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A theorem on Hermitian rank and mapping problems

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In this paper, we first prove a Huang's lemma type result. Then we discuss its applications in studying rigidity problems of mappings into indefinite hyperbolic spaces and bounded symmetric domains.

1. Introduction

It is a classical problem in several complex variables to understand proper holomorphic maps between complex unit balls since the pioneer work of Poincaré and Alexander (see [Al]). The classical result of Alexander asserts that any proper holomorphic self-mapping of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ must be an automorphism if $n \geq 2$. Since the work of Webster [W], much effort has also been made to study proper maps between unit balls of different dimensions. See [Fr], [CS], [St], [Hu], [HJY], [DX] and many references therein for research along this line. A seminal step toward understanding this problem was made by Huang in [Hu]. Huang proved when $n < N \leq 2n-2$, any proper holomorphic map F from $\mathbb{B}^n$ to $\mathbb{B}^N$ is totally geodesic with respect to the Bergman metrics if F extends C^2-smoothly up to some open piece of the boundary $\partial\mathbb{B}^n$. One crucial ingredient in his proof is an algebraic lemma (Lemma 3.2 in [Hu]), which is nowadays known as *Huang's lemma* in the field due to its wide applications. This lemma reveals the deep connection between the mapping problem in CR geometry and the rank problem in real algebraic geometry. Here we recall the definition of the rank of a real polynomial or more generally a real-valued real analytic function $R(z,\overline{z})$ at some point $z_0 \in \mathbb{C}$. Suppose $R(z,\overline{z})$ can be written as $R(z,\overline{z}) = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2, p, q \in \mathbb{Z}_{\geq 0}$, where f_i's and g_j's are holomorphic functions near z_0, and $f_1, \cdots, f_p, g_1, \cdots, g_q$ are linearly independent over $\mathbb{C}$. Then we say $R(z,\overline{z})$ is of finite rank and $r = p+q$ is called the rank of $R(z,\overline{z})$. We remark that the rank of $R(z,\overline{z})$ is independent of the choices of f_i's and g_j's. The rank of $R(z,\overline{z})$ is zero if and only if $R(z,\overline{z})$ is identically zero.

Huang's lemma can be stated as follows. Write $z = (z_1, \cdots, z_m)$ for the coordinates in $\mathbb{C}^m, m \geq 2$. Write $|z|$ for the Euclidean norm of z. Let $A(z, \overline{z})$ be a real analytic function near 0 such that

$$A(z, \overline{z})|z|^2 = \sum_{j=1}^{m-1} \psi_j(z)\overline{\phi_j(z)}, \tag{1.1}$$

where $\psi_j(z)$ and $\phi_j(z)$ are holomorphic functions near $0 \in \mathbb{C}^m$. Then $A(z, \overline{z})$ must be identically zero. In the particular case when $A(z, \overline{z})$ is real-valued, Huang's lemma implies the rank of $A(z, \overline{z})|z|^2$ cannot be less than m unless $A(z, \overline{z})$ is of rank zero. The importance of Huang's lemma lies in the fact that it provides an effective tool to detect the degeneration of CR second fundamental form of a CR maps between spheres (see [Hu] for more details). For more discussion on various versions of Hermitian rank problems and their connections to mapping problems, see [DL], [E1], [E2] and references therein. Recently, Ebenfelt systematically studied a rank problem (i.e., the sums of square problem introduced in [E2]. See also [E1]) in real algebraic geometry and discussed how it is related to a gap rigidity phenomenon (see Huang-Ji-Yin [HJY]) for proper maps between unit balls. Huang's lemma also plays an important role in the study of mapping problems into generalized balls or hyperquadrics. Recall the generalized ball $\mathbb{B}_l^n, 0 \leq l \leq n-1$, is defined as the following open subset of $\mathbb{P}^n$:

$$\mathbb{B}_l^n = \{[z_0, \cdots, z_n] \in \mathbb{P}^n : |z_0|^2 + \cdots + |z_l|^2 > |z_{l+1}|^2 + \cdots + |z_n|^2\}.$$

The generalized ball has an important geometric feature as it inherites a canonical metric that is invariant under the action of its automorphisms:

$$\omega_{\mathbb{B}_l^n} = -\sqrt{-1}\partial\bar{\partial}\log\Big(\sum_{j=0}^{l} |z_j|^2 - \sum_{j=l+1}^{n} |z_j|^2\Big).$$

When $l = 0$, the metric is identical with the (normalized) Poincaré metric on the unit ball. The generalized ball equipped with the metric $\omega_{\mathbb{B}_l^n}$ is often called the indefinite hyperbolic space. See [BH], [EHZ], [BEH] for many deep results on mappings into generalized balls or hyperquadrics, as well as various different versions of Huang's lemma and their applications. See also recent papers [HLTX1, HLTX2] and references therein. Roughly speaking, the complexity of proper holomorphic maps from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$ depends heavily on l and l'. We mention the following result of [HLTX2]. Here we say a holomorphic map F from an open subset V of $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$ is isometric if $F^*(\omega_{\mathbb{B}^N_{l'}}) = \omega_{\mathbb{B}^n_l}$ on V.

Theorem 0.1 *(Huang-Lu-Tang-Xiao [HLTX2]) Let $N \geq n \geq 3$, $1 \leq l \leq n-2, l \leq l' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}^n_l$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $U \cap \mathbb{B}^n_l$ is connected and $F(U \cap \mathbb{B}^n_l) \subseteq \mathbb{B}^N_{l'}$, $F(U \cap \partial\mathbb{B}^n_l) \subseteq \partial\mathbb{B}^N_{l'}$. Assume one of the following conditions holds:*

$$\begin{aligned}
&(1).\ \ l' < 2l, l' < n-1;\\
&(2).\ \ l' < 2l, N - l' < n;\\
&(3).\ \ N - l' < 2n - 2l - 1, l' < n-1;\\
&(4).\ \ N - l' < 2n - 2l - 1, N - l' < n.
\end{aligned}$$

Then F is an isometric embedding from $(U \cap \mathbb{B}^n_l, \omega_{\mathbb{B}^n_l})$ to $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$.

The main result of the paper is a Huang's lemma type theorem, i.e., Theorem 1. To explain our result, we first introduce some notations. Fix $0 \leq l \leq m$, we denote by $\delta_{j,l}$ the symbol which equals -1 when $1 \leq j \leq l$ and equals 1 otherwise. In particular, if $l = 0, \delta_{j,0}$ is identically one for all $j \geq 1$. Write $z = (z_1, \cdots, z_m)$ for the coordinates in $\mathbb{C}^m$. For $z, w \in \mathbb{C}^m$, we write $\langle z, w\rangle_l = \sum_{j=1}^m \delta_{j,l} z_j w_j$ and $|z|^2_l = \langle z, \overline{z}\rangle_l$. If $l = 0$, we have $|z|^2_0 = |z|^2$. Denote by $I_{l,m}$ the diagonal $m \times m$ matrix whose first l diagonal entries are -1 and the rest are 1. We are now at the position to introduce our main theorem.

Theorem 1. *Let $m \geq 3$ and $0 \leq l \leq m$. Let $\{\psi_j(z)\}_{j=1}^m$ and $\{\phi_j(z)\}_{j=1}^m$ be holomorphic functions in $z \in \mathbb{C}^m$ near 0. Assume there is a real-analytic function $A(z, \overline{z})$ near 0 such that*

$$A(z, \overline{z})|z|^2_l = \sum_{j=1}^m \psi_j(z)\overline{\phi_j(z)}. \tag{1.2}$$

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If $A(z, \overline{z}) \not\equiv 0$, then there exist holomorphic functions h_1, h_2 near 0, and $B, C \in GL(m, \mathbb{C})$ with $B\overline{C}^t = I_{l,m}$, such that $A(z, \overline{z}) = h_1(z)\overline{h_2(z)}$, and

$$(\psi_1, \cdots, \psi_m) = h_1(z)(z_1, \cdots, z_m)B; \quad (\phi_1, \cdots, \phi_m) = h_2(z)(z_1, \cdots, z_m)C.$$

Remark 1.1. If in addition $A(z, \overline{z})$ is real-valued in Theorem 1, then we can choose in such a way that $h_2 = h_1$ or $h_2 = -h_1$, and thus $A(z, \overline{z}) = \pm|h_1(z)|^2$ for some holomorphic function h_1 near 0.

The following result is an immediate consequence of Theorem 1.

Corollary 1.1. *Let m and l be as in Theorem 1. Let $0 \leq \tau^+, \tau^- \leq m$ such that $1 \leq \tau^+ + \tau^- \leq m$. Let $A(z, \overline{z})$ be a real-valued real analytic function near 0, and $\{a_i(z)\}_{i=1}^{\tau^-}, \{b_j(z)\}_{j=1}^{\tau^+}$ be two sets of holomorphic functions near 0 such that*

$$A(z, \overline{z})|z|_l^2 = -\sum_{i=1}^{\tau^-} |a_i(z)|^2 + \sum_{j=1}^{\tau^+} |b_j(z)|^2.$$

Then one of the following three mutually exclusive cases must hold:

1) $A(z, \overline{z}) \equiv 0$.
2) $A(z, \overline{z}) = |h(z)|^2$ *for some nonzero holomorphic function* $h(z)$ *and* $\tau^- = l, \tau^+ = m - l$.
3) $A(z, \overline{z}) = -|h(z)|^2$ *for some nonzero holomorphic function* $h(z)$ *and* $\tau^- = m - l, \tau^+ = l$.

Moreover, in case (2) and (3), $\{a_i(z), b_j(z)\}_{1 \leq i \leq \tau^-, 1 \leq j \leq \tau^+}$ *must be linearly independent over* $\mathbb{C}$.

We remark that Theorem 1 and Corollary 1.1 both fail if $m = 2$. For example, let $z = (z_1, z_2) \in \mathbb{C}^2$ and $A(z, \overline{z}) = |z_1|^2 - |z_2|^2$. Then $A(z, \overline{z})|z|^2 = |z_1|^4 - |z_2|^4$, and $A(z, \overline{z})$ does not satisfy the conclusions of Theorem 1 and Corollary 1.1. See also the following more general examples.

Example 1.1. 1) Let $z = (z_1, z_2) \in \mathbb{C}^2$ and $A(z, \overline{z}) = |z_1|^{2n-2} + |z_1|^{2n-4}|z_2|^2 + \cdots + |z_2|^{2n-2}$ for $n \geq 2$. Then $A(z, \overline{z})|z|_1^2 = |z_1|^{2n} - |z_2|^{2n}$.

2) Let $z = (z_1, z_2) \in \mathbb{C}^2$ and let $k \geq 2$. Note that there exists a unique real polynomial $A(z, \overline{z})$ such that $A(z, \overline{z})|z|^2 = |z_1|^{2^k} - |z_2|^{2^k}$, and $A(z, \overline{z})$ does not equal $\pm|h(z)|^2$ for any holomorphic function $h(z)$.

We remark that, when $0 < l < m$, one can also directly prove Corollary 1.1 by using the result of [BH] or [BEH]. Indeed, if $0 < l < m$, the map $(a_1(z), \cdots, a_{\tau^-}(z), b_1(z), \cdots, b_{\tau^+}(z))$ induces a holomorphic map sending the quadric $\{|z|_l^2 = 0\}$ to another quadric. Then similarly as in the proof of Lemma 2.3 in [BEH], one can reduce it to a mapping problem between hyperquadrics and so that the rigidity result in [BH] or [BEH] can be applied. This approach, however, does not work for the cases $l = 0$ and $l = m$.

We will prove Theorem 1 by reducing it to a mapping problem. One will see that the proof of Theorem 1 breaks down when $m = 2$ due to the failure of Poincaré type result in one dimensional case (see §2). Note Corollary 1.1 implies that, if $m \geq 3$ and the rank of $A(z,\overline{z})|z|_l^2$ is less than or equal to m, then $A(z,\overline{z})$ must be of rank either zero or one. We expect Theorem 1 and Corollary 1.1 to be useful in the future study of mapping problems in CR geometry. In particular, in this paper we will apply them to establish rigidity theorems (see Corollary 1.2 and 1.3) for mappings into indefinite hyperbolic spaces and bounded symmetric domains.

Corollary 1.2. *Let $n \geq 4, 0 \leq l \leq n-1$ and $0 \leq l' \leq 2n-2$. Let U be an open subset of $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}_l^n$ such that $U \cap \mathbb{B}_l^n$ is connected. Let $F : U \to \mathbb{P}^{2n-1}$ be a holomorphic map such that $F(U \cap \mathbb{B}_l^n) \subseteq \mathbb{B}_{l'}^{2n-1}$ and $F(U \cap \partial\mathbb{B}_l^n) \subseteq \partial\mathbb{B}_{l'}^{2n-1}$. If $l' \neq 2l$ and $l' \neq n-1$, then F is an isometric embedding from $(U \cap \mathbb{B}_l^n, \omega_{\mathbb{B}_l^n})$ to $(\mathbb{B}_{l'}^{2n-1}, \omega_{\mathbb{B}_{l'}^{2n-1}})$.*

We have the following remark and example regarding Corollary 1.2.

Remark 1.2. 1) Corollary 1.2 is optimal in the sense that the conclusion fails if either $l' = 2l$ or $l' = n-1$. Indeed, there is the well-known Whitney map if $l' = l = 0$. More generally, see Example 1.6 in [HLTX2] for the generalized Whitney maps in the case $l' = 2l > 0$, and Example 1.7 in [HLTX2] for the generalized Whitney maps in the case $l' = n-1$ with $1 \leq l \leq n-1$, and the following Example 1.2 for the generalized Whitney maps in the case $l' = n-1$ with $0 \leq l \leq n-2$.

2) In the special case $1 \leq l \leq n-2$, Corollary 1.2 follows also from Theorem 0.1 (i.e., Theorem 1.1 in [HLTX2]). Indeed, the assumption of Corollary 1.2 yields one of the four conditions holds in Theorem 0.1. It however does not cover the cases $l = 0$ and $l = n-1$. We also remark that to prove for these two cases, we don't need to use the full generality of Theorem 1 (or Corollary 1.1).

Example 1.2. Let $l \geq 0, k \geq 2$. Write $[w,z] = [w_0, w_1, \cdots, w_l, z_1, \cdots, z_k]$ for the homogeneous coordinates of $\mathbb{P}^{l+k}$ and

$$\mathbb{B}_l^{l+k} = \{[w,z] \in \mathbb{P}^{k+l} : \sum_{i=0}^{l} |w_i|^2 > \sum_{j=1}^{k} |z_j|^2\}.$$

Let $V = \mathbb{P}^{l+k} \setminus \{z_1 = z_k = 0\}$ and $H : V \to \mathbb{P}^{2k+2l-1}$ be defined as follows:

$$\begin{aligned} H([w,z]) = [&w_0 z_k, w_1 z_k, \cdots, w_l z_k, z_1^2, z_1 z_2, \cdots, z_1 z_{k-1}, \\ &z_2 z_k, z_3 z_k, \cdots, z_k^2, w_0 z_1, w_1 z_1, \cdots, w_l z_1]. \end{aligned}$$

Notice that $|H|^2_{l+k} = (|z_k|^2 - |z_1|^2)(-\sum_{i=0}^{l} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Thus H maps $V \cap \partial\mathbb{B}^{l+k}_l$ to $\partial\mathbb{B}^{2l+2k-1}_{l+k-1}$. In particular, set $V_+ := \{[w,z] \in V : |z_k| > |z_1|\}$. Then H maps $V_+ \cap \mathbb{B}^{l+k}_l$ to $\mathbb{B}^{2l+2k-1}_{l+k-1}$ and maps $V_+ \cap \partial\mathbb{B}^{l+k}_l$ to $\partial\mathbb{B}^{2l+2k-1}_{l+k-1}$. Hence the conclusion in Corollary 1.2 fails if $l' = n - 1$.

Corollary 1.2 can be applied to study proper maps from the unit ball to classical domains. The study of holomorphic maps from the unit ball to higher rank classical domain was initiated by Mok [M] and later investigated in [CM], [Ch], [UWZ], [XY1], [XY2] and [X], etc. In particular, Yuan and the author [XY1] studied holomorphic proper maps from the unit ball to the type IV classical domains (also called the Lie ball). Recall the Lie ball D^{IV}_N in $\mathbb{C}^N (N \geq 2)$ is defined by

$$D^{IV}_N = \{Z = (z_1, \cdots, z_N) \in \mathbb{C}^N : Z\overline{Z}^t < 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

We normalize the Bergman metric on $\mathbb{B}^n$ and D^{IV}_N so that the minimal disc is of constant Gaussian curvature -2. Denote by $\omega_{\mathbb{B}^n}$ and $\omega_{D^{IV}_N}$ the two normalized Bergman metrics of $\mathbb{B}^n$ and D^{IV}_N, respectively. We say a holomorphic map $F : \mathbb{B}^n \to D^{IV}_N$ is an isometric embedding or simply an isometry if $F^*(\omega_{D^{IV}_N}) = \omega_{\mathbb{B}^n}$. The following result follows from the work in [XY1] and [X]: Let F be a holomorphic proper map from $\mathbb{B}^n$ to the Lie ball $D^{IV}_N (5 \leq n+1 \leq N \leq 2n-3)$ that is $C^{N-n+1}-$smooth up to some open piece of $\partial\mathbb{B}^n$. Then F is an isometric embedding with $F^*(\omega_{D^{IV}_N}) = \omega_{\mathbb{B}^n}$. Furthermore, counterexamples were given in [XY1] to illustrate such rigidity result fails if $N \geq 2n$, no matter what boundary regularity is assumed. Yuan and the author thus raised the question to understand whether the rigidity still holds in the remaining cases $N = 2n-2$ and $N = 2n-1$. In the last part of the paper, we apply Corollary 1.2 to give an affirmative answer to this question in the case $N = 2n-2$.

Corollary 1.3. *Let F be a holomorphic proper map from $\mathbb{B}^n (n \geq 4)$ to D^{IV}_{2n-2} that extends $C^{n-1}-$smoothly across some open piece of $\partial\mathbb{B}^n$. Then F is an isometric embedding (with respect to the normalized Bergman metrics).*

The paper is organized as follows. Section 1 includes the proof of Theorem 1 and Corollary 1.1, except that a technical lemma (i.e., Lemma 2.2) will be established in Section 4. We prove Corollary 1.2 and Corollary 1.3 in Section 3.

2. Proof of Theorem 1

In this section, we give a proof of Theorem 1. As was mentioned, we will reduce it to a mapping problem between complex quadrics in $\mathbb{P}^m \times \mathbb{P}^m$. We recall the following result (Lemma 2.1) due to Chern-Ji (see [CJ1], [CJ2]), which is a well-known generalization of Poincaré type theorem to Segre families. Let $[z] = [z^0, \cdots, z^m] \in \mathbb{P}^m$ and $[\xi] = [\xi_0, \cdots, \xi_m] \in \mathbb{P}^m$. Let $\mathcal{M} \subseteq \mathbb{P}^m \times \mathbb{P}^m$ be defined by

$$\mathcal{M} := \{([z], [\xi]) \in \mathbb{P}^m \times \mathbb{P}^m : \sum_{j=0}^{m} z_j \overline{\xi}_j = 0\}.$$

Lemma 2.1. *(Lemma 3.1 in [CJ2]) Let $U, \widetilde{U}$ and $V, \widetilde{V}$ be connected open subsets of $\mathbb{P}^m_z$ and $\mathbb{P}^m_\xi (m \geq 2)$, respectively. Assume $(U \times V) \cap \mathcal{M} \neq \emptyset$. If $f : U \to \widetilde{U}$ and $g : V \to \widetilde{V}$ are biholomorphic maps such that*

$$f \times g\,((U \times V) \cap \mathcal{M}) \subseteq \mathcal{M},$$

then f and g are restrictions of elements of $\mathrm{PGL}(m+1, \mathbb{C})$.

This result is, however, not sufficient for our application to prove Theorem 1. We will need Lemma 2.2, which is a more general version of Lemma 2.1. It proves a Poincaré type result for holomorphic maps from a degenerate complex quadric. See other types of generalization of Lemma 2.1 in [Zh] and references therein.

Write $w = (w_0, \cdots, w_{m-1}) \in \mathbb{C}^m$, and $\eta = (\eta_0, \cdots, \eta_{m-1}) \in \mathbb{C}^m$. And define

$$\mathcal{M}_0 = \left\{ (w, \eta) \in \mathbb{C}^m \times \mathbb{C}^m : \sum_{j=1}^{m-1} w_j \overline{\eta}_j + 1 = 0 \right\};$$

$$\mathcal{M}_1 = \left\{ (w, \eta) \in \mathbb{C}^m \times \mathbb{C}^m : \sum_{j=1}^{m-2} w_j \overline{\eta}_j + w_{m-1} + \overline{\eta}_{m-1} = 0 \right\}.$$

Note $\mathcal{M}_0$ and $\mathcal{M}_1$ are degenerate in the sense that their defining functions do not depend on w_0, η_0. Write $\chi = (\chi_1, \cdots, \chi_{m-1})$ and $\tau = (\tau_1, \cdots, \tau_{m-1})$.

Set

$$\hat{\mathcal{M}}_0 = \left\{ (\chi, \tau) \in \mathbb{C}^{m-1} \times \mathbb{C}^{m-1} : \sum_{j=1}^{m-1} \chi_j \overline{\tau}_j + 1 = 0 \right\};$$

$$\hat{\mathcal{M}}_1 = \left\{ (\chi, \tau) \in \mathbb{C}^{m-1} \times \mathbb{C}^{m-1} : \sum_{j=1}^{m-2} \chi_j \overline{\tau}_j + \chi_{m-1} + \overline{\tau}_{m-1} = 0 \right\}.$$

We are now in a position to formulate Lemma 2.2.

Lemma 2.2. *(a). Let $U \subseteq \mathbb{C}^m_w, V \subseteq \mathbb{C}^m_\eta (m \geq 3)$ be connected open subsets of $\mathbb{C}^m$ with $(U \times V) \cap \mathcal{M}_0 \neq \emptyset$. Let $f(w) = (f_1(w), \cdots, f_{m-1}(w)), g(\eta) = (g_1(\eta), \cdots, g_{m-1}(\eta))$ be holomorphic maps in U and V respectively. Assume f, g are nondegenerate in $(w_1, \cdots, w_{m-1})$ and $(\eta_1, \cdots, \eta_{m-1})$, respectively. That is, the matrices $\left(\frac{\partial f_i}{\partial w_j}\right)_{1 \leq i \leq m-1, 1 \leq j \leq m-1}$ and $\left(\frac{\partial g_i}{\partial \eta_j}\right)_{1 \leq i \leq m-1, 1 \leq j \leq m-1}$ are nondegenerate everywhere in U and V, respectively. Assume $f \times g$ sends $\mathcal{M}_0 \cap (U \times V)$ to $\hat{\mathcal{M}}_0$. Then f, g do not depend on the variables w_0 and η_0, respectively. Moreover, f, g extend to holomorphic linear fractional maps in $(w_1, \cdots, w_{m-1})$ and $(\eta_1, \cdots, \eta_{m-1})$, respectively.*

(b). The statement in part (a) still holds if $\mathcal{M}_0$ is replaced by $\mathcal{M}_1$ or $\hat{\mathcal{M}}_0$ is replaced by $\hat{\mathcal{M}}_1$.

We will postpone the proof of Lemma 2.2 to Section 4 and concentrate on the proof of Theorem 1 here. For that we first need to establish the following key proposition for the polynomial case.

Proposition 2.1. *Let $z = (z_1, \cdots, z_m), m \geq 3$. Let $\psi(z) = (\psi_1, \cdots, \psi_m)$ and $\phi(z) = (\phi_1, \cdots, \phi_m)$ be holomorphic polynomial map from $\mathbb{C}^m$ to $\mathbb{C}^m$. Assume $A(z, \overline{z})$ is a polynomial in $(z, \overline{z})$ such that*

$$A(z, \overline{z}) |z|^2_l = \sum_{j=1}^{m} \psi_j(z) \overline{\phi_j(z)}. \tag{2.1}$$

If $A(z, \overline{z}) \not\equiv 0$, then there exist holomorphic polynomials $h_1(z), h_2(z)$ and $B, C \in GL(m, \mathbb{C})$ with $B\overline{C}^t = I_{l,m}$, such that $A(z, \overline{z}) = h_1(z)\overline{h_2(z)}$, and

$$\psi(z) = h_1(z)(z_1, \cdots, z_m)B; \qquad \phi(z) = h_2(z)(z_1, \cdots, z_m)C. \tag{2.2}$$

Proof. We first prove Proposition 2.1 under the following additional assumption.

Assumption (*): Suppose $\psi_j(0) = 0$ and $\phi_j(0) = 0$ for all $1 \le j \le m$.

Recall a holomorphic map $\varphi = (\varphi_1, \cdots, \varphi_m)$ defined near $p \in \mathbb{C}^m$ is called nondegenerate at p if the Jacobian matrix $\left(\frac{\partial \varphi_i}{\partial z_j}\right)_{1 \le i,j \le m}$ is invertible at p. We will proceed in two different cases.

Case I: We first suppose either ψ or ϕ is degenerate everywhere. Without loss of generality, assume ψ is degenerate everywhere. Then it follows from Huang's proof of his original lemma (see Lemma 3.2 in [Hu]) that $A(z, \overline{z}) \equiv 0$. For the self-containedness of this paper, we sketch a proof here. Write $\xi = (\xi_1, \cdots, \xi_m)$. We first complexify (2.1) to obtain

$$A(z, \overline{\xi})\langle z, \overline{\xi}\rangle_l = \sum_{j=1}^{m} \psi_j(z)\overline{\phi_j(\xi)}, \qquad \forall z, \xi \in \mathbb{C}^m. \tag{2.3}$$

Note we can assume $\psi_j \not\equiv 0$ for every j (Otherwise, it is reduced to the case of Huang's original lemma, i.e., Lemma 3.2 in [Hu]). Then by the degeneracy of ψ, we can find some point $z = p$ near 0 such that

(1). $\psi_j(p) = \epsilon_j \neq 0$ for at least one j; and

(2). $V_p = \{z \approx p : \psi_j(z) = \psi_j(p), \forall\ 1 \le j \le m\}$ defines a complex variety of dimension at least 1 near p.

Since $\psi_j(0) = 0$ and $\epsilon_j \neq 0$, we see V_p cannot contain any complex line passing through the origin. Hence there is a point $p^* \in V_p$ such that V_p contains a complex curve C^* near p^* which is parametrized by an equation of the form:

$$z(t) = p^* + vt + o(t). \tag{2.4}$$

Here $\{p^*, v\}$ are independent vectors and $|t| < 1$. Note for each $z \in C^*$ and ξ with $\langle z, \overline{\xi}\rangle_l = 0$, by (2.3) we have $\sum_{j=1}^m \overline{\epsilon_j}\phi_j(\xi) = 0$. Also (2.4) implies all such ξ fill in an open subset of $\mathbb{C}^m$. We see $\sum_{j=1}^m \overline{\epsilon_j}\phi_j(z) \equiv 0$. Then (2.1) is reduced to

$$A(z, \overline{z})|z|_l^2 = \sum_{j=1}^{m-1} \left(\psi_j(z) - \frac{\epsilon_j}{\epsilon_m}\psi_m(z)\right) \overline{\phi_j(z)}.$$

Then it follows from Lemma 3.2 in [Hu] that $A(z, \overline{z}) \equiv 0$. This contradicts with the assumption.

Case II: We then suppose both ψ and ϕ are of generically full rank. Equivalently, at a generic point z^* (respectively, a generic point ξ^*), ψ (respectively, ϕ) is a local biholomorphism. Assume $A(z,\overline{z})$ has bidegree (d_0,d_1) in $(z,\overline{z})$ i.e., the highest degree in z (respectively, in $\overline{z}$) equals d_0 (respectively, equals d_1). Write $d_2=\max\{d_0,d_1\}$. Assume the highest degree of $\psi_j(z)$ and $\phi_j(z), 1\leq j\leq m$, is $\hat{d}_2$. Then $\hat{d}_2\geq d_2+1$. Write $d=\hat{d}_2-1$. Write $\widetilde{z}=(z_0,z)\in\mathbb{C}\times\mathbb{C}^m$ and set $\hat{A}(\widetilde{z},\overline{\widetilde{z}})=|z_0|^{2d}A(\frac{z}{z_0},\overline{\frac{z}{z_0}})$ and $\hat{\psi}_j(\widetilde{z})=z_0^{d+1}\psi(\frac{z}{z_0}),\hat{\phi}_j(\widetilde{z})=z_0^{d+1}\phi(\frac{z}{z_0})$ for all j. Note $\hat{A}(\widetilde{z},\overline{\widetilde{z}})$ and $\hat{\psi}_j(\widetilde{z}),\hat{\phi}_j(\widetilde{z})$ are all homogeneous polynomials. Moreover, by homogenizing (2.1), we obtain

$$\hat{A}(\widetilde{z},\overline{\widetilde{z}})|z|_l^2=\sum_{j=1}^m\hat{\psi}_j(\widetilde{z})\overline{\hat{\phi}_j(\widetilde{z})}.$$

Writing $\widetilde{\xi}=(\xi_0,\xi)=(\xi_0,\xi_1,\cdots,\xi_m)\in\mathbb{C}\times\mathbb{C}^m$, we complexify the above equation to get

$$\hat{A}(\widetilde{z},\overline{\widetilde{\xi}})\langle z,\overline{\xi}\rangle_l=\sum_{j=1}^m\hat{\psi}_j(\widetilde{z})\overline{\hat{\phi}_j(\widetilde{\xi})},\qquad \widetilde{z},\widetilde{\xi}\in\mathbb{C}^{m+1}. \tag{2.5}$$

Write $\hat{\psi}=(\hat{\psi}_1,\cdots,\hat{\psi}_m)$ and $\hat{\phi}=(\hat{\phi}_1,\cdots,\hat{\phi}_m)$. Since $\psi(z)$ and $\psi(\xi)$ are of generically full rank, we see that $\hat{\psi}$ and $\hat{\phi}$ have the following property.

Nondegeneracy Property : For any fixed $z_0^*\neq 0,\hat{\psi}(z_0^*,z)$ is of generically full rank in z near 0; for any fixed $\xi_0^*\neq 0,\hat{\phi}(\xi_0^*,\xi)$ is of generically full rank in ξ.

In particular, the nondegeneracy property implies every $\hat{\psi}_j$ and $\hat{\phi}_j$ are not identically zero. Write $\mathcal{N}=\{(\widetilde{z},\widetilde{\xi})\in\mathbb{C}^{m+1}\times\mathbb{C}^{m+1}:\langle z,\xi\rangle_l=0\}$. Pick some small open subsets $G\subseteq\mathbb{C}^{m+1}_{\widetilde{z}},W\subseteq\mathbb{C}^{m+1}_{\widetilde{\xi}}$ such that $\hat{\psi}_m(\widetilde{z})\neq 0$ in G and $\hat{\phi}_m(\widetilde{\xi})\neq 0$ in W, and $\mathcal{N}\cap(G\times W)\neq\emptyset$. We can also assume G does not intersect with $\{z_0z_m=0\}$ and W does not intersect with $\{\xi_0\xi_m=0\}$. Moreover, by the *nondegeneracy property*, shrinking G and W if necessary, we can assume the following hold:

The map $(\frac{\hat{\psi}_1}{\hat{\psi}_m},\cdots,\frac{\hat{\psi}_{m-1}}{\hat{\psi}_m},\hat{\psi}_m)(\widetilde{z})$ is of full rank in $z=(z_1,\cdots,z_m)$ everywhere in G; and the map $(\frac{\hat{\phi}_1}{\hat{\phi}_m},\cdots,\frac{\hat{\phi}_{m-1}}{\hat{\phi}_m},\hat{\phi}_m)(\widetilde{\xi})$ is of full rank in $\xi=(\xi_1,\cdots,\xi_m)$ everywhere in W.

Consequently, writing $\widetilde{\psi}_i=\frac{\hat{\psi}_i}{\hat{\psi}_m}$ for $1\leq i\leq m-1$, the rank of $\left(\frac{\partial\widetilde{\psi}_j}{\partial z_k}\right)_{1\leq i\leq m-1,1\leq k\leq m}$ equals $m-1$ in G. Hence, shrinking G if necessary, there exists some $1 \leq j_1 \leq m$, such that

$$\left(\frac{\partial \widetilde{\psi}_i}{\partial z_k}\right)_{1\leq i\leq m-1, 1\leq k\neq j_1\leq m} \quad \text{is nondegenerate everywhere in } G. \tag{2.6}$$

Similarly, We write $\widetilde{\phi} = \frac{\hat{\phi}_i}{\hat{\phi}_m}$ for $1 \leq i \leq m-1$. By shrinking W if necessary, there is some $1 \leq j_2 \leq m$, such that

$$\left(\frac{\partial \widetilde{\phi}_j}{\partial z_k}\right)_{1\leq j\leq m-1, 1\leq k\neq j_2\leq m} \quad \text{is nondegenerate everywhere in } W. \tag{2.7}$$

Now set

$$\Psi(\widetilde{z}) = (\widetilde{\psi}_1(\widetilde{z}), \cdots, \widetilde{\psi}_{m-1}(\widetilde{z}), 1), \text{ for } \widetilde{z} \in G;$$
$$\Phi(\widetilde{\xi}) = (\widetilde{\phi}_1(\widetilde{\xi}), \cdots, \widetilde{\phi}_{m-1}(\widetilde{\xi}), 1), \text{ for } \widetilde{\xi} \in W.$$

We have the following claim:

Claim. *The maps $\Psi(\widetilde{z})$ and $\Phi(\widetilde{\xi})$ are independent of the variables z_0 and ξ_0, respectively. Moreover, they are linearly fractional in z and ξ, respectively.*

Proof of Claim. We have two cases depending on whether j_1 and j_2 are equal. We will only prove for the case $j_1 = j_2$ and the proof of the other case is similar. Without loss of generality, assume $j_1 = j_2 = m$. By rescaling G and W, we can assume $\{z_m = 1\} \times \{\xi_m = 1\}$ intersects $\mathcal{N} \cap (G \times W)$. Write $G_0 = \{[\widetilde{z}] = [z_0, \cdots, z_m] \in \mathbb{P}^m : (z_0, \cdots, z_m) \in G\}$, and $W_0 = \{[\widetilde{\xi}] = [\xi_0, \cdots, \xi_m] \in \mathbb{P}^m : (\xi_0, \cdots, \xi_m) \in W\}$. Notice by homogeneity, Ψ (respectively, Φ) induces a map $[\Psi]$ (respectively, $[\Phi]$) from G_0 (respectively, from W_0) to $\mathbb{P}^m$. Moreover, by (2.5) we see

$$\hat{A}(\widetilde{z}, \overline{\widetilde{\xi}})\langle z, \overline{\xi}\rangle_l = \widetilde{\psi}_m(\widetilde{z})\overline{\widetilde{\phi}_m(\widetilde{\xi})}\langle \Psi(\widetilde{z}), \overline{\Phi(\widetilde{\xi})}\rangle \text{ for } \widetilde{z} \in G, \widetilde{\xi} \in W.$$

Consequently, $[\Psi] \times [\Phi]$ maps an open piece of $\mathcal{H}$ to $\hat{\mathcal{M}}$. Here $\mathcal{H} = \{([\widetilde{z}], [\widetilde{\xi}]) \in \mathbb{P}^m \times \mathbb{P}^m : \langle z, \overline{\xi}\rangle_l = \sum_{j=1}^m \delta_{j,l} z_j \overline{\xi_j} = 0\}$, and $\hat{\mathcal{M}} = \{([\chi], [\tau]) \in \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} : \sum_{j=1}^m \chi_j \overline{\tau_j} = 0\}$ with $[\chi] = [\chi_1, \cdots, \chi_m], [\tau] = [\tau_1, \cdots, \tau_m]$.

Note G_0 and W_0 are contained in the affine cells $\{[\widetilde{z}] : z_m \neq 0\} \subset \mathbb{P}^m$ and $\{[\widetilde{\xi}] : \xi_m \neq 0\} \subset \mathbb{P}^m$, respectively. We will use the standard nonhomogeneous coordinates on these affine cells:

$$(z_0, \cdots, z_{m-1}) \to [z_0, \cdots, z_{m-1}, 1];$$
$$\text{and} \quad (\xi_0, \cdots, \xi_{m-1}) \to [\xi_0, \cdots, \xi_{m-1}, 1].$$

Moreover, the images of G_0 and W_0 under $[\Psi]$ and $[\Phi]$, are contained in the affine cells $\{[\chi] : \chi_m \neq 0\}$ and $\{[\tau] : \tau_m \neq 0\}$, respectively. We again use the standard nonhomogeneous coordinates on these affine cells:

$$(\chi_1, \cdots, \chi_{m-1}) \to [\chi_1, \cdots, \chi_{m-1}, 1];$$
$$\text{and} \quad (\tau_1, \cdots, \tau_{m-1}) \to [\tau_1, \cdots, \tau_{m-1}, 1].$$

We still denote the maps by Ψ and Φ in these local coordinates. Then (Ψ, Φ) maps (an open piece of) $\mathcal{H}_0 = \{(z_0, \cdots, z_{m-1}), (\xi_0, \cdots, \xi_{m-1}) : \sum_{j=1}^{m-1} \delta_{j,l} z_j \overline{\xi_j} + \delta_{m,l} = 0\} \subset \mathbb{C}^m \times \mathbb{C}^m$ to $\hat{\mathcal{M}}_0 = \{\sum_{j=1}^{m-1} \chi_j \overline{\tau_j} + 1 = 0\} \subset \mathbb{C}^{m-1} \times \mathbb{C}^{m-1}$. Moreover, by (2.6) and (2.7), Ψ and Φ are nondegenerate in $(z_1, \cdots, z_{m-1})$ and $(\xi_1, \cdots, \xi_{m-1})$, respectively. Then it follows from Lemma 2.2 (Note we can apply a linear change of coordinates in z to transform $\mathcal{H}_0$ into $\mathcal{M}_0$ and therefore reduce it to the setting of Lemma 2.2) that Ψ and Φ are independent of the variables z_0 and ξ_0, respectively, and they are linear fractional in z and ξ. Hence we obtain the desired conclusion. If $j_1 \neq j_2$, say $j_1 = m-1, j_2 = m$, a similar argument together with Lemma 2.2 will also yield the conclusion. This finishes the proof of the claim. $\square$

It follows from the above claim and the nondegeracy condition (2.6) that there are some matrix $B \in GL(m; \mathbb{C})$ and a (nonzero) linear function $L_1(z)$ in z such that $(\widetilde{\psi}_1(\widetilde{z}), \cdots, \widetilde{\psi}_{m-1}(\widetilde{z}), 1)$ equals $\frac{1}{L_1(z)}(z_1, \cdots, z_m)B$ (By the above claim, Ψ does not depend on z_0). Consequently, we have

$$L_1(z)(\hat{\psi}_1(\widetilde{z}), \cdots, \hat{\psi}_m(\widetilde{z})) = \hat{\psi}_m(\widetilde{z})(z_1, \cdots, z_m)B, \quad \forall \widetilde{z} \in \mathbb{C}^{m+1}. \tag{2.8}$$

Similarly, there exists some matrix $C \in GL(m; \mathbb{C})$ and some nonzero linear function $L_2(\xi)$ in ξ such that

$$L_2(\xi)(\hat{\phi}_1(\widetilde{\xi}), \cdots, \hat{\phi}_m(\widetilde{\xi})) = \hat{\phi}_m(\widetilde{\xi})(\xi_1, \cdots, \xi_m)C, \quad \forall \widetilde{\xi} \in \mathbb{C}^{m+1}. \tag{2.9}$$

It then follows that

$$L_1(z)\overline{L_2(\xi)} \sum_{j=1}^{m} \hat{\psi}_j(\widetilde{z}) \overline{\hat{\phi}_j(\widetilde{\xi})} = \hat{\psi}_m(\widetilde{z}) \overline{\hat{\phi}_m(\widetilde{\xi})}(z_1, \cdots, z_m) B \overline{C^t} (\overline{\xi_1}, \cdots, \overline{\xi_m})^t.$$

By (2.5), the above quantity vanishes on $\langle z, \overline{\xi} \rangle_l = 0$. This implies $B\overline{C^t} = \lambda I_{l,m}$ for nonzero λ. By choosing a different L_1, we can make $\lambda = 1$. Consequently,

$$L_1(z)\overline{L_2(\xi)} \sum_{j=1}^{m} \hat{\psi}_j(\widetilde{z})\overline{\hat{\phi}_j(\widetilde{\xi})} = \hat{\psi}_m(\widetilde{z})\overline{\hat{\phi}_m(\widetilde{\xi})}\langle z, \overline{\xi} \rangle_l.$$

Combining this with (2.5), we obtain for all $\widetilde{z}, \widetilde{\xi} \in \mathbb{C}^{m+1}$,

$$L_1(z)\overline{L_2(\xi)}\hat{A}(\widetilde{z}, \overline{\widetilde{\xi}})\langle z, \overline{\xi} \rangle_l = \hat{\psi}_m(\widetilde{z})\overline{\hat{\phi}_m(\widetilde{\xi})}\langle z, \overline{\xi} \rangle_l.$$

The above is then reduced to

$$L_1(z)\overline{L_2(\xi)}\hat{A}(\widetilde{z}, \overline{\widetilde{\xi}}) = \hat{\psi}_m(\widetilde{z})\overline{\hat{\phi}_m(\widetilde{\xi})}, \quad \forall \widetilde{z}, \widetilde{\xi} \in \mathbb{C}^{m+1}. \tag{2.10}$$

Note (2.10) implies that $\hat{\psi}_m(\widetilde{z})$ vanishes on $\{L_1(z) = 0\}$. This further implies there is some holomorphic polynomial $p_1(\widetilde{z})$ such that $\hat{\psi}_m(\widetilde{z}) = L_1(z)p_1(\widetilde{z})$. Similarly, we have $\hat{\phi}_m(\widetilde{\xi})$ vanishes on $\{L_2(\xi) = 0\}$, and $\hat{\phi}_m(\widetilde{\xi}) = L_2(\xi)p_2(\widetilde{\xi})$ for some holomorphic polynomial $p_2(\widetilde{\xi})$. Then it follows from (2.10) that $\hat{A}(\widetilde{z}, \overline{\widetilde{z}}) = p_1(\widetilde{z})\overline{p_2(\widetilde{z})}$. Finally we let $z_0 = 1$ and write $h_1(z) = p_1(1, z), h_2(\xi) = p_2(1, \xi)$ to obtain that $A(z, \overline{z}) = h_1(z)\overline{h_2(z)}$. Furthermore, (2.8) and (2.9) are reduced to

$$(\hat{\psi}_1(\widetilde{z}), \cdots, \hat{\psi}_m(\widetilde{z})) = p_1(\widetilde{z})(z_1, \cdots, z_m)B;$$
$$(\hat{\phi}_1(\widetilde{\xi}), \cdots, \hat{\phi}_m(\widetilde{\xi})) = p_2(\widetilde{\xi})(\xi_1, \cdots, \xi_m)C.$$

We again let $z_0 = 1$ and $\xi_0 = 1$ to get

$$(\psi_1(z), \cdots, \psi_m(z)) = h_1(z)(z_1, \cdots, z_m)B;$$
$$(\phi_1(\xi), \cdots, \phi_m(\xi)) = h_2(\xi)(\xi_1, \cdots, \xi_m)C.$$

This finishes the proof of Proposition 2.1 under the additional assumption (*).

To prove Proposition 2.1 in the general case, we multiple $|z_1|^2$ to both sides of (2.1) and obtain:

$$A^*(z, \overline{z})|z|_l^2 = \sum_{j=1}^{m} \psi_j^*(z)\overline{\phi_j^*(z)}.$$

Here $A^*(z, \overline{z}) = |z_1|^2 A(z, \overline{z})$ and $\psi_j^*(z) = z_1\psi_j(z), \phi_j^*(z) = z_1\phi_j(z)$ for all $1 \leq j \leq m$. Then ψ^* and ϕ^* satisfy the assumption (*). By what we have proved, we see there exist holomorphic polynomials $h_1^*(z), h_2^*(z)$ and $B^*, C^* \in GL(m, \mathbb{C})$ with $B^*\overline{C^*}^t = I_{l,m}$, such that $A^*(z, \overline{\xi}) = h_1^*(z)\overline{h_2^*(\xi)}$, and

$$\psi^*(z) = h_1^*(z)zB^*; \qquad \phi^*(z) = h_2^*(z)zC^*.$$

Since $A^*(z, \overline{\xi}) = 0$ on $\{z_1 = 0\}$ and on $\{\xi_1 = 0\}$, we have $h_1^*(z) = z_1 h_1(z)$ and $h_2^*(z) = z_1 h_2(z)$ for some polynomials h_1 and h_2. Consequently, $A(z, \overline{\xi}) = h_1(z)\overline{h_2(\xi)}$, and

$$\psi(z) = h_1(z)zB^*; \qquad \phi(z) = h_2(z)zC^*.$$

This proves Proposition 2.1 in the general case.

□

Finally we apply Proposition 2.1 to derive Theorem 1.

Proof of Theorem 1: We first fix some notations. We set, for $k, j \geq 0$, $A^{(k,j)}(z, \overline{z})$ to be the truncated Taylor polynomial of $A(z, \overline{z})$ to the order (k, j) in $(z, \overline{z})$. More precisely, writing $A(z, \overline{z}) = \sum_{|\alpha|,|\beta| \geq 0} a_{\alpha\beta} z^\alpha \overline{z^\beta}$ near 0, let $A^{(k,j)}(z, \overline{z})$ equal to the sum of terms $a_{\alpha\beta} z^\alpha \overline{z^\beta}$ with $|\alpha| \leq k, |\beta| \leq j$. Similarly for $k \geq 0$, we set $\psi^{(k)}(z)$ and $\phi^{(k)}(z)$ to be truncated Taylor polynomials at degree k of $\psi(z)$ and $\phi(z)$, respectively. Then it follows from the assumption (1.2) that

$$A^{(d,d)}(z, \overline{z})|z|_l^2 = \sum_{j=1}^{m} \psi^{(d+1)}(z)\overline{\phi^{(d+1)}(z)}. \tag{2.11}$$

Since $A(z, \overline{z}) \not\equiv 0$, we have $A^{(d,d)}(z, \overline{z}) \not\equiv 0$ for sufficiently large d. We conclude by Proposition 2.1 that, for every sufficiently large d, there are holomorpihc polynomials $h_{1,d}$ $h_{2,d}$, and $B_d, C_d \in GL(m, \mathbb{C})$ with $B_d C_d = I_{l,m}$, such that

$$A^{(d,d)}(z, \overline{z}) = h_{1,d}(z)\overline{h_{2,d}(z)}; \tag{2.12}$$

$$\begin{aligned} \psi^{(d+1)}(z) &= h_{1,d}(z)(z_1, \cdots, z_m)B_d; \\ \phi^{(d+1)}(z) &= h_{2,d}(z)(z_1, \cdots, z_m)C_d. \end{aligned} \tag{2.13}$$

We pick a small open ball U centered at 0 in $\mathbb{C}^m$ such that $A^{(d,d)}(z, \overline{\xi})$ converges uniformly to $A(z, \overline{\xi})$ in $U \times U$, and $\psi^{(d+1)}(z)$ (respectively,

$\phi^{(d+1)}(z)$) converges uniformly to $\psi(z)$ (respectively, $\phi(z)$) on U. Consequently, $\{A^{(d,d)}(z,\overline{\xi})\}_{d=1}^{\infty}$ is uniformly bounded on $U \times U$. Since $A(z,\overline{z})$ is not identically zero, there exists some $z^* \in U$ such that $A(z^*,\overline{z^*}) = c_0 \neq 0$. We can normalize $h_{1,d}$ and $h_{2,d}$ such that $|h_{1,d}(z^*)| \geq \frac{|c_0|}{2}$ and $h_{2,d}(z^*) = 1$ for every sufficiently large d. We complexify (2.12) to obtain

$$A^{(d,d)}(z,\overline{\xi}) = h_{1,d}(z)\overline{h_{2,d}(\xi)}, \text{ for } (z,\xi) \in U \times U. \tag{2.14}$$

We set $\xi = z^*$ in the above equation to see $\{h_{1,d}(z)\}_{d=1}^{\infty}$ is uniformly bounded on U. Similarly, $\{h_{2,d}(z)\}_{d=1}^{\infty}$ is also uniformly bounded on U. By Montel's theorem, passing to a subsequence if necessary, we can assume $h_{1,d}(z)$ and $h_{2,d}(z)$ converge uniformly on compact subsets of U. Denote their limits by $h_1(z), h_2(z)$, respectively, which are holomorphic functions on U. We then let $d \to \infty$ in (2.12) to see $A(z,\overline{z}) = h_1(z)\overline{h_2(z)}$ near 0. Note by normalization, $h_2(z^*) = 1$ and $h_1(z^*) = c_0 \neq 0$.

Next since $h_{1,d}, h_{2,d}$ converge to h_1, h_2, respectively, uniformly on compact subsets of U, we can then find a small ball $B(z^*,r) \subset\subset U$ of radius r centered at z^* such that $|h_{1,d}(z)| \geq \frac{|c_0|}{2}$ and $|h_{2,d}(z)| \geq \frac{1}{2}$ in $B(z^*,r)$ for all sufficiently large d. Since $\psi^{(d+1)}(z)$ also converges to $\psi(z)$ uniformly on $B(z^*,r)$, we see $\{\psi^{(d+1)}(z)\}_{d=1}^{\infty}$ is uniformly bounded on $B(z^*,r)$. It then follows from (2.13) that $\{zB_d = (z_1,\cdots,z_m)B_d\}_{d=1}^{\infty}$ is uniformly bounded on $z \in B(z^*,r)$. This implies $\{B_d\}_{d=1}^{\infty}$ is bounded in $GL(m,\mathbb{C})$. A similar argument yields that $\{C_d\}_{d=1}^{\infty}$ is also bounded in $GL(m,\mathbb{C})$. Thus by passing to subsequences if necessary, we can assume B_d, C_d converge to B, C, respectively. Since $B_dC_d = I_{l,m}$, we have $BC = I_{l,m}$ and thus $B, C \in GL(m,\mathbb{C})$. We finally let $d \to \infty$ in (2.13) to obtain the last two equations in Theorem 1. This finishes the proof of Theorem 1. □

To see the conclusion in Remark 1.1, we assume $A(z,\overline{z})$ is real-valued and need to show that $h_1\overline{h_2} = \pm|h|^2$ for some holomorphic function h near 0. This follows easily from the elementary lemma:

Lemma 2.3. *Let h_1, h_2 be holomorphic functions on an open connected set $U \subseteq \mathbb{C}^n$. Assume $h_1\overline{h_2}$ is real-valued in U, then $h_1 \equiv 0$ or $h_2 = ch_1$ for some real number c.*

Proof of Lemma 2.3: Note by the assumption $h_1\overline{h_2} = \overline{h_1}h_2$. If h_1 is not identically zero, then we can divide by $|h_1|^2$ to obtain $\overline{(\frac{h_2}{h_1})} = \frac{h_2}{h_1}$ around a generic point $z \in U$. Then the conclusion follows from the open mapping theorem. □

3. Proof of Corollary 1.2 and Corollary 1.3

We will apply Corollary 1.1 in [HLTX1] to prove Corollary 1.2. Recall Theorem 1 in [HLTX1] implies that, under the setting of Corollary 1.2, F is an isometric embedding if and only if F is CR transversal at $F(q)$ for some point $q \in U \cap \partial\mathbb{B}_l^n$ and F has zero geometric rank near q along $U \cap \partial\mathbb{B}_l^n$. See [HLTX1] for the definition of the geometric rank of a CR transversal map from $\partial\mathbb{B}_l^n$ to $\partial\mathbb{B}_{l'}^N$. Note by the assumption of Corollary 1.2, we have either $l' < n-1$ or $2n-2-l' < n-1$. Then it already follows from Lemma 4.1 of [BH] (or Theorem 1.1 in [BER]) that F is CR transversal at $F(q)$ for a generic point $q \in U \cap \partial\mathbb{B}_l^n$. Fix such a point $q = q_0$. By the proceeding argument, to establish Corollary 1.2, it suffices to show F has zero geometric rank near q_0.

Proposition 3.1. *The map F has zero geometric rank near q_0 along $U \cap \partial\mathbb{B}_l^n$.*

We first recall certain notations and terminologies which will be used in the proof of Proposition 3.1. Let $\delta_{j,l}$ and $|\cdot|_l^2$ be as defined in §1 (see the paragraph above Theorem 1). Assume $l' \geq l$. We denote by $\delta_{j,l,l',n}$ the symbol which takes value -1 when $1 \leq j \leq l$ or $n \leq j \leq n + l' - l - 1$ and 1 otherwise. When $l' = l$, $\delta_{j,l,l,n}$ is the same as $\delta_{j,l}$. For $0 \leq l \leq n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}_l^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > \sum_{j=1}^{n-1} \delta_{j,l}|z_j|^2\}.$$

The boundary of $\mathbb{S}_l^n$ is the standard hyperquadrics $\mathbb{H}_l^n$ given by $\mathrm{Im}(w) = \sum_{j=1}^{n-1} \delta_{j,l}|z_j|^2$. We also define for $l \leq l' \leq N-1$,

$$\mathbb{S}_{l,l',n}^N = \{(Z, W) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(W) > \sum_{j=1}^{N-1} \delta_{j,l,l',n}|Z_j|^2\}.$$

We similarly define $\mathbb{S}_{l'}^N, \mathbb{H}_{l'}^N, \mathbb{H}_{l,l',n}^N$. Now for $(z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n$, let $\Psi(z, w) = [i + w, 2z, i - w] \in \mathbb{P}^n$. Then Ψ is the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}_l^n$ and its boundary $\mathbb{H}_l^n$ onto $\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$ and $\partial\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$, respectively.

Proof of Proposition 3.1: By composing F with automorphisms of $\mathbb{B}_l^n$ and $\mathbb{B}_{l'}^{2n-1}$ if necessary, we assume $q_0 = [1, 0, \cdots, 0, 1] \in \partial\mathbb{B}_l^n$ and $F(q_0) = [1, 0, \cdots, 0, 1] \in \partial\mathbb{B}_{l'}^{2n-1}$. Recall Ψ is the aforementioned Cayley transformation from $\mathbb{S}_l^n$ to $\mathbb{B}_l^n$ with $\Psi(0) = q_0$, and we denote by Φ the Cayley transformation from $\mathbb{S}_{l,l',n}^{2n-1}$ to $\mathbb{B}_{l'}^{2n-1}$. Write $\widetilde{F} := \Phi^{-1} \circ F \circ \Psi$. By the definition of the geometric rank (see Section 3 in [HLTX1]), $\widetilde{F}$ is of geometric rank zero at p if and only if F is so at $\Psi(p)$. Thus it suffices to prove the new map $\widetilde{F}$ has zero geometric rank near 0. To make the notations simple, we still write the new map as F instead of $\widetilde{F}$. That is, F is now a holomorphic map from a neighborhood V of some point $p_0 = 0 \in \mathbb{H}_l^n$ to $\mathbb{C}^{2n-1}$, satisfying $F(V \cap \mathbb{S}_l^n) \subseteq \mathbb{S}_{l,l',n}^{2n-1}$ and $F(V \cap \mathbb{H}_l^n) \subseteq \mathbb{H}_{l,l',n}^{2n-1}$. Shrinking V if necessary, we additionally assume $M_1 := V \cap \mathbb{H}_l^n$ is connected and F is CR transversal on M_1. Next for each $p \in M_1$, we associate it with a map F_p defined as in [BH, HLTX1]. See (3.2) in [HLTX1]. Furthermore, we normalize F_p into F_p^*, F_p^{**} as defined in (3.9) and (3.13) of [HLTX1], respectively. As in [HLTX1], F_p^{**} sends 0 to 0, and maps $\mathbb{H}_l^n$ (respectively, $\mathbb{S}_l^n$) to $\mathbb{H}_{l,l',n}^N$(respectively, $\mathbb{S}_{l,l',n}^{2n-1}$) near 0.

We now pause to recall some notations for functions of weighted degree from [Hu, BH].We parameterize $\mathbb{H}_l^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i\sum_{j=1}^{n-1} \delta_{j,l}|z_j|^2)$. We assign the weight of z to be 1, and assign the weight of u (and thus w) to be 2 . For a smooth function $h(z, \bar{z}, u)$ defined in a neighborhood W of 0 in $\mathbb{H}_l^n$, we say it is of quantity $O_{wt}(s)$ for $0 \leq s \in \mathbb{N}$, if $\frac{h(tz, t\bar{z}, t^2u)}{t^s}$ is bounded for (z, u) on any compact subset of W and t close to 0. Moreover, for a smooth function $h(z, \bar{z}, u)$ on W, we denote by $h^{(k)}(z, \bar{z}, u)$ the sum of terms of weighted degree k in the Taylor expansion of h about 0. And $h^{(k)}(z, \bar{z}, u)$ also sometimes denotes a weighted homogeneous polynomial of degree k, if h is not specified. When $h^{(k)}(z, \bar{z}, u)$ extends to a holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z, w)$ or $h^{(k)}(z)$ if it depends only on z.

Write $F_p^{**} = (f_p^{**}, \phi_p^{**}, g_p^{**})$, where f_p^{**}, ϕ_p^{**} both have $n-1$ components, and g_p^{**} is a scalar function. Under the notations above, F_p^{**} satisfies the following normalization by [BH].

Lemma 3.1. *(Lemma 2.2 in [BH]) Write $(z, w) = (z_1, \cdots, z_{n-1}, w)$ for the coordinates of $\mathbb{C}^n$. For each $p \in M_1$, F_p^{**} satisfies the normalization condition:*

$$\begin{cases} f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z)w + O_{wt}(4) \\ \phi_p^{**} = \phi_p^{**(2)}(z) + O_{wt}(3) \\ g_p^{**} = w + O_{wt}(5), \end{cases}$$

with

$$\langle \bar{z}, a_p^{**(1)}(z)\rangle_l |z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2,\ \tau = l' - l. \tag{3.1}$$

If we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$ for an $(n-1)\times(n-1)$ matrix $\mathcal{A}(p)$, then by [HLTX1] the geometric rank of F at p is defined as the rank of the matrix $\mathcal{A}(p)$. Set $A_p(z,\overline{z}) = \langle \bar{z}, a_p^{**(1)}(z)\rangle_l$, which is a real polynomial. By (3.1), we have

$$A_p(z,\overline{z})|z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2.$$

Note $\phi_p^{**(2)}(z)$ has $n-1$ components and by the assumption of Corollary 1.2, $\tau \neq l$ and $\tau \neq n-1-l$. Then by Corollary 1.1, we have $A_p(z,\overline{z}) \equiv 0$ and thus F has geometric rank zero at p. Since p is arbitrary, we conclude F has zero geometric rank near 0, and finish the proof of Proposition 3.1. □

Remark 3.1. When $l=0$ or $l=n-1$, we indeed don't need to use the full generality of Corollary 1.1 to conclude $A_p(z,\overline{z})=0$ in the above. Instead it suffices to use a much weaker version of Corollary 1.1 where a_i's and b_j's are assumed to be quadratic homogeneous polynomials. For that the readers are referred to the proof of Lemma 3.1 in [HJ] and we omit the details here.

Proof of Corollary 1.2. The result follows from Theorem 1 in [HLTX1] and Proposition 3.1. □

We next prove Corollaries 1.3.

Proof of Corollary 1.3. First since F extends $C^{n-1}-$smoothly up to some open piece of the boundary, we conclude by Theorem 3 in [X] that F is algebraic. Consequently F extends holomorphically across a generic boundary point $p \in \partial\mathbb{B}^n$. And we can find a small neighborhood U of p such that $U \cap \mathbb{B}^n$ is connected, $F(U\cap\mathbb{B}^n) \subseteq D_{2n-2}^{IV}$ and $F(U\cap\partial\mathbb{B}^n) \subseteq \partial D_{2n-2}^{IV}$. Then Corollary 1.3 follows from Corollary 1.2 and an identical argument as in the proof of Theorem 1.1 in [XY1]. This establishes Corollary 1.3. □

4. Proof of Lemma 2.2

We prove Lemma 2.2 in this section. We first note the following linear fractional map gives a local biholomorphic map from $\mathcal{M}_1$ to $\mathcal{M}_0$:

$$\hat{w}_0 = w_0, \quad \hat{w}_i = \frac{\sqrt{2}w_i}{w_{m-1}+1}, 1 \le i \le m-2, \quad \hat{w}_{m-1} = \frac{1-w_{m-1}}{w_{m-1}+1};$$

$$\hat{\eta}_0 = \eta_0, \quad \hat{\eta}_i = \frac{\sqrt{2}\eta_i}{\eta_{m-1}+1}, 1 \le i \le m-2, \hat{\eta}_{m-1} = \frac{\eta_{m-1}-1}{\eta_{m-1}+1}.$$

Similarly, $\hat{\mathcal{M}}_0$ and $\hat{\mathcal{M}}_1$ are locally biholomorphic by a linear fractional map. Thus it suffices to prove Lemma 2.2 only for the map $f \times g$ from $\mathcal{M}_1$ to $\hat{\mathcal{M}}_0$. We fix a point $(p, q) \in \mathcal{M}_1 \cap (U \times V)$. Note that $(0, 0) \in \mathcal{M}_1$ and there is a biholomorphic map $(\varphi_1(w), \varphi_2(\eta))$ in a neighborhood of $(0, 0)$ that sends $(0, 0)$ to (p, q) and maps an open piece of $\mathcal{M}_1$ near $(0, 0)$ to $\mathcal{M}_1 \cap (U \times V)$. Indeed, writing $p = (p_0, p', p_{m-1}) = (p_0, p_1, \cdots, p_{m-2}, p_{m-1}), q = (q_0, q', q_{m-1}) = (q_0, q_1, \cdots, q_{m-2}, q_{m-1})$, and $w' = (w_1, \cdots, w_{m-2}), \eta' = (\eta_1, \cdots, \eta_{m-2})$, we can take

$$\begin{aligned}
\varphi_1(w) &= (w_0 + p_0, w' + p', w_{m-1} + p_{m-1} - \langle w', \overline{q'} \rangle); \\
\varphi_2(\eta) &= (\eta_0 + q_0, \eta' + q', \eta_{m-1} + q_{m-1} - \langle \eta', \overline{p'} \rangle).
\end{aligned}$$

Hence, by composing f, g with φ_1, φ_2 if necessary, we can just assume $(p, q) = (0, 0)$. For $\eta = (\eta_0, \cdots, \eta_{m-1})$, write $L_i^\eta = \frac{\partial}{\partial w_i} - \overline{\eta_i} \frac{\partial}{\partial w_{m-1}}, 1 \le i \le m-2$. Then $\{L_i^\eta\}_{i=1}^{m-2}$ gives a set of holomorphic tangent vector fields along $\mathcal{M}_1$. Set

$$D_\eta(w) := \begin{vmatrix} f \\ L_1^\eta f \\ \cdots \\ L_{m-2}^\eta f \end{vmatrix}.$$

Here $|\cdot|$ denotes the determinant of a square matrix. Note $D_\eta(w)$ is independent of η_0. We will show the following nondegeneracy property of $D_\eta(w)$.

Lemma 4.1. *There exists $(p^*, q^*) \in \mathcal{M}_1 \cap (U \times V)$ such that $D_\eta(w) \neq 0$ at $(w, \eta) = (p^*, q^*)$.*

Proof. By the definitions of $D_\eta(w)$ and $L_i^\eta, 1 \le i \le m-2$, we see $D_\eta(w)$ is linear in each $\overline{\eta_i}$. More precisely, $D_\eta(w) = -\sum_{i=1}^{m-2} B_i(w)\overline{\eta_i} + B_0(w)$. Here

$$B_0(w) = \begin{vmatrix} f(w) \\ \frac{\partial f}{\partial w_1}(w) \\ \cdots \\ \frac{\partial f}{\partial w_{m-2}}(w) \end{vmatrix} \tag{4.1}$$

and for each $1 \le i \le m-2, B_j(w)$ equals the determinant in (4.1) with the $(j+1)$−st row (i.e., $\frac{\partial f(w)}{\partial w_j}$) replaced by $\frac{\partial f(w)}{\partial w_{m-1}}$.

Recall by assumption of Lemma 2.2, if we write $B(w) = \begin{vmatrix} \frac{\partial f}{\partial w_1}(w) \\ \cdots \\ \frac{\partial f}{\partial w_{m-1}}(w) \end{vmatrix}$, then $B(w)$ is everywhere nonzero in U.

Claim. There is some $0 \leq j_0 \leq m-2$, such that $B_{j_0}(w) \not\equiv 0$.

Proof of Claim. Suppose $B_j(w) \equiv 0$ for all $0 \leq j \leq m-2$. Then by the fact that $B(w) \neq 0$ in U and Lemma 4.7 in [BX], we conclude $f \equiv 0$ in U. This is a contradiction. $\square$

By the claim, we can find some $0 \leq j_0 \leq m-2$, and some $p^* = (p_0^*, \cdots, p_{m-1}^*)$ near 0 such that $B_{j_0}(p^*) \neq 0$. If $j_0 \neq 0$, then we can find a number $\eta_{j_0}^* \approx 0$ such that $B_{j_0}(p^*)\eta_{j_0}^* + B_0(p^*) \neq 0$. Set $q^* = (0, 0, \cdots, 0, \eta_{j_0}^*, 0, \cdots, 0, -\overline{p_{m-1}^*} - \overline{p_{j_0}^*}\eta_{j_0}^*) \in \mathbb{C}^m$, where $\eta_{j_0}^*$ is at the $(j_0 + 1)-$st position. Then we have $(p^*, q^*) \in \mathcal{M}_1$ and $D_{q^*}(p^*) \neq 0$. If $j_0 = 0$, then we can find some $p^* = (p_0^*, \cdots, p_{m-1}^*)$ near 0 such that $B_0(p^*) \neq 0$. Pick $q^* = (0, \cdots, 0, -\overline{p_{m-1}^*})$, so that $(p^*, q^*) \in \mathcal{M}_1$ and we have $D_{q^*}(p^*) \neq 0$. This proves Lemma 4.1. $\square$

By Lemma 4.1, we can shrink U, V and assume $D_\eta(w)$ is everywhere nonzero in $U \times V$. By assumption, we have

$$\langle f(w), \overline{g(\eta)} \rangle = -1 \text{ on } \mathcal{M}_1 \cap (U \times V). \tag{4.2}$$

We then apply $L_i^\eta, 1 \leq i \leq m-2$, to (4.2) and obtain

$$\langle L_i^\eta f(w), \overline{g(\eta)} \rangle = 0 \text{ on } \mathcal{M}_1 \cap (U \times V), 1 \leq i \leq m-2. \tag{4.3}$$

Fix $p = (p_0, \cdots, p_{m-1}) \in U$ near p^*. Write $Q_p = \{\eta \in \mathbb{C}^m : \sum_{j=1}^{m-2} p_j \overline{\eta_j} + p_{m-1} + \overline{\eta_{m-1}} = 0\}$. Putting together equations (4.2) and (4.3) and evaluating at $w = p$, we get

$$\begin{pmatrix} f(p) \\ L_1^\eta f(p) \\ \cdots \\ L_{m-2}^\eta f(p) \end{pmatrix} \overline{g^t(\eta)} = \begin{pmatrix} -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \qquad \eta \in Q_p \cap V. \tag{4.4}$$

Here g^t denotes the column vector-valued function obtained by taking the transpose of g. Note $\frac{\partial}{\partial \eta_0}$ is tangent to Q_p. We apply $\frac{\partial}{\partial \eta_0}$ to (4.4) to get

$$\begin{pmatrix} f(p) \\ L_1^\eta f(p) \\ \cdots \\ L_{m-2}^\eta f(p) \end{pmatrix} \overline{\frac{\partial g^t}{\partial \eta_0}(\eta)} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \qquad \eta \in Q_p \cap V.$$

Since $D_\eta(p) \neq 0$, the matrix on the left hand side of the above equation is nondegenerate. Hence we must have $\frac{\partial g}{\partial \eta_0}(\eta) = 0$ for $\eta \in Q_p \cap V$. Note that we can vary p near p^* and Q_p will fill in an open subset of $\mathbb{C}^m_\eta$. This implies $\frac{\partial g}{\partial \eta_0} \equiv 0$ in V and thus g is independent of η_0. Similarly, we can prove f is independent of w_0. Once we know that f and g only depende on the variables $w_1, \cdots, w_{m-1}$ and $\eta_1, \cdots, \eta_{m-1}$, respectively, it is reduced to the case where the original result of Chern-Ji (Lemma 2.1) can be applied. We thus see the fractional linearity of f and g. This proves Lemma 2.2.

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