mathrm{d}x}{1+x} = \log\left(\frac{1+\frac{s}{n}}{1+\frac{s}{n+1}}\right).$$

Since

$$\frac{1+\frac{1}{n}}{1+\frac{1}{n+s}} = \frac{\frac{n+1}{n}}{\frac{n+s+1}{n+s}} = \frac{\frac{n+s}{n}}{\frac{n+s+1}{n+1}} = \frac{1+\frac{s}{n}}{1+\frac{s}{n+1}},$$

it follows that

$$\mu\big(\left[\tfrac{1}{n+s}, \tfrac{1}{n}\right]\big) = \mu\big(\left[\tfrac{s}{n+1}, \tfrac{s}{n}\right]\big).$$

Summing over $n \geqslant k$ finishes the proof of (6.4). □

Proof of Proposition 6.3. Consider the set $\tilde{E}_{n,m} := \bigcap_{i=0}^{n-1} T^{-i} B_m^c$. Since $E_{n,m} = T^{-1}\tilde{E}_{n,m}$, and since the Gauß measure μ is invariant under T, it follows that $\mu(E_{n,m}) = \mu(\tilde{E}_{n,m})$. Thus it suffices to prove (6.2) with $E_{n,m}$ replaced by $\tilde{E}_{n,m}$. We will also make use of the fact that for any $a \in [0,1]$ and any measurable $A \subset [0,a]$ one has

$$\frac{\lambda(A)}{(1+a)(\log 2)} \leqslant \mu(A) \leqslant \frac{\lambda(A)}{\log 2}. \tag{6.5}$$

Let us prove (6.2) (with $E_{n,m}$ replaced by $\tilde{E}_{n,m}$) by induction on n. If $n = 1$ then (6.2) says

$$1 - \frac{2}{k_m} \leqslant 1 - \mu(B_m) \leqslant 1 - \frac{1}{k_m + 1}.$$

The validity of this statement for all $m \in \mathbb{N}$ is straightforward to check using $\lambda(B_m) = 1/k_m$ and (6.5).

Next, let $n \geqslant 1$ and assume (6.2) has already been proven for n. Our goal is to show that it also holds for $n+1$. Using $\tilde{E}_{n+1,m} = B_m^c \cap T^{-1}\tilde{E}_{n,m}$, we get

$$\begin{aligned}\mu(\tilde{E}_{n+1,m}) &= \mu(B_m^c \cap T^{-1}\tilde{E}_{n,m})\\ &= \mu(T^{-1}\tilde{E}_{n,m}) - \mu(B_m \cap T^{-1}\tilde{E}_{n,m})\\ &= \mu(\tilde{E}_{n,m}) - \mu(B_m \cap T^{-1}\tilde{E}_{n,m})\\ &= \mu(\tilde{E}_{n,m}) - \mu(M_{1/k_m}\tilde{E}_{n,m}),\end{aligned}$$

where the last equality follows from Lemma 6.4. Using (6.5), we can derive an upper bound on the last term of the above displayed equation as follows:

$$\begin{aligned}\mu(\tilde{E}_{n,m}) - \mu(M_{1/k_m}\tilde{E}_{n,m}) &\leqslant \mu(\tilde{E}_{n,m}) - \frac{\lambda(M_{1/k_m}\tilde{E}_{n,m})}{(1+\frac{1}{k_m})(\log 2)}\\ &= \mu(\tilde{E}_{n,m}) - \frac{\frac{1}{k_m}\lambda(\tilde{E}_{n,m})}{(1+\frac{1}{k_m})(\log 2)}\\ &= \mu(\tilde{E}_{n,m}) - \frac{1}{k_m+1}\cdot\frac{\lambda(\tilde{E}_{n,m})}{(\log 2)}\\ &\leqslant \mu(\tilde{E}_{n,m}) - \frac{1}{k_m+1}\cdot\mu(\tilde{E}_{n,m})\\ &= \mu(\tilde{E}_{n,m})\left(1 - \frac{1}{k_m+1}\right).\end{aligned}$$

We conclude that

$$\mu(\tilde{E}_{n+1,m}) \leqslant \mu(\tilde{E}_{n,m})\left(1 - \frac{1}{k_m+1}\right). \tag{6.6}$$

In a similar fashion, we can also obtain a lower bound:

$$\begin{aligned}\mu(\tilde{E}_{n,m}) - \mu(M_{1/k_m}\tilde{E}_{n,m}) &\geqslant \mu(\tilde{E}_{n,m}) - \frac{\lambda(M_{1/k_m}\tilde{E}_{n,m})}{(\log 2)}\\ &= \mu(\tilde{E}_{n,m}) - \frac{\frac{1}{k_m}\lambda(\tilde{E}_{n,m})}{(\log 2)}\\ &= \mu(\tilde{E}_{n,m}) - \frac{2}{k_m}\cdot\frac{\lambda(\tilde{E}_{n,m})}{2(\log 2)}\\ &\geqslant \mu(\tilde{E}_{n,m}) - \frac{2}{k_m}\cdot\mu(\tilde{E}_{n,m}) = \mu(\tilde{E}_{n,m})\left(1 - \frac{2}{k_m}\right).\end{aligned}$$

It follows that

$$\mu(\tilde{E}_{n+1,m}) \geqslant \mu(\tilde{E}_{n,m})\left(1 - \frac{2}{k_m}\right). \tag{6.7}$$

Combining (6.6) and (6.7) with the induction hypothesis finishes the proof. □

Corollary 6.5. *For every $\varepsilon > 0$ the inequalities*

$$\big(1 - \mu(B_m)\big)^{2(\log 2)(1+\varepsilon)m} \leqslant \mu(E_m) \leqslant \big(1 - \mu(B_m)\big)^{(\log 2)(1-\varepsilon)m}$$

hold for all but finitely many $m \in \mathbb{N}$.

Proof. It follows from (1.19) that

$$\frac{1}{k_m} = e^{(\log 2)\mu(B_m)} - 1.$$

Since $e^x - 1 = x + \mathrm{O}(x^2)$, we obtain

$$\frac{1}{k_m} = (\log 2)\mu(B_m) + \mathrm{O}(\mu(B_m)^2).$$

This also gives

$$\frac{1}{k_m + 1} = (\log 2)\mu(B_m) + \mathrm{O}(\mu(B_m)^2).$$

Consequently, there exists a positive constant C such that

$$1 - 2(\log 2)\mu(B_m) - C\mu(B_m)^2 \leqslant 1 - \frac{2}{k_m}$$

and

$$1 - \frac{1}{k_m + 1} \leqslant 1 - (\log 2)\mu(B_m) + C\mu(B_m)^2$$

From (6.2) we then obtain

$$\begin{aligned} \big(1 - 2(\log 2)\mu(B_m) - C\mu(B_m)^2\big)^n &\leqslant \mu(E_{n,m}) \\ &\leqslant \big(1 - (\log 2)\mu(B_m) + C\mu(B_m)^2\big)^n. \end{aligned} \tag{6.8}$$

The claim now follows by taking $m = n$ in (6.8) and applying the inequalities

$$(1-x)^{r(1+\varepsilon)} \leqslant 1 - rx - Cx^2 \text{ and } (1-x)^{r(1-\varepsilon)} \geqslant 1 - rx + Cx^2,$$

which hold for all $r > 0$ and all sufficiently small positive x. □

Remark 6.6. We are also interested in estimating the measure of sets of the form E_{m_{j-1},m_j}. In particular, if

$$m_j = \lfloor b^j \rfloor$$

for some $b > 1$, then the right-hand inequality in (6.8) yields

$$\text{(6.9)} \qquad \begin{aligned} \mu(E_{m_{j-1},m_j}) &\leqslant \big(1 - (\log 2)\mu(B_{m_j}) + \mathrm{O}(\mu(B_{m_j})^2)\big)^{m_{j-1}} \\ &\leqslant \big(1 - \mu(B_{m_j})\big)^{m_j (\log 2)/c} \end{aligned}$$

as long as $c > b$.

Proof of Theorem 1.6. We begin with the case where there exists $\sigma < 1$ such that $k_m \leqslant \frac{\sigma m}{\log\log m}$ for all but finitely many $m \in \mathbb{N}$. Let σ' be any number satisfying $\sigma < \sigma' < 1$, and define $C := \frac{\sigma'}{\sigma \log 2}$. Since $\mu(B_m) \geqslant \frac{\sigma'}{k_m \log 2}$ for all but finitely many m, it follows that

$$\mu(B_m) \geqslant \frac{\sigma' \log\log m}{\sigma m \log 2} = \frac{C \log\log m}{m}.$$

Then, repeating an analogous argument to the one used in the proof of Corollary 1.3, we can show that

$$\sum_{j \in \mathbb{N}} \big(1 - \mu(B_{m_j})\big)^{m_j (\log 2)/c} < \infty$$

for any $b, c \in [1, C)$ with $b < c$, where $m_j = \lfloor b^j \rfloor$. In view of (6.9), this means that $\sum_{j\in\mathbb{N}} \mu(E_{m_{j-1},m_j}) < \infty$. Using Lemma 2.3, we conclude that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ has full measure.

Next, we deal with the case when there exists $\tau > 0$ such that $k_m \geqslant (\log m)^\tau$ for all but finitely many $m \in \mathbb{N}$. Set $F(m) := \frac{1}{(\log m)^{1+\tau/2}\mu(B_m)}$, and note that $F(m)$ converges to ∞ as $m \to \infty$, because of the assumption that $k_m \geqslant (\log m)^\tau$. Moreover, F satisfies (1.8) by construction and, as explained at the beginning of this section, (1.7) is satisfied for $\eta(m) =$

$\mathrm{O}\Big(\exp\big(-C\sqrt{F(m)}\,\big)\Big)$. Here it is important that $\lim_{m\to\infty} F(m)=\infty$, since this implies $\lim_{m\to\infty}\eta(m)=0$. In light of Corollary 6.5 we have

$$\sum_{m=1}^{\infty}\frac{\big(1-\mu(B_m)\big)^{(\log 2)(1-\varepsilon)m}}{m}<\infty \implies \sum_{m=1}^{\infty}\frac{\mu(E_m)^{1-\frac{\varepsilon}{2}}}{m}<\infty$$

as well as

$$\sum_{m=1}^{\infty}\frac{\big(1-\mu(B_m)\big)^{2(\log 2)(1+\varepsilon)m}}{m}=\infty \implies \sum_{m=1}^{\infty}\frac{\mu(E_m)}{m}=\infty.$$

Hence Theorem 1.6 follows directly from Theorem 1.1. □

We omit the proof of Corollary 1.7, since, with the help of (1.19), it can be derived from Theorem 1.6 in the same way that Corollary 1.3 was derived from Theorem 1.2.

7. Further explorations and open questions

There are still a multitude of intriguing questions surrounding the behavior of eventually always hitting sets. We begin with the following.

Question 7.1. *Is it possible to upgrade Theorem 1.1 to include necessary and sufficient conditions for $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ to be a null/co-null set, perhaps with some mixing condition different from* (1.7)–(1.8)*?*

Another intriguing question concerns rotations on the torus. Fix $\alpha\in[0,1)$ and consider the shrinking target system where X equals the torus $\mathbb{T}$, the transformation is given by $T(x)=x+\alpha \bmod 1$, μ is Lebesgue measure, and

$$\mathcal{B}=(B_n) \text{ with } B_n:=\{x\in\mathbb{T}:\|x\|_{\mathbb{T}}<\psi(n)\}, \tag{7.1}$$

where $\psi\colon\mathbb{N}\to[0,1]$ is some non-increasing function. In this case, the set of eventually always hitting points can be written as

$$\mathcal{H}_{\text{e.a.}}(\mathcal{B})=\Big\{y\in\mathbb{T}:\min_{1\leqslant k\leqslant n}\|k\alpha-y\|_{\mathbb{T}}<\psi(n)\text{ eventually always}\Big\}. \tag{7.2}$$

In [KL] the Hausdorff dimension of $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ was computed for the cases where $\psi(n)=n^{-\tau}$ for some $\tau>0$. Closely related to the study of (7.2) are also questions regarding inhomogeneous versions of Dirichlet's classical approximation theorem addressed in [KW, KK].

As was mentioned in Section 1, Kurzweil [Ku] proved that when α is badly approximable the hitting set $\mathcal{H}_{\text{i.o.}}(\mathcal{B})$ for $\mathcal{B}$ as in (7.1) obeys the zero–one law

$$\sum_n \psi(n) \begin{cases} < \infty \\ = \infty \end{cases} \iff \mathcal{H}_{\text{i.o.}}(\mathcal{B}) \text{ has } \begin{cases} \text{zero} \\ \text{full} \end{cases} \text{ measure.}$$

More recently, an extension of Kurzweil's result to arbitrary $\alpha \in [0, 1)$ was given by Fuchs and Kim [FK]:

$$\sum_{k=1}^{\infty} \left(\sum_{n=q_k}^{q_{k+1}-1} \min\{\psi(n), \|q_k\alpha\|_{\mathbb{T}}\} \right) \begin{cases} < \infty \\ = \infty \end{cases}$$
$$\iff \mathcal{H}_{\text{i.o.}}(\mathcal{B}) \text{ has } \begin{cases} \text{zero} \\ \text{full} \end{cases} \text{ measure,}$$

where p_k/q_k denote the principal convergents of α.

By Corollary 2.2, we know that $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ also obeys a zero–one law. This leads to the following question.

Question 7.2. *For a fixed α (at least in the case when α is badly approximable) what are necessary and sufficient conditions on ψ so that the set $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ as in* (7.2) *is a null set (or co-null set respectively)?*

Another classical type of shrinking target systems are β-transformations. Let $X = \mathbb{T}$ and, for $\beta > 1$, consider the map $T_\beta(x) = \beta x \bmod 1$ alongside the shrinking targets on the torus $\mathbb{T}$ given by (7.1). In this set-up,

$$\text{(7.3)} \qquad \mathcal{H}_{\text{e.a.}}(\mathcal{B}) = \big\{ y \in \mathbb{T} : \min_{1 \leqslant k \leqslant n} \|T_\beta^k(y)\|_{\mathbb{T}} < \psi(n) \text{ eventually always} \big\}.$$

The Hausdorff dimension of the set $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ in (7.3) was studied in [BL]. Unlike rotation by α, the map T_β is highly mixing, which suggests the following question.

Question 7.3. *Does T_β and $\mathcal{B}$ as above satisfy condition* (1.7)*, perhaps with some additional assumptions on ψ?*

An affirmative answer to Question 7.3 could lead to a better understanding of necessary and sufficient conditions for $\mathcal{H}_{\text{e.a.}}(\mathcal{B})$ in (7.3) to have full or zero measure respectively, similar in spirit to Theorems 1.4 and 1.6.

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