

Whitney-Graustein homotopy of locally convex curves via a curvature flow

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Let X_0, \tilde{X} be two smooth, closed and locally convex curves in the plane with same winding number. A curvature flow with a nonlocal term is constructed to evolve X_0 into \tilde{X} . It is proved that this flow exists globally, preserves both the local convexity and the elastic energy of the evolving curve. If the two curves have same elastic energy then the curvature flow deforms the evolving curve into the target curve \tilde{X} as time tends to infinity.

1. Introduction

In 1937, H. Whitney [32] showed that two smooth and closed curves in the Euclidean plane may be smoothly deformed to each other if and only if the two curves have same winding number. Whitney in his paper said that this result and its proof had been suggested by W. C. Graustein, so this fact is called Whitney-Graustein Theorem now.

Since 1980, geometers have created different kinds of curvature flows to study the deformation of curves, surfaces and higher dimensional manifolds. The developments of these curvature flows play very important roles in geometry and topology. Apart from those higher dimensional arts, there are some profound and essential results on the curvature flow of curves, such as the curve shortening flow by Gage [12, 13], Gage-Hamilton [16] and Grayson [21, 22], the expanding flow by Chow-Liou-Tsai [7] and Tsai [28], the anisotropic flow by Chou-Zhu [8, 9] and the applications of curve flows to classical geometry by Angenent [5] and so on. In this situation, S. T. Yau in 2007 [25] asked that whether one can use a parabolic curvature flow method to evolve one curve to another. An answer to Yau's question is a realization of Whitney-Graustein differential homotopy for closed curves via a curvature flow.

In order to settle this problem, Lin and Tsai in their paper [25] defined a new parabolic model to evolve one convex curve to another. They showed that if two convex curves have same length then their flow can deform one

curve into the other, provided that the flow exists globally. Later, Tsai [29] found the blow-up phenomenon of this linear flow. Following Gage [15] and Gage-Li [17], Pan and Yang [26] in 2017 studied a nonlocal flow which evolves convex curves into a given centrosymmetric convex one. In the same time, inspired by Lin-Tsai [25] and Chou-Zhu [8, 9], Gao and Zhang [20] generalized Gage's area-preserving flow [14] and proved that the generalized flow exists globally and, up to a rescaling, evolves one convex curve to another given one. So the convex case of Yau's above question of evolving one curve to another has been solved.

In this paper, the author continues to study Yau's question for the case of locally convex curves. If a C^2 and closed curve in the plane has positive (relative) curvature everywhere then it is called locally convex. Apart from convex curves, there are uncountably many other locally convex curves. This is a very special phenomenon in the planar geometry, because compact and locally convex hypersurfaces in higher dimensional Euclidean spaces are all convex ones (see Hadamard's theorem [23] or [24]). Due to this reason, the curvature flows of locally convex curves in the plane arose some particular interests in the past several years (see Chen-Wang-Yang [6], Wang-Li-Chao [30], Wang-Wo-Yang [31]). Locally convex curves also play an important role in understanding the asymptotic behavior of the famous curve shortening flow (see Abresch-Langer [1], Altschuler [2], Angenent [4] and Epstein-Gage [11]) and its generalization (see Andrews [3], Chou-Zhu [10]).

Let $X_0 : [0, L_0] \rightarrow \mathbb{R}^2 (s \rightarrow (x(s), y(s)))$ be a C^2 and closed curve in the plane, where s is the arc length parameter and L_0 is the length. Denote by $\{T, N\}$ the Frenet frame of this curve, i.e., for each s , the ordered pair $(T(s), N(s))$ determines a positive orientation of the plane. Denote by $\kappa_0(s) := \langle \frac{dT(s)}{ds}, N(s) \rangle$ the curvature of the curve X_0 . The winding number of X_0 is defined by

$$(1.1) \quad m(X_0) := \frac{1}{2\pi} \int_0^{L_0} \kappa_0(s) ds.$$

If X_0 is locally convex then $\kappa_0(s)$ is positive everywhere and $m(X_0)$ is a positive integer. The elastic energy of the curve (see [18], [27]) is defined by

$$E(X_0) := \int_0^{L_0} (\kappa_0(s))^2 ds.$$

The function $p_0(s) := -\langle X_0(s), N(s) \rangle$ is called the support function of the curve. Let θ be the tangent angle, i.e., the angle from positive direction of x -axis to the unit tangent vector. Since $\frac{d\theta}{ds} = \kappa_0(s)$ is positive for each s ,

θ can be used as a parameter of X_0 . And the curve $X_0 : mS^1 \rightarrow \mathbb{E}^2(\theta \mapsto (x(\theta), y(\theta)))$ is a C^2 mapping from the m -fold circle to the plane. If the locally convex curve X_0 is parametrised by the tangent angle θ then the Frenet formula is as follows

$$(1.2) \quad \frac{dT}{d\theta}(\theta) = N(\theta), \quad \frac{dN}{d\theta}(\theta) = -T(\theta).$$

Furthermore, differentiating the support function gives us

$$(1.3) \quad \frac{dp_0}{d\theta}(\theta) = \langle X_0(\theta), T(\theta) \rangle, \quad \frac{d^2p_0}{d\theta^2}(\theta) = \rho_0(\theta) - p(\theta),$$

where $\rho_0(\theta) = \frac{1}{\kappa_0(\theta)}$ is the radius of the curvature.

Let $X_0, \tilde{X} : mS^1 \rightarrow \mathbb{E}^2$ be two smooth and locally convex curves in the plane with same winding number m . Denote by

$$X : mS^1 \times [0, \omega) \rightarrow \mathbb{E}^2((\theta, t) \mapsto (x(\theta, t), y(\theta, t)))$$

a family of locally convex curves with $X(\theta, 0) = X_0(\theta)$, where θ is the tangent angle. Since the locally convex curve \tilde{X} has the same winding number with $X(\cdot, t)$, θ can also be used as a parameter for this curve. Let $p(\theta, t)$ be the support function of the curve $X(\cdot, t)$ and let $\rho(\theta, t)$ be its radius of curvature. Denote by $\tilde{p}(\theta)$ and $\tilde{\rho}(\theta)$ the support function and the radius of curvature of the curve \tilde{X} , respectively. In order to answer Yau's question for the case of locally convex curves, the next curvature flow is introduced:

$$(1.4) \quad \begin{cases} \frac{\partial X}{\partial t}(\theta, t) = \alpha(\theta, t)T(\theta, t) \\ \quad + [2p(\theta, t) - \rho(\theta, t) - 2\tilde{p}(\theta, t) + \tilde{\rho}(\theta, t) + f(t)]N(\theta, t) \\ \quad \text{in } mS^1 \times (0, \omega), \\ X(\theta, 0) = X_0(\theta) \text{ on } mS^1, \end{cases}$$

where the coefficient of the tangent component is formulated as

$$(1.5) \quad \alpha(\theta, t) = -\frac{\partial}{\partial \theta}(2p(\theta, t) - \rho(\theta, t) - 2\tilde{p}(\theta, t) + \tilde{\rho}(\theta, t))$$

and the nonlocal term is given by

$$(1.6) \quad f(t) = \frac{1}{\int_{mS^1} \kappa^2 d\theta} \left[\int_{mS^1} \kappa^2 \left(\frac{\partial^2 \rho}{\partial \theta^2} - \frac{d^2 \tilde{\rho}}{d\theta^2} \right) d\theta - \int_{mS^1} \kappa^2 (\rho - \tilde{\rho}) d\theta \right].$$

Remark 1.1. *The local convexity of the evolving curve is preserved under the flow (see Lemma 2.9), so the tangent angle θ can be used as a parameter of the evolving curve and the evolution equation (1.4) is well defined for $t > 0$.*

Remark 1.2. *In contrast to the flow by Lin-Tsai [25], the support functions p and \tilde{p} and a complicated nonlocal term $f(t)$ are used in the flow equation (1.4). Without these terms, one can not expect the global existence and the convergence of the flow (1.4) with a generic initial locally convex curve X_0 . See a blow-up example of the flow by Lin-Tsai [25] in the paper [29].*

The evolution equation (1.4) is a completely nonlinear parabolic system for the evolving curve $X(\theta, t) = (x(\theta, t), y(\theta, t))$, where $(\theta, t) \in mS^1 \times [0, \omega)$. The purpose of this paper is to partially answer Yau's question by understanding the asymptotic behavior of the evolving curve $X(\cdot, t)$. As an application in the field of topology, this curvature flow can be used to realize Whitney-Graustein differential homotopy for locally convex curves. The main result of this paper is as follows.

Theorem 1.3. *Let X_0 and \tilde{X} be two smooth and locally convex curves. The flow (1.4) with initial X_0 and target \tilde{X} exists globally, preserves both the positivity of the curvature and the elastic energy of the evolving curve $X(\cdot, t)$. If X_0 and \tilde{X} have same elastic energy, then $X(\cdot, t)$ converges, in the sense of C^∞ metric, to the target curve \tilde{X} as time $t \rightarrow +\infty$.*

The key idea in the proof of Theorem 1.3 is to reduce the nonlinear system to the evolution equation of the radius of the curvature ρ (see (2.7)). It is a semilinear equation with nonlinear part contained in the integral term $f(t)$. The short time existence and the global existence of the flow (1.4) are proved in Section 2. The convergence of this nonlocal flow is proved in Section 3. An example is presented in Section 4.

2. Existence

2.1. Short Time Existence

In this subsection, we prove that the flow (1.4) has a smooth solution on the domain $mS^1 \times [0, t_0)$, where t_0 is a positive number.

Suppose there is a family of locally convex curves evolving under the flow (1.4). Set $\beta(\theta, t) = 2p(\theta, t) - \rho(\theta, t) - 2\tilde{p}(\theta, t) + \tilde{\rho}(\theta, t) + f(t)$. By direct calculations, one has the following evolution equations.

Lemma 2.1. *Applying the equations (1.14)-(1.17) in the book [10], one obtains*

$$(2.1) \quad \frac{\partial T}{\partial t} = \left(\alpha\kappa + \frac{\partial\beta}{\partial s} \right) N = \left(\alpha + \frac{\partial\beta}{\partial\theta} \right) \kappa N,$$

$$(2.2) \quad \frac{\partial N}{\partial t} = - \left(\alpha\kappa + \frac{\partial\beta}{\partial s} \right) T = - \left(\alpha + \frac{\partial\beta}{\partial\theta} \right) \kappa T,$$

$$(2.3) \quad \frac{\partial\theta}{\partial t} = \alpha\kappa + \frac{\partial\beta}{\partial s} = \left(\alpha + \frac{\partial\beta}{\partial\theta} \right) \kappa,$$

$$(2.4) \quad \frac{\partial\kappa}{\partial t} = \kappa^2 \left(-\frac{\partial^2\rho}{\partial\theta^2} + \frac{d^2\tilde{\rho}}{d\theta^2} + \rho - \tilde{\rho} - f(t) \right).$$

By the choice of the tangent component (1.5), we know $\alpha = -\frac{\partial\beta}{\partial\theta}$. So both the Frenet frame $\{T, N\}$ and the tangent angle θ are independent of the time:

$$(2.5) \quad \frac{\partial T}{\partial t} \equiv 0, \quad \frac{\partial N}{\partial t} \equiv 0, \quad \frac{\partial\theta}{\partial t} \equiv 0.$$

Using the above evolution equations, one can compute the evolution equation of the support function:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial t} \langle X, N \rangle = - \left\langle \frac{\partial X}{\partial t}, N \right\rangle = -\beta.$$

So it follows from (1.3) that

$$(2.6) \quad \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial\theta^2}(\theta, t) - p(\theta, t) + 2\tilde{p}(\theta) - \tilde{\rho}(\theta) - f(t)$$

Using the equation (2.4) and the fact that the radius of the curvature $\rho = \frac{1}{\kappa}$, we have

$$(2.7) \quad \frac{\partial\rho}{\partial t}(\theta, t) = \frac{\partial^2\rho}{\partial\theta^2}(\theta, t) - \frac{\partial^2\tilde{\rho}}{\partial\theta^2}(\theta) - \rho(\theta, t) + \tilde{\rho}(\theta) - f(t).$$

Since the function $\rho(\theta, t)$ determines the shape of the evolving curve, the flow (1.4) can be reduced to the equation (2.7) with initial $\rho(\theta, 0) = \rho(\theta)$ in some small time interval.

Lemma 2.2. *The flow (1.4) is equivalent to the equation (2.7) with initial $\rho(\theta, 0) = \rho_0(\theta)$ on some domain $mS^1 \times [0, t_0)$.*

Proof. Let $X(\cdot, t)$ be a family of locally convex curves evolving under the flow (1.4). We immediately have the evolution equation (2.7).

On the other hand, suppose we have a smooth function $\rho(\theta, t)$ satisfying the evolution equation (2.7) with initial $\rho(\theta, 0) = \rho_0(\theta) > 0$ which is the radius of curvature of a given locally convex curve X_0 . By the continuity of ρ , there is a positive t_0 such that $\rho(\theta, t) > 0$ for all $(\theta, t) \in mS^1 \times [0, t_0)$. Then one can construct a family of locally convex curves:

$$(2.8) \quad X(\theta, t) = \int_0^\theta \rho(\hat{\theta}, t) T(\hat{\theta}) d\hat{\theta},$$

where $(\theta, t) \in mS^1 \times [0, t_0)$. Using the equation (2.7), one can check that the $X(\cdot, t)$, up to a parallel movement, satisfies the equation (1.4). So we have done. □

Lemma 2.3. *The flow (1.4) has a unique smooth solution in some time interval $[0, \omega)$.*

Proof. The equation (2.7) is semilinear with a non-local term $f(t)$. The equation is uniformly parabolic, so this Cauchy problem has a unique smooth solution in a short time interval.

Since the initial value ρ_0 is smooth and the higher order derivative evolves according to a linear equation

$$(2.9) \quad \frac{\partial^{k+1}\rho}{\partial\theta^k\partial t} = \frac{\partial^{k+2}\rho}{\partial\theta^{k+2}} - \frac{d^{k+2}\tilde{\rho}}{d\theta^{k+2}} - \frac{\partial^k\rho}{\partial\theta^k} + \frac{d^k\tilde{\rho}}{d\theta^k},$$

the higher order regularity of $\rho(\theta, t)$ can be obtained by the classical theory of linear parabolic equations. Using the equation (2.8), we know that the evolving curve $X(\cdot, t)$ is smooth. □

Under the flow (1.4), the length of the evolving curve satisfies that

$$\frac{dL}{dt}(t) = -L(t) + \tilde{L} - 2\pi f(t),$$

where \tilde{L} is the length of \tilde{X} and the nonlocal term $f(t)$ is given by (1.6). In fact, the nonlocal term $f(t)$ is rather complicated. So the length $L(t)$ of the evolving curve has no explicit solution.

Remark 2.4. *In the previous studies [19, 25], the Fourier series expansion is applied to study curvature flows. Under the flow (1.4), both the evolution*

equation (2.6) and (2.7) are semilinear. The nonlinear term $f(t)$ makes the method of solving linear equations with constant coefficients by the Fourier series expansion do not work here.

Remark 2.5. *Although the initial value $\rho(\theta, 0) = \rho_0(\theta)$ of the equation (2.7) is positive on mS^1 , there is a lack of maximum principle for the equation (2.7). Until now we do not know whether $\rho(\theta, t)$ is always positive or not. Once it is proved that $\rho(\theta, t) > 0$ holds for all $t \in (0, +\infty)$, one can further show that the flow (1.4) exists globally. We leave this part to the next subsection.*

2.2. Global Existence

In this subsection, it is proved that the flow (1.4) exists on time interval $[0, +\infty)$. We shall show that the curvature of the evolving curve has both uniformly lower and upper bounds and the evolving curve $X(\cdot, t)$ is smooth for each $t \in (0, +\infty)$.

Lemma 2.6. *If the flow (1.4) preserves the local convexity of the evolving curve, then the elastic energy is fixed as time goes.*

Proof. Under the flow (1.4), the curvature $\kappa(\theta, t)$ of the curve $X(\cdot, t)$ satisfies the equation (2.4). So the elastic energy of the evolving curve satisfies that

$$\begin{aligned} \frac{dE}{dt}(t) &= \frac{d}{dt} \int_{X(\cdot, t)} \kappa^2(s, t) ds = \frac{d}{dt} \int_{mS^1} \kappa(\theta, t) d\theta \\ &= \int_{mS^1} \kappa^2 \left(-\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \tilde{\rho}}{\partial \theta^2} + \rho - \tilde{\rho} + f(t) \right) d\theta. \end{aligned}$$

By the choice of the nonlocal term $f(t)$, we have $\frac{dE}{dt} \equiv 0$. □

To make the statement brief in the following proofs, we introduce two functions for $t \geq 0$. Now define

$$\rho_{\max}(t) := \max\{\rho(\theta, t) | \theta \in mS^1\}, \quad \rho_{\min}(t) := \min\{\rho(\theta, t) | \theta \in mS^1\}.$$

Corollary 2.7. *If the flow (1.4) preserves the local convexity of the evolving curve, then*

$$(2.10) \quad \rho_{\min}(t) \leq \frac{2m\pi}{E} \leq \rho_{\max}(t).$$

Proof. The elastic energy is preserved under the flow (1.4), so

$$E = \int_{mS^1} \kappa(\theta,t)d\theta = \int_{mS^1} \frac{1}{\rho(\theta,t)}d\theta \leq \frac{2m\pi}{\rho_{\min}(t)},$$

i.e., $\rho_{\min}(t) \leq \frac{2m\pi}{E}$. Similarly, we have $\frac{2m\pi}{E} \leq \rho_{\max}(t)$. □

The following Harnack estimate is a key step towards the proof of the global existence of the flow (1.4). By this estimate, one may control the curvature uniformly.

Lemma 2.8. (*Harnack estimate*) *If the flow (1.4) preserves the local convexity of the evolving curve, then there exists a constant H independent of time such that*

$$(2.11) \quad \rho_{\max}(t) \leq \rho_{\min}(t) \cdot H.$$

Proof. Set $u(\theta, t) = \rho(\theta, t) - \tilde{\rho}(\theta)$. By the evolution equation of ρ (or see the equation (2.9)), $\frac{\partial^i u}{\partial \theta^i}$ satisfies

$$\frac{\partial}{\partial t} \frac{\partial^i u}{\partial \theta^i} = \frac{\partial^{i+2} u}{\partial \theta^{i+2}} - \frac{\partial^i u}{\partial \theta^i}$$

So the function $\left(\frac{\partial^i u}{\partial \theta^i}\right)^2$ evolves according to

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial^i u}{\partial \theta^i}\right)^2 &= \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^i u}{\partial \theta^i}\right)^2 - 2 \left(\frac{\partial^{i+2} u}{\partial \theta^{i+2}}\right)^2 - 2 \left(\frac{\partial^i u}{\partial \theta^i}\right)^2 \\ &\leq \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^i u}{\partial \theta^i}\right)^2 - 2 \left(\frac{\partial^i u}{\partial \theta^i}\right)^2. \end{aligned}$$

Applying the maximum principle, one obtains that

$$(2.12) \quad \left(\frac{\partial^i u}{\partial \theta^i}\right)^2 \leq C_i e^{-2t},$$

where

$$C_i = \max \left\{ \left(\frac{\partial^i u}{\partial \theta^i}(\theta, 0)\right)^2 \mid \theta \in mS^1 \right\}$$

is a constant depending on the initial curve X_0 . Since

$$\left|\frac{\partial^i \rho}{\partial \theta^i}\right| \leq \left|\frac{\partial^i \tilde{\rho}}{\partial \theta^i}\right| + \left|\frac{\partial^i u}{\partial \theta^i}\right| \leq \left|\frac{\partial^i \tilde{\rho}}{\partial \theta^i}\right| + \sqrt{C_i},$$

there is a constant, denoted by M_i , independent of time such that

$$(2.13) \quad \left| \frac{\partial^i \rho}{\partial \theta^i} \right| \leq M_i,$$

where $i = 1, 2, 3, \dots$.

Fixed the time t . Suppose the function $\rho(\theta, t)$ attains its minimum and maximum at points θ_1, θ_2 , respectively. Compute

$$\ln \rho_{\max}(t) - \ln \rho_{\min}(t) = \int_{\theta_1}^{\theta_2} \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} d\theta \leq \int_{\theta_1}^{\theta_2} \frac{1}{\rho} \left| \frac{\partial \rho}{\partial \theta} \right| d\theta \leq M_1 E,$$

where the constant M_1 is given by (2.13). Therefore, choosing $H = e^{M_1 E}$ can give us the estimate (2.11). □

Lemma 2.9. *The flow (1.4) preserves the local convexity of the evolving curve.*

Proof. By the continuity of the evolving curve, there exists $t_0 > 0$ such that $X(\cdot, t)$ is locally convex on the time interval $[0, t_0)$. In this same time interval, it follows from Lemma 2.8 and Corollary 2.7,

$$(2.14) \quad \rho_{\max}(t) \leq \rho_{\min}(t)H \leq \frac{2m\pi}{E}H.$$

So the curvature of the evolving curve satisfies that

$$\kappa(\theta, t) \geq \kappa_{\min}(t) = \frac{1}{\rho_{\max}(t)} \geq \frac{E}{2m\pi H} > 0.$$

As the flow exists, the evolving curve is always locally convex. □

Theorem 2.10. *The flow (1.4) exists on the time interval $[0, +\infty)$.*

Proof. By (2.13), all higher derivatives $\frac{\partial^i \rho}{\partial \theta^i}$ is uniformly bounded. It suffices to show that $\rho(\theta, t) > 0$ for all $(\theta, t) \in mS^1 \times [0, +\infty)$. The Harnack estimate (2.14) and the second inequality of (2.10) imply that

$$(2.15) \quad \rho_{\min}(t) \geq \rho_{\max}(t) \frac{1}{H} \geq \frac{2m\pi}{EH}.$$

Hence, the curvature $\kappa = \frac{1}{\rho}$ never blows up as time goes. Until now we have shown that the function $\rho(\theta, t)$ is uniformly bounded and positive on the

domain $mS^1 \times [0, +\infty)$. So the velocity of the flow (1.4) $\frac{\partial X}{\partial t}$ is smooth for all $t > 0$. Integrating the equation (1.4) gives us

$$(2.16) \quad X(\theta, t) = X_0(\theta) + \int_0^t \frac{\partial X}{\partial t}(\theta, \tilde{t}) d\tilde{t}.$$

We know the evolving curve $X(\cdot, t)$ is smooth for every $t \in [0, +\infty)$. The flow (1.4) exists globally. □

There are two key steps in the proof of Theorem 2.10. The first one is that the derivative $\frac{\partial \rho}{\partial \theta}$ has a uniform bound under the flow (see the equation (2.13)). This fact holds because we have the term $2p(\theta, t) - 2\tilde{p}(\theta)$ in the flow equation (1.4). The second step is that the flow preserves the elastic energy. The construction of the nonlocal term $f(t)$ in the flow equation guarantees this property.

3. Convergence

We first prove the convergence of the radius of curvature. Then we show the convergence of the support function, which implies the convergence of the evolving curve $X(\cdot, t)$ as the time $t \rightarrow +\infty$.

Lemma 3.1. *Under the flow (1.4), the radius of the curvature $\rho(\theta, t)$ of the evolving curve converges as $t \rightarrow +\infty$. If X_0 and \tilde{X} have same elastic energy then*

$$(3.1) \quad \lim_{t \rightarrow +\infty} \rho(\theta, t) = \tilde{\rho}(\theta).$$

Proof. By (2.11), (2.13) and (2.14), both ρ and $|\frac{\partial \rho}{\partial \theta}|$ are uniformly bounded by constants. There exists a convergent subsequence $\rho(\theta, t_i)$ as $t_i \rightarrow +\infty$. Let $\rho_\infty(\theta)$ be the limit of $\rho(\theta, t_i)$. By (2.13), $\rho_\infty(\theta)$ is smooth and for any positive integer k ,

$$(3.2) \quad \lim_{t_i \rightarrow +\infty} \frac{\partial^k \rho}{\partial \theta^k}(\theta, t_i) = \frac{d^k \rho_\infty}{d\theta^k}(\theta).$$

It follows from (2.12) that, for every fixed $\theta \in mS^1$,

$$(3.3) \quad \lim_{t \rightarrow +\infty} \frac{\partial \rho}{\partial \theta}(\theta, t) = \frac{d\tilde{\rho}}{d\theta}(\theta).$$

So, combining (3.2) and (3.3), we have

$$\frac{d\rho_\infty}{d\theta}(\theta) = \lim_{t_i \rightarrow +\infty} \frac{\partial \rho}{\partial \theta}(\theta, t_i) = \frac{d\tilde{\rho}}{d\theta}(\theta).$$

By (2.14) and (2.15), the limit function $\rho_\infty(\theta)$ is uniformly bounded. There is a constant c_0 such that

$$(3.4) \quad \rho_\infty(\theta) = \tilde{\rho}(\theta) + c_0,$$

where $\theta \in mS^1$. Recall that the flow (1.4) preserves the elastic energy E . The constant c_0 is uniquely determined by

$$(3.5) \quad E = \int_{mS^1} \frac{1}{\tilde{\rho}(\theta) + c_0} d\theta.$$

Since every convergent subsequence of $\rho(\cdot, t)$ tends to the fixed limit $\tilde{\rho}(\cdot) + c_0$, $\rho(\cdot, t)$ itself converges to the same limit:

$$(3.6) \quad \lim_{t \rightarrow +\infty} \rho(\theta, t) = \tilde{\rho}(\theta) + c_0.$$

Suppose X_0 and \tilde{X} have same elastic energy. Noticing that the flow (1.4) preserves the elastic energy, one gets from (3.4) and (3.5) that

$$(3.7) \quad \begin{aligned} \int_{mS^1} \frac{1}{\tilde{\rho}(\theta) + c_0} d\theta &= \int_{mS^1} \frac{1}{\rho_\infty(\theta)} d\theta \\ &= E = \int_{mS^1} \frac{1}{\rho_0(\theta)} d\theta = \int_{mS^1} \frac{1}{\tilde{\rho}(\theta)} d\theta. \end{aligned}$$

Comparing the both sides, one obtains that the constant $c_0 = 0$. By (3.6), $\rho(\theta, t)$ converges to $\tilde{\rho}(\theta)$ as $t \rightarrow +\infty$. It follows from (2.13) and (2.12) that this convergence is in the sense of C^∞ metric, i.e., we have the limit

$$\lim_{t \rightarrow +\infty} \frac{\partial^i \rho}{\partial \theta^i}(\theta, t) = \frac{d^i \tilde{\rho}}{d\theta^i}(\theta),$$

for each integer $i = 1, 2, \dots$. □

As a corollary of the convergence of $\rho(\theta, t)$, we know that the nonlocal term $f(t)$ (see (1.6)) in the flow (1.4) converges:

$$\lim_{t \rightarrow +\infty} f(t) = 0.$$

Theorem 3.2. *The evolving curve under flow (1.4) converges to the target curve \tilde{X} , if X_0 and \tilde{X} have same elastic energy.*

Proof. By (1.2) and (1.3), the evolving $X(\cdot, t)$ is uniquely determined by its support function

$$X(\theta, t) = \frac{\partial p}{\partial \theta}(\theta, t)T(\theta) - p(\theta)N(\theta).$$

So it suffices to prove the convergence of $p(\cdot, t)$. Under the flow (1.4), the support function of the evolving curve satisfies the equation (2.6). Mimicing the proof of Lemma 3.1, one can show that

$$(3.8) \quad \lim_{t \rightarrow +\infty} \frac{\partial^i p}{\partial \theta^i}(\theta, t) = \frac{d^i \tilde{p}}{d\theta^i}(\theta),$$

where $i = 1, 2, \dots$. Therefore the equations (3.6) and (3.8) together with the relation between p and ρ imply that

$$(3.9) \quad \lim_{t \rightarrow +\infty} p(\theta, t) = \tilde{p}(\theta) + c_0.$$

where constant c_0 is uniquely determined by (3.5).

If X_0 and \tilde{X} have same elastic energy then it follows from (3.7) that $c_0 = 0$. The convergence of the support function implies that the evolving curve $X(\cdot, t)$ converges to the curve \tilde{X} :

$$\lim_{t \rightarrow +\infty} X(\theta, t) = \tilde{X}(\theta).$$

The higher order convergence of the evolving curve follows from the C^∞ convergence of the support function directly. □

Combining Theorem 2.10 with Theorem 3.2, one immediately proves Theorem 1.3.

Corollary 3.3. *Let X_0 be an initial locally convex curve with winding number m and elastic energy E . Let Y_0 be a target convex curve with elastic energy $\frac{E}{m}$. The flow (1.4) can deform X_0 into the m -fold convex curve mY_0 as $t \rightarrow +\infty$.*

Remark 3.4. *Let X_0 and Y_0 be any two locally convex curves with same winding number. There is a proper rescaling of Y_0 , denoted by \tilde{Y} , such that X_0 and \tilde{Y} have same elastic energy. By Theorem 1.3, X_0 can be deformed*

into \tilde{Y} under the flow (1.4). Therefore, up to a rescaling, any two locally convex curves with same winding number can be evolved into each other via the curvature flow (1.4).

4. An Example

In this section, an example of the flow (1.4) is investigated. Let \tilde{X} be a locally convex curve with support function

$$\tilde{p}(\theta) = 10 + 10 \sin\left(\frac{2}{3}\theta\right) + 10 \sin\left(\frac{2}{3}\theta\right),$$

where $\theta \in [0, 6\pi]$. With the help of MATLAB, one can calculate that the minimum value of its radius of the curvature is 2.1433. We choose a locally convex curve X_0 with support function

$$p_0(\theta) = 11.6736 + 9 \sin\left(\frac{4}{3}\theta\right) + 9 \sin\left(\frac{4}{3}\theta\right),$$

where $\theta \in [0, 6\pi]$. The minimum value of its radius of the curvature is 1.7741. The two curves have same winding number 3 and same elastic energy 3.0469. See the figures (up to proper rotations) of the two curves in Figure 1.

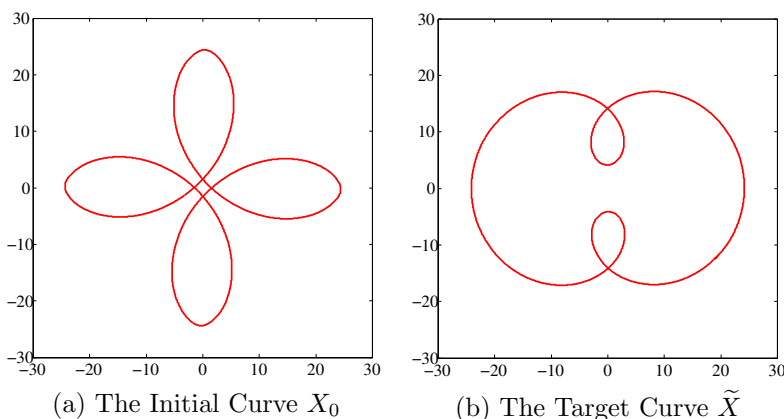


Figure 1. The Initial Curve And The Target Curve

Example 4.1. Let \tilde{X} be the target curve and let X_0 evolve according to the flow (1.4). By Theorem 1.3, the evolving curve $X(\cdot, t)$ is smooth on the time

Time t	$\frac{L(t)}{6\pi}$	$\rho_{\min}(t)$	$\rho_{\max}(t)$	Elastic Energy
0	11.6736	1.7741	21.5731	3.0469
0.01	11.4599	1.632	21.1711	3.0469
0.05	10.8917	1.3325	19.9101	3.0469
0.1	10.4986;	1.2312	18.7768	3.0469
0.2	10.073	1.2673	17.2338	3.0469
0.4	9.76318	1.5216	15.8235	3.0469
0.6	9.7429	1.7638	15.7577	3.0469
1	9.8708	2.0262	16.8691	3.0469
2	9.9905	2.1369	17.7900	3.0469
4	9.99999	2.1433	17.8565	3.0469
$+\infty$	10	2.1432	17.8567	3.0469

Table 1. Some Geometric Quantities of the Evolving Curve

interval $[0, +\infty)$ and it converges to the curve \tilde{X} as $t \rightarrow +\infty$. Some evolving curves (up to proper rotations) are presented in Figure 2 and some relative geometric quantities are given in Table 1.

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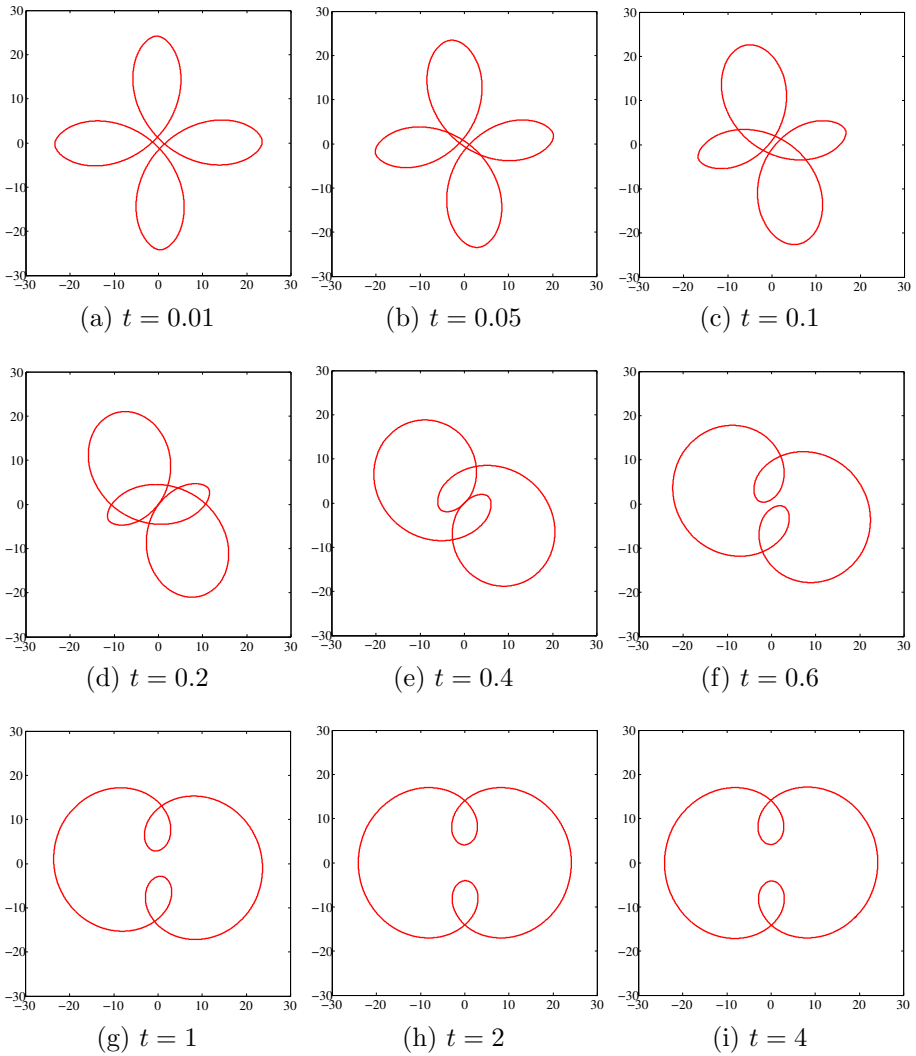


Figure 2. Some Time Shots of the Evolution

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