

On the Bogomolov-Gieseker inequality for hypersurfaces in the projective spaces

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We investigate the stronger form of the Bogomolov-Gieseker inequality on smooth hypersurfaces in the projective space of any degree and dimension. The main technical tool is the theory of tilt-stability conditions in the derived category.

1. Introduction

1.1. Motivation and results

One of the most important theorems in the study of vector bundles on algebraic varieties in characteristic zero is the following Bogomolov-Gieseker (BG) inequality [Bog78, Gie79]:

$$(1.1) \quad \frac{H^{n-2} \operatorname{ch}_2(E)}{H^n \operatorname{ch}_0(E)} \leq \frac{1}{2} \left(\frac{H^{n-1} \operatorname{ch}_1(E)}{H^n \operatorname{ch}_0(E)} \right)^2,$$

where E is a slope semistable vector bundle on a polarized smooth projective variety (X, H) . Often it does not give a sharp bound. Indeed, it is easy to obtain stronger inequalities on del Pezzo surfaces or K3 surfaces, simply by using Serre duality and the Riemann-Roch theorem. Finding a sharp bound is the same problem as the classification of the Chern characters of slope semistable sheaves. Such a study goes back to the work of Drezet–Le Potier [DLP85] on the projective plane, and is recently developing in relations with Bridgeland stability conditions, see e.g., [FLZ21, LR21].

However, only a few results are known for general type surfaces or higher dimensional varieties. In this paper, we prove several results in this direction:

Theorem 1.1 (Theorem 4.3, Corollary 4.4, Theorem 6.2). *Let $k = \bar{k}$ be an algebraically closed field of arbitrary characteristic. Let E be a slope*

semistable torsion free sheaf on \mathbb{P}_k^3 with slope $\mu(E) \in [0, 1]$. Then the following inequality holds:

$$\frac{\text{ch}_2(E)}{\text{ch}_0(E)} \leq \Theta(\mu(E)),$$

where the function $\Theta: [0, 1] \rightarrow \mathbb{R}$ is defined as follows:

$$\Theta(t) := \begin{cases} -t/4 & (t \in [0, 1/2]) \\ 5t/4 - 3/4 & (t \in (1/2, 1]). \end{cases}$$

Theorem 1.2 (Theorems 5.1, Theorem 6.3). Let $k = \bar{k}$ be an algebraically closed field of characteristic zero. Let $S_d^n \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree $d \geq 1$, dimension $n \geq 2$, H the restriction of the hyperplane class on \mathbb{P}_k^{n+1} to S_d^n . Let E be a slope semistable sheaf with slope $\mu_H(E) \in [0, 1]$. Then we have the inequality

$$\frac{H^{n-2} \text{ch}_2(E)}{H^n \text{ch}_0(E)} \leq \Xi(\mu_H(E)),$$

where we define the function $\Xi: [0, 1] \rightarrow \mathbb{R}$ as

$$\Xi(t) := \begin{cases} \frac{1}{3}t^2 - \frac{1}{12}t & (t \in [0, 1/2]) \\ \frac{1}{3}t^2 + \frac{5}{12}t - \frac{1}{4} & (t \in [1/2, 1]). \end{cases}$$

Moreover, when $n = 2$, the result also holds in positive characteristic.

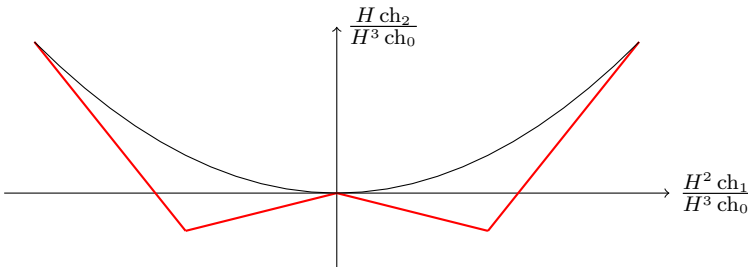


Figure 1: strong BG inequality on \mathbb{P}^3 .

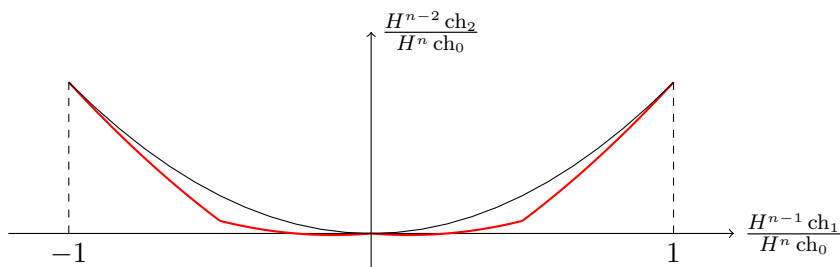


Figure 2: strong BG inequality on hypersurfaces.

1.2. Idea of proof

The key ingredient of the proofs is the theory of *tilt-stability* in the derived category (see Section 2 for the definition and basic properties). In particular, we use (1) *the restriction technique for tilt-stability*, and (2) *the generalized BG type inequality* on \mathbb{P}^3 .

(1) The restriction result for tilt-stability (Lemma 3.3), first found by Feyzbakhsh [Fey16] and Li [Li19a], enables us to reduce the problems to the surface case. In contrast to the usual effective restriction theorem for slope semistability (see e.g. [Lan10]), we are often allowed to cut by a hyperplane of degree one, not its higher multiples. By this observation, we are able to deduce Theorem 1.2 for S_d^n from the case of the surface S_d^2 of the same degree. Similarly, we obtain the result on \mathbb{P}^3 by restricting to the low degree hypersurfaces, but to obtain the strong result as in Theorem 1.1, we use the quadric and quartic surfaces, not only \mathbb{P}^2 .

(2) The generalized BG type inequality is the inequality for the Chern characters of tilt-semistable objects on $D^b(\mathbb{P}^3)$, involving the third part of the Chern character, which depends on the parameter $(\beta, \alpha) \in \mathbb{R}^2$ of tilt-stability conditions (see Theorem 2.3). Let us call this inequality as $BG(\beta, \alpha)$.

For a slope semistable sheaf $E \in \text{Coh}(S_d^2)$, it is known that the torsion sheaf $\iota_* E \in \text{Coh}(\mathbb{P}^3)$ is tilt-semistable when the parameter α is sufficiently large. The next important step is to analyze the tilt-stability of $\iota_* E$ when we decrease the parameter α , since if it remains tilt-semistable for the smaller α , we get the better inequality $BG(\beta, \alpha)$. To get the better bound on α , we use Theorem 1.1 in a crucial way.

1.3. Complete intersections

One may wish to generalize Theorem 1.2 to all complete intersections in the projective space. For this, we need the conjectural ch_3 -inequality $BG(\beta, \alpha)$ on threefolds $S_d^3 \subset \mathbb{P}^4$ for $d \geq 2$. At this moment, the conjecture is solved only when $d \leq 5$ (cf. [Li19a, Li19b]).

1.4. Relation with the existing works

There are several works investigating strong forms of the BG inequality. In [Har78], Hartshorne obtains a sharp bound for possible Chern characters of semistable vector bundle of rank two on \mathbb{P}^3 . In [Li19b], Li proves the stronger BG inequality for all Fano threefolds of Picard rank one, in particular for \mathbb{P}^3 . Our Theorem 1.1 is stronger than that.

In [SS18], Schmidt-Sung obtain a sharp bound for rank two vector bundles on hypersurfaces in \mathbb{P}^3 with a similar method as in this paper. Theorem 1.2 for higher rank bundles is completely new. See also [MS11, MS18, Tod14] for other results concerning rank two vector bundles.

In [Kos20a, Li19a], stronger BG inequalities on certain classes of Calabi-Yau threefolds are studied, with a different argument using the restriction to complete intersection curves. The advantage of our approach is the fact that we can uniformly treat hypersurfaces of all degrees at the same time, with short and simple computations.

See also [Bay18, BL17, Fey19, Fey20, FL18, FT21, FT20] for other interesting applications of the wall-crossing in tilt-stability.

1.5. Plan of the paper

The paper is organized as follows. Until Section 5, we work over an algebraically closed field of characteristic zero. In Section 2, we recall basic notions in the theory of tilt-stability. In Section 3, we summarize results obtained via wall-crossing arguments in tilt-stability. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2. Finally in Section 6, we discuss the case of positive characteristic.

Notation and Convention. Until Section 5, we work over an algebraically closed field of characteristic zero, while in Section 6 we work in positive characteristic. We use the following notations:

- $\text{Coh}(X)$: the category of coherent sheaves on a variety X .

- $D^b(X) := D^b(\text{Coh}(X))$: the bounded derived category of coherent sheaves.
- $\text{ch}^\beta = (\text{ch}_0^\beta, \text{ch}_1^\beta, \dots, \text{ch}_n^\beta) := e^{-\beta H} \cdot \text{ch}$: the β -twisted Chern character for a real number $\beta \in \mathbb{R}$ and an ample divisor H .

2. Preliminaries

Until the end of Section 5, we work over an algebraically closed field of characteristic zero. In this section, we quickly recall the notion of tilt-stability on the derived categories and its properties. See [BMS16, BMT14, Li19a, MS17] for the details. Let X be a smooth projective variety of dimension $n \geq 2$, and H an ample divisor on X . Let us fix real numbers $\beta, \alpha \in \mathbb{R}$ with $\alpha > \beta^2/2$.

2.1. Definition of tilt-stability

Let us first define a slope function on the category $\text{Coh}(X)$ as

$$\mu_H := \frac{H^{n-1} \text{ch}_1}{H^n \text{ch}_0} : \text{Coh}(X) \rightarrow \mathbb{R} \cup \{+\infty\}.$$

We define the notion of μ_H -stability (or *slope stability*) for coherent sheaves in the usual way. We are then able to construct a new heart in the derived category $D^b(X)$ using the notion of *torsion pair* and *tilting* (cf [HRS96]). Let us define full subcategories $\mathcal{T}_\beta, \mathcal{F}_\beta \subset \text{Coh}(X)$ as follows:

$$\begin{aligned} \mathcal{T}_\beta &:= \langle T \in \text{Coh}(X) : T \text{ is } \mu_H\text{-semistable with } \mu_H(T) > \beta \rangle, \\ \mathcal{F}_\beta &:= \langle F \in \text{Coh}(X) : F \text{ is } \mu_H\text{-semistable with } \mu_H(F) \leq \beta \rangle. \end{aligned}$$

Here, for a set of objects $S \subset \text{Coh}(X)$, we denote by $\langle S \rangle \subset \text{Coh}(X)$ the extension closure of S . By the existence of Harder-Narasimhan filtrations with respect to slope stability, the pair $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ is a torsion pair on $\text{Coh}(X)$. Hence the category

$$\text{Coh}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle \subset D^b(X),$$

defined as the extension closure of $\mathcal{F}_\beta[1] \cup \mathcal{T}_\beta$ in $D^b(X)$, is the heart of a bounded t-structure on $D^b(X)$. We now define a new slope function on

$\text{Coh}^\beta(X)$ as

$$\nu_{\beta,\alpha} := \frac{H^{n-2} \text{ch}_2 - \alpha H^n \text{ch}_0}{H^{n-1} \text{ch}_1 - \beta H^n \text{ch}_0} : \text{Coh}^\beta(X) \rightarrow \mathbb{R} \cup \{+\infty\},$$

and the notion of $\nu_{\beta,\alpha}$ -stability (or *tilt-stability*) for objects in $\text{Coh}^\beta(X)$ as similar to μ_H -stability. See for example [Li19a, Section 2] for basic properties of tilt-stability.

2.2. Bogomolov-Gieseker type inequalities

Here we recall several variants of Bogomolov-Gieseker (BG) type inequalities. For an object $E \in D^b(X)$, we define

$$\begin{aligned} \Delta(E) &:= (\text{ch}_1(E))^2 - 2 \text{ch}_0(E) \text{ch}_2(E), \\ \overline{\Delta}_H(E) &:= (H^{n-1} \text{ch}_1(E))^2 - 2H^n \text{ch}_0(E)H^{n-2} \text{ch}_2(E). \end{aligned}$$

The following is the classical BG inequality:

Theorem 2.1 ([Bog78, Gie79, Lan04]). *Every μ_H -semistable torsion free sheaf E satisfies the inequality*

$$\overline{\Delta}_H(E) \geq H^{n-2} \Delta(E) \geq 0.$$

It is known that tilt-stable objects also satisfy the BG inequality:

Theorem 2.2 ([BMS16, Theorem 3.5]). *Every tilt-semistable object $E \in D^b(X)$ satisfies the inequality*

$$\overline{\Delta}_H(E) \geq 0.$$

The following generalized BG type inequality on \mathbb{P}^3 plays a crucial role in this paper:

Theorem 2.3 ([BMT14, Mac14]). *Let $\alpha, \beta \in \mathbb{R}$ be real numbers with $\alpha > \beta^2/2$. For every $\nu_{\beta,\alpha}$ -semistable object $E \in \text{Coh}^\beta(\mathbb{P}^3)$, we have the inequality*

$$(2\alpha - \beta^2) \overline{\Delta}_H(E) + 4 \left(H \text{ch}_2^\beta(E) \right)^2 - 6H^2 \text{ch}_1^\beta(E) \text{ch}_3^\beta(E) \geq 0.$$

3. Wall-crossing arguments

In this section, we summarize various wall-crossing arguments in tilt-stability, developed in [BMS16, Fey16, Li19b, Kos20a], and others. As in the previous section, we denote by X a smooth projective variety of dimension $n \geq 2$, and by H an ample divisor on X .

We put $\mathcal{S} := \{(\beta, \alpha) : \alpha > \beta^2/2\} \subset \mathbb{R}^2$, and call it as a *space of tilt-stability conditions*. Recall that a *wall* for an object $E \in D^b(X)$ with respect to tilt-stability is defined as a connected component of solutions $(\beta, \alpha) \in \mathcal{S}$ of an equation $\nu_{\beta, \alpha}(F) = \nu_{\beta, \alpha}(E)$ for an inclusion $F \subset E$ in the tilted heart $\text{Coh}^\beta(X)$, where $F \in \text{Coh}^\beta(X)$ is a tilt-semistable object. It is easy to see that a wall W for E is a line segment, and satisfies one of the following two properties:

- 1) W passes through the point $p_H(E)$ when $\text{ch}_0(E) \neq 0$,
- 2) W has a fixed slope $H^{n-2} \text{ch}_2(E)/H^{n-1} \text{ch}_1(E)$ when $\text{ch}_0(E) = 0$.

Here, for an object $E \in D^b(X)$ with $\text{ch}_0(E) \neq 0$, we define the point $p_H(E) \in \mathbb{R}^2$ as follows:

$$p_H(E) := \left(\frac{H^{n-1} \text{ch}_1(E)}{H^n \text{ch}_0(E)}, \frac{H^{n-2} \text{ch}_2(E)}{H^n \text{ch}_0(E)} \right).$$

3.1. Restriction lemma for tilt-stability

First let us recall Feyzbakhsh’s restriction lemma from [Fey16] (see also [Li19a, Lemma 5.1]):

Lemma 3.1 ([Fey16, Corollary 4.3]). *Let $d \geq 1$ be a positive integer, $\alpha > 0$ a positive real number. Let $E \in \text{Coh}^0(X)$ be a slope stable reflexive sheaf. Suppose that the following conditions hold:*

- we have $E(-dH)[1] \in \text{Coh}^0(X)$,
- the objects $E, E(-dH)[1] \in \text{Coh}^0(X)$ are $\nu_{0, \alpha}$ -stable,
- we have the equality $\nu_{0, \alpha}(E) = \nu_{0, \alpha}(E(-dH)[1])$.

Then for any irreducible hypersurface $Y_d \in |dH|$, the restriction $E|_{Y_d}$ is $\mu_{H_{Y_d}}$ -stable.

We use the following terminology:

Definition 3.2. Fix an integer $d \geq 0$. We say that a function $f: [0, 1] \rightarrow \mathbb{R}$ is *star-shaped along the line $\beta = d$* if the following condition holds: for every real number $t \in [0, 1]$, the line segment connecting the point $(t, f(t))$ and the point $(d, d^2/2)$ is above the graph of f .

We will use the following variant of [Li19a, Proposition 5.2]:

Lemma 3.3 (cf. [Li19a, Proposition 5.2]). *Let $d \geq 1$ be an integer, and $f: [0, 1] \rightarrow \mathbb{R}$ be a star-shaped function along the lines $\beta = 0, d$ with $f(0) = 0, f(1) = 1/2$, satisfying*

$$t^2 - \frac{d}{2}t \leq f(t) \leq \frac{1}{2}t^2$$

for every $t \in [0, 1]$. Assume that there exist objects $E' \in D^b(X)$ satisfying the following conditions:

- (a) E' is either $\nu_{0,\alpha}$ -semistable for some $\alpha > 0$, or $\nu_{d,\alpha'}$ -semistable for some $\alpha' > d^2/2$.
- (b) $\mu_H(E') \in [0, 1]$, $\frac{H^{n-2} \text{ch}_2(E')}{H^{n-1} \text{ch}_0(E')} > f(\mu_H(E'))$.

Then we can choose such an object E so that the restriction $E|_{Y_d}$ of E to an irreducible hypersurface $Y_d \in |dH|$ is $\mu_{H_{Y_d}}$ -semistable.

Proof. By Theorem 2.2, every tilt-semistable object E satisfies the inequality $\overline{\Delta}_H(E) \geq 0$. Hence we may choose an object E which has the minimum discriminant $\overline{\Delta}_H$ among those satisfying the conditions (a) and (b). We claim that such an object E is $\nu_{0,\alpha}$ -stable for all $\alpha > 0$, and $\nu_{d,\alpha'}$ -stable for all $\alpha' > d^2/2$. Assume for a contradiction that there is a wall for E along the line $\beta = 0$. Then there exists a Jordan-Hölder factor F of E such that the point $p_H(F)$ lies on the line segment connecting $p_H(E)$ and $(0, \alpha)$ for some $\alpha > 0$. As we assume the function f is star-shaped along the line $\beta = 0$, the object F also satisfies the conditions (a) and (b). Moreover, we have $\overline{\Delta}_H(F) < \overline{\Delta}_H(E)$ by [BMS16, Corollary 3.10], which contradicts the minimality assumption of $\overline{\Delta}(E)$. Similarly, we can see that the object E cannot be destabilized along the line $\beta = d$ or the vertical wall $\beta = \mu_H(E)$.

Hence the objects $E, E(-dH)[1] \in \text{Coh}^0(X)$ are $\nu_{0,\alpha}$ -stable for all $\alpha > 0$. In particular, the $\nu_{0,\alpha}$ -stability of E for $\alpha \gg 0$ implies that E is a coherent sheaf. Moreover, as in the first paragraph of the proof of [Li19a, Lemma 5.1], the $\nu_{0,\alpha}$ -stability of $E(-dH)[1]$ implies that E is reflexive. Note also that, by the assumption $t^2 - \frac{d}{2}t \leq f(t)$, the line passing through the points $p_H(E)$

and $p_H(E(-dH))[1]$ intersects with the α -axis at $(0, \alpha_0)$ for some positive real number α_0 . Hence the assumptions of Lemma 3.1 are satisfied, and we can conclude that the restriction $E|_{Y_d}$ is slope semistable. \square

3.2. Strong BG inequalities for tilt-stable objects

Let \mathcal{D} be a set of objects $E \in \text{Coh}^0(X)$ satisfying one of the following conditions:

- $E \in \text{Coh}(X)$ and it is μ_H -semistable with $\text{ch}_0(E) > 0$,
- $\mathcal{H}^{-1}(E)$ is μ_H -semistable and $\dim \mathcal{H}^0(E) \leq n - 2$.

We will also use the following lemma:

Proposition 3.4 ([Kos20a, Proposition 2.5]). *Let $f: [0, 1] \rightarrow \mathbb{R}$ be a star-shaped function along the line $\beta = 0$. Assume that for every object $E \in \mathcal{D}$, the inequality*

$$\frac{H^{n-2} \text{ch}_2(E)}{H^n \text{ch}_0(E)} \leq f(\mu_H(E))$$

holds. Then for every $\alpha > 0$ and $\nu_{0,\alpha}$ -semistable object E with $\text{ch}_0(E) \neq 0$, the same inequality holds.

3.3. Bounding first walls for torsion sheaves

In the proof of Theorem 1.2, it is important to bound the first possible wall for sheaves supported on divisors. The following lemma is a useful general fact:

Lemma 3.5 ([Kos20a, Lemma 3.6], [Li19a]). *Let $Y_d \in |dH|$ be a smooth hypersurface of degree $d \geq 1$, denote by $\iota: Y_d \hookrightarrow X$ the embedding. Let $E \in \text{Coh}(Y_d)$ be a $\mu_{H_{Y_d}}$ -semistable torsion free sheaf.*

*Assume that there exists a wall for $\iota_*E \in \text{Coh}^0(X)$ with respect to tilt-stability with end points $(\beta_1, \alpha_1), (\beta_2, \alpha_2)$ satisfying $\beta_1 < 0 < \beta_2$. Then we have $\beta_2 - \beta_1 \leq d$.*

Proof. The same proof as in [Kos20a, Lemma 3.6] works, where the author considers the case of $d = 6$. \square

4. BG inequality on the projective space

In this section, we investigate the stronger form of the BG inequality on the three dimensional projective space. For a hypersurface $S_d \subset \mathbb{P}^3$ of degree $d \in \mathbb{Z}_{>0}$, we denote by $H := H_{\mathbb{P}^3}|_{S_d}$ the restriction of the hyperplane on \mathbb{P}^3 to the surface S_d . The following two lemmas are well-known:

Lemma 4.1 (cf. [Rud94]). *Let $S_2 \subset \mathbb{P}^3$ be a smooth quadric hypersurface. Let $E \in \text{Coh}(S_2)$ be a torsion free μ_H -semistable sheaf with slope $\mu_H(E) \in [0, 1]$. Then the inequality*

$$\frac{\text{ch}_2(E)}{H^2 \text{ch}_0(E)} \leq \Gamma(\mu_H(E))$$

holds, where we define the function $\Gamma: [0, 1] \rightarrow \mathbb{R}$ as follows:

$$\Gamma(t) := \begin{cases} -t/2 & (t \in [0, 1/2)) \\ 0 & (t = 1/2) \\ 3t/2 - 1 & (t \in (1/2, 1]). \end{cases}$$

Proof. For $i = 1, 2$, let h_i be divisors on $S_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_{S_2}(h_1) = \mathcal{O}_{S_2}(1, 0)$, $\mathcal{O}_{S_2}(h_2) = \mathcal{O}_{S_2}(0, 1)$. Note that we have $\mu_H(\mathcal{O}_{S_2}(h_i)) = 1/2$ and $\text{ch}_2(\mathcal{O}_{S_2}(h_i)) = 0$.

Let $E \in \text{Coh}(S_2)$ be a slope stable vector bundle with $\mu_H(E) \in [0, 1/2]$, not isomorphic to $\mathcal{O}_{S_2}(h_1)$ nor $\mathcal{O}_{S_2}(h_2)$. By stability of E and Serre duality, we have

$$\text{hom}(\mathcal{O}_{S_2}(h_i), E) = 0 = \text{ext}^2(\mathcal{O}_{S_2}(h_i), E).$$

Hence by the Riemann-Roch theorem, we have

$$\begin{aligned} 0 &\geq -\text{ext}^1(\mathcal{O}_{S_2}(h_i), E) = \chi(\mathcal{O}_{S_2}(h_i), E) \\ &= \int_{S_2} \text{ch}(E) \cdot (1, H - h_i, 0) \\ &= \text{ch}_2(E) + (H - h_i) \text{ch}_1(E). \end{aligned}$$

Summing up these inequalities for $i = 1, 2$, we get

$$(4.1) \quad 2 \text{ch}_2(E) \leq -(2H - (h_1 + h_2)) \text{ch}_1(E) = -H \text{ch}_1(E)$$

as required. When $\mu_H(E) \in [1/2, 1]$, we get the required inequality by applying the inequality (4.1) to the bundle $E^\vee(H)$. □

Lemma 4.2. *Let $S_4 \subset \mathbb{P}^3$ be a smooth quartic hypersurface. Let $E \in \text{Coh}(S_4)$ be a torsion free μ_H -semistable sheaf with slope $\mu_H(E) = 1/2$. Then the inequality*

$$\frac{\text{ch}_2(E)}{H^2 \text{ch}_0(E)} \leq -\frac{1}{8}.$$

holds.

Proof. We may assume that E is a slope stable vector bundle with $\text{ch}_1(E) = \text{ch}_0(E)H/2$. Recall that we have $\text{ch}(E) \in H_{\text{alg}}^*(X, \mathbb{Z})$, so we have $\text{ch}_0(E) = 2a$ for some positive integer $a \in \mathbb{Z}_{>0}$ and $\text{ch}_2(E) \in \mathbb{Z}$. First we claim that the bundle E is non-spherical. If otherwise, we have

$$2 = \chi(E, E) = 2 \text{ch}_0(E)^2 - \text{ch}_0(E)^2 + 2 \text{ch}_0(E) \text{ch}_2(E)$$

and hence

$$\text{ch}_2(E) = \frac{1}{\text{ch}_0(E)} - \frac{\text{ch}_0(E)}{2} = \frac{1}{2a} - a \notin \mathbb{Z},$$

which is a contradiction. Now for a non-spherical stable bundle E , we have the inequality $0 \geq \chi(E, E)$, from which we deduce the required inequality. \square

Let us define a periodic function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ with $\gamma(t + 1) = \gamma(t)$ as follows:

$$\gamma(t) := \begin{cases} \frac{1}{2}t^2 + \frac{1}{4}t & (t \in [0, 1/2]), \\ \frac{1}{2}t^2 - \frac{5}{4}t + \frac{3}{4} & (t \in [1/2, 1]). \end{cases}$$

We then define a function $\Theta: \mathbb{R} \rightarrow \mathbb{R}$ as $\Theta(t) := t^2/2 - \gamma(t)$.

Now we are ready to prove the following stronger BG inequality on \mathbb{P}^3 :

Theorem 4.3. *Let $E \in \text{Coh}^0(\mathbb{P}^3)$ be a $\nu_{0,\alpha}$ -semistable object for some positive real number $\alpha > 0$, with slope $\mu_H(E) \in [0, 1]$. Then the inequality*

$$(4.2) \quad \frac{H \text{ch}_2(E)}{H^3 \text{ch}_0(E)} \leq \Theta(\mu_H(E))$$

holds.

Proof. Assume for a contradiction that there exists a $\nu_{0,\alpha}$ -semistable object E with $\mu_H(E) \in [0, 1]$ violating the inequality (4.2). Note that the function $\Theta: [0, 1] \rightarrow \mathbb{R}$ is star-shaped along the lines $\beta = 0, 2, 4$, and satisfies the inequality $t^2 - dt/2 \leq \Theta(t)$ for all $t \in [0, 1]$ and $d \geq 2$. Hence by Lemma 3.3,

we may assume that the restrictions $E|_{S_d}$ are slope semistable vector bundle on general hypersurfaces $S_d \subset \mathbb{P}^3$ of degree $d = 2, 4$. Note that we have $\mu_H(E|_{S_d}) = \mu_H(E)$. When $\mu_H(E) \neq 1/2$ (resp. $\mu_H(E) = 1/2$), we get a contradiction by Lemma 4.1 (resp. Lemma 4.2). \square

Corollary 4.4. *Every torsion free slope semistable sheaf $E \in \text{Coh}(\mathbb{P}^3)$ satisfies the inequality*

$$\frac{H \text{ch}_2(E)}{H^3 \text{ch}_0(E)} \leq \Theta(\mu_H(E)).$$

In particular, we have a continuous family of tilt-stability parametrized by pairs (β, α) of real numbers satisfying $\alpha > \Theta(\beta)$.

Proof. By definition, the function Θ satisfies $\Theta(t + 1) = \Theta(t) + t + 1/2$ for all $t \in \mathbb{R}$. Hence it is enough to prove the assertion for semistable sheaves E with slope $\mu_H(E) \in [0, 1]$, which directly follows from Theorem 4.3. \square

5. BG inequality on hypersurfaces

The goal of this section is to prove the following theorem:

Theorem 5.1. *Let $S_d^n \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 1$, dimension $n \geq 2$. Let E be a $\nu_{0,\alpha}$ -semistable object for some $\alpha > 0$, with slope $\mu_H(E) \in [0, 1]$. Then we have the inequality*

$$(5.1) \quad \frac{H^{n-2} \text{ch}_2(E)}{H^n \text{ch}_0(E)} \leq \Xi(\mu_H(E)),$$

where we define the function $\Xi: [0, 1] \rightarrow \mathbb{R}$ as

$$\Xi(t) := \begin{cases} \frac{1}{3}t^2 - \frac{1}{12}t & (t \in [0, 1/2]) \\ \frac{1}{3}t^2 + \frac{5}{12}t - \frac{1}{4} & (t \in [1/2, 1]) \end{cases}$$

First we reduce the problem to the case of surfaces:

Lemma 5.2. *Assume that every slope stable sheaf E on surfaces S_d^2 with slope $\mu_H(E) \in [0, 1/2]$ satisfies the inequality (5.1). Then Theorem 5.1 holds.*

Proof. First note that the function Ξ is star-shaped along the lines $\beta = 0, 1$, and satisfies $t^2 - t/2 \leq \Xi(t)$. Hence by Lemma 3.3 and induction on $n \geq 2$, it is enough to prove the assertion for $n = 2$.

As we assume that the inequality (5.1) holds for every slope semistable sheaf $E \in \text{Coh}(S_d^2)$ with $\mu_H(E) \in [0, 1/2]$, it also holds when $\mu_H(E) \in [1/2, 1]$, by applying the assumed inequality to $E^\vee(H)$. Now the same inequality (5.1) holds for every $\nu_{0,\alpha}$ -semistable objects by Proposition 3.4. \square

In the following, we fix a smooth hypersurface $S_d^2 \subset \mathbb{P}^3$ of degree $d \geq 1$, and denote by $\iota: S_d^2 \hookrightarrow \mathbb{P}^3$ the embedding.

Lemma 5.3. *Let $E \in D^b(S_d^2)$ be an object and put $r := \text{ch}_0(E), a := H \text{ch}_1(E)/d, b := \text{ch}_2(E)$. Then we have*

$$(5.2) \quad \text{ch}(\iota_* E) = \left(0, drH, \left(a - \frac{d}{2}r \right) dH^2, b - \frac{d^2}{2}a + \frac{d^3}{6}r \right).$$

Proof. Using Grothendieck-Riemann-Roch theorem for embeddings, we have

$$\iota_* \left((r, \text{ch}_1(E), b) \cdot \text{td}_{S_d^2} \right) = \text{ch}(\iota_* E) \text{td}_{\mathbb{P}^3}.$$

Combining with the facts

$$(5.3) \quad \begin{aligned} \text{td}_{S_d^2} &= \left(1, \left(2 - \frac{d}{2} \right) HS, \frac{d^3}{6} - d^2 + \frac{11}{6}d \right), \\ \text{td}_{\mathbb{P}^3} &= \left(1, 2H, \frac{11}{6}H^2, 1 \right), \end{aligned}$$

the straightforward computation yields the result. \square

Lemma 5.4. *Let $E \in \text{Coh}(S_d^2)$ be a slope semistable vector bundle on S_d^2 with slope $\mu := \mu_H(E) \in [0, 1/2]$. Then the sheaf $\iota_* E \in \text{Coh}^0(\mathbb{P}^3)$ is $\nu_{0,\alpha}$ -semistable for all positive $\alpha \geq \alpha_\mu$, where the real number α_μ is defined as follows:*

$$(5.4) \quad \alpha_\mu := -\mu^2 + \frac{2d-1}{4}\mu.$$

Proof. Let W be a wall for $\iota_* E$ with respect to $\nu_{\beta,\alpha}$ -stability. Note that the wall W is the line segment with slope $\mu - d/2$. Note also that by Corollary 4.4, the end points of the wall W are on the graph of Θ . Let $\beta_1 < 0 < \beta_2$ be their β -coordinates.

By Lemma 3.5, we have $\beta_2 - \beta_1 \leq d$ and hence the slope of the line passing through the points $(\beta_2, \Theta(\beta_2)), (\beta_2 - d, \Theta(\beta_2 - d))$ should be smaller

than or equal to that of W , i.e., $\beta_2 - d/2 \leq \mu - d/2$. Hence every wall for ι_*E should be below the line

$$y_\mu = (\mu - d/2)(x - \mu) + \Theta(\mu).$$

By computing $\alpha_\mu := y_\mu(0)$, we get the result. □

Proposition 5.5. *Let $E \in \text{Coh}(S_d^2)$ be a slope semistable vector bundle on S_d^2 with slope $\mu := \mu_H(E) \in [0, 1/2]$. Then the inequality (5.1) holds.*

Proof. By Lemma 5.4, the sheaf $\iota_*E \in \text{Coh}^0(\mathbb{P}^3)$ is ν_{0,α_μ} -semistable. Hence by the generalized BG type inequality Theorem 2.3, we get

$$(5.5) \quad 2\alpha_\mu \bar{\Delta}_H(\iota_*E) + 4(H \text{ch}_2(\iota_*E))^2 - 6H^2 \text{ch}_1(\iota_*E) \text{ch}_3(\iota_*E) \geq 0.$$

Let us put $r := \text{ch}_0(E)$, $a := H \text{ch}_1(E)/d$, $b := \text{ch}_2(E)$. Note that we have $H_S^2 = d$ and $\mu = a/r$. Using the equations (5.2) and (5.4), and dividing the inequality (5.5) by d^2r^2 , we get the inequality

$$-2\mu^2 + \left(d - \frac{1}{2}\right)\mu + 4\left(\mu - \frac{d}{2}\right)^2 - 6\left(\frac{b}{dr} - \frac{d}{2}\mu + \frac{d^2}{6}\right) \geq 0,$$

which is equivalent to the desired inequality (5.1). □

We are now able to finish the proof of Theorem 5.1:

Proof of Theorem 5.1. The assertion follows from Lemma 5.2 and Proposition 5.5. □

6. Positive characteristics

In this section, we work over an algebraically closed field k of positive characteristic. In positive characteristic, the main difference with the case of characteristic zero is that the classical BG inequality does not hold in general. The failure of the BG inequality is related to the failure of Kodaira vanishing in positive characteristic (cf. [Muk13, Ray78]).

Before discussing about Theorems 1.1 and 1.2, let us recall the result of Langer [Lan04]. Let X be a smooth projective variety of dimension $n \geq 2$, and H an ample divisor on X . We say that a coherent sheaf $E \in \text{Coh}(X)$ is *strongly μ_H -semistable* if for every $e > 0$, the sheaf $F^{e*}E$ is μ_H -semistable, where $F: X \rightarrow X$ is the absolute Frobenius morphism. We have the following result:

Theorem 6.1 ([Lan04, Corollary 2.4, Theorem 3.2]). *Assume that we have $\mu_{H,\max}(\Omega_X) \leq 0$. Then every μ_H -semistable torsion free sheaf E is strongly μ_H -semistable. Moreover, the usual BG inequality holds:*

$$H^{n-2}\Delta(E) \geq 0.$$

In particular, the usual BG inequality holds on the projective space \mathbb{P}^3 . Hence all the results in Section 4 hold true also in positive characteristic, without any modification:

Theorem 6.2. *Theorem 1.1 holds true in arbitrary characteristic.*

For the arguments in Section 5, the only issue in positive characteristic is Lemma 5.2, for which we used Lemma 3.3. To apply Lemma 3.3 to hypersurfaces $S_d^n \subset \mathbb{P}_k^{n+1}$, we need the usual BG inequality on S_d^n , which is not known in positive characteristic. The other arguments in Section 5 also work in positive characteristic, and hence:

Theorem 6.3. *When $n = 2$, Theorem 1.2 holds in arbitrary characteristic.*

Corollary 6.4. *Let k be an algebraically closed field of positive characteristic. Let $S_d^2 \subset \mathbb{P}^3$ be a smooth hypersurface of degree $d \geq 1$, denote by H the restriction of the hyperplane class on \mathbb{P}^3 . Then for every torsion free μ_H -semistable sheaf $E \in \text{Coh}(S_d^2)$, we have*

$$\overline{\Delta}_H(E) \geq 0.$$

Remark 6.5. In a subsequent paper [Kos20b], we will prove the usual BG inequality (with respect to $\overline{\Delta}_H$) on S_d^n for $n \geq 3$. The key input in [Kos20b] is Corollary 6.4 above. Hence Theorem 1.2 will be true for all S_d^n in arbitrary characteristic.

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