Math. Res. Lett.
Volume 30, Number 4, 1063–1080, 2023

3-manifolds that bound no definite 4-manifold

Marco Golla and Kyle Larson

We produce a rational homology 3-sphere that does not smoothly bound either a positive *or* negative definite 4-manifold. Such a 3-manifold necessarily cannot be rational homology cobordant to a Seifert fibered space or any 3-manifold obtained by Dehn surgery on a knot. The proof requires an analysis of short characteristic covectors in bimodular lattices.

1. Introduction

By a definite 4-manifold we mean a smooth, compact, oriented 4-manifold, possibly with boundary, whose intersection form is positive or negative definite. Many familiar classes of 3-manifolds are known to bound either positive or negative definite 4-manifolds. These include, for example, any Seifert fibered rational homology sphere or any 3-manifold obtained by nonzero surgery on a knot in S^3. Lens spaces and those 3-manifolds which are the double covers of S^3 branched over an alternating knot are examples of 3-manifolds that bound positive *and* negative definite 4-manifolds.

However, one can often argue that a given 3-manifold cannot smoothly bound *both* positive and negative definite 4-manifolds. For example, consider the Poincaré homology sphere P, oriented as the link of the Brieskorn singularity. Since its Frøyshov h-invariant (or equivalently, the Heegaard Floer correction term d) is positive, it cannot bound a positive definite 4-manifold (see [5, Theorem 3] or [19, Corollary 9.8]). On the other hand, P does bound negative definite 4-manifolds, e.g., either as -1-surgery on the left-handed trefoil or as the boundary of the negative E_8-plumbing.

To obstruct a 3-manifold from bounding *either* positive or negative definite 4-manifolds is rather more difficult. Frøyshov has announced the first examples of integral homology spheres that bound no definite 4-manifolds, provided that the first homology of the putative definite 4-manifold does not contain 2-torsion [4]. His argument depends on the fact that the h-invariant and the q_2-invariant arising from instanton homology are linearly independent. Recent work of Nozaki, Sato, and Taniguchi [17], using filtered

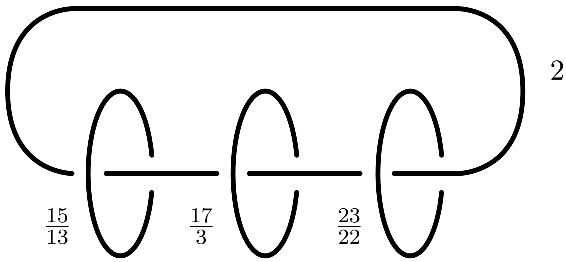

Figure 1: A surgery description of $Y(2; \frac{15}{13}, \frac{17}{3}, \frac{23}{22})$.

instanton Floer homology, gives examples of integral homology spheres that bound no definite 4-manifolds, without any restrictions on the torsion in homology. Any *rational* homology sphere rational homology cobordant to one of the examples of [17] evidently has the same property.

The main result of this paper is the following.

Theorem 1.1. *There exist rational homology spheres that are not rational homology cobordant to an integral homology sphere and that bound no definite* 4*-manifold.*

The authors understand that Matthew Hedden has produced similar (unpublished) examples. As far as we understand, such a result is not currently accessible with the techniques of [17], since filtered instanton Floer homology is only defined for integral homology spheres.

The proof of our theorem instead combines an inequality of Ozsváth and Szabó [19] relating correction terms and the squares of first Chern classes of spin^c structures together with an analysis of short characteristic covectors in bimodular lattices, using work of Elkies [2] on unimodular lattices.

Let us give a specific example: let N denote the 3-manifold $N := 3P\#\overline{Y}$, i.e. the sum of three copies of the Poincaré sphere P and the Seifert fibered space $\overline{Y} = Y(2; \frac{15}{13}, \frac{17}{3}, \frac{23}{22})$. (See Figure 1.) Note that $H_1(N) \cong \mathbb{Z}/2\mathbb{Z}$, so N cannot be homology cobordant to an integral homology sphere. We will prove below that N cannot bound a definite 4-manifold.

One can obtain other examples satisfying various properties by constructing 3-manifolds that are integral or rational homology cobordant to N. For example, work of Livingston [12] implies that N is integral homology cobordant to an irreducible 3-manifold, which can further be assumed to be hyperbolic by work of Myers [14]. From what we discussed above we get the following immediate corollaries.

Corollary 1.2. *N is not rational homology cobordant to any Seifert fibered space.*

Examples of integral homology spheres that are not integral homology cobordant to any Seifert fibered space have been produced by Stoffregen [22] and Frøyshov [4]; previously, rational homology spheres that are not *integral* homology cobordant to any Seifert fibered space appeared in [1].

Corollary 1.3. *N is not rational homology cobordant to any 3-manifold obtained by Dehn surgery on a knot in S^3.*

It is often quite difficult to prove that a 3-manifold cannot be obtained by surgery on a knot (see, for example, [8]), and most of the known obstructions are not preserved under rational homology cobordism. However, in [7] examples of 3-manifolds Y with $b_1(Y) = 1$ are constructed that are not rational homology cobordant to 0-surgery on a knot.

In Section 4 we will give a more direct proof that the 3-manifold $\overline{Y}$ is not obtained as Dehn surgery along a knot in S^3, and use $\overline{Y}$ to produce another example of a *spineless* 4-manifold (see [11] and [6] for previous work on the subject—Hayden and Piccirillo's results in particular are much stronger than ours).

Organization of the paper

In Section 2 we study bimodular lattices and their short characteristic covectors. In Section 3 we describe an obstruction for a rational homology sphere Y with $H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ to bound a definite 4-manifold. In Section 4, we show that the manifold N described above satisfies the conditions of the obstruction from Section 3.

2. Characteristic covectors of bimodular lattices

In this section, a lattice Λ will be a subgroup $\Lambda \subset \mathbb{R}^m$ isomorphic to $\mathbb{Z}^m$, and such that, with respect to the Euclidean scalar product on $\mathbb{R}^m$ one has $v \cdot w \in \mathbb{Z}$ for each $v, w \in \Lambda$. A lattice is said to be *minimal* if it contains no vector of square 1.

Note that by our definition lattices correspond to positive definite integral quadratic forms. Some authors use the 'lattice' to denote any integral quadratic form, even one that is not positive definite.

We denote with Λ' the *dual lattice* of Λ, i.e. the set of vectors in $\mathbb{R}^m$ that pair integrally with each element in Λ. Note that $\Lambda \subset \Lambda'$, and that Λ' is *not* a lattice according the definition above, unless $\Lambda = \Lambda'$. (There are always vectors in Λ' that square to a rational if the containment is strict.) Each element in Λ' is a rational linear combination of vectors in Λ. We implicitly identify the $\mathbb{Q}$-span of Λ with $\Lambda \otimes \mathbb{Q}$, which therefore we view as a subset of $\mathbb{R}^n$. The index $[\Lambda' : \Lambda]$ is called the *determinant* of Λ, and is denoted by $\det \Lambda$. If $\det \Lambda = 1$ (or, equivalently, $\Lambda = \Lambda'$) we say that Λ is *unimodular* (or *self-dual*). We use the notation $\langle \xi, v \rangle$ to denote the pairing of a covector $\xi \in \Lambda'$ and a vector $v \in \Lambda$ (which is simply the scalar product in $\mathbb{R}^n$).

We denote with $\mathrm{Char}(\Lambda)$ the subset of Λ' comprising all ξ such that $\langle \xi, v \rangle \equiv v^2 \pmod 2$. Each such ξ is a *characteristic covector*. (Again, when Λ is unimodular ξ is actually an element in Λ, but we prefer to talk about covectors to emphasize that we are thinking about the dual.)

Throughout the section, the letter L will always denote an integral lattice of rank n and determinant 2 (a *bimodular lattice*), A will be an auxiliary lattice of determinant 2, and M_A will be the lattice $L \oplus A$. Note that ${M_A}'/M_A \cong L'/L \oplus A'/A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has a unique metabolizer (i.e. a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ that is isotropic with respect to the induced $\mathbb{Q}/\mathbb{Z}$-valued bilinear form induced by the product on M_A), so that M_A is an index-2 sublattice of a unimodular lattice U_A. (See, for example, [10] for a topologically-minded treatment.) In fact, U_A is simply $L \oplus A \cup (L' \setminus L) \times (A' \setminus A) \subset L' \oplus A'$. With a slight abuse of notation, we view L and A as subsets of U_A. It is easy to see that $L = A^\perp$ and $A = L^\perp$. Since neither L nor A have a metabolizer, note that $L \otimes \mathbb{Q} \cap U_A = L$ and $A \otimes \mathbb{Q} \cap U_A = A$. Note that U_A is uniquely determined by both L and A, but we do not make the dependency on L explicit in the notation.

Dualizing, $U'_A \cong U_A$ is an index-2 subset of ${M_A}' = L' \oplus A'$. Moreover, the restriction maps $U'_A \to L'$ and $U'_A \to A'$ are both onto. In the same way, the restriction maps $\mathrm{Char}(U_A) \to \mathrm{Char}(L)$ and $\mathrm{Char}(U_A) \to \mathrm{Char}(A)$ are onto.

Two choices for A stand out. We can choose $A = L$, or $A = A_1$, where A_1 is the rank-1 lattice generated by a vector of square 2 (i.e. a *root*). We write U and M instead of U_{A_1} and M_{A_1}. Call r one of the two generators of the auxiliary lattice A_1. By construction, U contains r, a vector of norm 2.

Lemma 2.1. *Let $\xi \in \mathrm{Char}(L)$ be a characteristic covector. Then ξ^2 is an integer, and $\xi^2 \equiv n \pm 1 \pmod 8$.*

Proof. Choose a characteristic covector $\xi_U \in \mathrm{Char}(U)$ that extends ξ, and call $\xi_A \in \mathrm{Char}(A_1)$ its restriction to A_1. Then $\xi_U^2 \equiv \mathrm{rk}\, U = n + 1 \pmod 8$, and $\xi_U^2 = \xi^2 + \xi_A^2$. Since $\xi_A^2 \equiv 0, 2 \pmod 8$ by direct verification, the lemma follows. $\square$

We call $\mathrm{Char}_\pm(L)$ the set of characteristic covectors ξ of L for which $\xi^2 \equiv n \pm 1 \pmod 8$. Let us go back to the case of A arbitrary. Since A is a determinant-2 lattice, $\mathrm{Char}_\pm(A)$ are defined, too.

Lemma 2.2. *$\xi_u \in U'_A$ is a characteristic covector of U_A if and only if there exists a sign s such that $\xi_u|_L \in \mathrm{Char}_s(L)$ and $\xi_u|_A \in \mathrm{Char}_{-s}(A)$.*

Recall that if Λ is a lattice, then $\mathrm{Char}(\Lambda)$ is affine over $2\Lambda'$ (i.e. $\mathrm{Char}(\Lambda) = \xi + 2\Lambda'$ for any $\xi \in \mathrm{Char}(\Lambda)$). An easy extension of this fact is the observation that $\mathrm{Char}_\pm(L)$ is affine over $2L$ (and not over $2L'$), and that translations by elements in $2L' \setminus 2L$ swap $\mathrm{Char}_+(L)$ and $\mathrm{Char}_-(L)$.

Proof. Call $\xi_\ell = \xi_u|_L$ and $\xi_a = \xi_u|_A$. The 'only if' direction is clear, since $\xi_\ell^2 + \xi_a^2 = \xi_u^2 \equiv n + \mathrm{rk}\, A \pmod 8$.

Let us look at the 'if' direction. Now, let $C := \mathrm{Char}_+(L) \times \mathrm{Char}_-(A) \cup \mathrm{Char}_-(L) \times \mathrm{Char}_+(A) \subset M'$. By the 'only if' direction above, C contains $\mathrm{Char}(U_A)$, which is an affine space over $2U'_A$. On the other hand, $\mathrm{Char}_\pm(L) \times \mathrm{Char}_\mp(A)$ is also affine over $2L + 2A = 2M_A$, and, by the remark below the statement, if $v \in U_A \setminus (L \oplus A) = (L' \setminus L) \times (A' \setminus A)$ and $\xi \in \mathrm{Char}_\pm(L) \times \mathrm{Char}_\mp(A)$ then $\xi + 2v \in \mathrm{Char}_\mp(L) \times \mathrm{Char}_\pm(A)$, so that C is affine over $2U'_A$, too. In particular, C and $\mathrm{Char}(U_A)$ are both affine subspaces over $2U'_A$, and since $\mathrm{Char}(U_A) \subset C$ then they are equal. $\square$

We denote with I^m the lattice $\mathbb{Z}^m \subset \mathbb{R}^n$. This is the unique unimodular lattice that has an orthonormal basis (and we will say it is *diagonal*). The lattice $\Delta_n := I^{n-1} \oplus A_1$ is the unique bimodular lattice of rank n whose intersection form is diagonal (i.e. which has an orthogonal basis).

Lemma 2.3. *The lattice U is diagonal if and only if L is.*

Proof. Suppose that U is diagonal, with basis $e_0, \dots, e_n$. Then the generator r of A_1 is a root in U. In particular, up to re-indexing the generators of U (and possibly flipping signs), $r = e_0 - e_1$. Now, $L \cong \langle r \rangle^\perp$, but $\langle r \rangle^\perp$ is spanned by $e_0 + e_1, e_2, \dots, e_n$, and in particular it is isomorphic to Δ_n.

If $L = \Delta_n$, then L embeds in I^{n+1} as we have just seen; however, U is uniquely determined by L, so $U \cong I^{n+1}$. $\square$

We recall a result of Elkies on characteristic covectors in unimodular lattices.

Theorem 2.4 ([2]). *Let Λ be unimodular lattice of rank m. Then*

$$\min_{\xi\in\mathrm{Char}(\Lambda)} \xi^2 \leq m,$$

and the equality is attained if and only if Λ is the diagonal lattice I^m.

We find it convenient to introduce the notation for the defect of a lattice. The *defect* $d(\Lambda)$ of Λ is:

$$d(\Lambda) = \min_{\xi\in\mathrm{Char}(\Lambda)} \frac{\xi^2 - \operatorname{rk}\Lambda}{4}$$

Note that Elkies' theorem can be rephrased as saying that a unimodular lattice has non-positive defect, and that the defect is 0 if and only if the lattice is diagonal.

When L is bimodular, using Lemma 2.1 we can identify *two* defects, denoted with $d_\pm(L)$:

$$d_\pm(L) = \min_{\xi\in\mathrm{Char}_\pm(L)} \frac{\xi^2 - \operatorname{rk} L}{4}$$

Note that $d_\pm(L) \equiv \pm\frac{1}{4} \pmod 2$.

We start by recasting Lemma 2.3 in terms of $d_\pm$.

Lemma 2.5. *The lattice L is diagonal if and only if $d_\pm(L) = \pm\frac{1}{4}$.*

Proof. Suppose that $d_\pm(L) = \pm\frac{1}{4}$. Since $d_\pm(A_1) = \pm\frac{1}{4}$, $d(U) = \min\{d_+(L) + d_-(A_1), d_-(L) + d_+(A_1)\} = 0$. By Elkies' theorem U is diagonal, and by Lemma 2.3 so is L.

On the other hand, a direct computation shows that $d_\pm(\Delta_n) = \pm\frac{1}{4}$. □

The main result of this section is a bimodular version of of Elkies' theorem and gives another characterization of Δ_n (see [18] for a different characterization).

Theorem 2.6. *For every bimodular lattice L, $d_+(L) + d_-(L) \leq 0$. Moreover, if equality is attained, L is diagonal, and in particular $d_\pm(L) = \pm\frac{1}{4}$.*

As mentioned above, a root in a lattice Λ is a vector of square 2; we say that Λ is a *root lattice* if it is rationally spanned by its set of roots. To a collection $R \subset \Lambda$ of $\mathbb{Q}$-linearly independent roots we associate an edge-weighted graph $G(R)$ as follows: the set of vertices of $G(R)$ is R, and there is an edge joining $r, s \in R$ with weight $r \cdot s$ if $r \cdot s \neq 0$. Note that if $r, s \in R$ are distinct, they are linearly independent, so by Cauchy–Schwarz $r \cdot s \in \{-1, 0, 1\}$.

We will make the choice of L itself as an auxiliary lattice in the proof. To make the notation lighter, we denote U_L by D, and we call it the *double* of L. To distinguish between the two summands in $M_L \cong L \oplus L$, we call B its second summand; however, we drop the dependency the auxiliary lattice from the notation. In summary, we have $M = L \oplus B$ as an index-2 sublattice of D, and L and B are isomorphic and they are viewed as a pair of orthogonal sublattices in M and in D.

We call $\pi_\ell\colon M \to L$ and $\pi_b\colon M \to B$ the two orthogonal projections, and $\rho_\ell\colon D' \to L'$ and $\rho_b\colon D' \to B'$ the two restriction maps. In fact, π_ℓ and ρ_ℓ are both restrictions of a linear map $D \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ (and similarly for B).

Proof. We can assume, without loss of generality, that L is minimal (i.e. it contains no vectors of norm 1): indeed, it is easy to verify that $d_\pm(L) = d_\pm(L \oplus I^m)$, and L is diagonal if and only if $L \oplus I^m$ is.

Call n the rank of L. Consider now D, the double of L. D is a unimodular lattice, so $d(D) \leq 0$. By Lemma 2.2, $d(D) = \min\{d_+(L) + d_-(B), d_-(L) + d_+(B)\} = d_+(L) + d_-(L)$, which proves the first assertion.

Let us now suppose that $d_+(L) + d_-(L) = 0$; again by Elkies' theorem, this implies that D is the diagonal lattice I^{2n}. Call $e_1, \dots, e_{2n}$ an orthonormal basis of D.

We claim that L is a root lattice.

Since L is minimal, $e_i \in D \setminus M$. However, since M has index 2 in D, $2e_i \in M$. We also know that $2e_i \notin L \cup B$: indeed, as mentioned at the beginning of the section, $L \otimes \mathbb{Q} \cap D = L$, so if $2e_i \in L$, then also $e_i \in L$. (By symmetry, this proves the statement for B as well.)

This implies that $r_i := \pi_\ell(2e_i)$ and $\pi_b(2e_i)$ are two non-zero vectors whose squares sum to $(2e_i)^2 = 4$; since L is minimal, they both have square 2. Since the collection $\{2e_i\}$ is a rational basis of D, the collection $\{r_i\}$ is a set of roots that rationally spans L, which proves the claim.

If $d_\pm(L) = \pm\frac{1}{4}$, by Lemma 2.5 L is diagonal.

Suppose now $d_\pm(L) \neq \pm\frac{1}{4}$, so that in particular $|d_+(L)| = |d_-(L)| =: d \geq \frac{7}{4}$. Consider the characteristic covector $\xi_0 = e_1 + \cdots + e_{2n} \in \mathrm{Char}(D)$, and, for each i, the characteristic covector $\xi_i = \xi_0 - 2e_i \in \mathrm{Char}(D)$. Note that ξ_i is norm-minimizing among all characteristic covectors in D for $i = 0, \dots, 2n$, and so that its restrictions $\lambda_i = \rho_\ell(\xi_i) \in L'$ and $\beta_i = \rho_b(\xi_i)$ are characteristic and they minimize the norm *in their congruence class.*

That is, if $\lambda_i \in \mathrm{Char}_+(L)$, then λ_i minimizes the norm among all elements in $\mathrm{Char}_+(L)$; in this case, $\beta_i \in \mathrm{Char}_-(B)$ and β_i minimizes the norm in $\mathrm{Char}_-(B)$.

Without loss of generality, let us suppose that $\lambda_0 \in \mathrm{Char}_+(L)$. The key observation is that $\lambda_i \in \mathrm{Char}_-(L)$ for each $i = 1, \dots, 2n$. This follows from the fact, observed above, that $e_i \notin L$, so that $\pi_\ell(2e_i) \in 2L' \setminus 2L$, and in particular $\pi_\ell(2e_i)$ swaps $\mathrm{Char}_+(L)$ and $\mathrm{Char}_-(L)$.

Now, since $\lambda_i \in \mathrm{Char}_-(L)$ for each $i > 0$ is a norm-minimizer in its class:

$$|\lambda_0^2 - \lambda_i^2| = 8d \geq 14.$$

However,

$$\lambda_0^2 - \lambda_i^2 = 2\langle \lambda_0, r_i \rangle - r_i^2,$$

so that for each $i > 0$:

$$|\langle \lambda_0, r_i \rangle| \geq 4d - 1 \geq 3.$$

Pick a subset $J \subset \{1, \dots, 2n\}$ of indices such that $R = \{r_j \mid j \in J\}$ is a rational basis for L; this in particular means that $|J| = n$. Up to relabelling, let us assume $J = \{1, \dots, n\}$. We claim that $G(R)$ is bipartite.

To see this, we will prove that all cycles in $G(R)$ have even length. More precisely, say that a cycle $r_1, \dots, r_k$ is *minimal* if there is an edge between r_i and r_j if and only if $|i - j| = 1$ or $\{i, j\} = \{1, k\}$. We claim that all minimal cycles in $G(R)$ have length 4. (See Remark 2.7 below for the necessity of this assumption.) This implies that $G(R)$ is bipartite, since every cycle in $G(R)$ can be decomposed into minimal cycles, and if all minimal cycles have even length, then so do all other cycles.

Let us now prove that all minimal cycles have length 4. Assume by contradiction that there is a minimal cycle $C \subset G(R)$ of length $k \geq 3, k \neq 4$. Up to another relabelling, let us assume that C comprises $r_1, \dots, r_k$ in this order. Up to replacing r_i with $-r_i$ for some values of i, we can assume that all edges $(r_1, r_2), \dots, (r_{k-1}, r_k)$ are labelled with -1. Under this assumption, (r_k, r_1) has to be labelled by $+1$, for otherwise $(r_1 + \cdots + r_k)^2 = 0$, which would contradict the fact that R is a linearly independent set. Now recall that $R \subset L \subset D \cong I^{2n}$ comprises elements of square 2. So there is a basis $f_1, \dots, f_{2n}$ of D such that $r_1 = f_1 - f_2, \dots, r_{k-1} = f_{k-1} - f_k$, and $r_k = f_k + f_1$; but then $r_1 + \cdots + r_k = 2f_1 \in L$, which implies $f_1 \in L$ since $L \otimes \mathbb{Q} \cap D = L$, and this contradicts the minimality of L.

Since $G(R)$ is bipartite there is a subset $R' \subset R$, indexed by $J' \subset J$, containing $\lceil \frac{n}{2} \rceil$ roots that are pairwise orthogonal.

Now, by Bessel's inequality:

$$\lambda_0^2 \geq \sum_{j\in J'} \frac{\langle \lambda_0, r_j\rangle^2}{r_j^2} \geq \left\lceil \frac{n}{2} \right\rceil \cdot \frac{9}{2} > 2n,$$

which contradicts the fact that $\lambda_0^2 \leq \lambda_0^2 + \beta_0^2 = \xi_0^2 = 2n$. □

Remark 2.7. There exists, in fact, a 4-cycle of roots in a diagonal lattice for which the argument above does not hold. The quadruple of linearly independent vectors

$$(r_1, r_2, r_3, r_4) = (f_1 - f_2, f_2 - f_3, -f_1 - f_2, -f_1 + f_4),$$

defines a minimal 4-cycle of roots in I^4 (with orthonormal basis $f_1, \dots, f_4$) such that $r_1 + \cdots + r_4 = -f_1 - f_2 - f_3 + f_4$ is a primitive vector. We do not know whether this cycle embeds in a lattice of determinant 2 that is also rationally generated by roots.

3. The obstruction

In this section we discuss a topological application of Theorem 2.6. We start with an algebraic topology lemma.

Lemma 3.1. *Let X be a compact, oriented 4-manifold with boundary Y, a closed 3-manifold with $H_1(Y)$ finite of square-free order. Then $|\det Q_X| = |H_1(Y)|$ and all spin^c structures on Y extend to X.*

Proof. Let us look at the long exact sequence for the pair (X, Y):

$$0 \to H^2(X,Y) \to H^2(X) \to H^2(Y) \to H^3(X,Y) \to H^3(X) \to 0.$$

All spin^c structures on Y extend if and only if the restriction map $H^2(X) \to H^2(Y)$ is onto. Since $H^2(Y)$ is finite, $H^3(X,Y)$ and $H^3(X)$ have the same rank, b_3; call B and A their torsion subgroups, respectively. For the same reason, $H^2(X,Y)$ and $H^2(X)$ have the same rank, b_2; by the universal coefficient theorem and Poincaré–Lefschetz duality, their torsion subgroups are isomorphic to A and B, respectively. Since torsion can only map to torsion, call τ_i the map obtained by restricting $\pi_i^* : H^i(X,Y) \to H^i(X)$ to the torsion subgroup, and then projecting the target to the torsion subgroup; we regard τ_2 as a map $\tau_2 : A \to B$, and τ_3 as a map $\tau_3 : B \to A$. Note that, since π_3^* is onto and $H^2(Y)$ is torsion, the induced map on the quotient

$H^3(X,Y)/B \to H^3(X)/A$ is injective (in fact, an isomorphism). Finally, the map $H^2(X,Y)/A \to H^2(X)/B$ is represented by the intersection form Q_X of X.

With this in place, we can then apply the nine lemma to

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & H^3(X,Y) \arrow[r] \arrow[d] & H^3(X,Y)/B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & H^3(X) \arrow[r] & H^3(X)/A \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & H^2(X,Y) \arrow[r] \arrow[d] & H^2(X,Y)/A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & H^2(X) \arrow[r] & H^2(X)/B \arrow[r] & 0 \end{tikzcd}$$

to obtain that $\ker \pi_3^*$ is a group G of order $|\ker \tau_3| = |B|/|A|$, and that $\operatorname{coker} \pi_2^*$ is a group H of order $|\operatorname{coker} \tau_2| \cdot |\operatorname{coker} Q_X| = (|B|/|A|) \cdot |\det Q_X|$.

It follows that we can extract a short exact sequence of finite groups:

$$0 \to H \to H_1(Y) \to G \to 0,$$

from which

$$|H_1(Y)| = |H| \cdot |G| = \frac{|B|^2}{|A|^2} |\det Q_X|.$$

Since we assumed that $H_1(Y)$ has square-free order, we conclude that $|A| = |B|$ and $|H_1(Y)| = |\det Q_X|$; each of these conclusions imply that $H^2(X) \to H^2(Y)$ is onto. □

To a closed, oriented, spin^c rational homology 3-sphere $(Y, \mathfrak{t})$, Ozsváth and Szabó [21] associate a family of invariants, collectively called *Heegaard Floer homology.*

In [19], Ozszáth and Szabó extracted from the Heegaard Floer homology package a rational number $d(Y, \mathfrak{t})$, called the *correction term* of $(Y, \mathfrak{t})$, that is an invariant under spin^c rational homology cobordism, and reduces modulo 2 to the rho-invariant $\rho(Y, \mathfrak{t})$. Recall that the *rho-invariant* of $(Y, \mathfrak{t})$ is defined as:

$$\rho(Y, \mathfrak{t}) = \frac{c_1(\mathfrak{s})^2 - \sigma(W)}{4} \in \mathbb{Q}/2\mathbb{Z},$$

where $(W, \mathfrak{s})$ is any compact spin^c 4–manifold whose boundary is $(Y, \mathfrak{t})$.

In the remainder of the section Y will always denote a closed, oriented 3-manifold with $H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$; we say that Y is a *homology* $\mathbb{RP}^3$. Hence Y has exactly two spin^c structures.

Our argument will depend on the value of these correction terms. First we pin down their value modulo 2.

Proposition 3.2. *We can label the two spin^c structures on Y as $\mathfrak{t}_+$ and $\mathfrak{t}_-$, so that $d(Y, \mathfrak{t}_\pm) \equiv \pm\frac{1}{4} \pmod 2$.*

We start with a preliminary lemma.

Lemma 3.3. *Let W be a cobordism from an integral homology sphere Z to a homology $\mathbb{RP}^3$, Y. Then both spin^c structures on Y extend to W.*

Proof. Carve an open neighborhood of a path from Z to Y into W, to obtain a 4-manifold X with boundary $Y\#(-Z)$. The statement is equivalent to the fact that both spin^c structures on $\partial X = Y\#(-Z)$ extend to X, which follows from Lemma 3.1. $\square$

Fix a spin^c structure $\mathfrak{t}$ on Y and a simply-connected 4-manifold X with spin^c structure $\mathfrak{s}$, such that $\partial X = Y$ and $\mathfrak{s}|_Y = \mathfrak{t}$. Recall now that the d-invariant $d(Y, \mathfrak{t})$ reduces modulo 2 to the rho-invariant $\rho(Y, \mathfrak{t}) \in \mathbb{Q}/2\mathbb{Z}$. The rho-invariant is defined as $\rho(Y, \mathfrak{t}) \equiv \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{4} \pmod 2$; it follows from the definition that if $(W, \mathfrak{s})$ is a spin^c cobordism from $(Y, \mathfrak{t})$ to $(Y', \mathfrak{t}')$, then $\rho(Y', \mathfrak{t}') - \rho(Y, \mathfrak{t}) \equiv \frac{c_1(\mathfrak{s})^2 - \sigma(W)}{4} \pmod 2$. In particular, integral homology spheres Z have $\rho(Z, \mathfrak{t}) = 0$, since they bound spin manifolds (which have signature divisible by 8, by the van der Blij Lemma [13, Section II.5], and which have a spin^c structure with trivial first Chern class).

Proof of Proposition 3.2. Since $\rho(Y, \mathfrak{t})$ lifts to $d(Y, \mathfrak{t})$, the statement clearly reduces to showing that $\rho(Y, \mathfrak{t}_\pm) \equiv \pm\frac{1}{4} \pmod 2$. This is what we set out to prove, by finding a suitable cobordism from Y to an integral homology sphere.

Pick a knot K in Y such that $[K] \neq 0 \in H_1(Y)$. Then there exists a slope γ such that the result of Dehn surgery along K with slope γ is an integral homology sphere $Z_0 := Y_\gamma(K)$. Let $K_0 \subset Z_0$ denote the dual knot. Since $H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ and Z_0 is an integer homology sphere, the surgery on K_0 that returns Y must have slope $2/q$ for some odd integer q.

We can write $2/q$ as a (negative) continued fraction $2/q = [0, -\frac{q+1}{2}, -2]$, so that Y can be represented by the surgery diagram in Figure 2. It is easy to see that the 3-manifold obtained by doing surgery on the 0- and $-\frac{q+1}{2}$-framed components is again an integral homology sphere, which we denote

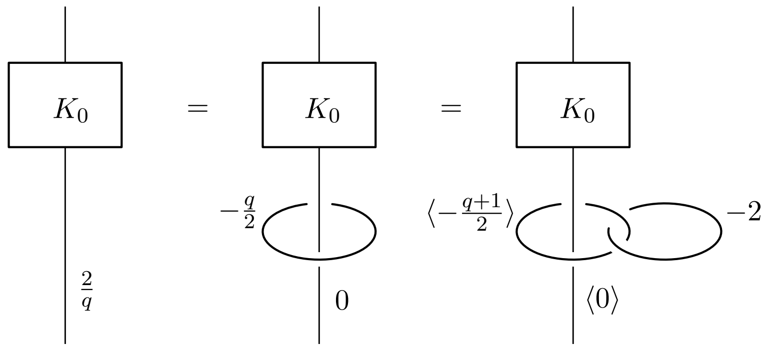

Figure 2: Going from Z_0 to Z. Recall that the knot K_0 lives in Z_0. The rightmost picture represents the cobordism from Z to Y. Here we used the braced framing notation: namely, the surgery diagram comprising the components with braced framings describes Z, i.e. the lower boundary component of the cobordism, and the non-braced ones represent actual handle attachments for the cobordism.

by Z, and that the cobordism W from Z to Y given by the -2-framed 2-handle is negative definite. Moreover, since W is obtained by attaching a single 2-handle to an integral homology sphere, $H_1(W) = H_3(W) = 0$; in particular, both spin^c structures on Y extend to W.

Since W is the trace of a 2-handle attachment along a knot in an integral homology sphere with framing -2, the two spin^c structures $\mathfrak{s}_+$ and $\mathfrak{s}_-$, with Chern classes 0 and $2\gamma \in H^2(W;\mathbb{Z}) \equiv \mathbb{Z}\cdot\gamma$ respectively, have $c_1(\mathfrak{s}_+)^2 = 0$, $c_1(\mathfrak{s}_-^2) = -2$. By the cobordism formula mentioned above, letting $\mathfrak{t}_\pm$ be the restriction of $\mathfrak{s}_\pm$ to Y, we get:

$$\rho(Y,\mathfrak{t}_+) = \frac{c_1(\mathfrak{s}_+)^2 - \sigma(W)}{4} + \rho(Z,\mathfrak{t}) \equiv +\frac{1}{4} \pmod 2,$$
$$\rho(Y,\mathfrak{t}_-) = \frac{c_1(\mathfrak{s}_-)^2 - \sigma(W)}{4} + \rho(Z,\mathfrak{t}) \equiv -\frac{1}{4} \pmod 2,$$

thus concluding the proof. □

Proposition 3.2 justifies the following definition.

Definition 3.4. For Y a homology $\mathbb{RP}^3$, we set $d_{\pm 1/4}(Y) = d(Y,\mathfrak{t}_\pm)$.

Note that the labelling is chosen so that $d_{\pm 1/4}(Y) \equiv \pm\frac{1}{4} \pmod 2$; observe also that since $d(Y,\mathfrak{t}) = -d(-Y,\mathfrak{t})$, we have that $d_{\pm 1/4}(-Y) = -d_{\mp 1/4}(Y)$.

Now suppose that Y bounds a *positive definite* 4-manifold W. In this context we have the following inequality.

Theorem 3.5 **([19]).** *For each spin^c structure $\mathfrak{s}$ on W with $\mathfrak{s}|_Y = \mathfrak{t}$, we have*

$$\frac{c_1(\mathfrak{s})^2 - b_2(W)}{4} \geq d(Y, \mathfrak{t}).$$

Moreover, the two sides of the inequality are congruent modulo 2.

We are now ready to give a topological translation of Theorem 2.6.

Proposition 3.6. *Let Y be a homology $\mathbb{RP}^3$. If Y bounds a positive definite 4-manifold, then $d_{1/4}(Y) + d_{-1/4}(Y) \leq 0$. Moreover, if equality is attained, then $d_{\pm 1/4}(Y) = \pm\frac{1}{4}$.*

Proof. Suppose that Y bounds a positive definite 4-manifold W, and let L be the lattice $(H_2(W;\mathbb{Z})/\mathrm{Tor}, Q_W)$. By Lemma 3.1, L is a positive definite lattice of determinant 2, and the first Chern class gives a surjection $c_1\colon \mathrm{Spin}^c(W) \to \mathrm{Char}(L)$. Call $n = b_2(W) = \mathrm{rk}\, L$.

By the last statement in Theorem 3.5, using the labelling of Proposition 3.2, we see that $\mathfrak{s} \in \mathrm{Spin}^c(W)$ restricts to $\mathfrak{t}_\pm$ if and only if $c_1(\mathfrak{s}) \in \mathrm{Char}_\pm(L)$. Let $\xi_+ \in \mathrm{Char}_+(L)$ and $\xi_- \in \mathrm{Char}_-(L)$ be characteristic covectors with minimal square; note that there exist spin^c structures $\mathfrak{s}_\pm$ on W such that $c_1(\mathfrak{s}_\pm) = \xi_\pm$, and that $\mathfrak{s}_\pm$ restricts to $\mathfrak{t}_\pm$.

Then using Theorem 3.5 and Theorem 2.6 we get

$$d_{1/4}(Y) + d_{-1/4}(Y) \leq \frac{\xi_+^2 - n}{4} + \frac{\xi_-^2 - n}{4} = d_+(L) + d_-(L) \leq 0,$$

proving the first part of the theorem. Furthermore, if $d_{1/4}(Y) + d_{-1/4}(Y) = 0$, then the above inequality forces $d_+(L) + d_-(L) = 0$, and so by Theorem 2.6 we get that $d_\pm(L) = \pm\frac{1}{4}$. This, in turn, together with Theorem 3.5, forces $d_{\pm 1/4}(Y) = \pm\frac{1}{4}$. □

Remark 3.7. It follows from the proof of Proposition 3.2 that every homology $\mathbb{RP}^3$ Y bounds a negative definite *topological* 4-manifold. In that proof, we exhibited a negative definite homology cobordism W from an integral homology sphere Z to Y. By Freedman's work [3], Z bounds a contractible topological 4-manifold W', and gluing W' and W along Z gives a negative definite topological 4-manifold bounding Y.

4. The example

Recall that we defined $\overline{Y}$ as the Seifert fibered space $Y(2; \frac{15}{13}, \frac{17}{3}, \frac{23}{22})$ and $N = 3P\#\overline{Y}$, where P is the Poincaré homology sphere, oriented as the boundary of the negative E_8-plumbing; equivalently, P is the Brieskorn sphere $\Sigma(2,3,5)$.

We start by computing the correction terms of $\overline{Y}$.

Proposition 4.1. *$d_{-1/4}(\overline{Y}) = -\frac{17}{4}$ and $d_{1/4}(\overline{Y}) = -\frac{31}{4}$.*

Proof. Since $-\overline{Y}$ is the boundary of a negative definite plumbing with a single bad vertex, we can compute these correction terms using Çağrı Karakurt's implementation [9] of Némethi's formula [15, Section 11.13], which, in turn, is a generalization of Ozsváth and Szabó's algorithm from [20]. □

Remark 4.2. Némethi computes of the d-invariant of a Seifert fibered space as a sum of two terms; the first summand is expressed in terms of Dedekind–Rademacher sums associated to the Seifert parameters [15, Section 11.9], while the second depends on the minimum of a certain function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. The function is eventually increasing, and the minimum is contained in a bounded interval $[0, N]$, where N can be chosen to be the product of the multiplicities of the fibers.

Furthermore, in principle the computation of the correction terms of $\overline{Y}$ could be done in other ways: either by computing the minimal squares in the lattice associated to the canonical negative plumbing of $-\overline{Y}$ [20, Corollary 1.5] or by following the entire algorithm in [20].

We are now ready to prove our main result; more precisely, we will prove that N does not bound a definite 4-manifold.

Proof of Theorem 1.1. By additivity of correction terms, and since $d(P, \mathfrak{t}) = 2$ for the unique spin^c structure $\mathfrak{t}$ on P, we know that $d_{\pm 1/4}(N) = 3d(P, \mathfrak{t}) + d_{\pm 1/4}(\overline{Y})$. By the previous proposition, we get $d_{\pm 1/4}(N) = \mp\frac{7}{4}$.

Proposition 3.6 now implies that N cannot bound a positive definite 4-manifold. Reversing orientation and again applying Proposition 3.6 shows that N cannot bound a negative definite 4-manifold either. □

We conclude with two observations about $\overline{Y}$ and spineless 4-manifolds.

Proposition 4.3. *Let Σ be an integral homology sphere. The 3-manifold $\overline{Y}\#\Sigma$ is not integral homology cobordant to a 3-manifold obtained as Dehn surgery along a knot in S^3.*

Proof. If we let $Y = S^3_{2/q}(K)$ with $q > 0$; then, by [16, Proposition 1.4 and Lemma 2.4]:

$$d_{\pm 1/4}(Y) \in \left\{-2V_0(K) + \frac{1}{4}, -2V_1(K) + \frac{1}{4}\right\} \Longrightarrow d_{1/4}(Y) \geq -2V_0(K) + \frac{1}{4},$$
$$d_{-1/4}(Y) = -2V_0(K) - \frac{1}{4},$$

so that in particular $d_{1/4}(Y) - d_{-1/4}(Y) \geq \frac{1}{2}$. However

$$d_{1/4}(\overline{Y}\#\Sigma) - d_{-1/4}(\overline{Y}\#\Sigma) = d_{1/4}(-(\overline{Y}\#\Sigma)) - d_{-1/4}(-(\overline{Y}\#\Sigma)) = -\frac{7}{2},$$

so $\pm(\overline{Y}\#\Sigma)$ cannot be integrally homology cobordant to a positive surgery along a knot in S^3. □

The following remark was suggested to the authors by Adam Levine.

Remark 4.4. Note that the previous proposition implies that, for any integral homology sphere Σ, $\overline{Y}\#\Sigma$ cannot bound a simply-connected 4-manifold with $H_2(W) = \mathbb{Z}$ that has a spine, i.e. the generator of $H_2(W)$ is represented by a PL-sphere. (Note that under these assumptions W is homotopy equivalent to a 2-sphere.)

Closely following Levine and Lidman's approach [11], we produce a homotopy S^2 whose boundary is $\overline{Y}\#\Sigma$ for some homology sphere Σ, which is necessarily going to be spineless. We sketch the construction, which is very similar to [11].

The key observation is that there is an integral homology sphere $-\Sigma$ such that $\overline{Y}$ is obtained as integral surgery along a knot in $-\Sigma$. For example, we can choose Σ to be the Brieskorn sphere $\Sigma(15, 17, 181)$. Indeed, the negative plumbing graph of $\Sigma(15, 17, 181)$ is obtained by adding a single vertex to the negative plumbing graph of $-\overline{Y}$, which exhibits $\overline{Y}$ as surgery along a singular fiber of $-\Sigma(15, 17, 181)$.

By [11, Lemma 3.2 and Proposition 3.1], the 4-manifold obtained from the trace of this surgery and carving a path in $\Sigma \times I$ is a homotopy S^2 whose boundary is $\overline{Y}\#\Sigma$.

Acknowledgements

We thank Paolo Aceto for bringing this problem to our attention, and Adam Levine, Brendan Owens, Kim Frøyshov, Matthew Hedden, Arunima Ray, and András Stipsicz for useful conversations. We would like to thank the referee for their comments on the first draft of the paper. MG thanks the Rényi Institute for their hospitality at the beginning of this project.

References

[1] Tim D. Cochran and Daniel Tanner, *Homology cobordism and Seifert fibered 3-manifolds*, Proc. Amer. Math. Soc. **142** (2014), no. 11, 4015–4024.

[2] Noam D. Elkies, *A characterization of the* $\mathbf{Z}^n$ *lattice*, Math. Res. Lett. **2** (1995), no. 3, 321–326.

[3] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differ. Geom. **17** (1982), 357–453 (English).

[4] Kim A. Frøyshov, *Mod 2 instanton homology and 4-manifolds with boundary*. `arXiv:2307.03950`.

[5] ———, *Equivariant aspects of Yang-Mills Floer theory*, Topology **41** (2002), no. 3, 525–552.

[6] Kyle Hayden and Lisa Piccirillo, *The trace embedding lemma and spinelessness*, 2019. To appear in J. Differential Geom. Preprint available at `arXiv:1912.13021`.

[7] Matthew Hedden, Min Hoon Kim, Thomas E. Mark, and Kyungbae Park, *Irreducible 3-manifolds that cannot be obtained by 0-surgery on a knot*, Trans. Amer. Math. Soc. **372** (2019), no. 11, 7619–7638.

[8] Jennifer Hom, Çağrı Karakurt, and Tye Lidman, *Surgery obstructions and Heegaard Floer homology*, Geom. Topol. **20** (2016), no. 4, 2219–2251.

[9] Çağrı Karakurt, *Magma code*. `http://web0.boun.edu.tr/cagri.karakurt/HFNem2.zip`.

[10] Kyle Larson, *Lattices and correction terms*. In: Fernández de Bobadilla, J., László, T., Stipsicz, A.(eds) Singularities and Their Interaction with

Geometry and Low Dimensional Topology, pp. 247–257, Birkhäuser, 2021.

[11] Adam Simon Levine and Tye Lidman, *Simply connected, spineless 4-manifolds*, Forum Math. Sigma **7** (2019), Paper No. e14, 11.

[12] Charles Livingston, *Homology cobordisms of* 3*-manifolds, knot concordances, and prime knots*, Pacific J. Math. **94** (1981), no. 1, 193–206.

[13] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[14] Robert Myers, *Homology cobordisms, link concordances, and hyperbolic* 3*-manifolds*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 271–288.

[15] András Némethi, *On the Ozsváth–Szabó invariant of negative definite plumbed 3-manifolds*, Geom. Topol. **9** (2005), no. 2, 991–1042.

[16] Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in* S^3, J. Reine Angew. Math. **706** (2015), 1–17.

[17] Yuta Nozaki, Kouki Sato, and Masaki Taniguchi, *Filtered instanton Floer homology and the homology cobordism group*, 2019. Preprint available at `arXiv:1905.04001`.

[18] Brendan Owens and Sašo Strle, *A characterization of the* $\mathbb{Z}^n \oplus \mathbb{Z}(\delta)$ *lattice and definite nonunimodular intersection forms*, Amer. J. Math. **134** (2012), no. 4, 891–913.

[19] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.

[20] ———, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. **7** (2003), 185–224 (electronic).

[21] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.

[22] Matthew Stoffregen, *Pin(2)-equivariant Seiberg-Witten Floer homology of Seifert fibrations*, Compos. Math. **156** (2020), no. 2, 199–250.

Laboratoire de Mathématiques Jean Leray
CNRS and Nantes Université, 44322 Nantes, France
E-mail address: marco.golla@univ-nantes.fr

Department of Mathematics, University of Georgia
Athens, GA 30602, USA
E-mail address: kyle.larson@uga.edu

Received January 7, 2021
Accepted July 28 2022