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An example of non-Kähler Calabi–Yau fourfold

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We show that there exists a non-Kähler Calabi–Yau fourfold, constructing an example by smoothing a normal crossing variety.

1. Introduction

In this note, a *Calabi–Yau manifold* is a simply-connected compact complex manifold with trivial canonical class and $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < \dim M$. $K3$ surfaces are Calabi–Yau twofolds in this definition and they are all Kähler ([24]), forming a single irreducible smooth family of dimension 20 ([14, 15]). Hence, they are all diffeomorphic. The moduli spaces of projective $K3$ surfaces are a countable union of analytic subspaces inside of the family of all $K3$ surfaces.

In higher dimensions, the situation is a bit different. Kähler Calabi–Yau manifolds are necessarily projective when their dimensions are greater than two and there exist non-Kähler Calabi–Yau threefolds. A number of non-homeomorphic Kähler Calabi–Yau threefolds have been constructed ([1, 2, 17]) and it is still an open problem whether there are finitely many non-homeomorphic Kähler Calabi–Yau threefolds or not. On the other hand, there are found infinitely many non-Kähler, non-homeomorphic Calabi–Yau threefolds ([5, 8, 9, 12]). In an effort to understand the situation in a similar way to that of the $K3$ surfaces, Reid conjectured that there still may be a single irreducible moduli space of Calabi–Yau threefolds, such that any Kähler (thus, projective) Calabi–Yau threefold is the small resolution of a degeneration of this family and that any two Kähler Calabi–Yau threefolds may be related by deformations, small resolutions and their inverses through non-Kähler Calabi–Yau threefolds, although they are non-homeomorphic ([22]). — figuratively speaking, projective Calabi–Yau threefolds may appear as scattered islands in the sea of (generically non-projective) Calabi–Yau threefolds just as projective $K3$ surfaces appear as scattered islands in the sea of (generically non-projective) $K3$ surfaces. This speculation demonstrates a role of non-Kähler Calabi–Yau manifolds in understanding Kähler ones.

In dimension four, the situation is even more obscure. A huge number of non-homeomorphic Kähler Calabi–Yau fourfolds can be constructed as complete intersections in toric varieties. However, to the best of the author's knowledge, not a single example of non-Kähler Calabi–Yau fourfold has been found. The purpose of this note is to construct an example of non-Kähler Calabi–Yau fourfold. Hence, we establish the following:

Theorem 1.1. *There exists a non-Kähler Calabi–Yau fourfold.*

We note that some examples of simply-connected non-Kähler compact holomorphic symplectic fourfolds with trivial canonical class and $H^2(M, \mathcal{O}_M) \neq 0$ were constructed ([4, 11]). We also remark that interests in non-Kähler Calabi–Yau manifolds in other directions also have been growing rapidly ([6, 10, 21, 25]).

We shall construct our example by smoothing a normal crossing variety. By smoothing, we mean the reverse process of the semistable degeneration of a manifold to a normal crossing variety. If a normal crossing variety is the central fiber of a semistable degeneration of Calabi–Yau manifolds, it can be regarded as a member in a deformation family of those Calabi–Yau manifolds. So building a normal crossing variety smoothable to a Calabi–Yau manifold can be regarded as building a deformation type of Calabi–Yau manifolds. The construction by smoothing is intrinsically up to deformation.

The structure of this note is as follows.

We start Section 2 by introducing two background materials – a smoothing theorem and a Calabi–Yau threefold. The smoothing theorem will be used to smooth a normal crossing variety to a non-Kähler Calabi–Yau fourfold and the Calabi–Yau threefold will be used as a building block in constructing the normal crossing variety in the next sections.

In Section 3, we build a smoothable normal crossing variety. The construction, starting from the Calabi–Yau threefold introduced in Section 2, involves several steps of taking quotients and blow-ups of varieties.

In Section 4, we showed that the fourfold, which is a smoothing of the normal crossing variety constructed in Section 3, is a non-Kähler Calabi–Yau fourfold and calculate its topological Euler number.

After finishing the manuscript, the author received an e-mail from Taro Sano, informing that he constructed non-Kähler Calabi–Yau manifolds of dimension higher than three by smoothing method ([23]).

2. A smoothing theorem and a Calabi–Yau threefold

By a variety, we mean a reduced complex analytic space. Let $\mathcal{X} = X_1 \cup X_2$ be a variety whose irreducible components are two smooth varieties X_1 and X_2. $\mathcal{X}$ is called a *normal crossing variety* if, near any point $p \in X_1 \cap X_2$, $\mathcal{X}$ is locally isomorphic to

$$\{(x_0, x_1, \cdots, x_n) \in \mathbb{C}^{n+1} | x_{n-1}x_n = 0\}$$

with p corresponding to the origin and X_1, X_2 locally corresponding to the hypersurfaces $x_{n-1} = 0, x_n = 0$ respectively in $\mathbb{C}^{n+1}$. Note that the variety $D_{\mathcal{X}} := X_1 \cap X_2$ is smooth. Suppose that there is a proper map $\varsigma : \mathfrak{X} \to \Delta$ from a complex manifold $\mathfrak{X}$ onto the unit disk $\Delta = \{t \in \mathbb{C} | \|t\| \leq 1\}$ such that the fiber $\mathfrak{X}_t = \varsigma^{-1}(t)$ is a smooth manifold for every $t \neq 0$ and $\mathfrak{X}_0 = \mathcal{X}$. We denote a generic fiber $\mathfrak{X}_t$ ($t \neq 0$) by $M_{\mathcal{X}}$ and we say that $\mathcal{X}$ is a semistable degeneration of a smooth manifold $M_{\mathcal{X}}$ and that $M_{\mathcal{X}}$ is a *semistable smoothing* (simply smoothing) of $\mathcal{X}$.

We will use the following result from [12] and [7] as a generalization of results in [13].

Theorem 2.1. *Let $\mathcal{X} = X_1 \cup X_2$ be a normal crossing variety of dimension $n \geq 3$ whose irreducible components are two smooth compact varieties X_1 and X_2 such that $D_{\mathcal{X}}$ is smooth. Assume that*

1) $\omega_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$,
2) $H^{n-1}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$, $H^{n-2}(X_i, \mathcal{O}_{X_i}) = 0$ *for* $i = 1, 2$,
3) $N_{D_{\mathcal{X}}/X_1} \otimes N_{D_{\mathcal{X}}/X_2}$ *on* $D_{\mathcal{X}}$ *is trivial.*

Then $\mathcal{X}$ is smoothable to a complex manifold $M_{\mathcal{X}}$.

This smoothing theorem was proved firstly in [13] with an assumption of Kählerness condition and the condition was removed in [12] and [7]. We will also need the following lemma (Lemma 3.14 in [12]) in the proof of Theorem 4.1.

Lemma 2.2. *With conditions in Theorem 2.1, assume further that X_1, X_2 are projective and $M_{\mathcal{X}}$, $D_{\mathcal{X}}$ are a projective Calabi–Yau n-fold and a Calabi–Yau $(n-1)$-fold respectively. Then there exists a big line bundle on $\mathcal{X}$.*

We will build a normal crossing variety and smooth it to a Calabi–Yau fourfold by applying Theorem 2.1 and show, using Lemma 2.2, that the

Calabi–Yau fourfold is non-Kähler. To make the normal crossing variety $\mathcal{X} = X_1 \cup X_2$, we need to construct the varieties X_1, X_2 first. X_1 and X_2 will be built from Beauville's Calabi–Yau threefold and two involutions on it.

We briefly recall Beauville's Calabi–Yau threefold. Let $\zeta = e^{2\pi\frac{\sqrt{-1}}{3}}$. By E_ζ, we denote the elliptic curve whose period is ζ and by $E_\zeta^3/\langle\zeta\rangle$ the quotient of the product manifold E_ζ^3 by the scalar multiplication by ζ. Let

$$Q_0 = 0,\ Q_1 = \frac{1-\zeta}{3},\ Q_2 = \frac{-1+\zeta}{3}$$

in E_ζ. These are exactly the fixed points of the scalar multiplication by ζ on E_ζ. For $i, j, k = 0, 1, 2$, let

$$Q_{i,j,k} = (Q_i, Q_j, Q_k) \in E_\zeta^3$$

and let $\overline{Q}_{i,j,k}$ be its image in $E_\zeta^3/\langle\zeta\rangle$. Then $\overline{Y} = E_\zeta^3/\langle\zeta\rangle$ has singularities of type $\frac{1}{3}(1,1,1)$ at $\overline{Q}_{i,j,k}$'s and the blow-up $Y \to \overline{Y}$ at these 27 singular points gives a Calabi–Yau threefold Y. This is a Kähler Calabi–Yau threefold with Hodge numbers $h^{1,2}(Y) = 0$ and $h^{1,1}(Y) = 36$, which was originally found by Beauville ([3]).

3. Construction of a normal crossing variety $\mathcal{X} = X_1 \cup X_2$

Consider two 3×3 matrices in $\mathrm{GL}_3(\mathbb{Z}[\zeta])$

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that A_i^2 is the 3×3 identity matrix and $\det(A_i) = -1$ for $i = 1, 2$. The matrix A_i induces an involution σ_i on E_ζ^3. Note that σ_i^* acts as multiplication by -1 on $H^{3,0}\left(E_\zeta^3\right)$. Noting that the subgroup of $\mathrm{GL}_3(\mathbb{Z}[\zeta])$ that is generated by A_1, A_2 is infinite, we make the following remark which will play a key role in showing the non-Kählerness in the proof of Theorem 4.1.

Remark 3.1. The group of automorphisms of E_ζ^3 that is generated by σ_1, σ_2 is infinite.

There is a unique involution ρ_i on Y such that the following diagram commutes:

$$\begin{array}{ccc} E_\zeta^3 & \xrightarrow{\sigma_i} & E_\zeta^3 \\ \downarrow & & \downarrow \\ \overline{Y} & & \overline{Y} \\ \uparrow & & \uparrow \\ Y & \xrightarrow{\rho_i} & Y \end{array}$$

We note that ρ_i^* also acts as multiplication by -1 on $H^{3,0}(Y)$ and the fixed locus S_i of ρ_i is a smooth surface (a disjoint union of irreducible surfaces). Now we move on to the construction of smooth varieties X_1, X_2. We borrow a method of construction from [16] (§4 in [16]) that makes use of non-symplectic involutions on $K3$ surfaces.

Let $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be any involution fixing two distinct points and consider a quotient $\overline{X}_i = (Y \times \mathbb{P}^1)/(\rho_i \times \psi)$. Then the singular locus of $\overline{X}_i$ is a product of smooth surfaces and ordinary double points, resulting from the fixed locus of ρ_i. Let $X_i \to \overline{X}_i$ be the blow-up along the singular locus of $\overline{X}_i$. It is elementary to check that X_i is smooth. Choose a point $p \in \mathbb{P}^1$ such that $p \neq \psi(p)$. Let D_i' be the image of $Y \times \{p\}$ in $\overline{X}_i$ and D_i be the inverse image of D_i' in X_i. Then D_i is isomorphic to Y and it is an anticanonical divisor of X_i whose normal bundle N_{D_i/X_i} in X_i is trivial. Since Y is projective, all the varieties $Y \times \mathbb{P}^1$, $\overline{X}_i$ and X_i are projective.

We summarize our notations, including ones to be defined:

- $\zeta = e^{2\pi\frac{\sqrt{-1}}{3}}$.
- $E_\zeta = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\zeta)$ is the elliptic curve with period ζ.
- $\overline{Y} = E_\zeta^3/\langle\zeta\rangle$.
- $\phi : \widetilde{Y} \to E_\zeta^3$ is the blow-up at the 27 points of $Q_{i,j,k}$'s.
- $\eta : \widetilde{Y} \to Y$ is the map induced by $E_\zeta^3 \to \overline{Y}$.
- $Y \to \overline{Y}$ is the blow-up of $\overline{Y}$ at its singular points. Y is a Kähler Calabi–Yau threefold.
- $\sigma_i : E_\zeta^3 \to E_\zeta^3$ is the involution induced by the matrix A_i for $i = 1, 2$.
- $\rho_i : Y \to Y$ is the involution induced by σ_i for $i = 1, 2$.
- $S_i = Y^{\rho_i}$ is the fixed locus of ρ_i for $i = 1, 2$.

- $\check{S}_i = \phi(\eta^{-1}(S_i))$, the fixed locus of σ_i for $i = 1, 2$.
- $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ is an involution fixing two distinct points $q_1, q_2 \in \mathbb{P}^1$.
- $\overline{X}_i = (Y \times \mathbb{P}^1)/(\rho_i \times \psi)$ for $i = 1, 2$.
- $X_i \to \overline{X}_i$ is the blow-up along the singular locus of $\overline{X}_i$ for $i = 1, 2$.
- $\widetilde{X_i} \to Y \times \mathbb{P}^1$ is the blow-up of $Y \times \mathbb{P}^1$ along the surface $S_i \times \{q_1, q_2\}$.
- D_i': the image of $Y \times \{p\}$ in $\overline{X}_i$ for a point $p \in \mathbb{P}^1$ with $p \neq \psi(p)$ for $i = 1, 2$.
- D_i : the inverse image of D_i' in X_i for $i = 1, 2$. $D_i \simeq Y$ and $D_i \in |-K_{X_i}|$.
- $X_i^* = X_i - D_i$.
- $\mathcal{X} = X_1 \cup X_2$ is the normal crossing variety of X_1, X_2, made by gluing along their isomorphic smooth anticanonical sections D_1, D_2.
- $D_{\mathcal{X}} = X_1 \cap X_2$.
- $M_{\mathcal{X}}$ is a smoothing of a normal crossing variety $\mathcal{X} = X_1 \cup X_2$.

We make a normal crossing variety $\mathcal{X} = X_1 \cup X_2$ by gluing transversally along D_1 and D_2, (see §2 (especially Corollary 2.4) of [12] for details of the gluing process). Then $D_{\mathcal{X}} := X_1 \cap X_2$ is a copy of Y. Since $D_{\mathcal{X}} = X_1 \cap X_2$ is an anticanonical divisor of both X_1 and X_2, $\omega_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$. Note that $N_{D_{\mathcal{X}}/X_1} \otimes N_{D_{\mathcal{X}}/X_2}$ is trivial. Let $X_i^* = X_i - D_i$. For varieties X_1, X_2 and $\mathcal{X} = X_1 \cup X_2$ constructed in this section, we gather some of their properties:

Proposition 3.2. 1) *X_1 and X_2 are projective.*

2) *$\omega_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}$.*

3) *$N_{D_{\mathcal{X}}/X_1} \otimes N_{D_{\mathcal{X}}/X_2}$ on $D_{\mathcal{X}}$ is trivial.*

4) *Both X_i and X_i^* are simply-connected for $i = 1, 2$.*

5) *$H^k(X_i, \mathcal{O}_{X_i}) = 0$ for $i = 1, 2$, $k = 1, 2, 3, 4$.*

6) *$H^k(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ for $k = 1, 2, 3$.*

Proof. The properties (1), (2), (3) are already shown.

We show the property (4). One can obtain X_i differently. Let q_1, q_2 be the fixed points of ψ. Let $\widetilde{X_i}$ be the blow-up of $Y \times \mathbb{P}^1$ along the surface $S_i \times \{q_1, q_2\}$. Then the involution on $Y \times \mathbb{P}^1$ induces an involution on $\widetilde{X_i}$, whose fixed locus is the exceptional divisor over $S_i \times \{q_1, q_2\}$. The quotient of $\widetilde{X_i}$ by the involution is isomorphic to X_i. This may be summarized by the diagram,

$$\begin{tikzcd} \widetilde{X_i} \arrow[r, "h"] \arrow[d] & X_i \arrow[d] \\ Y \times \mathbb{P}^1 \arrow[r] & \overline{X}_i, \end{tikzcd}$$

where $h : \widetilde{X_i} \to X_i$ is the quotient map. Let $f : \widehat{X_i} \to X_i$ be the universal covering. The fourfold $\widetilde{X_i}$ is simply-connected, therefore the map $h : \widetilde{X_i} \to X_i$ lifts to a map $g : \widetilde{X_i} \to \widehat{X_i}$ so that there is a commutative diagram:

$$\begin{tikzcd} & \widehat{X_i} \arrow[d, "f"] \\ \widetilde{X_i} \arrow[ur, "g"] \arrow[r, "h"'] & X_i. \end{tikzcd}$$

Since the involution $\rho_i \times \psi$ has fixed points, the induced involution on $\widetilde{X_i}$ also has fixed points and hence the branch locus of the double covering $h : \widetilde{X_i} \to X_i$ is not empty. Choose a point b in the branch locus of the map h. Then the set $h^{-1}(\{b\})$ $(= g^{-1}\left(f^{-1}(\{b\})\right))$ is composed of a single point. Hence the set $f^{-1}(\{b\})$ is also composed of a single point, which implies that f is of degree one because f is an étale covering. We conclude that the universal covering $f : \widehat{X_i} \to X_i$ is an isomorphism. Now $\widehat{X_i}$ is isomorphic to X_i and so X_i is simply-connected.

There is a natural projection: $\overline{X}_i = (Y \times \mathbb{P}^1)/(\rho_i \times \psi) \to Y/\rho_i$. Let ν be the composition of $X_i \to \overline{X}_i$ and $(Y \times \mathbb{P}^1)/(\rho_i \times \psi) \to Y/\rho_i$. Let $x \in Y/\rho_i$ be a point in the branch locus of the map $Y \to Y/\rho_i$. Then $\nu^{-1}(x)$ is a union of three smooth rational curves, one of which (denoted by l) crosses D_i transversely at a single point and the other two are disjoint from D_i, resulting from the blow-up. Since X_i and D_i are simply connected, the fundamental group $\pi_1(X_i^*)$ of X_i^* is generated by a loop around D_i. We can assume that the loop is contained in $l^* = l - D_i$. Since the loop can be contracted to a point in l^*, X_i^* is simply-connected.

We move on to the property (5). Since

$$\dim H^k(X_i, \mathcal{O}_{X_i}) \leq \dim H^k(\widetilde{X_i}, \mathcal{O}_{\widetilde{X_i}}) = \dim H^k(Y \times \mathbb{P}^1, \mathcal{O}_{Y \times \mathbb{P}^1}) = 0$$

for $k = 1, 2$, we have

$$H^k(X_i, \mathcal{O}_{X_i}) = 0$$

for $k = 1, 2$.

Note that X_i has an effective anticanonical divisor D_i, which is a Calabi–Yau threefold. Hence, we have

$$H^4(X_i, \mathcal{O}_{X_i}) \simeq H^0(X_i, \Omega^4_{X_i}) = 0.$$

Taking the cohomology of the structure sheaf sequence,

$$0 \to \mathcal{O}_{X_i}(K_{X_i}) \to \mathcal{O}_{X_i} \to \mathcal{O}_{D_i} \to 0,$$

we obtain an exact sequence

$$\begin{aligned} H^3(X_i, \mathcal{O}_{X_i}(K_{X_i})) \to H^3(X_i, \mathcal{O}_{X_i}) \to H^3(D_i, \mathcal{O}_{D_i}) \to \\ H^4(X_i, \mathcal{O}_{X_i}(K_{X_i})) \to H^4(X_i, \mathcal{O}_{X_i}) = 0. \end{aligned}$$

Since, by Serre duality,

$$H^3(X_i, \mathcal{O}_{X_i}(K_{X_i})) \simeq H^1(X_i, \mathcal{O}_{X_i}) = 0,$$

$$\dim H^4(X_i, \mathcal{O}_{X_i}(K_{X_i})) = \dim H^0(X_i, \mathcal{O}_{X_i}) = 1$$

and

$$\dim H^3(D_i, \mathcal{O}_{D_i}) = 1,$$

we have $\dim H^3(X_i, \mathcal{O}_{X_i}) = 0$.

Finally, we show the property (6). From the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{D_{\mathcal{X}}} \to 0,$$

we obtain an exact sequence

$$H^{k-1}(D_{\mathcal{X}}, \mathcal{O}_{D_{\mathcal{X}}}) \to H^k(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^k(X_1, \mathcal{O}_{X_1}) \oplus H^k(X_2, \mathcal{O}_{X_2}).$$

Since

$$H^{k-1}(D_{\mathcal{X}}, \mathcal{O}_{D_{\mathcal{X}}}) = H^k(X_1, \mathcal{O}_{X_1}) = H^k(X_2, \mathcal{O}_{X_2}) = 0$$

for $k = 2, 3$, we have

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^3(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0.$$

Moreover, the exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^0(X_1, \mathcal{O}_{X_1}) \oplus H^0(X_2, \mathcal{O}_{X_2}) \to H^0(D_{\mathcal{X}}, \mathcal{O}_{D_{\mathcal{X}}}) \\ \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^1(X_1, \mathcal{O}_{X_1}) \oplus H^1(X_2, \mathcal{O}_{X_2}) = 0 \end{aligned}$$

gives $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$.

$\square$

4. The example

By Theorem 2.1 with the properties in Proposition 3.2, one can show that the normal crossing variety $\mathcal{X}$, constructed in Section 3, is smoothable to a smooth fourfold $M_{\mathcal{X}}$ with trivial canonical class. We can also check that $H^i(M_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = 0$ for $i = 1, 2, 3$ by the upper semicontinuity theorem with the property (6) in Proposition 3.2.

Theorem 4.1. *$M_{\mathcal{X}}$ is a non-Kähler Calabi–Yau fourfold.*

Proof. We only need to show that $M_{\mathcal{X}}$ is simply-connected and non-Kähler.

First, we show the simply-connectedness. We can obtain the topological type of $M_{\mathcal{X}}$ by pasting X_1^* and X_2^*. One can regard the normal bundle $N_{D_{\mathcal{X}}/X_i}$ as a complex manifold containing $D_{\mathcal{X}}$. Then

$$N^*_{D_{\mathcal{X}}/X_i} := N_{D_{\mathcal{X}}/X_i} - D_{\mathcal{X}}$$

is a $\mathbb{C}^*$-bundle over $D_{\mathcal{X}}$, where $\mathbb{C}^* := \mathbb{C} - \{0\}$. The triviality property on $N_{D_{\mathcal{X}}/X_1} \otimes N_{D_{\mathcal{X}}/X_2}$ implies the map

$$\varphi : N^*_{D_{\mathcal{X}}/X_1} \to N^*_{D_{\mathcal{X}}/X_2},$$

locally defined by

$$(x \in \mathbb{C}^*, y \in D_{\mathcal{X}}) \mapsto (1/x, y)\,,$$

is globally well-defined and an isomorphism. Note that $D_{\mathcal{X}}$ in X_i has a neighborhood U_i that is homeomorphic to $N_{D_{\mathcal{X}}/X_i}$. Let $U_i^* = U_i - D_{\mathcal{X}}$. Then the map φ induces a homeomorphism between U_1^* and U_2^*. One can construct a manifold M' by pasting together X_1^* and X_2^* along U_1^* and U_2^* with the homeomorphism. The manifold M' is homeomorphic to $M_{\mathcal{X}}$. Note that X_1^*, X_2^* are simply-connected (the property (4) in Proposition 3.2). Hence, by Seifert–van Kampen theorem, M' is simply-connected.

For the non-Kählerness, suppose that $M_{\mathcal{X}}$ is Kähler, then it is necessarily projective. Note that $D_{\mathcal{X}}$, X_1 and X_2 are all projective (the property (1) in Proposition 3.2). Hence, by Lemma 2.2, there exists a big line bundle $\mathcal{L}$ on $\mathcal{X}$. Let h_i be the big divisor class in $\mathrm{Pic}(X_i)$ corresponding to $\mathcal{L}|_{X_i}$, then $h_1|_{D_{\mathcal{X}}}$ is linearly equivalent to $h_2|_{D_{\mathcal{X}}}$. Note that $D_{\mathcal{X}}$ is a copy of Y. Let us denote the divisor class in $\mathrm{Pic}(Y)$ of $h_1|_{D_{\mathcal{X}}}$, $h_2|_{D_{\mathcal{X}}}$ by $\hat{h}$. Chasing the construction of X_1 and X_2, one can check that $\hat{h}$ belongs to $\mathrm{Pic}(Y)^{\rho_1^*} \cap \mathrm{Pic}(Y)^{\rho_2^*}$, where $\mathrm{Pic}(Y)^{\rho_i^*}$ is the subgroup of $\mathrm{Pic}(Y)$ that consists of the classes invariant under ρ_i^*.

The linear system $|D_{\mathcal{X}}|$ is base-point free and it gives a fibration $X_i \to \mathbb{P}^1$ and $D_{\mathcal{X}}$ is one of its generic fibers. Hence $\hat{h}$ is a big divisor of Y (see, for example, Corollary 2.2.11 in [18]). Let $\phi : \widetilde{Y} \to E_\zeta^3$ be the blow-up at the 27 points of $Q_{i,j,k}$'s and $\eta : \widetilde{Y} \to Y$ be the map induced by $E_\zeta^3 \to \overline{Y}$ such that the diagram commutes:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\eta} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ E_\zeta^3 & \longrightarrow & \overline{Y} \end{array}$$

It is not hard to check that $\check{h} = \phi_*(\eta^*(\hat{h}))$ is a big divisor of E_ζ^3 and the class $\check{h}$ belongs to $\mathrm{NS}(E_\zeta^3)^{\sigma_1^*} \cap \mathrm{NS}(E_\zeta^3)^{\sigma_2^*}$. Note that any big divisor of the abelian variety E_ζ^3 is ample. However, the group of automorphisms of E_ζ^3 that is generated by σ_1, σ_2 is infinite (Remark 3.1) and so $\mathrm{NS}(E_\zeta^3)^{\sigma_1^*} \cap \mathrm{NS}(E_\zeta^3)^{\sigma_2^*}$ does not contain an ample class. Therefore, we have a contradiction and $M_{\mathcal{X}}$ should be non-Kähler. □

Topological invariants of $M_{\mathcal{X}}$ can be calculated from the topological manifold M' in the proof of Theorem 4.1. For example, the topological Euler number $\chi(M_{\mathcal{X}})$ of $M_{\mathcal{X}}$ is

$$\chi(M_{\mathcal{X}}) = \chi(M') = \chi(X_1^*) + \chi(X_2^*) - \chi(U_1).$$

Note

$$\chi(X_i) = \chi(X_i^*) + \chi(D_i) = \chi(X_i^*) + \chi(Y)$$

and $\chi(U_1) = \chi(S^1)\chi(D_1) = 0$. Hence,

$$\chi(M_{\mathcal{X}}) = \chi(X_1) + \chi(X_2) - 2\chi(Y).$$

The topological Euler characteristic of $\widetilde{X}_i$ is

$$\chi(\widetilde{X}_i) = 2\chi(Y) + 2\chi(S_i)$$

On the other hand,

$$2\chi(X_i) - 4\chi(S_i) = \chi(\widetilde{X}_i)$$

and so

$$\begin{aligned}\chi(X_i) &= \frac{1}{2}\left(4\chi(S_i) + \chi(\widetilde{X}_i)\right)\\ &= 2\chi(S_i) + \chi(Y) + \chi(S_i)\\ &= \chi(Y) + 3\chi(S_i).\end{aligned}$$

Let $\check{S}_i = \phi(\eta^{-1}(S_i))$. Note that $\check{S}_i$ is the fixed locus of σ_i. Let $\Theta = \{Q_{i,j,k} \mid i,j,k = 0,1,2\}$. One can easily check

$$\chi(S_i) = 2\left|\check{S}_i \cap \Theta\right| = 18,$$

where $\left|\check{S}_i \cap \Theta\right|$ is the number of points in $\check{S}_i \cap \Theta$.

Therefore, the topological Euler number $\chi(M_{\mathcal{X}})$ of $M_{\mathcal{X}}$ is

$$\begin{aligned}\chi(M_{\mathcal{X}}) &= \chi(X_1) + \chi(X_2) - 2\chi(Y)\\ &= \chi(Y) + 3\chi(S_1) + \chi(Y) + 3\chi(S_2) - 2\chi(Y)\\ &= 3\chi(S_1) + 3\chi(S_2)\\ &= 108.\end{aligned}$$

The pair of matrices A_1, A_2 in Section 2, which can be used in the construction, is obviously not unique. Any pair of 3×3 matrices A_1, A_2 that satisfies the following conditions,

1) $A_i \in \mathrm{GL}_3(\mathbb{Z}[\zeta])$,
2) A_i^2 is the 3×3 identity matrix,
3) $\det(A_i) = -1$,
4) ρ_i is not fixed-free and its fixed locus is a smooth surface and
5) The subgroup of $\mathrm{GL}_3(\mathbb{Z}[\zeta])$ containing both A_1, A_2 is infinite,

gives rise to a non-Kähler Calabi–Yau fourfolds through the construction of Sections 3 and 4, where the conditions guarantee, respectively,

1) A_i induces an automorphism of E_ζ^3 which induces an automorphism of Y.
2) A_i induces an involution of E_ζ^3 which induces an involution of Y,
3) ρ_i^* acts as multiplication by -1 on $H^{3,0}(Y)$ so that X_i has an anti-canonical section isomorphic to Y,

4) both X_i and X_i^* are simply-connected, which leads to the simply-connectedness of $M_{\mathcal{X}}$ and

5) the group of automorphisms of E_ζ^3 that is generated by σ_1, σ_2 is infinite, which eventually leads to the non-Kählerness of $M_{\mathcal{X}}$.

The author could not obtain other non-Kähler Calabi–Yau fourfolds of different topological Euler numbers although he tried many pairs of such matrices. The author suspects that all the pairs of such matrices may give rise to non-Kähler Calabi–Yau fourfolds of the same topological Euler number ($= 108$).

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