

Moduli spaces of semistable pairs on projective Deligne-Mumford stacks

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We generalize the construction of a moduli space of semistable pairs parametrizing isomorphism classes of morphisms from a fixed coherent sheaf to any sheaf with fixed Hilbert polynomial under a notion of stability to the case of projective Deligne-Mumford stacks. We study the deformation and obstruction theories of stable pairs, and then prove the existence of virtual fundamental classes for some cases of dimension two and three. This leads to a definition of Pandharipande-Thomas invariants on three-dimensional smooth projective Deligne-Mumford stacks.

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1. Introduction

Gromov-Witten theory, Donaldson-Thomas theory and Pandharipande-Thomas theory are three important approaches to curve counting in enumerative geometry via intersection theories on moduli spaces of stable maps, ideal sheaves and stable pairs respectively. It is interesting that

these theories are conjectured to be related to each other. The conjecture of Gromov-Witten/Donaldson-Thomas correspondence was proposed by MNOP in [37, 38], which predicted that the partition function of GW theory can be equated with the one of DT theory by a change of variables for smooth projective 3-folds. The GW/DT correspondence has been proved in several cases (e.g., [14, 37–40, 46]) with a formal reduced theory for DT side. In [53], the authors developed a PT theory providing a geometric interpretation of the reduced DT theory, and the DT/PT correspondence for Calabi-Yau 3-folds was proposed to be viewed as a wall-crossing formula in the derived category. This DT/PT correspondence has been proved in several approaches [10, 62, 64]. On the other hand, the GW/PT correspondence has also been treated in [42, 45, 51, 52].

As a generalization of the above manifold cases, the orbifold GW theory and orbifold DT theory for smooth projective Deligne-Mumford stacks have been investigated in [2, 15] and [13, 20, 68] respectively. The orbifold GW/DT correspondence was proved in some cases (e.g. [56–58, 70, 71]). It is natural and expected to have an orbifold PT theory together with the orbifold GW/PT or DT/PT correspondence. In [7], the authors follow Toda’s method [64] of applying the notion of a torsion pair to obtain a stacky version of PT stable pairs, and then combine the motivic Hall algebra (cf. [11]) and Behrend’s constructible function [8] to define orbifold PT invariants for smooth projective Calabi-Yau 3-orbifolds. The orbifold DT/PT correspondence are proved there for some special cases. However, without Calabi-Yau condition, it is well known that one may construct a perfect obstruction theory on some moduli space to define a virtual fundamental class [9, 34], which can be integrated to obtain orbifold PT invariants. For this purpose, we shall first aim to provide an alternative construction of moduli spaces of PT stable pairs in the stacky sense.

To achieve this goal, there is a need for another way to generalize the notion of PT stable pairs to the stacky case. Now we begin with the notion of stable pairs for the case of a smooth projective variety X . Based on the work of Le Potier [32], Pandharipande and Thomas [53] define a stable pair (F, s) where F is a pure sheaf of dimension one and the section $s : \mathcal{O}_X \rightarrow F$ has cokernel with 0-dimensional support. In a more general setting, the author in [67] defines a δ -semistable pair (F, φ) consisting of a coherent sheaf F on X and a morphism $\varphi : F_0 \rightarrow F$ under a notion of stability depending on some choice of a parameter $\delta \in \mathbb{Q}[m]$ where F_0 is any fixed coherent sheaf. It is shown in [36] that the notion of δ -stable pairs (F, φ) defined in [67] actually generalizes the one of PT stable pairs for the case when $\deg \delta \geq \deg P = 1$ where P is the Hilbert polynomial of F . Therefore, it is

natural to generalize the notion of stability [36, 67] to the case of projective Deligne-Mumford stacks to obtain the stacky PT stable pairs.

Observe that the stability condition used in [36, 67] is different from the one defined in [22, 23] due to the different framing, where the latter has been naturally generalized to the stacky case [12] using the modified Hilbert polynomial defined in [43]. In order to construct moduli spaces of the stacky PT stable pairs, we generalize the construction of moduli spaces of δ -(semi)stable pairs in [36, 67] to the case of projective Deligne-Mumford stacks by means of techniques developed in [12, 23, 43]. We state this result explicitly as follows. Let \mathcal{X} be a projective Deligne-Mumford stack of dimension d over an algebraically closed field k of characteristic zero with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_{\mathcal{X}}(1))$, see Definition 2.9. Assume \mathcal{F}_0 is any fixed coherent sheaf on \mathcal{X} . Let δ be any given stability parameter which is zero or a rational polynomial with positive leading coefficient and P any given polynomial of degree $\deg P \leq d$. We have a contravariant functor

$$\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta) : (\text{Sch}/k)^\circ \rightarrow (\text{Sets})$$

where if S is a k -scheme of finite type, $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta)(S)$ is the set of isomorphism classes of flat families of δ -(semi)stable pairs (\mathcal{F}, φ) on \mathcal{X} with modified Hilbert polynomial P parametrized by S , see Definition 4.8. A central result for the existence of (fine) moduli space for this moduli functor is obtained in the following

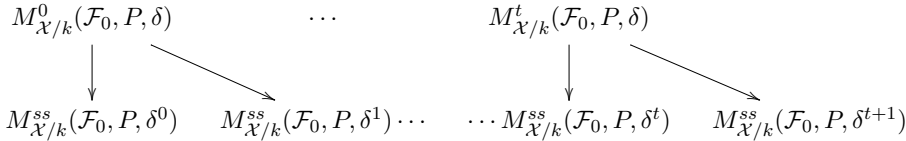
Theorem 1.1 (see **Theorem 4.10** and **Theorem 4.24**). *There is a projective scheme $M^{ss} := M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ which is a moduli space for the moduli functor $\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$. Moreover, there is an open subscheme $M^s := M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$ of M^{ss} which is a fine moduli space for the moduli functor $\mathcal{M}_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$.*

This theorem provides a moduli space $M^{(s)s}$ of δ -(semi)stable pairs on \mathcal{X} with modified Hilbert polynomial P and implies that the quasi-projective scheme M^s has equipped with a universal family. Given a coherent sheaf \mathcal{F}_0 and a polynomial P , moduli spaces $M^{(s)s}$ of δ -(semi)stable pairs depend on the stability parameter δ . As in [67, Section 5] for the case of smooth projective varieties, we have a chamber structure of the stability parameter for the variation of moduli spaces $M^{(s)s}$.

Theorem 1.2 (see Theorem 4.26). *There are finitely many critical values $\delta^1, \dots, \delta^t \in \mathbb{Q}[m]$ satisfying*

$$\delta^0 := 0 < \delta^1 < \dots < \delta^t < \delta^{t+1} := +\infty$$

such that we have the chamber structure of the stability parameter as follows:



where $M_{\mathcal{X}/k}^i(\mathcal{F}_0, P, \delta) := M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta) = M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$ for some $\delta \in (\delta^i, \delta^{i+1})$, $i = 0, \dots, t$.

Since the moduli space M^s contains as a special case the moduli space of stacky PT stable pairs mentioned above, in order to construct a perfect obstruction theory, we next consider the deformation and obstruction theory of stable pairs in M^s . Suppose that \mathcal{X} is a smooth projective Deligne-Mumford stack. Let $[(\mathcal{F}, \varphi)]$ be a point in $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$ where $\varphi : \mathcal{F}_0 \rightarrow \mathcal{F}$. Suppose that Art_k is the category of Artinian local k -algebras with residue field k . For $A, A' \in \text{Ob Art}_k$ and let the short exact sequence

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$$

be a small extension, that is, $\mathfrak{m}_A \cdot I = 0$. Assume $\check{\varphi}_A : \mathcal{F}_0 \otimes_k A \rightarrow \mathcal{F}_A$ is a morphism over $\mathcal{X}_A := \mathcal{X} \times_{\text{Spec } k} \text{Spec } A$ extending φ , where \mathcal{F}_A is a coherent sheaf flat over A . Let $\mathbf{I}^\bullet := \{\mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}\}$ and $\mathbf{I}_A^\bullet := \{\mathcal{F}_0 \otimes_k A \xrightarrow{\check{\varphi}_A} \mathcal{F}_A\}$ be the complexes concentrated in degree 0 and 1. Then we have $\mathbf{I}_A^\bullet \otimes k = \mathbf{I}^\bullet$. As obtained in [36, Theorem 1.2] for the deformation and obstruction theory of stable pairs on smooth projective varieties, we generalize this result to our stacky case.

Theorem 1.3 (see Theorem 5.1). *For a given small extension $0 \rightarrow I \rightarrow A' \xrightarrow{\sigma} A \rightarrow 0$, there is a class*

$$\text{ob}(\check{\varphi}_A, \sigma) \in \text{Ext}^1(\mathbf{I}^\bullet, \mathcal{F} \otimes I)$$

such that there exists a flat extension of $\check{\varphi}_A$ over $\mathcal{X}_{A'}$ if and only if $\text{ob}(\check{\varphi}_A, \sigma) = 0$. If $\text{ob}(\check{\varphi}_A, \sigma) = 0$, the space of extensions is a torsor under $\text{Hom}(\mathbf{I}^\bullet, \mathcal{F} \otimes I)$.

With this result, one can show the existence of virtual fundamental classes for the 2-dimensional case which is a generalization of [36, Theorem 1.3].

Theorem 1.4 (see Theorem 5.12). *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 2 over \mathbb{C} . Let \mathcal{F}_0 be a torsion free sheaf and $\deg \delta \geq \deg P = 1$. Then the moduli space $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{F}_0, P, \delta)$ of δ -stable pairs admits a virtual fundamental class.*

While for the case of dimension three, it is unsuitable to apply Theorem 5.1 to obtain some two-term complex for the obstruction theory (cf. Lemma 5.13 and Remark 5.14). Alternatively, we follow Inaba’s approach to the deformation and obstruction theory in [27, Section 2]. Explicitly, we derive a stacky version of [27, Proposition 2.3] as follows.

Theorem 1.5 (see Theorem 5.5). *For a given small extension $0 \rightarrow I \rightarrow A' \xrightarrow{\sigma} A \rightarrow 0$, there is an element*

$$\omega(\mathbf{I}_A^\bullet) \in \text{Ext}^2(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$$

such that there exists a flat extension of $\check{\varphi}_A$ over $\mathcal{X}_{A'}$ if and only if $\omega(\mathbf{I}_A^\bullet) = 0$. If $\omega(\mathbf{I}_A^\bullet) = 0$, then the space of extensions form a torsor under $\text{Ext}^1(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$.

The similar result as above also holds in the case of a square zero extension (cf. Theorem 5.6). As an application, we study the case when $\dim \mathcal{X} = 3$, $\mathcal{F}_0 = \mathcal{O}_{\mathcal{X}}$ and polynomials δ, P satisfying $\deg \delta \geq \deg P = 1$. The corresponding moduli space $\overline{M}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)$ parametrizes orbifold PT stable pairs (cf. Lemma 2.26 and Remark 2.27). Furtherly, we consider the subfunctor $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ of $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)$, which associates for any \mathbb{C} -scheme S of finite type with the set of isomorphism classes of flat families of δ -(semi)stable pairs (\mathcal{F}, φ) with the fixed numerical class $\beta \in N_{\leq 1}(\mathcal{X})$ parametrized by S where $P_{\mathcal{E}}(\beta) = P$, see Section 5.3 for more details. Similarly, there is a projective scheme $\overline{M}_\beta^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ ($\subseteq \overline{M}^s$), which is a fine moduli space for $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$, see Theorem 5.20. Assume $\overline{\mathbf{I}}^\bullet$ and $\overline{\mathbf{I}}_\beta^s$ are the universal complexes on $\mathcal{X} \times \overline{M}^s$ and $\mathcal{X} \times \overline{M}_\beta^s$ respectively. Let $\overline{\pi}_{\overline{M}^s} : \mathcal{X} \times \overline{M}^s \rightarrow \overline{M}^s$, $\overline{\pi}_{\mathcal{X}} : \mathcal{X} \times \overline{M}^s \rightarrow \mathcal{X}$, $\tilde{\pi}_{\overline{M}_\beta^s} : \mathcal{X} \times \overline{M}_\beta^s \rightarrow \overline{M}_\beta^s$, and $\tilde{\pi}_{\mathcal{X}} : \mathcal{X} \times \overline{M}_\beta^s \rightarrow \mathcal{X}$ be the projections. To assert the existence of virtual fundamental classes of \overline{M}^s and \overline{M}_β^s , we construct the perfect obstruction theories with fixed determinant for these moduli spaces in the sense of [9].

Theorem 1.6 (see **Theorem 5.17**, **Corollary 5.18** and **Theorem 5.21**). *Let \mathcal{X} be a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} . Assume that polynomials δ and P satisfy $\deg \delta \geq \deg P = 1$. The maps*

$$\Phi : E^\bullet := R\bar{\pi}_{\overline{M}^s*}(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2] \rightarrow \mathbb{L}_{\overline{M}^s}.$$

and

$$\Phi_\beta : E_\beta^\bullet := R\tilde{\pi}_{\overline{M}_\beta^s*}(R\mathcal{H}om(\tilde{\mathbb{I}}_\beta^\bullet, \tilde{\mathbb{I}}_\beta^\bullet)_0 \otimes \tilde{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2] \rightarrow \mathbb{L}_{\overline{M}_\beta^s}.$$

are perfect obstruction theories for \overline{M}^s and \overline{M}_β^s respectively. And there exist virtual fundamental classes $[\overline{M}^s]^{\text{vir}} \in A_{\text{vdim}_1}(\overline{M}^s)$ and $[\overline{M}_\beta^s]^{\text{vir}} \in A_{\text{vdim}_2}(\overline{M}_\beta^s)$ where $\text{vdim}_1 = \text{rk}(E^\bullet)$ and $\text{vdim}_2 = \text{rk}(E_\beta^\bullet)$. Here, $\mathbb{L}_{\overline{M}^s}$ and $\mathbb{L}_{\overline{M}_\beta^s}$ are truncated cotangent complexes of \overline{M}^s and \overline{M}_β^s respectively.

By integrations against virtual fundamental classes of \overline{M}_β^s , one can define the orbifold descendent PT theory (cf. Definition 5.23) for 3-dimensional smooth projective Deligne-Mumford stacks. In particular, when \mathcal{X} is a Calabi-Yau 3-orbifold, our definition of orbifold PT invariants of \mathcal{X} is corresponding to the one in [7] (cf. Definition 5.25 and Remark 5.26) by applying [8, Theorem 4.18] for some moduli space with a symmetric perfect obstruction theory.

It is also natural to study the stacky version of higher rank PT stable pairs [61] if $\dim \mathcal{X} = 3$, $\mathcal{F}_0 = \mathcal{O}_{\mathcal{X}}(-n)^{\oplus r}$ and polynomials δ, P satisfying $\deg \delta \geq \deg P = 1$. As the orbifold PT invariants defined here is absolute, we will explore the relative orbifold PT theory and its degeneration formula elsewhere based on the work of [33, 42] as in the case of relative DT theory [68]. One may study the issue of rationality for the partition function of orbifold PT invariants (cf. Definition 5.23) as in [10, 48–50, 64], and furtherly investigate the conjecture of orbifold GW/PT or DT/PT correspondence as in [7, 69, 70], or more generally the stacky version of GW/PT or DT/PT correspondence proved in [10, 42, 45, 51, 64].

This paper is organized as follows. In Section 2, we recall the definitions of projective Deligne-Mumford stacks with polarizations and the modified Hilbert polynomial together with some relevant facts, and then generalize the notion of semistable pairs and some related results in [36, 67] to the stacky case. And we recollect the definition of set-theoretic families of coherent sheaves and several boundedness results in [43] and give the notion of flat families of pairs. We devote Section 3 to prove the boundedness of

families of δ -(semi)stable pairs on projective Deligne-Mumford stacks. In Section 4, we prove Theorem 1.1 generalizing the construction of moduli spaces of δ -(semi)stable pairs in [36, 67] and then describe the variation of moduli spaces along of the change of the stability parameter δ . In Section 5, we present two deformation and obstruction theories of stable pairs which are useful for proving the existence of virtual fundamental classes in some 2-dimensional and 3-dimensional cases respectively, and then provide a definition of orbifold Pandharipande-Thomas invariants.

2. Semistable pairs on projective Deligne-Mumford stacks

In this section, we first recollect some preliminaries of projective Deligne-Mumford stacks and properties of coherent sheaves on them, and then recall the definition of the modified Hilbert polynomial and generalize the notion of stability in [36, 67] to give the definition of δ -(semi)stable pairs and some related properties. We also recollect some boundedness results for set-theoretic families of coherent sheaves and introduce the definition of flat families of pairs used in the rest sections. For the theory of stacks, one may refer to [18, 31, 66] for more details.

2.1. Preliminaries on projective Deligne-Mumford stacks

In this subsection, we recall the definition of projective Deligne-Mumford stacks and some relevant properties of coherent sheaves on them (see [3, 4, 28, 30, 43, 47]). We first make some convention as in [12, 43] here as follows. Let k be an algebraically closed field of characteristic zero. All schemes are assumed to be noetherian over k , and every variety is a reduced separated scheme of finite type over k . Denote by S the generic base scheme of finite type over k . For a Deligne-Mumford S -stack and its related properties, one may refer to [18] or [66, Appendix]. Every Deligne-Mumford S -stack is assumed to be a separated noetherian Deligne-Mumford stack of finite type over S , and when $S = \text{Spec } k$, it is called a Deligne-Mumford stack (over k). Let $p : \mathcal{X} \rightarrow S$ be the structure morphism of \mathcal{X} . By an orbifold we mean a smooth Deligne-Mumford stack of finite type over k with generically trivial stabilizer. By [28, Corollary 1.3 (1)] for a Deligne-Mumford S -stack \mathcal{X} , we have a separated algebraic space X and a morphism $\pi : \mathcal{X} \rightarrow X$. More explicitly, the following properties are stated in [4, Theorem 2.2.1]:

- (i) The morphism $\pi : \mathcal{X} \rightarrow X$ is proper and quasi-finite.

- (ii) If \mathfrak{f} is an algebraically closed field, the map $\mathcal{X}(\mathfrak{f})/\text{Isom} \rightarrow X(\mathfrak{f})$ is a bijection.
- (iii) If $\widehat{S} \rightarrow S$ is a flat morphism of schemes, and suppose that $Y \rightarrow \widehat{S}$ is an algebraic space and $\mathcal{X} \times_S \widehat{S} \rightarrow Y$ is a morphism, then $\mathcal{X} \times_S \widehat{S} \rightarrow Y$ factors uniquely as $\mathcal{X} \times_S \widehat{S} \rightarrow X \times_S \widehat{S} \rightarrow Y$.
- (iv) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.
Here, we call X a coarse moduli space (or moduli space) of \mathcal{X} , and when X is a scheme, we call it a moduli scheme.

It is shown in [3] that Deligne-Mumford S -stacks with the above convention are tame. This implies three useful properties: (i) the functor $\pi_* : \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(X)$ is exact and carries coherent sheaves to coherent sheaves by [4, Lemma 2.3.4]; (ii) $\pi_* \mathcal{F}$ is flat over S if $\mathcal{F} \in \text{QCoh}(\mathcal{X})$ is a flat sheaf over S by [43, Corollary 1.3 (3)]; (iii) for any quasi-coherent sheaf \mathcal{F} , we have $H^\bullet(\mathcal{X}, \mathcal{F}) \cong H^\bullet(X, \pi_* \mathcal{F})$ by [43, Remark 1.4 (3)]. Here, QCoh denotes the category of quasi-coherent sheaves.

To begin with, we introduce the following

Definition 2.1. ([43, Definitions 2.4 and 2.6]) *Let \mathcal{X} be a Deligne-Mumford S -stack with moduli space $\pi : \mathcal{X} \rightarrow X$. Let \mathcal{E} be a locally free sheaf on \mathcal{X} . Define two functors*

$$\begin{aligned}
 F_{\mathcal{E}} : \text{QCoh}(\mathcal{X}) &\rightarrow \text{QCoh}(X), & \mathcal{F} &\mapsto \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}); \\
 G_{\mathcal{E}} : \text{QCoh}(X) &\rightarrow \text{QCoh}(\mathcal{X}), & F &\mapsto \pi^* F \otimes \mathcal{E}.
 \end{aligned}$$

A locally free sheaf \mathcal{E} is said to be a generator for the quasi-coherent sheaf \mathcal{F} if the left adjoint of the identity $\pi_(\mathcal{F} \otimes \mathcal{E}^\vee) \xrightarrow{id} \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee)$, i.e.,*

$$\theta_{\mathcal{E}}(\mathcal{F}) : G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) = \pi^* \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}$$

is surjective. It is a generating sheaf for \mathcal{X} if it is a generator for every quasicoherent sheaf on \mathcal{X} .

Obviously the functor $F_{\mathcal{E}}$ is exact since two functors $\mathcal{H}om(\mathcal{E}, \cdot)$ and π_* are exact. It is shown in [43, Remark 2.5 (1)] that $G_{\mathcal{E}}$ is exact if the morphism π is flat. And this holds when \mathcal{X} is a flat gerbe over a scheme or a root stack. Compared with $\theta_{\mathcal{E}}$, one can also define the right adjoint of the identity $\pi^* F \otimes \mathcal{E} \xrightarrow{id} \pi^* F \otimes \mathcal{E}$ as

$$\varphi_{\mathcal{E}}(F) : F \rightarrow \pi_* \mathcal{H}om(\mathcal{E}, \pi^* F \otimes \mathcal{E}) = F_{\mathcal{E}}(G_{\mathcal{E}}(F)) = F \otimes \pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$$

where the last equality is obtained by the projection formula in the following lemma. Thus $\varphi_{\mathcal{E}}$ is the map given by tensoring a section with the identity endomorphism, which implies that $\varphi_{\mathcal{E}}$ is injective.

Lemma 2.2. *Let \mathcal{F} be a quasi-coherent sheaf on \mathcal{X} and M a quasi-coherent sheaf on X . Then we have a projection formula*

$$\pi_*(\pi^*M \otimes \mathcal{F}) = M \otimes \pi_*\mathcal{F}.$$

Moreover, it is functorial that if $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ is a morphism of quasi-coherent sheaves on \mathcal{X} and $\psi : M \rightarrow M'$ is a morphism of quasi-coherent sheaves on X , then

$$\pi_*(\pi^*\psi \otimes \phi) = \psi \otimes \pi_*\phi.$$

Proof. See the proofs in [47, Corollary 5.4] and [43, Lemma 2.8]. □

Next, we recall some more results in [47, Section 5]. We start with the following

Proposition 2.3. ([47, Theorem 5.2]) *A locally free sheaf \mathcal{E} on \mathcal{X} is a generating sheaf if and only if for each geometric point of \mathcal{X} , the representation of the stabilizer group at that point on the fiber contains every irreducible representation.*

Definition 2.4. ([19, Definition 2.9]) *Let \mathcal{X} be a stack of finite type over a base scheme S . We say \mathcal{X} is a global quotient stack over S if it is isomorphic to a stack $[Z/G]$ where Z is an algebraic space of finite type over S and G is a flat group scheme over S which is a subgroup scheme of the general linear group scheme $\mathrm{GL}_{n,S}$ for some n .*

Theorem 2.5. *Let \mathcal{X} be a Deligne-Mumford S -stack which is a global quotient stack over S , we have*

- (i) *there is a locally free sheaf \mathcal{E} over \mathcal{X} which is a generating sheaf for \mathcal{X} ;*
- (ii) *let $\pi : \mathcal{X} \rightarrow X$ be the moduli space of \mathcal{X} and $f : X' \rightarrow X$ a morphism of algebraic spaces, then X' is the moduli space of $\mathcal{X}' := \mathcal{X} \times_X X'$. Moreover, let $p : \mathcal{X}' \rightarrow \mathcal{X}$ be the natural projection, then $p^*\mathcal{E}$ is a generating sheaf for \mathcal{X}' .*

Proof. See [47, Proposition 5.3] for the proof of (i), and use [3, Corollary 3.3 (a)] and [47, Lemma 1.3 and Theorem 5.5] for the second part. □

Now, we present the definitions of projective Deligne-Mumford stacks and a family of projective stacks as follows.

Definition 2.6. (see [30, Definition 5.5] and [43, Definition 2.23]) A Deligne-Mumford stack \mathcal{X} over k is a (quasi-)projective stack if \mathcal{X} admits a (locally) closed embedding into a smooth proper Deligne-Mumford stack and has a projective moduli scheme.

Let $p : \mathcal{X} \rightarrow S$ be a Deligne-Mumford S -stack which is a global quotient stack over S with a moduli scheme X . We call $p : \mathcal{X} \rightarrow S$ a family of projective stacks if p factorizes as $\pi : \mathcal{X} \rightarrow X$ followed by a projective morphism $\rho : X \rightarrow S$.

The notion of projective Deligne-Mumford stacks has some equivalent definitions.

Proposition 2.7 ([30, Theorem 5.3]). Let \mathcal{X} be a Deligne-Mumford stack over k . The following are equivalent:

- (i) \mathcal{X} is (quasi-)projective.
- (ii) \mathcal{X} has a (quasi-)projective moduli scheme and possesses a generating sheaf.
- (iii) \mathcal{X} is global quotient stack over k with a moduli space which is a (quasi-)projective scheme.

Remark 2.8. Proposition 2.7 shows that Theorem 2.5 holds for any projective Deligne-Mumford stack and every family of projective stacks. In particular, if $p : \mathcal{X} := [Z/G] \rightarrow S$ is a family of projective stacks, then for any geometric point $s \in S$, we have

- (i) the fibre $\mathcal{X}_s = [Z_s/G_s]$ is a projective Deligne-Mumford stack with a moduli scheme X_s which is the fibre of $\rho : X \rightarrow S$ over s ;
- (ii) there is a generating sheaf \mathcal{E} for the family of projective stacks $p : \mathcal{X} \rightarrow S$, and hence \mathcal{E}_s which is the fibre of \mathcal{E} over s is a generating sheaf for \mathcal{X}_s .

To conclude this subsection, we introduce the notion of polarizations for projective Deligne-Mumford stacks and the family of projective stacks [12, 43].

Definition 2.9. Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme X . A polarization of \mathcal{X} is a pair $(\mathcal{E}, \mathcal{O}_X(1))$ where \mathcal{E} is

a generating sheaf for \mathcal{X} and $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X relative to $\mathrm{Spec} k$.

Let $p : \mathcal{X} \rightarrow S$ be a family of projective stacks. A relative polarization of $p : \mathcal{X} \rightarrow S$ is a pair $(\mathcal{E}, \mathcal{O}_X(1))$ where \mathcal{E} is a generating sheaf for \mathcal{X} and $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X relative to S .

2.2. The modified Hilbert polynomial

In this subsection, we recall some properties of coherent sheaves on a projective Deligne-Mumford stack, and then present the definition of the modified Hilbert polynomial and its related stability [43]. Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$.

Definition 2.10 ([24, Definitions 1.1.1 and 1.1.2]). *Let \mathcal{F} be a coherent sheaf on \mathcal{X} . The support of \mathcal{F} , denoted by $\mathrm{Supp}(\mathcal{F})$, is defined by the closed substack associated to the ideal \mathcal{I} :*

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}).$$

The dimension of \mathcal{F} is defined as the dimension of its support. \mathcal{F} is called pure of dimension d if $\dim(\mathcal{G}) = d$ for every nonzero subsheaf $\mathcal{G} \subset \mathcal{F}$.

As in [24, Definition 1.1.4], there is a unique torsion filtration of any coherent sheaf \mathcal{F} :

$$0 \subseteq T_0(\mathcal{F}) \subseteq \dots \subseteq T_d(\mathcal{F}) = \mathcal{F}$$

where $d = \dim(\mathcal{F})$ and $T_i(\mathcal{F})$ is the maximal subsheaf of \mathcal{F} of dimension $\leq i$. Hence $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is zero or pure of dimension i . And \mathcal{F} is pure if and only if $T_{d-1}(\mathcal{F}) = 0$.

Definition 2.11. ([24, Definition 1.1.5]) *The saturation of a subsheaf $\mathcal{F}' \subset \mathcal{F}$ is the minimal subsheaf $\overline{\mathcal{F}'}$ containing \mathcal{F}' such that $\mathcal{F}/\overline{\mathcal{F}'}$ is pure of dimension $\dim(\mathcal{F})$ or zero.*

Remark 2.12. *As in [24], the saturation of \mathcal{F}' is also defined to be the kernel of the surjection*

$$\mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}' \rightarrow (\mathcal{F}/\mathcal{F}')/T_{\dim \mathcal{F}-1}(\mathcal{F}/\mathcal{F}').$$

Now we have the following properties of coherent sheaves on \mathcal{X} .

Lemma 2.13 (see [43, Lemma 3.4 and Proposition 3.6] and [12, Proposition 2.22]). *Let \mathcal{F} be a coherent sheaf on \mathcal{X} . Then we have*

- (i) $\pi(\text{Supp}(\mathcal{F})) = \pi(\text{Supp}(\mathcal{F} \otimes \mathcal{E}^\vee)) \supseteq \text{Supp}(F_{\mathcal{E}}(\mathcal{F}))$.
- (ii) $F_{\mathcal{E}}(\mathcal{F})$ is zero if and only if \mathcal{F} is zero.
- (iii) If \mathcal{F} is pure of dimension d , then $F_{\mathcal{E}}(\mathcal{F})$ is pure of the same dimension. Moreover, \mathcal{F} is pure if and only if $F_{\mathcal{E}}(\mathcal{F})$ is pure.

Lemma 2.14 ([43, Corollary 3.7 and 3.8]). *Let \mathcal{F} be a coherent sheaf on \mathcal{X} of dimension d . Then the functor $F_{\mathcal{E}}$ sends the torsion filtration $0 \subseteq T_0(\mathcal{F}) \subseteq \dots \subseteq T_d(\mathcal{F}) = \mathcal{F}$ of \mathcal{F} to the torsion filtration of $F_{\mathcal{E}}(\mathcal{F})$, i.e., $F_{\mathcal{E}}(T_i(\mathcal{F})) = T_i(F_{\mathcal{E}}(\mathcal{F}))$ for $0 \leq i \leq d$. Moreover, if \mathcal{F} is pure, then we have $\pi(\text{Supp}(\mathcal{F})) = \text{Supp}(F_{\mathcal{E}}(\mathcal{F}))$.*

Next, we have the following definition of the modified Hilbert polynomial.

Definition 2.15 ([43, Definition 3.10]). *Let \mathcal{F} be a coherent sheaf of dimension d on \mathcal{X} . The modified Hilbert polynomial of \mathcal{F} is defined as*

$$\begin{aligned} P_{\mathcal{E}}(\mathcal{F})(m) &:= \chi(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(m)) \\ &= \chi(X, F_{\mathcal{E}}(\mathcal{F})(m)) = P(F_{\mathcal{E}}(\mathcal{F})(m)). \end{aligned}$$

Since $\dim F_{\mathcal{E}}(\mathcal{F}) = \dim \mathcal{F} = d$ by Lemma 2.13, the modified Hilbert polynomial can be written as

$$P_{\mathcal{E}}(\mathcal{F})(m) = \sum_{i=0}^d \alpha_{\mathcal{E},i}(\mathcal{F}) \frac{m^i}{i!}.$$

The multiplicity of $F_{\mathcal{E}}(\mathcal{F})$ is defined by

$$r(F_{\mathcal{E}}(\mathcal{F})) = \alpha_{\mathcal{E},d}(\mathcal{F}).$$

As in [24, Definition 1.2.2], if $\dim \mathcal{F} = d = \dim X$, the rank of $F_{\mathcal{E}}(\mathcal{F})$ is defined by

$$\text{rk}(F_{\mathcal{E}}(\mathcal{F})) = \frac{\alpha_{\mathcal{E},d}(\mathcal{F})}{\alpha_d(\mathcal{O}_X)},$$

where $\alpha_d(\mathcal{O}_X)$ is the leading coefficient of Hilbert polynomial $\chi(\mathcal{O}_X(m)) = \sum_{i=0}^d \alpha_i(\mathcal{O}_X) \frac{m^i}{i!}$.

Definition 2.16 ([43, Definition 3.12 and 3.13]). *Define the reduced Hilbert polynomial of a coherent sheaf \mathcal{F} of dimension d as*

$$p_{\mathcal{E}}(\mathcal{F}) = \frac{P_{\mathcal{E}}(\mathcal{F})}{\alpha_{\mathcal{E},d}(\mathcal{F})},$$

and its slope as

$$\hat{\mu}_{\mathcal{E}}(\mathcal{F}) = \frac{\alpha_{\mathcal{E},d-1}(\mathcal{F})}{\alpha_{\mathcal{E},d}(\mathcal{F})}.$$

Then we have the following definition of stabilities with respect to $p_{\mathcal{E}}$ and $\hat{\mu}_{\mathcal{E}}$.

Definition 2.17. ([24, Definition 1.2.4]) *A coherent sheaf \mathcal{F} is $p_{\mathcal{E}}$ -semistable (respectively $\hat{\mu}_{\mathcal{E}}$ -semistable) if \mathcal{F} is pure and for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ one has $p_{\mathcal{E}}(\mathcal{F}') \leq p_{\mathcal{E}}(\mathcal{F})$ (respectively $\hat{\mu}_{\mathcal{E}}(\mathcal{F}') \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F})$), and it is $p_{\mathcal{E}}$ -stable (respectively $\hat{\mu}_{\mathcal{E}}$ -stable) if it is $p_{\mathcal{E}}$ -semistable (respectively $\hat{\mu}_{\mathcal{E}}$ -semistable) and the corresponding inequality is strict.*

As in [24, Theorem 1.6.7], for a pure sheaf \mathcal{F} , there is a unique Harder-Narasimhan filtration with respect to the $\hat{\mu}_{\mathcal{E}}$ -stability:

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_l = \mathcal{F}$$

such that the factors $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $i = 1, \dots, l$ are $\hat{\mu}_{\mathcal{E}}$ -semistable and

$$\hat{\mu}_{\mathcal{E},\max}(\mathcal{F}) := \hat{\mu}_{\mathcal{E}}(\mathcal{F}_1) > \hat{\mu}_{\mathcal{E}}(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \hat{\mu}_{\mathcal{E}}(\mathcal{F}_l/\mathcal{F}_{l-1}) := \hat{\mu}_{\mathcal{E},\min}(\mathcal{F}).$$

Similarly, one also has a unique Harder-Narasimhan filtration with respect to the $p_{\mathcal{E}}$ -stability.

2.3. Semistable pairs

We generalize the notion of semistable pairs on smooth projective varieties [36, 67] to the case of projective Deligne-Mumford stacks in this subsection. Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let \mathcal{F}_0 be a fixed coherent sheaf on \mathcal{X} and $\delta \in \mathbb{Q}[m]$ be 0 or a polynomial with positive leading coefficient.

Definition 2.18. *A pair (\mathcal{F}, φ) on \mathcal{X} consists of a coherent sheaf \mathcal{F} and a morphism $\varphi : \mathcal{F}_0 \rightarrow \mathcal{F}$. A morphism of pairs $\phi : (\mathcal{F}, \varphi) \rightarrow (\mathcal{G}, \psi)$ is a morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ such that there is an element $\lambda \in k$ making the*

following diagram commute

$$\begin{array}{ccc}
 \mathcal{F}_0 & \xrightarrow{\lambda \cdot \text{id}} & \mathcal{F}_0 \\
 \varphi \downarrow & & \downarrow \psi \\
 \mathcal{F} & \xrightarrow{\phi} & \mathcal{G}
 \end{array}$$

A subpair (\mathcal{F}', φ') of (\mathcal{F}, φ) consists of a coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ and a morphism $\varphi' : \mathcal{F}_0 \rightarrow \mathcal{F}'$ satisfying $i \circ \varphi' = \varphi$ if $\text{im } \varphi \subset \mathcal{F}'$, and $\varphi' = 0$ otherwise, where i denotes the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$. A quotient pair $(\mathcal{F}'', \varphi'')$ consists of a coherent quotient sheaf $q : \mathcal{F} \rightarrow \mathcal{F}''$ and a morphism $\varphi'' = q \circ \varphi : \mathcal{F}_0 \rightarrow \mathcal{F}''$.

A pair (\mathcal{F}, φ) is said to be of dimension d if $\dim \mathcal{F} = d$. We say a pair (\mathcal{F}, φ) is pure if \mathcal{F} is pure. Let P be a polynomial of degree d , we call a pair (\mathcal{F}, φ) of type P if $P_{\mathcal{E}}(\mathcal{F}) = P$.

Lemma 2.19 ([23, Lemma 1.5]). *The set $\text{Hom}((\mathcal{F}, \varphi), (\mathcal{G}, \psi))$ of morphisms of pairs is a linear subspace of $\text{Hom}(\mathcal{F}, \mathcal{G})$. If $\phi : (\mathcal{F}, \varphi) \rightarrow (\mathcal{G}, \psi)$ is an isomorphism, the factor λ in the definition satisfies $\lambda \in k^*$. In particular, the isomorphism $\phi_0 = \lambda^{-1}\phi$ satisfies $\phi_0 \circ \varphi = \psi$.*

Remark 2.20. *As in [12, Remark 3.5], we have the cartesian diagram*

$$\begin{array}{ccc}
 W & \longrightarrow & k \\
 \downarrow & & \downarrow \cdot \psi \\
 \text{Hom}(\mathcal{F}, \mathcal{G}) & \xrightarrow{\circ \varphi} & \text{Hom}(\mathcal{F}_0, \mathcal{G})
 \end{array}$$

and

$$W \cong \begin{cases} \text{Hom}((\mathcal{F}, \varphi), (\mathcal{G}, \psi)), & \text{if } \psi \neq 0; \\ \text{Hom}((\mathcal{F}, \varphi), (\mathcal{G}, \psi)) \times k, & \text{otherwise.} \end{cases}$$

Define the Hilbert polynomial of a pair (\mathcal{F}, φ) as

$$P_{(\mathcal{F}, \varphi)} = P_{\mathcal{E}}(\mathcal{F}) + \epsilon(\varphi)\delta$$

and the reduced Hilbert polynomial of this pair by

$$p_{(\mathcal{F}, \varphi)} = p_{\mathcal{E}}(\mathcal{F}) + \frac{\epsilon(\varphi)\delta}{r(F_{\mathcal{E}}(\mathcal{F}))}$$

where

$$\epsilon(\varphi) = \begin{cases} 1, & \text{if } \varphi \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.21. *As in [36], a short exact sequence of pairs,*

$$0 \rightarrow (\mathcal{F}', \varphi') \xrightarrow{i} (\mathcal{F}, \varphi) \xrightarrow{q} (\mathcal{F}'', \varphi'') \rightarrow 0$$

consists of a short exact sequence of sheaves $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ such that (\mathcal{F}', φ') is a subpair and $(\mathcal{F}'', \varphi'')$ is the corresponding quotient pair. Here, $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$, $\varphi'' = q \circ \varphi$ if $\varphi' = 0$, and $\varphi'' = 0$ if $\text{im } \varphi \subset \mathcal{F}'$. Since the modified Hilbert polynomial is additive in a short exact sequence of coherent sheaves [43, Remark 3.11 (2)] and $\epsilon(\varphi) = \epsilon(\varphi') + \epsilon(\varphi'')$, it is obviously that the Hilbert polynomial is also additive in a short exact sequence of pairs.

We present a definition of δ -(semi)stable pairs on a projective Deligne-Mumford stack.

Definition 2.22. *A pair (\mathcal{F}, φ) is δ -semistable if \mathcal{F} is pure and $p_{(\mathcal{F}', \varphi')} \leq p_{(\mathcal{F}, \varphi)}$ for every proper subpair (\mathcal{F}', φ') . We call (\mathcal{F}, φ) a δ -stable pair if it is δ -semistable and the inequality is strict.*

Remark 2.23. *As in [24, Proposition 1.2.6], Definition 2.22 can be shown to be equivalent to the statement: a pair (\mathcal{F}, φ) is δ -(semi)stable if and only if \mathcal{F} is pure and $p_{(\mathcal{F}'', \varphi'')} (\geq) p_{(\mathcal{F}, \varphi)}$ for every proper purely quotient pair $(\mathcal{F}'', \varphi'')$ of dimension $\dim \mathcal{F}$.*

Notice that when $\varphi = 0$ or $\delta = 0$, the δ -(semi)stability of a pair (\mathcal{F}, φ) is equivalent to $p_{\mathcal{E}}$ -(semi)stability of the coherent sheaf \mathcal{F} . We say a pair (\mathcal{F}, φ) is nondegenerate if $\varphi \neq 0$ as in [36]. Since most of the following results in this subsection are straightforward generalizations of those in [36, Section 2], some proofs are omitted.

Lemma 2.24. *When $\deg \delta \geq \deg P$, every nondegenerate δ -semistable pair of type P is δ -stable.*

Lemma 2.25. *Let \mathcal{F} be a pure coherent sheaf of dimension d with the modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$, and assume $\deg \delta \geq \deg P = d$. Then a pair (\mathcal{F}, φ) is δ -stable if and only if for every proper subpair (\mathcal{F}', φ') ,*

$$\frac{P_{\mathcal{E}}(\mathcal{F}')}{2r(F_{\mathcal{E}}(\mathcal{F}')) - \epsilon(\varphi')} < \frac{P_{\mathcal{E}}(\mathcal{F})}{2r(F_{\mathcal{E}}(\mathcal{F})) - \epsilon(\varphi)}.$$

Lemma 2.26. *If $\deg \delta \geq \deg P$, then a nondegenerate pair (\mathcal{F}, φ) of type P is δ -stable if and only if \mathcal{F} is pure and $\dim \operatorname{coker} \varphi < \deg P$.*

Remark 2.27. *Lemma 2.26 implies that when $\mathcal{F}_0 = \mathcal{O}_X$ and $\deg P = 1 \leq \deg \delta$, a nondegenerate δ -stable pair (\mathcal{F}, φ) of type P is a stable pair in the sense of Pandharipande-Thomas [53, Lemma 1.3].*

Lemma 2.28. *Suppose $\phi : (\mathcal{F}, \varphi) \rightarrow (\mathcal{G}, \psi)$ is a nonzero morphism of pairs. If (\mathcal{F}, φ) and (\mathcal{G}, ψ) are δ -semistable pairs of dimension d , then $p_{(\mathcal{F}, \varphi)} \leq p_{(\mathcal{G}, \psi)}$. Suppose (\mathcal{F}, φ) and (\mathcal{G}, ψ) are δ -stable with the same reduced Hilbert polynomial, then ϕ induces an isomorphism between \mathcal{F} and \mathcal{G} . In particular, we have $\operatorname{End}((\mathcal{F}, \varphi)) \cong k$ for a δ -stable pair (\mathcal{F}, φ) .*

Proposition 2.29 (Harder-Narasimhan filtration). *Let (\mathcal{F}, φ) be a pair and \mathcal{F} be pure of dimension d . Then there exists a unique filtration by subpairs*

$$0 \subsetneq (\mathcal{G}_1, \varphi_1) \subsetneq (\mathcal{G}_2, \varphi_2) \subsetneq \cdots \subsetneq (\mathcal{G}_l, \varphi_l) = (\mathcal{F}, \varphi)$$

such that the factors $\operatorname{gr}_i = (\mathcal{G}_i, \varphi_i) / (\mathcal{G}_{i-1}, \varphi_{i-1})$ for $i = 1, \dots, l$, are δ -semistable pair of dimension d with the reduced Hilbert polynomials p_{gr_i} satisfying

$$p_{\operatorname{gr}_1} > \cdots > p_{\operatorname{gr}_l}.$$

Proof. Given a pair (\mathcal{F}, φ) , any subpair (\mathcal{F}', φ') is actually determined by the subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ since φ' is determined by the given morphism φ and comparing \mathcal{F}' with $\operatorname{im} \varphi$ by definition of a subpair. With this point of view, the proof is completed by using the similar argument in the proof of [24, Theorem 1.3.4] with the (reduced) Hilbert polynomial of pairs and the multiplicity of $F_{\mathcal{E}}(\mathcal{G})$ for any subsheaf \mathcal{G} of \mathcal{F} . Alternatively, one may refer to the similar argument in the proof of [60, Theorem 1] for a pure sheaf as in [36, Proposition 2.12]. □

Remark 2.30. *For a nondegenerate pair (\mathcal{F}, φ) , let $\operatorname{gr}_i := (\operatorname{gr}_i \mathcal{F}, \operatorname{gr}_i \varphi)$ for $i = 1, \dots, l$, it is obviously that there is only one nonzero $\operatorname{gr}_i \varphi$ by the definition of subpairs and quotient pairs. When $\deg \delta \geq d$, only $\operatorname{gr}_1 \varphi = \varphi_1$ is nonzero.*

Proposition 2.31 (Jordan-Hölder filtration). *Let (\mathcal{F}, φ) be a δ -semistable pair. There is a filtration*

$$0 \subsetneq (\mathcal{F}_1, \varphi_1) \subsetneq (\mathcal{F}_2, \varphi_2) \subsetneq \cdots \subsetneq (\mathcal{F}_l, \varphi_l) = (\mathcal{F}, \varphi)$$

such that all factors $\text{gr}_i = (\mathcal{F}_i, \varphi_i)/(\mathcal{F}_{i-1}, \varphi_{i-1})$ for $i = 1, \dots, l$, are δ -stable with the same reduced Hilbert polynomial $p_{(\mathcal{F}, \varphi)}$. Here, $\text{gr}(\mathcal{F}, \varphi) = \bigoplus_i \text{gr}_i$ does not depend on the choice of filtration.

Remark 2.32. *As in [67], Jordan-Hölder filtration induces a homomorphism $\text{gr}(\varphi) : \mathcal{F}_0 \rightarrow \text{gr}(\mathcal{F}, \varphi)$ which is nontrivial for a nondegenerate pair (\mathcal{F}, φ) and its image is contained in exactly one summand of $\text{gr}(\mathcal{F}, \varphi)$.*

Definition 2.33. *Two δ -semistable pair $(\mathcal{F}_1, \varphi_1)$ and $(\mathcal{F}_2, \varphi_2)$ with the same reduced Hilbert polynomial are called S-equivalent if $\text{gr}(\mathcal{F}_1, \varphi_1) \cong \text{gr}(\mathcal{F}_2, \varphi_2)$.*

2.4. Families of coherent sheaves and pairs

We first recall the definition of a set-theoretic family of coherent sheaves as in [29, Section 1.12] and its relevant boundedness results in [43, Section 4], and then give the notion of flat families of pairs. Let $p : \mathcal{X} \rightarrow S$ be a family of projective stacks with a relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Suppose $s \in S$ and K is a field extension of residue field $k(s)$, a coherent sheaf on a fiber of p is defined to be a coherent sheaf \mathcal{F}_K on $\mathcal{X}_K := \mathcal{X} \times_S \text{Spec } K$. Given two extensions K_1 and K_2 , two coherent sheaves \mathcal{F}_{K_1} and \mathcal{F}_{K_2} are equivalent if there are $k(s)$ -homomorphisms of K_1, K_2 to a third extension K_3 such that $\mathcal{F}_{K_1} \otimes_{k(s)} K_3$ and $\mathcal{F}_{K_2} \otimes_{k(s)} K_3$ are isomorphic.

Definition 2.34 ([29, Section 1.12] or [43, Definitions 4.9 and 4.10]). *A set-theoretic family of coherent sheaves on $p : \mathcal{X} \rightarrow S$ is a set of coherent sheaves defined on the fibers of p . A set-theoretic family \mathfrak{F} of coherent sheaves on \mathcal{X} is bounded if there is an S -scheme T of finite type and a coherent sheaf \mathcal{G} on $\mathcal{X}_T := \mathcal{X} \times_S T$ such that every sheaf in \mathfrak{F} is contained in $\{\mathcal{G}|_{\mathcal{X} \times_S \text{Spec } k(t)} \mid t \in T\}$.*

Definition 2.35 (see [24, Definition 1.7.1 and 1.7.3] and [43, Definition 4.2]). *Let X be a projective scheme over k with a very ample line bundle $\mathcal{O}_X(1)$. A coherent sheaf F on X is said to be m -regular if for every $i > 0$ we have $H^i(X, F(m - i)) = 0$. The Mumford-Castelnuovo regularity of F is defined to be the least integer m such that F is m -regular. Let \mathcal{X} be a*

projective Deligne-Mumford stack over k with a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. A coherent sheaf \mathcal{F} on \mathcal{X} is defined to be m -regular if $F_{\mathcal{E}}(\mathcal{F})$ on X is m -regular. Denote by $\text{reg}_{\mathcal{E}}(\mathcal{F})$ the Mumford-Castelnuovo regularity of $F_{\mathcal{E}}(\mathcal{F})$.

A criterion for boundedness of a set-theoretic family of coherent sheaves is Kleiman criterion for stacks [43, Theorem 4.12]. It implies that if a set-theoretic family \mathfrak{F} is bounded, then the set of modified Hilbert polynomials $P_{\mathcal{E}_K}(\mathcal{F}_K)$ for $\mathcal{F}_K \in \mathfrak{F}$ is finite, and there exists an integer $m \geq 0$ such that every coherent sheaf \mathcal{F}_K is m -regular. Kleiman criterion for stacks is also used in proving the following stacky version of Grothendieck lemma.

Lemma 2.36 ([43, Lemma 4.13 and Remark 4.14]). *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let P be a polynomial of degree $d \in [0, \dim(X)]$ and $\bar{\rho}$ an integer. There exists a constant $C = C(P, \bar{\rho})$ such that if \mathcal{F} is a coherent sheaf of dimension d on \mathcal{X} with $P_{\mathcal{E}}(\mathcal{F}) = P$ and $\text{reg}_{\mathcal{E}}(\mathcal{F}) \leq \bar{\rho}$, then for every purely d -dimensional quotient \mathcal{F}' , we have $\hat{\mu}_{\mathcal{E}}(\mathcal{F}') \geq C$. Moreover, the family of purely d -dimensional quotients $\mathcal{F}'_i, i \in I$ (for some set of indices I) with $\hat{\mu}_{\mathcal{E}}(\mathcal{F}'_i)$ bounded from above is bounded.*

The similar statement as above is true, that is, for every purely saturated subsheaf \mathcal{F}' , the slope $\hat{\mu}_{\mathcal{E}}(\mathcal{F}')$ is bounded from above, and the family of pure subsheaves $\mathcal{F}'_i \subseteq \mathcal{F}, i \in I$ with $\hat{\mu}_{\mathcal{E}}(\mathcal{F}'_i)$ bounded from below such that the quotient $\mathcal{F}/\mathcal{F}'_i$ is pure of dimension d , is bounded.

Boundedness of a family of coherent sheaves on projective Deligne-Mumford stacks is proved to be equivalent to boundedness of the one on projective schemes under some conditions in the following

Proposition 2.37. ([43, Corollary 4.17]) *Let $p : \mathcal{X} \rightarrow S$ be a family of projective stacks with a relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let \mathfrak{F} be a set-theoretic family of coherent sheaves on the fibers of p . Then the family \mathfrak{F} is bounded if and only if $F_{\mathcal{E}}(\mathfrak{F})$ is bounded.*

Here is a useful boundedness result as follows.

Theorem 2.38 ([43, Theorem 4.27 (1)]). *Let $p : \mathcal{X} \rightarrow S$ be a family of projective stacks with a relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let P be a polynomial of degree d and μ_0 a real number. Every set-theoretic family $\mathcal{F}_i, i \in I$ (I is a set) of purely d -dimensional sheaves on the fiber of p with fixed modified Hilbert polynomial P such that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_i)) \leq \mu_0$ is bounded.*

The following bound of the number of global sections will be used in Section 3.

Lemma 2.39 ([43, Proposition 4.24 and Corollary 4.30]). *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. For any $\hat{\mu}_{\mathcal{E}}$ -semistable sheaf \mathcal{F} of dimension d on \mathcal{X} , we have*

$$h^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee) \leq \begin{cases} r \binom{\hat{\mu}_{\mathcal{E}}(\mathcal{F}) + \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}}{d}, \\ \text{if } \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \geq \frac{d+1}{2} - r^2 \\ 0, \text{ otherwise} \end{cases}$$

where $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + \tilde{m} \deg(\mathcal{O}_X(1))$, $r = r(F_{\mathcal{E}}(\mathcal{F}))$, $f(r) = -1 + \sum_{i=1}^r \frac{1}{i}$, and \tilde{m} is the integer making that $\pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})(\tilde{m})$ is generated by global sections.

Now, as in [36, 67], we introduce the notion of a flat family of pairs.

Definition 2.40. *A flat family (\mathcal{F}, φ) of pairs parametrized by a scheme S consists of a coherent sheaf \mathcal{F} on $\mathcal{X} \times S$ which is flat over S and a morphism $\varphi : \pi_{\mathcal{X}}^* \mathcal{F}_0 \rightarrow \mathcal{F}$, where $\pi_{\mathcal{X}} : \mathcal{X} \times S \rightarrow \mathcal{X}$ is the natural projection. Two families (\mathcal{F}, φ) and (\mathcal{G}, ψ) are isomorphic if there is an isomorphism $\Phi : \mathcal{F} \rightarrow \mathcal{G}$ such that $\Phi \circ \varphi = \psi$.*

Remark 2.41. *In Definition 2.40, the implicit λ -scaling of $\pi_{\mathcal{X}}^* \mathcal{F}_0$ as in Definition 2.18 for the isomorphism of two families has been absorbed in Φ as in Lemma 2.19. Compared with the notion of a flat family of framed sheaves defined in [12, Definition 3.16], one may alternatively define a flat family of pairs as follows. A flat family $(\mathcal{F}, L_{\mathcal{F}}, \phi_{\mathcal{F}})$ of pairs parameterized by a scheme S consists of a coherent sheaf \mathcal{F} on $\mathcal{X} \times S$ which is flat over S , a line bundle $L_{\mathcal{F}}$ on S , and a morphism $\phi_{\mathcal{F}} : L_{\mathcal{F}} \rightarrow \pi_{S*} \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{F})$, where $\pi_S : \mathcal{X} \times S \rightarrow S$ is the projection. Two families $(\mathcal{F}, L_{\mathcal{F}}, \phi_{\mathcal{F}})$ and $(\mathcal{G}, L_{\mathcal{G}}, \psi_{\mathcal{G}})$ are isomorphic if there are isomorphisms $\Phi : \mathcal{F} \rightarrow \mathcal{G}$ and $\Psi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ such that*

$$\psi_{\mathcal{G}} \circ \Psi = \widehat{\Phi} \circ \phi_{\mathcal{F}}$$

where

$$\widehat{\Phi} : \pi_{S*} \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{F}) \rightarrow \pi_{S*} \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{G})$$

is the isomorphism induced by Φ . For this definition, one can impose the λ -scaling on $\pi_{\mathcal{X}}^* \mathcal{F}_0$ for two isomorphic families which is also absorbed in Φ . As in [12, Remark 3.17], the morphism $\phi_{\mathcal{F}} : L_{\mathcal{F}} \rightarrow \pi_{S*} \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{F})$ may be taken as a nowhere vanishing morphism, i.e., the composition as follows

$$\begin{aligned} \pi_S^* L_{\mathcal{F}} \otimes \pi_{\mathcal{X}}^* \mathcal{F}_0 &\rightarrow \pi_S^* \pi_{S*} \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{F}) \otimes \pi_{\mathcal{X}}^* \mathcal{F}_0 \\ &\rightarrow \mathcal{H}om(\pi_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{F}) \otimes \pi_{\mathcal{X}}^* \mathcal{F}_0 \rightarrow \mathcal{F}. \end{aligned}$$

When $L_{\mathcal{F}}$ is trivial, the morphism $\phi_{\mathcal{F}}$ may be viewed as φ in Definition 2.40. However, we will adopt the notion of a flat family of pairs in Definition 2.40 (see Remark 4.23 for the reason).

3. Boundedness of the family of semistable pairs

One important step in constructing moduli spaces of semistable pairs is to prove the boundedness of the family of semistable pairs. Given a polynomial $P \in \mathbb{Q}[m]$ and a stability parameter $\delta \in \mathbb{Q}[m]$ which is zero or a polynomial with positive leading coefficient, we will prove in this section that the family of δ -semistable pairs of type P is bounded. In fact, we will generalize the boundedness results on smooth projective varieties in [67, Section 3] when $\deg \delta < \deg P$ and those in [36, Section 3] when $\deg \delta \geq \deg P$ to the case of projective Deligne-Mumford stacks. Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. We start with the case when $\deg \delta < \deg P$.

Lemma 3.1. *When $\deg \delta < \deg P$. Suppose that (\mathcal{F}, φ) is a nondegenerate δ -semistable pair with the modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$. Then $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$ is bounded above by a constant depending on P, \mathcal{F}_0 and X .*

Proof. We combine the arguments in the proofs of [67, Proposition 2.1] and [43, Proposition 4.24]. Suppose that \mathcal{F} is pure of dimension d , and let $\frac{\delta_1}{(d-1)!}$ be the coefficient of δ in degree $d - 1$. The assumption of δ implies that $\delta_1 \geq 0$. Let (\mathcal{F}', φ') be a subpair of (\mathcal{F}, φ) satisfying $\text{im } \varphi' \subset \mathcal{F}'$. By assumption we have

$$\hat{\mu}_{\mathcal{E}}(\mathcal{F}') + \frac{\delta_1}{r(F_{\mathcal{E}}(\mathcal{F}'))} \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + \frac{\delta_1}{r(F_{\mathcal{E}}(\mathcal{F}))}$$

Then $\hat{\mu}_{\mathcal{E}}(\mathcal{F}') \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F})$ since the exactness of the functor $F_{\mathcal{E}}$ implies $r(F_{\mathcal{E}}(\mathcal{F}')) \leq r(F_{\mathcal{E}}(\mathcal{F}))$. Now, let $\mathcal{F}' \subseteq \mathcal{F}$ be any subsheaf. Set $\mathcal{H} = \mathcal{F}' + \text{im } \varphi$

and $\mathcal{G} = \text{im}\varphi/(\mathcal{F} \cap \text{im}\varphi)$. Then \mathcal{G} is a quotient of \mathcal{F}_0 and we have a short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow 0.$$

Note that $\mathcal{H} \subseteq \mathcal{F}$ and $\text{im}\varphi \subseteq \mathcal{H}$, then $(\mathcal{H}, i \circ \varphi)$ is a subpair of (\mathcal{F}, φ) , where i denotes the inclusion $\mathcal{H} \hookrightarrow \mathcal{F}$. As above, we have $\hat{\mu}_{\mathcal{E}}(\mathcal{H}) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F})$. Since $F_{\mathcal{E}}(\mathcal{G})$ is a quotient of $F_{\mathcal{E}}(\mathcal{F}_0)$, we have $\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}_0)) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{G})$. Since the modified Hilbert polynomial $P_{\mathcal{E}}(\cdot)$ is additive in a short exact sequence, we have

$$\begin{aligned} \hat{\mu}_{\mathcal{E}}(\mathcal{F}') &= \frac{\hat{\mu}_{\mathcal{E}}(\mathcal{H})r(F_{\mathcal{E}}(\mathcal{H})) - \hat{\mu}_{\mathcal{E}}(\mathcal{G})r(F_{\mathcal{E}}(\mathcal{G}))}{r(F_{\mathcal{E}}(\mathcal{F}'))} \\ &\leq \frac{\hat{\mu}_{\mathcal{E}}(\mathcal{F})r(F_{\mathcal{E}}(\mathcal{H})) - \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}_0))r(F_{\mathcal{E}}(\mathcal{G}))}{r(F_{\mathcal{E}}(\mathcal{F}'))} \\ &= \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + (\hat{\mu}_{\mathcal{E}}(\mathcal{F}) - \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}_0))) \frac{r(F_{\mathcal{E}}(\mathcal{G}))}{r(F_{\mathcal{E}}(\mathcal{F}'))}. \end{aligned}$$

where \mathcal{F}' and \mathcal{H} are of dimension d , and $r(F_{\mathcal{E}}(\mathcal{G})) = \alpha_{\mathcal{E},d}(\mathcal{G}) \geq 0$ (it is zero if $\dim(F_{\mathcal{E}}(\mathcal{G})) < d$). Set

$$\tilde{C} := \max\{\hat{\mu}_{\mathcal{E}}(\mathcal{F}), \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + (\hat{\mu}_{\mathcal{E}}(\mathcal{F}) - \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}_0))) \cdot r(F_{\mathcal{E}}(\mathcal{F}))\}$$

which is a constant depending on P and \mathcal{F}_0 . Then $\hat{\mu}_{\mathcal{E}}(\mathcal{F}') \leq \tilde{C}$ for any subsheaf $\mathcal{F}' \subseteq \mathcal{F}$.

By Serre’s vanishing theorem, one can choose an integer \tilde{m} large enough such that $\pi_*\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(\tilde{m})$ is generated by global sections. Set $N = h^0(X, \pi_*\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(\tilde{m}))$. Let \overline{F} be the maximal destabilizing sheaf of $F_{\mathcal{E}}(\mathcal{F})$ with respect to ordinary $\hat{\mu}$ -stability. Then $\hat{\mu}(\overline{F}) = \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$. As in the argument of [43, Proposition 4.24], one has a surjection $\overline{F} \otimes \mathcal{O}_X(-\tilde{m})^{\oplus N} \rightarrow F_{\mathcal{E}}(\overline{F})$, where \overline{F} is a subsheaf of \mathcal{F} associated to \overline{F} by some transformation. Since $\overline{F} \otimes \mathcal{O}_X(-\tilde{m})^{\oplus N}$ is also $\hat{\mu}$ -semistable, then

$$\hat{\mu}(\overline{F}(-\tilde{m})) = \hat{\mu}(\overline{F} \otimes \mathcal{O}_X(-\tilde{m})^{\oplus N}) \leq \hat{\mu}(F_{\mathcal{E}}(\overline{F})) = \hat{\mu}_{\mathcal{E}}(\overline{F}) \leq \tilde{C}.$$

The above inequality implies that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq C$, where $C := \tilde{C} + \tilde{m} \deg(\mathcal{O}_X(1))$ is a constant depending on P , \mathcal{F}_0 and X . □

The remaining case is

Lemma 3.2. *When $\deg \delta \geq \deg P$. Suppose that (\mathcal{F}, φ) is a nondegenerate δ -semistable pair with the modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$. Then $\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}))$ is bounded below by a constant depending on P , \mathcal{F}_0 and X .*

Proof. Since the functor $F_{\mathcal{E}}$ is exact, from the short exact sequence

$$0 \rightarrow \text{im}\varphi \rightarrow \mathcal{F} \rightarrow \text{coker}\varphi \rightarrow 0$$

we have $F_{\mathcal{E}}(\text{coker}\varphi) \cong F_{\mathcal{E}}(\mathcal{F})/F_{\mathcal{E}}(\text{im}\varphi)$. By Lemma 2.24 and Lemma 2.26, $\dim \text{coker}\varphi < \deg P = \dim \mathcal{F}$. Then $\dim F_{\mathcal{E}}(\text{coker}\varphi) < \deg P = \dim F_{\mathcal{E}}(\mathcal{F})$ by Lemma 2.13 and Lemma 2.14. The exactness of $F_{\mathcal{E}}$ implies $F_{\mathcal{E}}(\varphi) : F_{\mathcal{E}}(\mathcal{F}_0) \twoheadrightarrow F_{\mathcal{E}}(\text{im}\varphi) \hookrightarrow F_{\mathcal{E}}(\mathcal{F})$ and $\text{im } F_{\mathcal{E}}(\varphi) = F_{\mathcal{E}}(\text{im}\varphi)$. The proof is completed by applying the argument about (E_0, E) in the proof of [36, Lemma 3.1] to $(F_{\mathcal{E}}(\mathcal{F}_0), F_{\mathcal{E}}(\mathcal{F}))$. \square

As the modified Hilbert polynomial is additive in a short exact sequence and $P_{\mathcal{E}}(\mathcal{F}) = P$ is fixed, bounding $\hat{\mu}_{\min}$ from below is equivalent to bounding $\hat{\mu}_{\max}$ from above. Hence the constant C can be chosen to be independent of δ such that $\hat{\mu}_{\max}$ is bounded above by C by Lemma 3.1 and Lemma 3.2. Using these two lemmas and Theorem 2.38 with $S = \text{Spec } k$, we have

Proposition 3.3. *Fix a modified Hilbert polynomial P and some δ . The set-theoretic family*

$$\{\mathcal{F} | (\mathcal{F}, \varphi) \text{ is a nondegenerate } \delta\text{-semistable pair of type } P\}$$

of coherent sheaves on \mathcal{X} is bounded.

Next, in order to apply GIT machinery, one may relate the semistability condition to the number of global sections of subsheaves as in [24, Theorem 4.4.1]. We need the following estimate for the number of global sections.

Lemma 3.4. *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a polarization $(\mathcal{E}, \mathcal{O}_{\mathcal{X}}(1))$. Let \mathcal{F} be a pure coherent sheaf of dimension d on \mathcal{X} , then*

$$\begin{aligned} \frac{h^0(X, F_{\mathcal{E}}(\mathcal{F})(m))}{r} &= \frac{h^0(\mathcal{X}, \mathcal{F} \otimes \pi^* \mathcal{O}_X(m) \otimes \mathcal{E}^{\vee})}{r} \\ &\leq \frac{r-1}{r} \left[\binom{\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) + m + C}{d} \right]_+ \\ &\quad + \frac{1}{r} \left[\binom{\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) + m + C}{d} \right]_+ \end{aligned}$$

where $r = r(F_{\mathcal{E}}(\mathcal{F}))$ and $C := \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}$. Here, $[x]_+ := \max\{0, x\}$ for any $x \in \mathbb{R}$, $f(r) = -1 + \sum_{i=1}^r \frac{1}{i}$ and the integer \tilde{m} is the same as in Lemma 2.39.

Proof. For a pure coherent sheaf \mathcal{F} of dimension d , we have the following Harder-Narasimhan filtration with respect to $\hat{\mu}_{\mathcal{E}}$ -stability

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \dots \subsetneq \mathcal{F}_l = \mathcal{F}$$

such that the factors $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $i = 1, \dots, l$ are $\hat{\mu}_{\mathcal{E}}$ -semistable of dimension d and

$$(1) \quad \hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_1)) > \hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_2/\mathcal{F}_1)) > \dots > \hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_l/\mathcal{F}_{l-1})).$$

For any $i = 1, \dots, l$, we have a short exact sequence

$$0 \rightarrow \mathcal{F}_{i-1} \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_i/\mathcal{F}_{i-1} \rightarrow 0.$$

Tensor with $\pi^* \mathcal{O}_X(m)$ and apply the exact functor $F_{\mathcal{E}}$, we obtain the following exact sequence

$$0 \rightarrow F_{\mathcal{E}}(\mathcal{F}_{i-1})(m) \rightarrow F_{\mathcal{E}}(\mathcal{F}_i)(m) \rightarrow F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m) \rightarrow 0.$$

Then for any $i = 1, \dots, l$, we have

$$\begin{aligned} r(F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m)) &= r(F_{\mathcal{E}}(\mathcal{F}_i)(m)) - r(F_{\mathcal{E}}(\mathcal{F}_{i-1})(m)), \\ h^0(X, F_{\mathcal{E}}(\mathcal{F}_i)(m)) - h^0(X, F_{\mathcal{E}}(\mathcal{F}_{i-1})(m)) &\leq h^0(X, F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m)). \end{aligned}$$

This implies that

$$\begin{aligned} r(F_{\mathcal{E}}(\mathcal{F})(m)) &= \sum_{i=1}^l r(F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m)), \\ h^0(X, F_{\mathcal{E}}(\mathcal{F})(m)) &\leq \sum_{i=1}^l h^0(X, F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m)). \end{aligned}$$

By a simple computation, we have for any pure coherent sheaf \mathcal{G} ,

$$\begin{aligned} r(F_{\mathcal{E}}(\mathcal{G})(m)) &= r(F_{\mathcal{E}}(\mathcal{G})); \\ \hat{\mu}_{\mathcal{E}}(\mathcal{G} \otimes \pi^* \mathcal{O}_X(m)) &= \hat{\mu}(F_{\mathcal{E}}(\mathcal{G})(m)) = \hat{\mu}(F_{\mathcal{E}}(\mathcal{G})) + m. \end{aligned}$$

Let $r_i =: r(F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1}))$ for $i = 1, \dots, l$. Since for each $1 \leq i \leq l$, the sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is $\hat{\mu}_{\mathcal{E}}$ -semistable of dimension d , $(\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes \pi^* \mathcal{O}_X(m)$ is also $\hat{\mu}_{\mathcal{E}}$ -semistable of dimension d . By Lemma 2.39, we have for each $i = 1, \dots, l$,

$$\begin{aligned} \frac{h^0(X, F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m))}{r_i} &= \frac{h^0(\mathcal{X}, (\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes \pi^* \mathcal{O}_X(m) \otimes \mathcal{E}^{\vee})}{r_i} \\ &\leq \left[\binom{\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})) + m + \tilde{m} \deg(\mathcal{O}_X(1)) + r_i^2 + f(r_i) + \frac{d-1}{2}}{d} \right]_+ \end{aligned}$$

Since $F_{\mathcal{E}}(\mathcal{F}_1) \hookrightarrow F_{\mathcal{E}}(\mathcal{F})$, we have

$$\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_1)) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_1)) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})).$$

Combining with (1), we have $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$ for $i = 1, \dots, l-1$ and $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}_l/\mathcal{F}_{l-1})) \leq \hat{\mu}(F_{\mathcal{E}}(\mathcal{F}))$. Then

$$\begin{aligned} \frac{h^0(X, F_{\mathcal{E}}(\mathcal{F})(m))}{r} &\leq \sum_{i=1}^l \frac{r_i}{r} \frac{h^0(X, F_{\mathcal{E}}(\mathcal{F}_i/\mathcal{F}_{i-1})(m))}{r_i} \\ &\leq \frac{r-1}{r} \left[\binom{\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) + m + C}{d} \right]_+ \\ &\quad + \frac{1}{r} \left[\binom{\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) + m + C}{d} \right]_+ \end{aligned}$$

where $C := \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}$. □

Now, we begin with the first case when $\deg \delta < \deg P$.

Lemma 3.5. *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Assume that $\deg \delta < \deg P$. Then there is an integer $m_0 > 0$, such that for any integer $m \geq m_0$ and any nondegenerate pair (\mathcal{F}, φ) satisfying that \mathcal{F} is a pure coherent sheaf of dimension d on \mathcal{X} with $P_{\mathcal{E}}(\mathcal{F}) = P$ and $r = r(F_{\mathcal{E}}(\mathcal{F}))$, the following properties are equivalent.*

- (i) *The pair (\mathcal{F}, φ) is δ -(semi)stable.*
- (ii) *$P(m) \leq h^0(F_{\mathcal{E}}(\mathcal{F})(m))$ and for any subpair (\mathcal{F}', φ') with $r(F_{\mathcal{E}}(\mathcal{F}')) = r'$ satisfying $0 < r' < r$,*

$$h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) + \epsilon(\varphi')\delta(m) \leq \frac{r'}{r} (P(m) + \epsilon(\varphi)\delta(m)).$$

(iii) For any quotient pair (\mathcal{G}, φ'') with $r(F_{\mathcal{E}}(\mathcal{G})) = r''$ satisfying $0 < r'' < r$,

$$\frac{r''}{r}(P(m) + \epsilon(\varphi)\delta(m)) \leq h^0(F_{\mathcal{E}}(\mathcal{G})(m)) + \epsilon(\varphi'')\delta(m).$$

Proof. We will deal with the δ -stable case, the δ -semistable case can be proved similarly.

(i) \Rightarrow (ii): By Proposition 3.3, the set-theoretic family of coherent sheaves on \mathcal{X} underlying δ -semistable pairs with fixed modified Hilbert polynomial P is bounded. By Kleiman criterion for stacks, there is an integer $m_0 > 0$ such that for any \mathcal{F} underlying a δ -stable pair (\mathcal{F}, φ) , we have $H^i(F_{\mathcal{E}}(\mathcal{F})(m)) = 0$ for all $i > 0$, and hence $P(m) = h^0(F_{\mathcal{E}}(\mathcal{F})(m))$. Since the pair (\mathcal{F}, φ) is δ -stable, by Lemma 3.1, there is a constant μ_0 depending on P , \mathcal{F}_0 and X such that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \mu_0$. As (\mathcal{F}', φ') is a subpair of (\mathcal{F}, φ) , we have $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}')) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \mu_0$. By Lemma 3.4 and the inequality $0 < r' < r$, there is a constant C depending on r and d such that

$$(2) \quad \frac{h^0(F_{\mathcal{E}}(\mathcal{F}'))(m)}{r'} \leq \frac{r-1}{r} \left[\binom{\mu_0 + m + C}{d} \right]_+ + \frac{1}{r} \left[\binom{\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) + m + C}{d} \right]_+.$$

We distinguish two cases:

- (a) $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) \geq r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r-1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2} + \delta_1)$,
- (b) $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) < r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r-1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2} + \delta_1)$,

where $\frac{\delta_1}{(d-1)!}$ is the coefficient of δ in degree $d-1$. Then $\delta_1 \geq 0$. In order to show (ii), we assume that \mathcal{F}' is saturated in \mathcal{F} since by Remark 2.12 the saturation $\overline{\mathcal{F}'} \supseteq \mathcal{F}'$ implies that $r(F_{\mathcal{E}}(\mathcal{F}')) = r(F_{\mathcal{E}}(\overline{\mathcal{F}'}))$ and $h^0(F_{\mathcal{E}}(\mathcal{F}'))(m) \leq h^0(F_{\mathcal{E}}(\overline{\mathcal{F}'})(m))$. The set-theoretic family of purely saturated subsheaves \mathcal{F}' of type (a) is bounded by Grothendieck's Lemma 2.36. By Kleiman criterion for stacks, the set of modified Hilbert polynomials are finite and enlarging m_0 if necessary, for $m \geq m_0$, we have $h^0(F_{\mathcal{E}}(\mathcal{F}'))(m) = P_{\mathcal{E}}(\mathcal{F}')(m)$ and by δ -stability of (\mathcal{F}, φ) ,

$$P_{\mathcal{E}}(\mathcal{F}') + \epsilon(\varphi')\delta < \frac{r'}{r}(P + \epsilon(\varphi)\delta) \\ \iff P_{\mathcal{E}}(\mathcal{F}')(m) + \epsilon(\varphi')\delta(m) < \frac{r'}{r}(P(m) + \epsilon(\varphi)\delta(m)).$$

For the subsheaves \mathcal{F}' of type (b), enlarging m_0 if necessary, it follows from (2)

$$\begin{aligned} \frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{r'} &\leq \frac{r-1}{r} \left[\binom{\mu_0 + m + C}{d} \right]_+ \\ &\quad + \frac{1}{r} \left[\binom{r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r-1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2} + \delta_1) + m + C}{d} \right]_+ \\ &= \frac{m^d}{d!} + \frac{m^{d-1}}{(d-1)!} \left(\frac{r-1}{r} \left(\mu_0 + C - \frac{d-1}{2} \right) + \frac{1}{r} \left(r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) \right. \right. \\ &\quad \left. \left. - (r-1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2} + \delta_1) + C - \frac{d-1}{2} \right) \right) + \dots \\ &= \frac{m^d}{d!} + \frac{m^{d-1}}{(d-1)!} (\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - \delta_1 - 1) + \dots \end{aligned}$$

where \dots denotes for some polynomial in m of degree $\leq d - 2$ with coefficients independent of \mathcal{F}' . Since

$$\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - \delta_1 - 1 + \frac{\epsilon(\varphi')}{r'} \delta_1 < \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) + \frac{\epsilon(\varphi)}{r} \delta_1,$$

by enlarging m_0 if necessary, we have

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) + \epsilon(\varphi')\delta(m)}{r'} < \frac{P(m) + \epsilon(\varphi)\delta(m)}{r}$$

for any $m \geq m_0$.

(ii) \Rightarrow (iii): For any quotient pair (\mathcal{G}, φ'') with multiplicity $r(F_{\mathcal{E}}(\mathcal{G})) = r''$ satisfying $0 < r'' < r$, we have a short exact sequence

$$0 \rightarrow (\mathcal{F}', \varphi') \rightarrow (\mathcal{F}, \varphi) \rightarrow (\mathcal{G}, \varphi'') \rightarrow 0.$$

Since the functor $F_{\mathcal{E}}$ is exact, we have $h^0(F_{\mathcal{E}}(\mathcal{G})(m)) \geq h^0(F_{\mathcal{E}}(\mathcal{F})(m)) - h^0(F_{\mathcal{E}}(\mathcal{F}')(m))$. Set $r' = r(F_{\mathcal{E}}(\mathcal{F}'))$, we have $r = r' + r''$. Also, we have $\epsilon(\varphi) = \epsilon(\varphi') + \epsilon(\varphi'')$. By (ii), we have

$$\begin{aligned} &h^0(F_{\mathcal{E}}(\mathcal{G})(m)) + \epsilon(\varphi'')\delta(m) \\ &\geq h^0(F_{\mathcal{E}}(\mathcal{F})(m)) + \epsilon(\varphi)\delta(m) - h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) - \epsilon(\varphi')\delta(m) \\ &> P(m) + \epsilon(\varphi)\delta(m) - \frac{r'}{r}(P(m) + \epsilon(\varphi)\delta(m)) \\ &= \frac{r''}{r}(P(m) + \epsilon(\varphi)\delta(m)). \end{aligned}$$

(iii) \Rightarrow (i): Fix a pair (\mathcal{F}, φ) with modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$. Let (\mathcal{G}, φ'') be any purely quotient pair of (\mathcal{F}, φ) with $0 < r(F_{\mathcal{E}}(\mathcal{G})) = r'' < r$, we distinguish all sheaves \mathcal{G} into two cases:

$$(\tilde{a}) \quad \hat{\mu}(F_{\mathcal{E}}(\mathcal{G})) > \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) + \frac{\delta_1}{r},$$

$$(\tilde{b}) \quad \hat{\mu}(F_{\mathcal{E}}(\mathcal{G})) \leq \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) + \frac{\delta_1}{r},$$

where $\frac{\delta_1}{(d-1)!}$ is the coefficient of δ in degree $d - 1$. For the case (\tilde{a}) , we have $p_{(\mathcal{F}, \varphi)} < p_{(\mathcal{G}, \varphi'')}$. For the case (\tilde{b}) , the set-theoretic family of purely d -dimensional quotient \mathcal{G} with $\hat{\mu}_{\mathcal{E}}(\mathcal{G})$ bounded from above is bounded by Grothendieck's Lemma 2.36. By Kleiman criterion for stacks, for large m we have $h^0(F_{\mathcal{E}}(\mathcal{G})(m)) = P_{\mathcal{E}}(\mathcal{G})(m)$, and by (iii) we have

$$\frac{r''}{r}(P(m) + \epsilon(\varphi)\delta(m)) < P_{\mathcal{E}}(\mathcal{G})(m) + \epsilon(\varphi'')\delta(m) \Leftrightarrow p_{(\mathcal{F}, \varphi)} < p_{(\mathcal{G}, \varphi'')}$$

Then the pair (\mathcal{F}, φ) is δ -stable by Remark 2.23. Thus all pairs (\mathcal{F}, φ) satisfying (iii) for large m are δ -stable pairs with fixed modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$, and hence the set-theoretic family of sheaves \mathcal{F} underlying pairs (\mathcal{F}, φ) satisfying (iii) for large m is bounded by Proposition 3.3. \square

The remaining case when $\deg \delta \geq \deg P$ is presented in the following

Lemma 3.6. *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Assume that $\deg \delta \geq \deg P$. Then there is an integer $m_0 > 0$, such that for any integer $m \geq m_0$ and any nondegenerate pair (\mathcal{F}, φ) satisfying that \mathcal{F} is a pure coherent sheaf of dimension d on \mathcal{X} with $P_{\mathcal{E}}(\mathcal{F}) = P$ and $r = r(F_{\mathcal{E}}(\mathcal{F}))$, the following properties are equivalent.*

(i) *The pair (\mathcal{F}, φ) is δ -stable.*

(ii) *$P(m) \leq h^0(F_{\mathcal{E}}(\mathcal{F})(m))$ and for any subpair (\mathcal{F}', φ') with $r(F_{\mathcal{E}}(\mathcal{F}')) = r'$ satisfying $0 < r' < r$,*

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{2r' - \epsilon(\varphi')} < \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m))}{2r - \epsilon(\varphi)}.$$

(iii) *For any quotient pair (\mathcal{G}, φ'') with $r(F_{\mathcal{E}}(\mathcal{G})) = r''$ satisfying $0 < r'' < r$,*

$$\frac{P(m)}{2r - \epsilon(\varphi)} < \frac{h^0(F_{\mathcal{E}}(\mathcal{G})(m))}{2r'' - \epsilon(\varphi'')}.$$

Proof. (i) \Rightarrow (ii): Using the same argument in the proof of Lemma 3.5 with the same notation μ_0 and C , we have $P(m) = h^0(F_{\mathcal{E}}(\mathcal{F})(m))$ for $m \geq m_0$ and

$\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}')) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \mu_0$ by Lemma 3.2. Then we also have the inequality (2). And all sheaves \mathcal{F}' underlying subpairs (\mathcal{F}', φ') are divided into the following two cases:

- (a) $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) \geq r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r - 1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2}),$
- (b) $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) < r \cdot \hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r - 1) \cdot \mu_0 - r \cdot (C - \frac{d-3}{2}).$

For type (a), one uses Grothendieck’s Lemma 2.36, Kleiman criterion for stacks and Lemma 2.25 to derive the desired inequality. For type (b), it is easy to show $\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{r'} < \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m))}{r}$ and then use the inequality $\frac{\epsilon(\varphi')}{r'} \leq \frac{\epsilon(\varphi)}{r}$ (see [36, Lemma 2.9 (2-1)]) to complete this part of proof.

(ii) \Rightarrow (iii): Follow the similar arguemnt in [36, Lemma 3.5].

(iii) \Rightarrow (i): Fix a nondegenerate pair (\mathcal{F}, φ) . As in [36, Lemma 3.5], let $\text{gr}_l := (\text{gr}_l \mathcal{F}, \text{gr}_l \varphi)$ be the last factor of Harder-Narasimhan filtration for (\mathcal{F}, φ) by Proposition 2.29, by (iii) we have

$$\frac{h^0(F_{\mathcal{E}}(\text{gr}_l \mathcal{F})(m))}{2r(F_{\mathcal{E}}(\text{gr}_l \mathcal{F})) - \epsilon(\text{gr}_l \varphi)} > \frac{P(m)}{2r - 1}.$$

For large m , we have $h^0(F_{\mathcal{E}}(\text{gr}_l \mathcal{F})(m)) = P_{\mathcal{E}}(\text{gr}_l \mathcal{F})(m)$ and hence

$$\frac{P_{\mathcal{E}}(\text{gr}_l \mathcal{F})(m)}{2r(F_{\mathcal{E}}(\text{gr}_l \mathcal{F})) - \epsilon(\text{gr}_l \varphi)} > \frac{P(m)}{2r - 1} \iff \frac{P_{\mathcal{E}}(\text{gr}_l \mathcal{F})}{2r(F_{\mathcal{E}}(\text{gr}_l \mathcal{F})) - \epsilon(\text{gr}_l \varphi)} > \frac{P}{2r - 1}.$$

Since both polynomials $\frac{P_{\mathcal{E}}(\text{gr}_l \mathcal{F})}{r(F_{\mathcal{E}}(\text{gr}_l \mathcal{F}))}$ and $\frac{P}{r}$ have the same leading coefficient, we have $\frac{\epsilon(\text{gr}_l \varphi)}{r(F_{\mathcal{E}}(\text{gr}_l \mathcal{F}))} \geq \frac{1}{r}$. Then $\epsilon(\text{gr}_l \varphi) = 1$, which implies $l = 1$ by Remark 2.30. Thus (\mathcal{F}, φ) is a nondegenerate δ -semistable pair, and hence it is δ -stable by Lemma 2.24. Thus all pairs (\mathcal{F}, φ) satisfying (iii) for large m are δ -stable pairs with fixed modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$, and hence the set-theoretic family of sheaves \mathcal{F} underlying pairs (\mathcal{F}, φ) satisfying (iii) for large m is bounded by Proposition 3.3. □

4. Construction of moduli spaces of semistable pairs

In this section, given a polynomial $P \in \mathbb{Q}[m]$ and a stability parameter $\delta \in \mathbb{Q}[m]$ which is zero or a polynomial with positive leading coefficient, we will construct a moduli space of δ -(semi)stable pairs of type P on a projective Deligne-Mumford stack \mathcal{X} over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. We will only consider the case of nondegenerate δ -(semi)stable pairs since otherwise these are the moduli spaces of

(semi)stable sheaves which have been constructed in [43, Section 6]. Actually, we generalize the construction of moduli spaces on smooth projective varieties in [36, 67] to the case of projective Deligne-Mumford stacks. We also give a description of variation of moduli spaces when the stability parameter δ changes.

4.1. The parameter space, group actions and linearizations

We first recall some notation and results in [24, 43, 47]. If $p : \mathcal{X} \rightarrow S$ is a family of projective stacks with a moduli scheme $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{\hat{p}} S$ and a relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let \mathcal{H} be a coherent sheaf on \mathcal{X} . Denote by $\underline{\text{Quot}}_{\mathcal{X}/S}(\mathcal{H}, P)$ the functor of quotients of \mathcal{H} with modified Hilbert polynomial P . It is shown in [43, Proposition 4.20] or [47, Proposition 6.2], the natural transformation $F_{\mathcal{E}}$ which maps $\underline{\text{Quot}}_{\mathcal{X}/S}(\mathcal{H}, P)$ to the ordinary Quot functor $\underline{\text{Quot}}_{\mathcal{X}/S}(F_{\mathcal{E}}(\mathcal{H}), P)$ is relatively representable by schemes and is a closed immersion. And this implies that $\widehat{\mathcal{Q}} := \underline{\text{Quot}}_{\mathcal{X}/S}(\mathcal{H}, P)$ which represents $\underline{\text{Quot}}_{\mathcal{X}/S}(\mathcal{H}, P)$ is a closed immersion of the ordinary Quot scheme $\mathcal{Q} := \underline{\text{Quot}}_{\mathcal{X}/S}(F_{\mathcal{E}}(\mathcal{H}), P)$ which represents $\underline{\text{Quot}}_{\mathcal{X}/S}(F_{\mathcal{E}}(\mathcal{H}), P)$, and hence $\widehat{\mathcal{Q}}$ is a projective scheme. Let $i : \widehat{\mathcal{Q}} \rightarrow \mathcal{Q}$ be the closed immersion. Let $\widehat{\mathcal{U}}$ be the universal quotient sheaf of $\widehat{\mathcal{Q}}$, that is, the morphism $\mathcal{O}_{\widehat{\mathcal{Q}}} \otimes \mathcal{H} \rightarrow \widehat{\mathcal{U}}$ is the universal quotient parameterized by $\widehat{\mathcal{Q}}$. And let \mathcal{U} be the universal quotient sheaf of \mathcal{Q} . As in the proof of [24, Proposition 2.2.5], for sufficiently large l , we have a closed immersion

$$\text{Quot}_{\mathcal{X}/S}(F_{\mathcal{E}}(\mathcal{H}), P) \xrightarrow{\xi_l} \text{Grass}_S(\hat{p}_*F_{\mathcal{E}}(\mathcal{H})(l), P(l))$$

and a Plücker embedding of the Grassmannian

$$\text{Grass}_S(\hat{p}_*F_{\mathcal{E}}(\mathcal{H})(l), P(l)) \xrightarrow{\xi_l} \widetilde{\mathbb{P}} := \mathbb{P}(\Lambda^{P(l)}(\hat{p}_*F_{\mathcal{E}}(\mathcal{H})(l))).$$

Then we have a class of very ample line bundles on \mathcal{Q} :

$$\zeta_l^* \xi_l^* \mathcal{O}_{\widetilde{\mathbb{P}}}(1) = \det(\hat{p}_{\mathcal{Q}*}(\mathcal{U}(l))).$$

where $\hat{p}_{\mathcal{Q}} : X_{\mathcal{Q}} := X \times_S \mathcal{Q} \rightarrow \mathcal{Q}$ is the natural projection. Then by [43, Proposition 6.2], a class of very ample line bundles on $\widehat{\mathcal{Q}}$ is constructed as follows

$$\mathcal{L}_l := \det(\hat{p}_{\widehat{\mathcal{Q}}*}(F_{\hat{p}_{\mathcal{X}}^*}(\widehat{\mathcal{U}})(l))) = i^* \zeta_l^* \xi_l^* \mathcal{O}_{\widetilde{\mathbb{P}}}(1)$$

where $\hat{p}_{\widehat{\mathcal{Q}}} : X_{\widehat{\mathcal{Q}}} = X \times_S \widehat{\mathcal{Q}} \rightarrow \widehat{\mathcal{Q}}$ and $\hat{p}_{\mathcal{X}} : \mathcal{X} \times_S \widehat{\mathcal{Q}} \rightarrow \mathcal{X}$ are the natural projections. In this section, we will consider the case when $S = \text{Spec } k$ and $\mathcal{H} := V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)$.

Using the boundedness results in Section 3, there exists an integer $\widehat{m} > 0$ such that for every integer $m \geq \widehat{m}$, we have

- (a) $F_{\mathcal{E}}(\mathcal{F}_0)$ is m -regular (by Serre vanishing theorem) and $\delta(m) > 0$ unless $\delta = 0$.
- (b) For any δ -semistable pair (\mathcal{F}, φ) of type P , the coherent sheaf \mathcal{F} is m -regular (by Proposition 3.3).
- (c) Equivalent properties in both Lemma 3.5 and Lemma 3.6 hold.

Now, we fix such an integer m , then for any δ -semistable pair (\mathcal{F}, φ) of type P , the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is globally generated and $h^0(F_{\mathcal{E}}(\mathcal{F})(m)) = P(m)$. Let $V = k^{\oplus P(m)}$. Then we have an isomorphism $\rho : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$, which induces the following quotient $\tilde{\rho} = \text{ev} \circ (\rho \otimes \text{id})$:

$$V \otimes \mathcal{O}_X(-m) \xrightarrow{\rho \otimes \text{id}} H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \otimes \mathcal{O}_X(-m) \xrightarrow{\text{ev}} F_{\mathcal{E}}(\mathcal{F}).$$

Since $G_{\mathcal{E}}$ is a right exact functor and the morphism $\theta_{\mathcal{E}}(\mathcal{F})$ is surjective, we have the following quotient $q := \theta_{\mathcal{E}}(\mathcal{F}) \circ G_{\mathcal{E}}(\tilde{\rho})$:

$$V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) = G_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m)) \xrightarrow{G_{\mathcal{E}}(\tilde{\rho})} G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\theta_{\mathcal{E}}(\mathcal{F})} \mathcal{F}.$$

This defines a point $[q]$ in $\text{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$. On the other hand, given a point in $\text{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$ which is represented by a quotient $V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \xrightarrow{\sigma} \mathcal{F}$, we have the composition $\tilde{\sigma} := F_{\mathcal{E}}(\sigma) \circ \varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m))$ as follows:

$$V \otimes \mathcal{O}_X(-m) \xrightarrow{\varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m))} F_{\mathcal{E}} \circ G_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m)) \xrightarrow{F_{\mathcal{E}}(\sigma)} F_{\mathcal{E}}(\mathcal{F})$$

which induces a morphism in cohomology $H^0(\tilde{\sigma}(m)) : V \rightarrow H^0(X, F_{\mathcal{E}}(\mathcal{F})(m))$. Define \mathbf{Q} to be the set of points $[\sigma]$ of $\text{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$ such that the induced morphism $H^0(\tilde{\sigma}(m))$ is an isomorphism where $\tilde{\sigma} := F_{\mathcal{E}}(\sigma) \circ \varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m))$. The proof in [43, Theorem 5.1] shows \mathbf{Q} is an open subscheme of the projective scheme $\text{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$.

Given a δ -semistable pair (\mathcal{F}, φ) , consider the morphism $\varphi : \mathcal{F}_0 \rightarrow \mathcal{F}$, by applying the functor $F_{\mathcal{E}}$, tensoring with $\mathcal{O}_X(m)$, and then applying the

global section functor, we get

$$H^0(F_{\mathcal{E}}(\varphi)(m)) : H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$$

Composing with $\rho^{-1} : H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \rightarrow V$, we have

$$a := \rho^{-1} \circ H^0(F_{\mathcal{E}}(\varphi)(m)) : H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \rightarrow V$$

Then we have the commutative diagram

$$\begin{array}{ccc} H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_X(-m) & \xrightarrow{\text{ev}} & F_{\mathcal{E}}(\mathcal{F}_0) \\ a \downarrow & & \downarrow F_{\mathcal{E}}(\varphi) \\ V \otimes \mathcal{O}_X(-m) & \xrightarrow{\tilde{\rho}} & F_{\mathcal{E}}(\mathcal{F}) \end{array}$$

where ev is the evaluation map, a on the left column denotes $a \otimes \text{id}$ and $\tilde{\rho} := \text{ev} \circ (\rho \otimes \text{id})$. By applying the functor $G_{\mathcal{E}}$ and then using the natural transformation $\theta_{\mathcal{E}}$, we have the commutative diagram

$$\begin{array}{ccc} H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) & \xrightarrow{\tilde{\text{ev}}} & \mathcal{F}_0 \\ a \downarrow & & \downarrow \varphi \\ V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) & \xrightarrow{q} & \mathcal{F} \end{array}$$

where a again denotes $a \otimes \text{id}$, $q := \theta_{\mathcal{E}}(\mathcal{F}) \circ G_{\mathcal{E}}(\tilde{\rho})$ and $\tilde{\text{ev}} := \theta_{\mathcal{E}}(\mathcal{F}_0) \circ G_{\mathcal{E}}(\text{ev})$. Notice that $a = 0$ if and only if $\varphi = 0$. Then $a \neq 0$ since the pair (\mathcal{F}, φ) is nondegenerate.

Define

$$\begin{aligned} \tilde{\mathcal{Q}} &:= \text{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P), \\ \mathbb{P} &:= \mathbb{P}(\text{Hom}(H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)), V)). \end{aligned}$$

Since the morphism $\tilde{\text{ev}}$ is surjective, assume that \mathcal{K}_0 is its kernel, then we have a short exact sequence:

$$0 \rightarrow \mathcal{K}_0 \xrightarrow{\iota} H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \xrightarrow{\tilde{\text{ev}}} \mathcal{F}_0 \rightarrow 0.$$

As in [67, Proposition 3.4], the composition $q \circ a$ factor through $\tilde{\text{ev}}$ if and only if $q \circ a \circ \iota = 0$, and there is a closed subscheme $\mathcal{Z} \subseteq \mathbb{P} \times \tilde{\mathcal{Q}}$ such that $([a], [q]) \in \mathcal{Z}$ if and only if $q \circ a \circ \iota = 0$. In particular, a point $([a], [q]) \in \mathcal{Z}$ corresponds to a pair (\mathcal{F}, φ) induced by $q \circ a$ factoring through $\tilde{\text{ev}}$.

Remark 4.1. *As in [12, Section 4.1], one has a flat family of pairs parameterized by $\mathcal{Z} \subseteq \mathbb{P} \times \tilde{\mathcal{Q}}$ as follows. From now on, we make the convention that the notation $p_{\bullet, \star}$ denotes the natural projection from \bullet to \star . Let $\tilde{i} : \mathcal{Z} \rightarrow \mathbb{P} \times \tilde{\mathcal{Q}}$ be the inclusion. Let $\tilde{q} : p_{\mathcal{X} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \rightarrow \tilde{\mathcal{F}}$ be the universal quotient family parameterized by $\tilde{\mathcal{Q}}$. Then there is a quotient morphism*

$$p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* : p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \rightarrow p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}.$$

The universal quotient family parameterized by \mathbb{P} is

$$\mathrm{Hom}(V \otimes \mathcal{O}_{\mathbb{P}}, H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_{\mathbb{P}}) \rightarrow \mathcal{O}_{\mathbb{P}}(1),$$

or

$$H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{\tilde{a}} V \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Then we have

$$\begin{aligned} & p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \otimes p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}} \\ & \xrightarrow{\mathrm{id}_{p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(\mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))} \otimes p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \tilde{a}} p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \otimes p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

and

$$p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^{\tilde{e}\mathrm{v}} : p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^*(H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \rightarrow p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^* \mathcal{F}_0.$$

Combined with the definition of \mathcal{Z} , we have a morphism

$$\begin{aligned} p_{\mathcal{X} \times \mathcal{Z}, \mathcal{X}}^* \mathcal{F}_0 &= (\mathrm{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^* \mathcal{F}_0 \\ &\rightarrow (\mathrm{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \otimes (\mathrm{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

which is obviously a flat family of pairs parameterized by \mathcal{Z} in the sense of Definition 2.40. In the sense of [12, Definition 3.16] (see also Remark 2.41), the flat family of pairs is given as

$$\left((\mathrm{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}, \tilde{i}^* p_{\mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(-1), \phi_{\tilde{\mathcal{F}}} \right)$$

where

$$\phi_{\tilde{\mathcal{F}}} : \tilde{i}^* p_{\mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(-1) \rightarrow p_{\mathcal{X} \times \mathcal{Z}, \mathcal{Z}}^* \mathrm{Hom} \left(p_{\mathcal{X} \times \mathcal{Z}, \mathcal{X}}^* \mathcal{F}_0, (\mathrm{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \right).$$

The subset of \mathcal{Z} in which every point $([a], [q])$ satisfying $[q] \in \mathbf{Q}$ and $\mathcal{F} = q(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))$ is pure forms an open subset. In fact, the first condition is an open condition mentioned above. With the notation in Remark 4.1, consider the universal quotient family parameterized by \mathcal{Z} :

$$p_{\mathcal{X} \times \mathcal{Z}, \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \rightarrow (\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}$$

The second condition is equivalent to the requirement that $(\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}$ is pure, which is an open condition by using [24, Proposition 2.3.1] for the sheaf $F_{p_{\mathcal{X} \times \mathcal{Z}, \mathcal{X}}^*} \left((\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \right)$ being of pure dimension (if and only if $(\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}$ is pure by Lemma 2.13), together with Grothendieck’s Lemma 2.36 and [43, Proposition 4.20]. Let \mathcal{Z}' be the closure of this open subset in \mathcal{Z} . We call \mathcal{Z}' the parameter space. Obviously, \mathcal{Z}' is a projective scheme.

Next, we will consider a $\text{GL}(V)$ -action on $\mathcal{Z}' \subseteq \mathbb{P} \times \tilde{\mathcal{Q}}$, which is induced by $\text{GL}(V)$ -actions on \mathbb{P} and $\tilde{\mathcal{Q}}$. We recall these actions in [24, Section 4.3] and [12, Section 4.1] as follows. Let $\tau : V \otimes \mathcal{O}_{\text{GL}(V)} \rightarrow V \otimes \mathcal{O}_{\text{GL}(V)}$ be the universal automorphism of V parameterized by $\text{GL}(V)$. By the universal property of $\tilde{\mathcal{Q}}$, we have a classifying morphism $\eta_1 : \tilde{\mathcal{Q}} \times \text{GL}(V) \rightarrow \tilde{\mathcal{Q}}$ which is the right action by composition $[q] \cdot g := [q \circ g]$, such that the following diagram commute

$$\begin{array}{ccc} p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) & \xrightarrow{(\text{id}_{\mathcal{X}} \times \eta_1)^* \tilde{q}} & (\text{id}_{\mathcal{X}} \times \eta_1)^* \tilde{\mathcal{F}} \\ p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \text{GL}(V)}^* \tau \downarrow & & \downarrow \Lambda_1 \\ p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X}}^*(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) & \xrightarrow{p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{q}} & p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \end{array}$$

where \tilde{q} is defined in Remark 4.1 and $\Lambda_1 : (\text{id}_{\mathcal{X}} \times \eta_1)^* \tilde{\mathcal{F}} \rightarrow p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}$ is an isomorphism. In Remark 4.1, the universal quotient family parameterized by \mathbb{P} is given as

$$H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{\tilde{a}} V \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Then we have the following surjective composition

$$\begin{aligned} H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_{\mathbb{P} \times \text{GL}(V)} & \xrightarrow{p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}}^* \tilde{a}} V \otimes p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \\ & \xrightarrow{p_{\mathbb{P} \times \text{GL}(V), \text{GL}(V)}^* \tau} V \otimes p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

which induces a classifying morphism $\eta_2 : \mathbb{P} \times \mathrm{GL}(V) \rightarrow \mathbb{P}$, where η_2 is the action by composition $[a] \cdot g = [g \circ a]$. Then we have an isomorphism $\Lambda_2 : \eta_2^* \mathcal{O}_{\mathbb{P}}(1) \rightarrow p_{\mathbb{P} \times \mathrm{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1)$ such that we have the following commutative diagram

$$\begin{array}{ccc}
 H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \otimes \mathcal{O}_{\mathbb{P} \times \mathrm{GL}(V)} & \xrightarrow{\eta_2^* \tilde{a}} & V \otimes \eta_2^* \mathcal{O}_{\mathbb{P}}(1) \\
 \downarrow p_{\mathbb{P} \times \mathrm{GL}(V), \mathbb{P}}^* \tilde{a} & & \downarrow p_{\mathbb{P} \times \mathrm{GL}(V), \mathrm{GL}(V)}^* \tau \otimes \Lambda_2 \\
 V \otimes p_{\mathbb{P} \times \mathrm{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) & \xrightarrow{p_{\mathbb{P} \times \mathrm{GL}(V), \mathrm{GL}(V)}^* \tau} & V \otimes p_{\mathbb{P} \times \mathrm{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1).
 \end{array}$$

Using the stacky version of the G -linearization of a coherent sheaf in [55, Example 4.3], it can be verified that Λ_1 is a $\mathrm{GL}(V)$ -linearization for $\tilde{\mathcal{F}}$ and Λ_2 is a $\mathrm{GL}(V)$ -linearization for $\mathcal{O}_{\mathbb{P}}(1)$.

Two morphisms η_1 and η_2 induce a $\mathrm{GL}(V)$ -action $\eta : \mathbb{P} \times \tilde{\mathcal{Q}} \times \mathrm{GL}(V) \rightarrow \mathbb{P} \times \tilde{\mathcal{Q}}$ and hence induces a natural $\mathrm{SL}(V)$ -action on $\mathbb{P} \times \tilde{\mathcal{Q}}$ pointwise as follows

$$([a], [q]) \cdot g = ([g^{-1} \circ a], [q \circ g])$$

where $g \in \mathrm{SL}(V)$ and $([a], [q]) \in \mathbb{P} \times \tilde{\mathcal{Q}}$. This is a right $\mathrm{SL}(V)$ -action. Obviously, \mathcal{Z} and \mathcal{Z}' are invariant under this action. For l sufficiently large, it is shown in [43, Lemma 6.3], the class of very ample invertible sheaves $\mathcal{L}_l := \det \left(p_{X \times \tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}^*} \left(F_{p_{X \times \tilde{\mathcal{Q}}, X}^*} \varepsilon(\tilde{\mathcal{F}})(l) \right) \right)$ carries a natural $\mathrm{GL}(V)$ -linearization which induces a $\mathrm{SL}(V)$ -linearization. Notice that Λ_2 induces a $\mathrm{SL}(V)$ -linearization for $\mathcal{O}_{\mathbb{P}}(1)$. Then for any two positive integers n_1 and n_2 , we have the very ample line bundles

$$\mathcal{O}_{\mathcal{Z}'}(n_1, n_2) := \mathcal{O}_{\mathbb{P}}(n_1) \boxtimes \mathcal{L}_l^{\otimes n_2} := p_{\mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(n_1) \otimes p_{\mathbb{P} \times \tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}}^* \mathcal{L}_l^{n_2}$$

which again carry natural $\mathrm{SL}(V)$ -linearizations.

4.2. GIT-stability and δ -stability

We have obtained $\mathrm{SL}(V)$ -linearized very ample line bundles $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ for l sufficiently large and any two integers n_1, n_2 , the next step is to consider the GIT (semi)stable points of \mathcal{Z}' and compare with δ -stability condition for the corresponding pairs. In this subsection, we first generalize the results which relate GIT-stability condition with δ -stability condition in [67, Section 4] when $\deg \delta < \deg P$ and then those in [36, section 4] when $\deg \delta \geq \deg P$ to the case of projective Deligne-Mumford stacks.

We recall the relation between linear subspaces of V and subsheaves of \mathcal{F} in [43, Remark 6.14], which is useful for the argument below. Given a quotient $q : V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \rightarrow \mathcal{F}$ and any linear subspace $V' \subseteq V$, we have the induced subsheaf $\mathcal{F}' := q(V' \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \subseteq \mathcal{F}$. On the other hand, given a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, we have a injective morphism $F_{\mathcal{E}}(\mathcal{F}')(m) \hookrightarrow F_{\mathcal{E}}(\mathcal{F})(m)$ and hence an inclusion $H^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \hookrightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$. Since we have the following composition $\tilde{q} := F_{\mathcal{E}}(q)(m) \circ \varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X) = (F_{\mathcal{E}}(q) \circ \varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X(-m)))(m)$:

$$V \otimes \mathcal{O}_X \xrightarrow{\varphi_{\mathcal{E}}(V \otimes \mathcal{O}_X)} F_{\mathcal{E}} \circ G_{\mathcal{E}}(V \otimes \mathcal{O}_X) \xrightarrow{F_{\mathcal{E}}(q)(m)} F_{\mathcal{E}}(\mathcal{F})(m),$$

then we have the following cartesian diagram:

$$\begin{CD} V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m)) @>>> H^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \\ @VVV @VVV \\ V @>H^0(\tilde{q})>> H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \end{CD}$$

where we use the notation $V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$ for $H^0(\tilde{q})^{-1}(H^0(F_{\mathcal{E}}(\mathcal{F}')(m)))$ as in [24, Lemma 4.4.6]. We call $V' := V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$ the linear space induced by \mathcal{F}' . If \mathcal{F}' is induced by $V' \subseteq V$, then $V' \subseteq V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$. And if $\mathcal{F}' \subseteq \mathcal{F}$ is an arbitrary subsheaf and $V' = V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$, then the subsheaf induced by V' is contained in \mathcal{F}' .

Since the parameter space \mathcal{Z}' is a closed set, we have to consider the more general class of sheaves in the following

Lemma 4.2. *If a pair (\mathcal{F}, φ) on \mathcal{X} of dimension d can be deformed to a pure pair, then there is a pure sheaf \mathcal{H} with a morphism $\psi : \mathcal{F} \rightarrow \mathcal{H}$ satisfying $\ker \psi = T_{d-1}(\mathcal{F})$ and $P_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{H})$.*

Proof. Since (\mathcal{F}, φ) can be deformed to a pure pair, there is a smooth connected curve C and a flat family $(\mathcal{F}_C, \varphi_C)$ on $\mathcal{X}_C := \mathcal{X} \times C$ such that $(\mathcal{F}_0, \varphi_0) \cong (\mathcal{F}, \varphi)$ for some closed point $0 \in C$ and $(\mathcal{F}_t, \varphi_t)$ is pure for any point $t \neq 0$. This implies that \mathcal{F} deforms into a pure sheaf. The proof is completed by [43, Lemma 6.10]. □

Now, we consider the case when $\deg \delta < \deg P$. In this case, as in [67, section 3], two positive integers n_1, n_2 are chosen to satisfy

$$(3) \quad \frac{n_1}{n_2} = \frac{P(l) \cdot \delta(m) - \delta(l) \cdot P(m)}{P(m) + \delta(m)}.$$

Proposition 4.3. *Assume the equality (3) holds. Let $([a], [q]) \in \mathcal{Z}'$ be a point with the corresponding pair (\mathcal{F}, φ) . For l sufficiently large, $([a], [q])$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if for any nonzero proper subsheaf \mathcal{F}' and the induced subspace $V' := V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$, we have an inequality of polynomials in l :*

$$P \cdot (\dim V' + \epsilon(\varphi')\delta(m)) + \delta \cdot (\dim V' - \epsilon(\varphi')P(m)) \leq P_{\mathcal{E}}(\mathcal{F}') \cdot (P(m) + \delta(m)).$$

where (\mathcal{F}', φ') is the induced subpair of (\mathcal{F}, φ) .

Proof. Let $([a], [q]) \in \mathcal{Z}'$ be represented by morphisms $a : H^0(F_{\mathcal{E}}(\mathcal{F}_0)(m)) \rightarrow V$ and $q : V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \rightarrow \mathcal{F}$. Let $\varphi : \mathcal{F}_0 \rightarrow \mathcal{F}$ be the corresponding pair. For any $[q] \in \tilde{\mathcal{Q}}$, we have a point $[F_{\mathcal{E}}(q)]$ in $\text{Quot}_{X/k}(V \otimes \pi_* \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(-m), P)$ and hence the set-theoretic family $\{F_{\mathcal{E}}(\mathcal{F}) \mid [q] \in \tilde{\mathcal{Q}}\}$ with fixed Hilbert polynomial P is bounded. Then we can choose an integer $l (\geq m)$ sufficiently large such that $F_{\mathcal{E}}(\mathcal{F})$ and $\pi_* \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(-m)$ are both l -regular for any $q \in \tilde{\mathcal{Q}}$. Using the same argument in the proof of [67, Proposition 4.1], the Hilbert-Mumford criterion shows the following:

$([a], [q])$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if for any nontrivial subspace $U \subseteq V$ we have

$$(4) \quad \dim U \cdot (n_2 P(l) - n_1) \leq P(m) \cdot (\dim(q'(U \otimes W))n_2 - \epsilon(U)n_1).$$

where $W := H^0(\pi_* \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(l - m))$, $q' := H^0(F_{\mathcal{E}}(q)(l)) : V \otimes W \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(l))$ and $\epsilon(U) = 1$ if $\text{ima} \subseteq U$ and 0 otherwise.

It is shown in the proof of [43, Lemma 6.15], for a fixed q , the family of subsheaves $\mathcal{F}_U \subseteq \mathcal{F}$ induced by a linear subspace $U \subseteq V$ is bounded since exact sequences of linear spaces split which implies that any subsheaf \mathcal{F}_U has the same regularity as \mathcal{F} . By Kleiman criterion for stacks, for l sufficiently large, all the subsheaves \mathcal{F}_U are l -regular, that is, all $F_{\mathcal{E}}(\mathcal{F}_U)$ are l -regular, and then we have $P_{\mathcal{E}}(\mathcal{F}_U)(l) = h^0(F_{\mathcal{E}}(\mathcal{F}_U)(l)) = \dim(q'(U \otimes W))$ for any \mathcal{F}_U . Using the similar argument in the proofs of [67, Propositions 4.3 and 4.4] and the equality (3), the criterion for GIT-(semi)stability above can be restated as:

$([a], [q])$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if for any nontrivial proper subspace $U \subseteq V$ and the induced sheaf $\mathcal{F}_U := q(U \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))$, we have an inequality of polynomials in l :

$$P \cdot (\dim U + \epsilon(\varphi|_U)\delta(m)) + \delta \cdot (\dim U - \epsilon(\varphi|_U)P(m)) \leq P_{\mathcal{E}}(\mathcal{F}_U) \cdot (P(m) + \delta(m)).$$

where $\varphi|_U = \varphi$ if $\text{im} \varphi \subseteq \mathcal{F}_U$ and $\varphi|_U = 0$ otherwise.

The proof is completed by using the similar argument in the proof of [67, Theorem 4.5] to drop the restriction to subsheaves \mathcal{F}_U induced by subspaces $U \subseteq V$ in the last statement of the criterion but instead use any subsheave \mathcal{F}' with the induced subspace $V' := V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$. \square

Theorem 4.4. *Assume $\deg \delta < \deg P$ and the equality (3) holds. For l sufficiently large, a point $([a], [q]) \in \mathcal{Z}'$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if the corresponding pair (\mathcal{F}, φ) is δ -(semi)stable and the map $H^0(\tilde{q}) : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ induced by q is an isomorphism.*

Proof. Let $([a], [q]) \in \mathcal{Z}'$ be a GIT-semistable point. Let $U \subseteq V$ be the kernel of $H^0(\tilde{q}) : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$, then it is shown in the proof of [43, Lemma 6.16] that

$$\mathcal{F}_U := q(U \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) = 0.$$

In fact, if $\mathcal{F}_U \neq 0$, we have $F_{\mathcal{E}}(\mathcal{F}_U) \neq 0$. Then $U \subseteq V \cap H^0(F_{\mathcal{E}}(\mathcal{F}_U)(m))$ and the cartesian diagram

$$\begin{array}{ccc} V \cap H^0(F_{\mathcal{E}}(\mathcal{F}_U)(m)) & \longrightarrow & H^0(F_{\mathcal{E}}(\mathcal{F}_U)(m)) \\ \downarrow & & \downarrow \\ V & \xrightarrow{H^0(\tilde{q})} & H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \end{array}$$

shows that the map $U \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F}_U)(m))$ is zero. Since $\mathcal{F}_U \neq 0$, we have a nonzero morphism $U \otimes \mathcal{O}_X \rightarrow \mathcal{F}_U \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(m)$ and hence $U \otimes \mathcal{O}_X \rightarrow F_{\mathcal{E}}(\mathcal{F}_U)(m)$ is nonzero by applying the exact functor π_* . This produces a contradiction. Thus, for l sufficiently large, $q'(U \otimes W) = 0$. By the inequality (4) and $P(l) > \frac{n_1}{n_2}$, one has $U = 0$. Then $H^0(\tilde{q}) : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ is injective.

The proof is completed by following the similar argument in the proofs of [67, Theorems 4.5 and 4.7] together with the corresponding Lemma 3.5, Lemma 4.2 and Proposition 4.3. \square

Next is the case when $\deg \delta \geq \deg P$ where as in [36, Section 4] we set

$$(5) \quad \frac{n_1}{n_2} = \frac{P(l)}{2r}.$$

Applying the proof of Proposition 4.7 with the equality (5), we have

Proposition 4.5. *Assume the equality (5) holds. Let $([a], [q]) \in \mathcal{Z}'$ be a point with the corresponding pair (\mathcal{F}, φ) . For l sufficiently large, $([a], [q])$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if for any nonzero proper subsheaf \mathcal{F}' and the induced subspace $V' := V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$, we have an inequality of polynomials in l :*

$$P \cdot ((2r - 1) \dim V' + P(m)\epsilon(\varphi'))(\leq) P_{\mathcal{E}}(\mathcal{F}') \cdot (2r \cdot P(m))$$

where (\mathcal{F}', φ') is the induced subpair of (\mathcal{F}, φ) .

Theorem 4.6. *Assume $\deg \delta \geq \deg P$ and the equality (5) holds. For l sufficiently large, a point $([a], [q]) \in \mathcal{Z}'$ is GIT-(semi)stable with respect to $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ if and only if the corresponding pair (\mathcal{F}, φ) is δ -(semi)stable and the map $H^0(\tilde{q}) : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ induced by q is an isomorphism.*

Proof. The proof follows from the similar argument in Theorem 4.4, but together with Lemma 3.6 and Proposition 4.5. □

Remark 4.7. *By Lemma 2.24, the statement in Theorem 4.6 actually prove that a GIT-stable point corresponds to a δ -stable pair.*

4.3. Moduli spaces of δ -(semi)stable pairs

In this subsection, we introduce a moduli functor of δ -(semi)stable pairs, and then prove that there is a moduli space for this functor. These results generalize the ones in [67, Theorem 3.8] and [36, Theorem 1.1] to the case of projective Deligne-Mumford stacks. We will adopt the techniques and ideas in [12, Section 4], most of which can be traced back to the theory for the case of projective schemes in [24, Chapter 4] (see also [22, 23]). Different from the definition of a flat family of pairs [12, Definition 3.16], we take Definition 2.40 for our families. With this notion of families, as in [24, Section 4.1], we give

Definition 4.8. *Define a functor*

$$\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta) : (\text{Sch}/k)^\circ \rightarrow (\text{Sets})$$

as follows. If S is a k -scheme of finite type, let $\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)(S)$ be the set of isomorphism classes of flat families of δ -semistable pairs (\mathcal{F}, φ) with modified Hilbert polynomial P parametrized by S , that is, such a flat family

(\mathcal{F}, φ) satisfies that for each point $s \in S$, the pair $(\mathcal{F}_s, \varphi|_{(\pi_{\mathcal{X}}^* \mathcal{F}_0)_s})$ is a δ -semistable pair with modified Hilbert polynomial $P_{\tilde{\pi}_s^* \mathcal{E}}(\mathcal{F}_s) = P$ where $\pi_{\mathcal{X}} : \mathcal{X} \times S \rightarrow \mathcal{X}$ and $\tilde{\pi}_s : \mathcal{X} \times \text{Spec}(k(s)) \rightarrow \mathcal{X}$ are the projections. And for every morphism of k -schemes $f : S' \rightarrow S$, we obtain a map

$$\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)(f) : \mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)(S) \rightarrow \mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)(S')$$

via pulling back \mathcal{F} and φ by $\text{id}_{\mathcal{X}} \times f$. If we take families of δ -stable pairs, we denote the corresponding subfunctor by $\mathcal{M}_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$.

Remark 4.9. By [43, Proposition 1.5] and Theorem 2.5, we have $F_{\pi_{\mathcal{X}}^* \mathcal{E}}(\mathcal{F})_s = F_{\tilde{\pi}_s^* \mathcal{E}}(\mathcal{F}_s)$. Then one has $P(F_{\pi_{\mathcal{X}}^* \mathcal{E}}(\mathcal{F})_s(m)) = P_{\tilde{\pi}_s^* \mathcal{E}}(\mathcal{F}_s)(m) = P(m)$ for each point $s \in S$ in the above definition.

Recall that a scheme M is called a (coarse) moduli space for the functor \mathcal{M} if it corepresents the functor \mathcal{M} and called a fine moduli space for \mathcal{M} if it represents \mathcal{M} . One can refer to [24, Sections 2.2, 4.1 and 4.6] for more details. In order to apply GIT to prove the existence of moduli spaces for functors $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta)$, we will consider the GIT-(semi)stable points.

Define $\mathcal{R}^{(s)s} \subseteq \mathcal{Z}' \subseteq \mathcal{Z}$ to be the subset of GIT-(semi)stable points $([a], [q])$ corresponding to δ -(semi)stable pairs by Theorem 4.4 and Theorem 4.6. As in [24, Section 4.2], the sets \mathcal{R}^s and \mathcal{R}^{ss} are open $\text{SL}(V)$ -invariant subset of \mathcal{Z}' . In particular, \mathcal{R}^s is an open subset of \mathcal{R}^{ss} . In Remark 4.1, we have obtained a flat family $(\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{Z}}, \check{\varphi}_{\mathcal{X} \times \mathcal{Z}})$ of pairs parametrized by \mathcal{Z} , where

$$\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{Z}} := (\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \otimes (\text{id}_{\mathcal{X}} \times \tilde{i})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1)$$

and $\check{\varphi}_{\mathcal{X} \times \mathcal{Z}} : p_{\mathcal{X} \times \mathcal{Z}, \mathcal{X}}^* \mathcal{F}_0 \rightarrow \check{\mathcal{F}}_{\mathcal{X} \times \mathcal{Z}}$. As in [12, Proposition 4.2], the scheme \mathcal{Z}' has the universal property due to the ones of \mathbb{P} and $\tilde{\mathcal{Q}}$. By pulling back the flat family $(\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{Z}}, \check{\varphi}_{\mathcal{X} \times \mathcal{Z}})$ to $\mathcal{R}^{(s)s}$, we get the universal family $(\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^{(s)s}}, \check{\varphi}_{\mathcal{X} \times \mathcal{R}^{(s)s}})$ of δ -semistable pairs with modified Hilbert polynomial P parameterized by $\mathcal{R}^{(s)s}$ where the morphism $\check{\varphi}_{\mathcal{X} \times \mathcal{R}^{(s)s}}$ is

$$\begin{aligned} \check{\varphi}_{\mathcal{X} \times \mathcal{R}^{(s)s}} &: p_{\mathcal{X} \times \mathcal{R}^{(s)s}, \mathcal{X}}^* \mathcal{F}_0 \rightarrow \check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^{(s)s}} \\ &:= (\text{id}_{\mathcal{X}} \times \tilde{i}_{(s)s})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \otimes (\text{id}_{\mathcal{X}} \times \tilde{i}_{(s)s})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

and $\tilde{i}_{(s)s} : \mathcal{R}^{(s)s} \rightarrow \mathbb{P} \times \tilde{\mathcal{Q}}$ are the inclusions.

Since $\text{SL}(V)$ -linearized ample line bundles $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)$ depend on choices of two positive integers n_1 and n_2 , we make the convention that they are

chosen to satisfy the equality (3) when $\deg \delta < \deg P$ and the equality (5) if $\deg \delta \geq \deg P$. By [24, Theorem 4.2.10], we have

Theorem 4.10. *There is a projective scheme $M^{ss} := M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ and a morphism $\Theta : \mathcal{R}^{ss} \rightarrow M^{ss}$ such that Θ is a universal good quotient for the $\mathrm{SL}(V)$ -action on \mathcal{R}^{ss} . And there is an open subscheme $M^s := M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta) \subseteq M^{ss}$ such that $\mathcal{R}^s = \Theta^{-1}(M^s)$ and $\Theta : \mathcal{R}^s \rightarrow M^s$ is a universal geometric quotient. Moreover, there is a positive integer \hat{l} and a very ample line bundle \mathbb{M} on M^{ss} such that $\mathcal{O}_{\mathcal{Z}'}(n_1, n_2)^{\otimes \hat{l}}|_{\mathcal{R}^{ss}} \cong \Theta^*(\mathbb{M})$.*

To obtain the similar results in [24, Theorem 4.3.3], we need the following semicontinuous result for the Hom group of pairs. The proof of this result is mainly based on the argument in that of [23, Lemma 3.4]. We will adopt the slightly modified proof of [12, Proposition A.2] as follows.

Lemma 4.11. *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a polarization $(\mathcal{E}, \mathcal{O}_{\mathcal{X}}(1))$. Let (\mathcal{F}, φ) and (\mathcal{G}, ψ) be two flat families of pairs over $\mathcal{X}_T := \mathcal{X} \times T$ parametrized by a scheme T of finite type over k . Then the function*

$$t \rightarrow \dim_{k(t)} \mathrm{Hom}_{\mathcal{X} \times \{t\}}((\mathcal{F}_t, \varphi_t), (\mathcal{G}_t, \psi_t))$$

is an upper semicontinuous function on T .

Proof. It suffices to prove the case when $T = \mathrm{Spec} A$ where A is a k -algebra of finite type. Since $p_T : \mathcal{X} \times T \rightarrow T$ is a family of projective stacks, we have the following locally free resolutions (see also the argument in Section 5.1):

$$\begin{aligned} p_{\mathcal{X}}^*(\mathcal{E})^{\oplus N_1} \otimes p_{\mathcal{X}}^* \circ \pi^* \mathcal{O}_X(-m_1) &\rightarrow p_{\mathcal{X}}^*(\mathcal{E})^{\oplus N_2} \otimes p_{\mathcal{X}}^* \circ \pi^* \mathcal{O}_X(-m_2) \rightarrow \mathcal{F} \rightarrow 0, \\ p_{\mathcal{X}}^*(\mathcal{E})^{\oplus \tilde{N}_1} \otimes p_{\mathcal{X}}^* \circ \pi^* \mathcal{O}_X(-\tilde{m}_1) &\rightarrow p_{\mathcal{X}}^*(\mathcal{E})^{\oplus \tilde{N}_2} \otimes p_{\mathcal{X}}^* \circ \pi^* \mathcal{O}_X(-\tilde{m}_2) \rightarrow p_{\mathcal{X}}^* \mathcal{F}_0 \rightarrow 0, \end{aligned}$$

where $p_{\mathcal{X}} : \mathcal{X} \times T \rightarrow \mathcal{X}$ is the natural projection, N_i, \tilde{N}_i are positive integers, and m_i, \tilde{m}_i are positive integers large enough for $i = 1, 2$. Then we have the exact sequences for any A -module M

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G} \otimes_A M) &\rightarrow M_{\mathcal{F}}^2 \otimes_A M \rightarrow M_{\mathcal{F}}^1 \otimes_A M, \\ 0 \rightarrow \mathrm{Hom}(p_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{G} \otimes_A M) &\rightarrow M_{\mathcal{F}_0}^2 \otimes_A M \rightarrow M_{\mathcal{F}_0}^1 \otimes_A M, \end{aligned}$$

where

$$M_{\mathcal{F}}^i := H^0(X \times T, F_{p_{\mathcal{X}}^*}(\mathcal{E})(\mathcal{G})^{\oplus N_i}(m_i)),$$

$$M_{\mathcal{F}_0}^i := H^0(X \times T, F_{p_{\mathcal{X}}^*}(\mathcal{E})(\mathcal{G})^{\oplus \tilde{N}_i}(\tilde{m}_i)),$$

and A -module $M_{\mathcal{F}}^i, M_{\mathcal{F}_0}^i$ are free for $i = 1, 2$ by the similar argument in the proof of [43, Lemma 6.18]. Define the following complexes of free A -modules concentrated in degree 0 and 1:

$$M_{\mathcal{F}}^{\bullet} : 0 \rightarrow M_{\mathcal{F}}^2 \rightarrow M_{\mathcal{F}}^1 \rightarrow 0,$$

$$M_{\mathcal{F}_0}^{\bullet} : 0 \rightarrow M_{\mathcal{F}_0}^2 \rightarrow M_{\mathcal{F}_0}^1 \rightarrow 0.$$

Then we have for any A -module M

$$\text{Hom}(\mathcal{F}, \mathcal{G} \otimes_A M) \cong H^0(M_{\mathcal{F}}^{\bullet} \otimes_A M),$$

$$\text{Hom}(p_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{G} \otimes_A M) \cong H^0(M_{\mathcal{F}_0}^{\bullet} \otimes_A M).$$

The morphism $\varphi : p_{\mathcal{X}}^* \mathcal{F}_0 \rightarrow \mathcal{F}$ induces a morphism of complexes, denoted again by $\varphi : M_{\mathcal{F}}^{\bullet} \rightarrow M_{\mathcal{F}_0}^{\bullet}$. Denote by A^{\bullet} the complex such that $A^0 = A$ and $A^i = 0$ for $i \neq 0$, then the morphism $\psi : p_{\mathcal{X}}^* \mathcal{F}_0 \rightarrow \mathcal{G}$ induces a morphism $\psi : A^{\bullet} \rightarrow M_{\mathcal{F}_0}^{\bullet}$. Therefore, we have a morphism $\Psi := (\varphi, -\psi) : M_{\mathcal{F}}^{\bullet} \oplus A^{\bullet} \rightarrow M_{\mathcal{F}_0}^{\bullet}$. Let $\text{Cone}(\Psi)^{\bullet}$ be the mapping cone of Ψ , we have a short exact sequence

$$0 \rightarrow M_{\mathcal{F}_0}^{\bullet} \rightarrow \text{Cone}(\Psi)^{\bullet} \rightarrow M_{\mathcal{F}}^{\bullet} \oplus A^{\bullet}[1] \rightarrow 0.$$

Then we obtain the following long exact sequence

$$0 \rightarrow h^{-1}(\text{Cone}(\Psi)^{\bullet} \otimes_A M) \rightarrow \text{Hom}(\mathcal{F}, \mathcal{G} \otimes_A M) \oplus M \rightarrow \text{Hom}(p_{\mathcal{X}}^* \mathcal{F}_0, \mathcal{G} \otimes_A M) \rightarrow \dots$$

For any $t \in T$ and $M = k(t)$, we have the following cartesian diagram

$$\begin{CD} h^{-1}(\text{Cone}(\Psi)^{\bullet} \otimes_A k(t)) @>>> k(t) \\ @VVV @VV\psi_t V \\ \text{Hom}(\mathcal{F}_t, \mathcal{G}_t) @>\circ\varphi_t>> \text{Hom}(\mathcal{F}_0, \mathcal{G}_t) \end{CD}$$

By Remark 2.20, we have

$$\begin{aligned} \dim_{k(t)} \operatorname{Hom}_{\mathcal{X} \times \{t\}}((\mathcal{F}_t, \varphi_t), (\mathcal{G}_t, \psi_t)) \\ = \dim_{k(t)} h^{-1}(\operatorname{Cone}(\Psi)^\bullet \otimes_A k(t)) - 1 + \epsilon(\psi_t). \end{aligned}$$

The proof is completed by using the fact that $\epsilon(\psi_t)$ is either zero for all $t \in T$ or never zero by assumption of flatness and the function $t \rightarrow \dim h^{-1}(\operatorname{Cone}(\Psi)^\bullet \otimes_A k(t))$ is upper semicontinuous by [12, Lemma A.1]. \square

As in [36, Section 4], we call a δ -semistable pair δ -polystable if it is isomorphic to the direct sum of δ -stable pair with the same reduced Hilbert polynomial. Notice that the assumption of nondegenerateness is not imposed on every summand of a nondegenerate δ -polystable pair. Using the similar argument in the proof of [23, Proposition 3.3] and Lemma 4.11, we have

Lemma 4.12. *Two points $([a_1], [q_1])$ and $([a_2], [q_2])$ are mapped to the same point in M^{ss} if and only if their corresponding δ -semistable pairs $(\mathcal{F}_1, \varphi_1)$ and $(\mathcal{F}_2, \varphi_2)$ are S -equivalent. The orbit of a point $([a], [q])$ is closed in \mathcal{R}^{ss} if and only if the corresponding pair (\mathcal{F}, φ) is δ -polystable.*

Remark 4.13. *Since $\Theta : \mathcal{R}^{ss} \rightarrow M^{ss}$ is a good quotient for the $\operatorname{SL}(V)$ -action on \mathcal{R}^{ss} , the statement that two points $([a_1], [q_1])$ and $([a_2], [q_2])$ are mapped to the same point in M^{ss} is equivalent to the one that the closure of orbits of two points $([a_1], [q_1])$ and $([a_2], [q_2])$ in \mathcal{R}^{ss} intersect.*

Next, we shall show that the projective scheme M^{ss} is a moduli space for the functor $\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ and the quasi-projective scheme M^s is a fine moduli space for the functor $\mathcal{M}_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Following the method in [12, Section 4.2], we first recall the definition of good moduli space [5] as follows.

Definition 4.14. ([5, Section 2]) *Let S be a scheme and $(\operatorname{Sch}/S)_{\text{Et}}$ a category of schemes over S with the global étale topology. An algebraic space over S is a sheaf of sets X on $(\operatorname{Sch}/S)_{\text{Et}}$ such that*

- (i) $\Delta_{X/S} : X \rightarrow X \times_S X$ is representable by schemes and quasi-compact.
- (ii) There exists an étale, surjective map $U \rightarrow X$, where U is a scheme.

An Artin stack over S is a stack \mathcal{X} over $(\operatorname{Sch}/S)_{\text{Et}}$ such that

- (i) $\Delta_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact.
- (ii) There exists a smooth, surjective map $X \rightarrow \mathcal{X}$, where X is an algebraic space.

Remark 4.15. *An Artin stack \mathcal{X} here is referred as an algebraic stack in [31, Definition 4.1]. It is a Deligne-Mumford stack over S when there exists an étale, surjective map from an algebraic space.*

Definition 4.16. *([5, Definition 3.1]) A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks over S is cohomologically affine if f is quasi-compact and the functor $f_* : \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(\mathcal{Y})$ is exact.*

Definition 4.17. *([5, Definition 4.1]) Let \mathcal{X} be an Artin stack and Y be an algebraic space over S . We say that a morphism $f : \mathcal{X} \rightarrow Y$ is a good moduli space if the following conditions are satisfied:*

- (i) *f is cohomologically affine.*
- (ii) *The natural morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

Definition 4.18. *([5, Definiton 7.1]) We call $f : \mathcal{X} \rightarrow Y$ a tame moduli space if*

- (i) *f is a good moduli space,*
- (ii) *for all geometric points $\text{Spec} k \rightarrow S$, the map $[\mathcal{X}(k)] \rightarrow Y(k)$ is a bijection of sets, where $[\mathcal{X}(k)]$ denotes the set of isomorphism classes of objects of $\mathcal{X}(k)$.*

We introduce the following Artin stacks of finite type:

$$\begin{aligned} \mathfrak{S}\mathfrak{R}^{(s)s} &:= [\mathcal{R}^{s(s)}/\text{SL}(V)], & \mathfrak{R}^{(s)s} &:= [\mathcal{R}^{s(s)}/\text{GL}(V)], \\ \mathfrak{P}\mathfrak{R}^{(s)s} &:= [\mathcal{R}^{s(s)}/\text{PGL}(V)], \end{aligned}$$

where $\mathfrak{P}\mathfrak{R}^{(s)s}$ is well-defined due to Lemma 4.21. Using the rigidification of a stack in [1, Section 5], it is easy to show that $\mathfrak{R}^{(s)s}$ is \mathbb{G}_m -gerbe on $\mathfrak{P}\mathfrak{R}^{(s)s}$ and $\mathfrak{S}\mathfrak{R}^{(s)s}$ is $\mu(V)$ -gerbe on $\mathfrak{P}\mathfrak{R}^{(s)s}$ where $\mu(V) \subseteq \text{SL}(V)$ denotes the group of the product of $\dim(V)$ -roots of unity and the identity matrix. Since $\mathcal{O}_{\mathbb{Z}'}(n_1, n_2)|_{\mathcal{R}^{ss}}$ is a $\text{SL}(V)$ -linearization on \mathcal{R}^{ss} , we denote by $\mathcal{O}(n_1, n_2)$ the corresponding line bundle on $\mathfrak{S}\mathfrak{R}^{ss}$. Combing with Theorem 4.10, an analogue of GIT in [5, Theorem 13.6] shows

Theorem 4.19. *There is a good moduli space $\Theta_{\mathfrak{S}} : \mathfrak{S}\mathfrak{R}^{ss} \rightarrow M^{ss}$ with $\mathcal{O}(n_1, n_2)^{\otimes i} = \Theta_{\mathfrak{S}}^*(\mathbb{M})$ and the morphism $\Theta_{\mathfrak{S}} : \mathfrak{S}\mathfrak{R}^s \rightarrow M^s$ is a tame moduli space.*

As in [12, Section 4.2], the morphism $\Theta_{\mathfrak{S}} : \mathfrak{S}\mathfrak{R}^{(s)s} \rightarrow M^{(s)s}$ induces the commutative diagram

$$\begin{array}{ccccc}
 \mathfrak{S}\mathfrak{R}^{(s)s} & \xrightarrow{\quad} & \mathfrak{P}\mathfrak{R}^{(s)s} & \xleftarrow{\quad} & \mathfrak{R}^{(s)s} \\
 & \searrow \Theta_{\mathfrak{S}} & \downarrow \Theta_{\mathfrak{P}} & \swarrow \Theta_{\mathfrak{S}} & \\
 & & M^{(s)s} & &
 \end{array}$$

and morphisms $\Theta_{\mathfrak{S}}$ and $\Theta_{\mathfrak{P}}$ also satisfy assertions stated in Theorem 4.19. Following [12, Section 4.2], we define $[\mathfrak{R}^{(s)s}]$ to be the contravariant functor such that for any scheme S of finite type, $[\mathfrak{R}^{(s)s}](S)$ is the set of isomorphism classes of objects of $\mathfrak{R}^{(s)s}(S)$. An object in $[\mathfrak{R}^{(s)s}](S)$ is an isomorphism class $[(\bar{q} : \bar{P} \rightarrow S, \phi : \bar{P} \rightarrow \mathcal{R}^{(s)s})]$, where $\bar{q} : \bar{P} \rightarrow S$ is a $\mathrm{GL}(V)$ -torsor over S and ϕ is a $\mathrm{GL}(V)$ -equivariant morphism. Here, $(\bar{q} : \bar{P} \rightarrow S, \phi : \bar{P} \rightarrow \mathcal{R}^{(s)s})$ and $(\bar{q}' : \bar{P}' \rightarrow S, \phi' : \bar{P}' \rightarrow \mathcal{R}^{(s)s})$ are called isomorphic objects if we have the following commutative diagram

$$\begin{array}{ccc}
 \bar{P} & \xrightarrow{\varsigma} & \bar{P}' \\
 \bar{q} \downarrow & & \downarrow \bar{q}' \\
 S & \xrightarrow{\mathrm{id}_S} & S
 \end{array}$$

where ς is an isomorphism of $\mathrm{GL}(V)$ -torsors compatible with ϕ and ϕ' . Now, we have

Theorem 4.20. *The functor $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta)$ is isomorphic to $[\mathfrak{R}^{(s)s}]$.*

Proof. Use the similar argument in the proof of [12, Theorem 4.12] with our notion of flat families of pairs in Definition 4.8 (see also Definition 2.40) and the universal family $(\tilde{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^{(s)s}}, \tilde{\varphi}_{\mathcal{X} \times \mathcal{R}^{(s)s}})$ of δ -(semi)stable pairs with modified Hilbert polynomial P parameterized by $\mathcal{R}^{(s)s}$. \square

Since the morphism $\Theta_{\mathfrak{S}} : \mathfrak{R}^{(s)s} \rightarrow M^{(s)s}$ factors through $\mathfrak{R}^{(s)s} \rightarrow [\mathfrak{R}^{(s)s}]$, we have the morphism $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta) \rightarrow M^{(s)s}$ by Theorem 4.20. By the Yoneda Lemma, we may take the scheme $M^{(s)s}$ as a functor $\mathrm{Mor}(-, M^{(s)s})$ and the morphism $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta) \rightarrow M^{(s)s}$ as a natural transformation $\mathcal{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{F}_0, P, \delta) \rightarrow \mathrm{Mor}(-, M^{(s)s})$. We need the following lemmas which are useful for proving the existence of a fine moduli space for $\mathcal{M}_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. First, as in [12, Lemma 4.5], we have

Lemma 4.21. *Let $([a], [q]) \in \mathcal{Z} \subseteq \mathbb{P} \times \tilde{\mathcal{Q}}$ be a closed point corresponding to a pair (\mathcal{F}, φ) such that $F_{\mathcal{E}}(\mathcal{F})(m)$ is globally generated and $H^0(\tilde{q}) : V \rightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ induced by $q : V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \rightarrow \mathcal{F}$ is an isomorphism. Then there exists a natural injective homomorphism $\text{Aut}(\mathcal{F}, \varphi) \rightarrow \text{GL}(V)$ whose image is precisely the stabilizer group $\text{GL}(V)_{([a], [q])}$ of the point $([a], [q])$.*

Proof. Notice morphisms of pairs in Definition 2.18 and use the similar argument in the proof of [24, Lemma 4.3.2]. □

Corresponding to [12, Proposition 4.14], we have

Lemma 4.22. *Let $(\tilde{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s}, \tilde{\varphi}_{\mathcal{X} \times \mathcal{R}^s})$ be the universal family of δ -semistable pairs with modified Hilbert polynomial P parameterized by \mathcal{R}^s . Then $\tilde{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s}$ is invariant with respect to the action of the center \mathbb{G}_m of $\text{GL}(V)$.*

Proof. Since the assumption of Lemma 4.21 is satisfied on \mathcal{R}^s by Theorem 4.4 and Theorem 4.6, the center \mathbb{G}_m of $\text{GL}(V)$ acts trivially on \mathcal{R}^s . This implies that the restriction of $\eta : \mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V) \rightarrow \mathbb{P} \times \tilde{\mathcal{Q}}$ to $\mathcal{R}^s \times \tilde{\mathbb{G}}_m$ is a trivial action where η is induced by $\text{GL}(V)$ -actions $\eta_1 : \tilde{\mathcal{Q}} \times \text{GL}(V) \rightarrow \tilde{\mathcal{Q}}$ and $\eta_2 : \mathbb{P} \times \text{GL}(V) \rightarrow \mathbb{P}$. Denote by $i_{\mathbb{G}_m} : \mathbb{G}_m \rightarrow \text{GL}(V)$ the inclusion. Then we have the following identities:

$$\begin{aligned} & p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}} \circ (\text{id}_{\mathcal{X}} \times \tilde{i}_s) \circ p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s} \\ &= (\text{id}_{\mathcal{X}} \times \eta_1) \circ p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V)} \circ (\text{id}_{\mathcal{X}} \times \tilde{i}_s \times i_{\mathbb{G}_m}) \\ &= p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}} \circ p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V)} \circ (\text{id}_{\mathcal{X}} \times \tilde{i}_s \times i_{\mathbb{G}_m}) \end{aligned}$$

and

$$\begin{aligned} & p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}} \circ (\text{id}_{\mathcal{X}} \times \tilde{i}_s) \circ p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s} \\ &= \eta_2 \circ p_{\mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathbb{P} \times \text{GL}(V)} \circ (\tilde{i}_s \times i_{\mathbb{G}_m}) \circ p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{R}^s \times \mathbb{G}_m} \\ &= p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}} \circ p_{\mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathbb{P} \times \text{GL}(V)} \circ (\tilde{i}_s \times i_{\mathbb{G}_m}) \circ p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{R}^s \times \mathbb{G}_m}. \end{aligned}$$

Then two isomorphisms $\Lambda_1 : (\text{id}_{\mathcal{X}} \times \eta_1)^* \tilde{\mathcal{F}} \rightarrow p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}}$ and $\Lambda_2 : \eta_2^* \mathcal{O}_{\mathbb{P}}(1) \rightarrow p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1)$ induces the following isomorphisms

$$\begin{aligned} \tilde{\Lambda}_1 &: p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \\ &= (\text{id}_{\mathcal{X}} \times \tilde{i}_s \times i_{\mathbb{G}_m})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V)}^* (\text{id}_{\mathcal{X}} \times \eta_1)^* \tilde{\mathcal{F}} \\ &\xrightarrow{\cong} (\text{id}_{\mathcal{X}} \times \tilde{i}_s \times i_{\mathbb{G}_m})^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V)}^* p_{\mathcal{X} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \\ &= p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \end{aligned}$$

and

$$\begin{aligned} \tilde{\Lambda}_2 &: p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \\ &= p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{R}^s \times \mathbb{G}_m}^* (\tilde{i}_s \times i_{\mathbb{G}_m})^* p_{\mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathbb{P} \times \text{GL}(V)}^* \eta_2^* \mathcal{O}_{\mathbb{P}}(1) \\ &\xrightarrow{\cong} p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{R}^s \times \mathbb{G}_m}^* (\tilde{i}_s \times i_{\mathbb{G}_m})^* p_{\mathbb{P} \times \tilde{\mathcal{Q}} \times \text{GL}(V), \mathbb{P} \times \text{GL}(V)}^* p_{\mathbb{P} \times \text{GL}(V), \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \\ &= p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

such that the following diagram commute

$$\begin{array}{ccc} p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X}}^* \mathcal{F}_0 & \xlongequal{\quad} & p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X}}^* \mathcal{F}_0 \\ \downarrow p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* \check{\varphi}_{\mathcal{X} \times \mathcal{R}^s} & & \downarrow p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* \check{\varphi}_{\mathcal{X} \times \mathcal{R}^s} \\ p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* \check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s} & \xrightarrow{\tilde{\Lambda}_1 \otimes \tilde{\Lambda}_2} & p_{\mathcal{X} \times \mathcal{R}^s \times \mathbb{G}_m, \mathcal{X} \times \mathcal{R}^s}^* \check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s} \end{array}$$

where

$$\begin{aligned} \check{\varphi}_{\mathcal{X} \times \mathcal{R}^s} &: p_{\mathcal{X} \times \mathcal{R}^s, \mathcal{X}}^* \mathcal{F}_0 \rightarrow \check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s} \\ &:= (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \otimes (\text{id}_{\mathcal{X}} \times \tilde{i}_s)^* p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1). \end{aligned}$$

Using the similar argument in the proof of [12, Proposition 4.14], one can show that both $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are the identity morphisms. This implies $\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s}$ is invariant with respect to the action of the center \mathbb{G}_m of $\text{GL}(V)$. \square

Remark 4.23. *In the proof of [12, Proposition 4.14], the assumption of irreducibility of \mathcal{X} is used to assert that their universal family of framed sheaves parameterized by \mathcal{R}^s can be $\text{PGL}(V)$ -linearized in order to show that moduli spaces of δ -stable framed sheaves are fine. While in our setting, the universal family $(\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s}, \check{\varphi}_{\mathcal{X} \times \mathcal{R}^s})$ of δ -stable pairs is $\text{PGL}(V)$ -linearized*

without this assumption since $\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^s}$ is invariant with respect to the action of the center \mathbb{G}_m of $\mathrm{GL}(V)$ by Lemma 4.22.

Now, we have

Theorem 4.24. *Let \mathcal{X} be a projective Deligne-Mumford stack of dimension d over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let \mathcal{F}_0 be any fixed coherent sheaf on \mathcal{X} , and let δ be any given stability parameter which is a rational polynomial with positive leading coefficient and P any given polynomial of degree $\deg P \leq d$. Then the projective scheme $M^{ss} := M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ is a moduli space for the moduli functor $\mathcal{M}_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ and the quasi-projective scheme $M^s := M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$ is a fine moduli space for the moduli functor $\mathcal{M}_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Moreover, the Artin stack \mathfrak{R}^s is a \mathbb{G}_m -gerbe over its moduli scheme M^s .*

Proof. The proof is completed by following the similar argument in [12, Theorem 4.15] but with our notion of universal families $(\check{\mathcal{F}}_{\mathcal{X} \times \mathcal{R}^{(s)s}}, \check{\varphi}_{\mathcal{X} \times \mathcal{R}^{(s)s}})$. The last statement is true since $\mathcal{R}^s \rightarrow M^s$ can be shown to be a $\mathrm{PGL}(V)$ -torsor. \square

Remark 4.25. *One may consider the construction of relative moduli spaces of δ -(semi)stable pairs by using [5, 59] for GIT construction in relative case and obtaining the relative version of [43, Proposition 4.24], Lemma 2.36 and Lemma 2.39 for the boundedness results. We will investigate this issue elsewhere which is essential to the study of relative orbifold PT theory and its degeneration formula.*

4.4. Variation of moduli spaces

For a given polynomial P and a fixed coherent sheaf \mathcal{F}_0 on \mathcal{X} , as in [67, Section 5], we will investigate the variation of moduli spaces $M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ when one changes the stability parameter $\delta \in \mathbb{Q}[m]$. To indicate the dependence on δ , we denote by $\mathcal{R}^{(s)s}(\delta)$ the subset of GIT-(semi)stable points $([a], [q]) \in \mathcal{Z}'$ corresponding to δ -(semi)stable pairs. We generalize [67, Theorem 5.5] to the case of projective Deligne-Mumford stacks as follows.

Theorem 4.26. *Let \mathcal{X} be a projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. There are finitely*

many critical values $\delta^1, \dots, \delta^t \in \mathbb{Q}[m]$ satisfying

$$\delta^0 := 0 < \delta^1 < \dots < \delta^t < \delta^{t+1} := +\infty$$

such that we have the following properties:

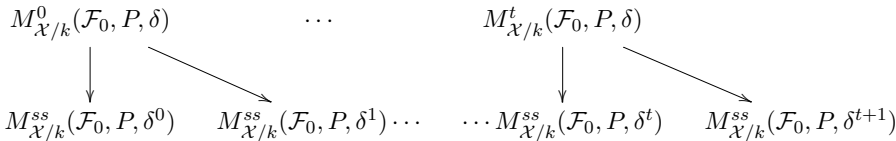
- (i) For $i = 0, \dots, t$ and $\delta, \delta' \in (\delta^i, \delta^{i+1})$, one has $\mathcal{R}^{(s)s}(\delta) = \mathcal{R}^{(s)s}(\delta')$.
- (ii) For $i = 0, \dots, t$ and $\delta \in (\delta^i, \delta^{i+1})$, we have

$$\mathcal{R}^{ss}(\delta) \subseteq \mathcal{R}^{ss}(\delta^i) \cap \mathcal{R}^{ss}(\delta^{i+1}), \quad \mathcal{R}^s(\delta) \supseteq \mathcal{R}^s(\delta^i) \cup \mathcal{R}^s(\delta^{i+1}).$$

- (iii) For $i = 0, \dots, t$ and $\delta \in (\delta^i, \delta^{i+1})$, one has

$$M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta) = M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta).$$

Moreover, we have the chamber structure of the stability parameter as follows:



where $M_{\mathcal{X}/k}^i(\mathcal{F}_0, P, \delta) := M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ for some $\delta \in (\delta^i, \delta^{i+1})$, $i = 0, \dots, t$.

Proof. With the notation in Remark 4.1, we have the following universal morphism

$$\begin{aligned}
 p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X}}^* (H^0(F\mathcal{E}(\mathcal{F}_0)(m)) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)) \otimes p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}} \\
 \rightarrow p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathcal{X} \times \tilde{\mathcal{Q}}}^* \tilde{\mathcal{F}} \otimes p_{\mathcal{X} \times \mathbb{P} \times \tilde{\mathcal{Q}}, \mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(1).
 \end{aligned}$$

As in [67, Lemma 5.1], one can show that the set-theoretic family of subsheaves $\text{im}\varphi$ from δ -semistable pairs (\mathcal{F}, φ) for all δ is bounded. Using this result together with Grothendieck’s Lemma 2.36 and Kleiman criterion for stacks, and following the similar argument in the proof of [67, Proposition 5.2], one has the following result:

There exists a rational polynomial δ_{\max} of degree $(\deg P - 1)$ such that for each $\delta > \delta_{\max}$ and each pair (\mathcal{F}, φ) , we have

$$(\mathcal{F}, \varphi) \text{ is } \delta\text{-semistable} \iff \mathcal{F} \text{ is pure and } \dim \text{coker} \varphi < \dim \mathcal{F}.$$

It is easy to verify that those δ satisfying $\deg \delta \geq \dim \mathcal{X}$ (or more generally $\deg \delta \geq \deg P$) are also allowed in the inequality $\delta > \delta_{\max}$.

The proof of [67, Lemma 5.3 and Corollary 5.4] also holds in our stacky version. We complete the proof by using the results above, Theorem 4.4 and Theorem 4.6. □

Remark 4.27. *Notice that a critical value is defined to be a value such that when δ crosses it, the moduli space $M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ changes. As in [67], the maps in the chamber structure above are determined by the property (ii) in Theorem 4.26 and the universal properties of the universal good quotient in Theorem 4.10. In addition, we have $\delta_{\max} \geq \delta^t$. One can choose δ_{\max} to be δ^t . By the properties (i), (iii) and Theorem 4.10, for any two parameters $\delta', \delta'' > \delta^t$, one has $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta') = M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta') \cong M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta'') = M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta'')$. This implies that as is pointed out in the introduction of [36], one can choose a stability parameter $\hat{\delta}$ of degree $(\deg P - 1)$ with $\hat{\delta} > \delta_{\max}$ such that the construction of moduli space $M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \hat{\delta})$ implies the existence of $M_{\mathcal{X}/k}^{ss}(\mathcal{F}_0, P, \delta)$ for any δ satisfying $\deg \delta \geq \deg P$. Furtherly, it is interesting to study the explicit wall-crossing behavior for variation of moduli spaces.*

5. Deformation-obstruction theories and virtual fundamental classes

We generalize the deformation and obstruction theory of δ -stable pairs for smooth projective varieties in [36, Theorem 1.2] and the existence of virtual fundamental classes of moduli spaces of δ -stable pairs for a smooth projective surface in [36, Theorem 1.3] to the case of smooth projective Deligne-Mumford stacks. We will generalize the deformation and obstruction theory developed in [27] to our case as an alternative approach, which is useful for proving the existence of virtual fundamental classes as in [25, 53] for the case of dimension three. Finally, we give a definition of a stacky version of Pandharipande-Thomas invariants.

5.1. Deformation and obstruction theories

It is proved in [30, Section 5] that the resolution property holds for a projective Deligne-Mumford stack \mathcal{X} , that is, any coherent sheaf admits a surjective morphism from a locally free coherent sheaf of finite rank. By inductive use of this property, every coherent sheaf on \mathcal{X} has a locally free resolution. It is interesting to know whether the locally free resolution is of finite length or not. It is shown in [12, Appendix B] that a smooth projective Deligne-Mumford stack is of the form $[Z/G]$ where Z is a smooth quasi-projective variety and G is a linear algebraic group, and hence any coherent sheaf on a smooth projective stack \mathcal{X} admits a finite resolution by locally free sheaves of finite rank by using [17, Proposition 5.1.28 and Theorem 5.1.30] and the fact that the category of coherent sheaves on $\mathcal{X} = [Z/G]$ is equivalent to the category of coherent G -equivariant sheaves on Z (see [12, Remark 2.17]). In this section, we will consider smooth projective Deligne-Mumford stacks.

Let \mathcal{X} be a smooth projective Deligne-Mumford stack over k with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Since \mathcal{E} is a generating sheaf for \mathcal{X} , the morphism

$$\pi^*(\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E} \rightarrow \mathcal{G}$$

is surjective for any coherent sheaf \mathcal{G} . As $\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})$ is a coherent sheaf on the projective scheme X , then we can take a positive integer $n_1 \geq m$ large enough such that $\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})(n_1)$ is generated by global sections where m is the integer chosen in the Section 4. Then we have the following surjective morphism

$$H^0(\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})(n_1)) \otimes \mathcal{O}_X(-n_1) \rightarrow \pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G}) \rightarrow 0.$$

Thus, we have the following surjective morphism

$$\begin{aligned} H^0(\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})(n_1)) \otimes \pi^*\mathcal{O}_X(-n_1) \otimes \mathcal{E} \\ \rightarrow \pi^*(\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})) \otimes \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0. \end{aligned}$$

Denote by $W_1 = H^0(\pi_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{G})(n_1))$, we have a surjection from a locally free sheaf $W_1 \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-n_1) \rightarrow \mathcal{G} \rightarrow 0$. Let $\mathcal{G}_1 = \ker(W_1 \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-n_1) \rightarrow \mathcal{G})$. Applying the same process above to \mathcal{G}_1 , one can get

$$W_2 \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-n_2) \twoheadrightarrow \mathcal{G}_1 \hookrightarrow W_1 \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-n_1) \twoheadrightarrow \mathcal{G}$$

or a locally free resolution

$$W_2 \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-n_2) \rightarrow W_1 \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-n_1) \rightarrow \mathcal{G} \rightarrow 0$$

where $n_1, n_2 \geq m$ are positive integers large enough and W_1, W_2 are vector spaces. The next step is again applying the same process to $\mathcal{G}_2 = \ker(W_2 \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-n_2) \rightarrow W_1 \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-n_1))$. By induction and the finite resolution property in [12, Lemma B.3], we construct a finite locally free resolution of \mathcal{G} (see also Lemma 5.9):

$$\cdots \rightarrow \mathcal{W}_{i+1} \rightarrow \mathcal{W}_i \rightarrow \cdots \rightarrow \mathcal{W}_1 \rightarrow \mathcal{G} \rightarrow 0$$

where $\mathcal{W}_i = W_i \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-n_i)$ with $n_i \geq m$ large enough and W_i is a vector space for any $i \geq 1$.

Let $\mathcal{A}rt_k$ be the category of Artinian local k -algebras with residue field k . For $A, A' \in \text{Ob} \mathcal{A}rt_k$ and let the short exact sequence

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$$

be a small extension, that is, $\mathfrak{m}_{A'} I = 0$. Let \mathcal{F}_0 be any fixed coherent sheaf on \mathcal{X} , and let δ be any rational polynomial with positive leading coefficient and P any given polynomial of degree $\deg P \leq d$. Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Suppose $\check{\varphi}_A : \mathcal{F}_0 \otimes A \rightarrow \mathcal{F}_A$ over $\mathcal{X}_A = \mathcal{X} \times \text{Spec } A$ is a flat extension of (\mathcal{F}, φ) where \mathcal{F}_A is flat over A . Let $\mathbf{I}^\bullet := \{\mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}\}$ and $\mathbf{I}_A^\bullet := \{\mathcal{F}_0 \otimes A \xrightarrow{\check{\varphi}_A} \mathcal{F}_A\}$ be the complexes concentrated in degree 0 and 1. Since $\mathcal{X}_A \rightarrow \text{Spec } A$ is a family of projective stacks by Theorem 2.5, the morphism

$$(\pi \times \text{id}_{\text{Spec } A})^* ((\pi \times \text{id}_{\text{Spec } A})_* \mathcal{H}om(p_{\mathcal{X}_A, \mathcal{X}}^* \mathcal{E}, \mathcal{F}_A)) \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \mathcal{E} \rightarrow \mathcal{F}_A$$

is surjective. Since any coherent sheaf on \mathcal{X}_A has a locally free resolution, together with using the finite resolution property for any fiber of $\mathcal{X}_A \rightarrow \text{Spec } A$, the similar argument as above shows that there exists a finite locally free resolution of \mathcal{F}_A :

$$\cdots \rightarrow \mathcal{W}_A^{i-1} \rightarrow \mathcal{W}_A^i \rightarrow \cdots \rightarrow \mathcal{W}_A^0 \rightarrow \mathcal{F}_A \rightarrow 0$$

where $\mathcal{W}_A^i = W^i \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \pi^* \mathcal{O}_X(-n_i)$ with $n_i \geq m$ large enough and W^i is a vector space for any $i \leq 0$. Again, we take a finite locally free

resolution of \mathcal{F}_0 as follows:

$$\dots \rightarrow \mathcal{V}^{i-1} \rightarrow \mathcal{V}^i \rightarrow \dots \rightarrow \mathcal{V}^0 \rightarrow \mathcal{F}_0 \rightarrow 0$$

where $\mathcal{V}^i = V^i \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_i)$ with $m_i \geq m$ large enough and V^i is a vector space for any $i \leq 0$. As in [36, Section 5A], lifting $\check{\varphi}_A : \mathcal{F}_0 \otimes A \rightarrow \mathcal{F}_A$ to a morphism of complexes $\check{\varphi}_A^\bullet : \mathcal{V}^\bullet \otimes A \rightarrow \mathcal{W}_A^\bullet$, we have the commutative diagram as follows

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{\mathcal{V}}^{-2} \otimes A} & \mathcal{V}^{-1} \otimes A & \xrightarrow{d_{\mathcal{V}}^{-1} \otimes A} & \mathcal{V}^0 \otimes A & \longrightarrow & \mathcal{F}_0 \otimes A \longrightarrow 0 \\ & & \downarrow \check{\varphi}_A^{-1} & & \downarrow \check{\varphi}_A^0 & & \downarrow \check{\varphi}_A \\ \dots & \xrightarrow{d_{\mathcal{W}}^{-2}} & \mathcal{W}_A^{-1} & \xrightarrow{d_{\mathcal{W}}^{-1}} & \mathcal{W}_A^0 & \longrightarrow & \mathcal{F}_A \longrightarrow 0. \end{array}$$

Notice that $n_i, m_i \geq m$, one has

$$\begin{aligned} H^i(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(n_j)) &= H^i(X, F_{\mathcal{E}}(\mathcal{F})(n_j)) = 0, \\ H^i(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(m_j)) &= H^i(X, F_{\mathcal{E}}(\mathcal{F})(m_j)) = 0, \end{aligned}$$

for all $i > 0$ and $j \leq 0$.

With these preparations, by the similar argument in [36, Section 5], we have the following straightforward generalization of [36, Theorem 1.2].

Theorem 5.1. *Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Let $\check{\varphi}_A : \mathcal{F}_0 \otimes_k A \rightarrow \mathcal{F}_A$ be a morphism over $\mathcal{X}_A = \mathcal{X} \times_{\text{Spec } k} \text{Spec } A$ extending φ , where \mathcal{F}_A is a coherent sheaf flat over A . Then for a given small extension $0 \rightarrow I \rightarrow A' \xrightarrow{\sigma} A \rightarrow 0$, there is a class*

$$\text{ob}(\check{\varphi}_A, \sigma) \in \text{Ext}^1(\mathbf{I}^\bullet, \mathcal{F} \otimes I)$$

such that there exists a flat extension of $\check{\varphi}_A$ over $\mathcal{X}_{A'}$ if and only if $\text{ob}(\check{\varphi}_A, \sigma) = 0$. If $\text{ob}(\check{\varphi}_A, \sigma) = 0$, the space of extensions is a torsor under $\text{Hom}(\mathbf{I}^\bullet, \mathcal{F} \otimes I)$.

Remark 5.2. *The deformation and obstruction theory in [36, Theorem 1.2] is analogous to the one in [61, Theorem 4.2] (see also [26, IV 3.2.12]) for any small extension $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$ when we view the 2-term complex as a mapping cone. Actually Theorem 5.1 provides a deformation and obstruction theory for the stacky version of higher rank Pandharipande-Thomas stable pairs [61].*

The deformation and obstruction theory in Theorem 5.1 is suitable for showing the existence of virtual fundamental classes for the case of dimension two in the next subsection, but it seems difficult to deal with 3-dimensional case (see Lemma 5.13 and Remark 5.14). Now, we take another approach to give a stacky version of the deformation and obstruction theory in [27, Section 2]. We start with

Lemma 5.3. *Suppose $B \in \text{ObArt}_k$, and let \mathcal{F}^\bullet be a bounded complex of coherent sheaves over $\mathcal{X}_B := \mathcal{X} \times \text{Spec } B$ where each \mathcal{F}^j is flat over B . Then there are a complex $V^\bullet = (V^i, d_{V^\bullet}^i)$ where $V^i = L_i \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_i)$ and a quasi-isomorphism $\phi^\bullet : V^\bullet \rightarrow \mathcal{F}^\bullet$ such that $(V^\bullet \rightarrow \mathcal{F}^\bullet, V^\bullet)$ satisfies the following two conditions:*

- (i) $H^c(\mathcal{X}_B, \mathcal{F}^j \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_i)) = 0$ for any i, j and any $c > 0$.
- (ii) Let $U^\bullet := \mathcal{F}^\bullet \oplus V^\bullet[1]$ be the mapping cone of ϕ^\bullet and set $W^i := \ker(U^i \rightarrow U^{i+1})$. Then the map $H^0(\mathcal{X}_B, U^{j-1} \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_j)) \rightarrow H^0(\mathcal{X}_B, W^j \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_j))$ is surjective for any j and $H^c(\mathcal{X}_B, W^j \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_i)) = 0$ for $i \leq j$ and $c > 0$.

Here, L_i is a vector space and m_i is a sufficiently large integer for any i .

Proof. We follow the similar argument in the poofs of [27, Proposition 1.1 and Remark 2.2]. Choose an integer l such that $\mathcal{F}^j = 0$ if $j > l$. By Theorem 2.5, the sheaf $p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}$ is a generating sheaf for \mathcal{X}_B . Then we have the surjective morphism

$$(\pi \times \text{id}_{\text{Spec } B})^*(F_{p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}}(\mathcal{F}^l)) \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E} \rightarrow \mathcal{F}^l.$$

Choose a positive integer m_l sufficiently large such that $F_{p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}}(\mathcal{F}^l) \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{O}_X(m_l)$ is generated by global sections, and then we have the following surjective morphism

$$\begin{aligned} V^l &:= H^0(X_B, F_{p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}}(\mathcal{F}^l) \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{O}_X(m_l)) \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_l) \\ &\rightarrow \mathcal{F}^l \end{aligned}$$

where $X_B := X \times \text{Spec } B$. Set $L_l := H^0(X_B, F_{p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}}(\mathcal{F}^l) \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{O}_X(m_l))$, $\widetilde{W}^l := \mathcal{F}^l$ and $\widetilde{W}^{l-1} := \ker(\mathcal{F}^{l-1} \oplus V^l \rightarrow \widetilde{W}^l)$. Inductively define $V^i := L_i \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_i)$ and $\widetilde{W}^{i-1} := \ker(\mathcal{F}^{i-1} \oplus V^i \rightarrow \widetilde{W}^i)$ where

$L_i := H^0(X_B, F_{p_{\mathcal{X}_B, \mathcal{X}}}^* \mathcal{E}(\widetilde{W}^i) \otimes p_{\mathcal{X}_B, X}^* \mathcal{O}_X(m_i))$ and m_i is a positive integer sufficiently large. Notice that for any coherent sheaf \mathcal{G} on \mathcal{X} and any i, j , we have

$$\begin{aligned} H^j(\mathcal{X}_B, \mathcal{G} \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_i)) \\ = H^j(X_B, F_{p_{\mathcal{X}_B, \mathcal{X}}}^* \mathcal{E}(\mathcal{G}) \otimes p_{\mathcal{X}_B, X}^* \mathcal{O}_X(m_i)). \end{aligned}$$

Actually one can inductively choose m_i sufficiently large to satisfy the condition (i) and

$$H^c(\mathcal{X}_B, \widetilde{W}^j \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \mathcal{E}^\vee \otimes p_{\mathcal{X}_B, \mathcal{X}}^* \pi^* \mathcal{O}_X(m_i)) = 0, \quad \text{for } i \leq j \text{ and } c > 0.$$

The similar argument in the proof of [27, Proposition 1.1] shows one can define the complex V^\bullet with $V^i = 0$ for $i > l$ and $d_{V^\bullet}^i : V^i \rightarrow \widetilde{W}^i \rightarrow V^{i+1}$ such that there is a quasi-isomorphism $\phi^\bullet : V^\bullet \rightarrow \mathcal{F}^\bullet$. It is easy to verify that $\widetilde{W}^i = W^i := \ker(U^i \rightarrow U^{i+1})$. Then we have the surjection $U^{j-1} = \mathcal{F}^{j-1} \oplus V^j \rightarrow W^j$ for any j which implies the surjectivity in the condition (ii) by the definition of V^\bullet . □

Remark 5.4. *Using the similar argument in the proof of [27, Lemma 2.1], the condition (i) in Lemma 5.3 corresponding to the condition (*) in [27, Lemma 2.1] implies that there are bijective canonical homomorphisms for any $c > 0$ as follows*

$$H^c(\text{Hom}^\bullet(V^\bullet, \mathcal{F}^\bullet)) \rightarrow \text{Ext}_{\mathcal{X}_B}^c(V^\bullet, \mathcal{F}^\bullet).$$

And the condition (ii) in Lemma 5.3 corresponding to the condition (L_0) in [27, Lemma 2.1] shows that the canonical homomorphisms

$$H^c(\text{Hom}^\bullet(V^\bullet, V^\bullet)) \rightarrow H^c(\text{Hom}^\bullet(V^\bullet, \mathcal{F}^\bullet))$$

are surjective for $c \geq 0$ and bijective for $c > 0$.

Then, we have

Theorem 5.5. *Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Let $\check{\varphi}_A : \mathcal{F}_0 \otimes_k A \rightarrow \mathcal{F}_A$ be a morphism over $\mathcal{X}_A = \mathcal{X} \times_{\text{Spec } k} \text{Spec } A$ extending φ , where \mathcal{F}_A is a coherent sheaf flat over A . Let $\mathbf{I}^\bullet = \{\mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}\}$ and $\mathbf{I}_A^\bullet = \{\mathcal{F}_0 \otimes_k A \xrightarrow{\check{\varphi}_A} \mathcal{F}_A\}$ be the complexes concentrated in degree 0 and 1.*

Then for a given small extension $0 \rightarrow I \rightarrow A' \xrightarrow{\sigma} A \rightarrow 0$, there is an element

$$\omega(\mathbf{I}_A^\bullet) \in \text{Ext}^2(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$$

such that there exists a flat extension of $\check{\varphi}_A$ over $\mathcal{X}_{A'}$ if and only if $\omega(\mathbf{I}_A^\bullet) = 0$. If $\omega(\mathbf{I}_A^\bullet) = 0$, then the space of extensions form a torsor under $\text{Ext}^1(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$.

Proof. Notice that the bounded complex we consider here is a two-term complex \mathbf{I}_A^\bullet with $\mathbf{I}^\bullet = \mathbf{I}_A^\bullet \otimes k$ from a stable pair $(\mathcal{F}_A, \check{\varphi}_A)$ and the isomorphic relations in Definition 2.40 is contained in the equivalent relations in [27, Definition 0.1]. Following the similar argument in the proof of [27, Proposition 2.3] and using Lemma 5.3 and Remark 5.4 to show that the space of extensions is pseudo-torsor under $\text{Ext}^1(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$. The proof is completed by using the similar argument in the proof of [25, Corollary 3.4] or [35, Corollary 3.2.12] to show it is actually a torsor under $\text{Ext}^1(\mathbf{I}^\bullet, \mathbf{I}^\bullet \otimes I)$. \square

Actually, the above result can be generalized to the case of a square zero extension as follows.

Theorem 5.6. *Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/k}^s(\mathcal{F}_0, P, \delta)$. Let $\check{\varphi}_A : \mathcal{F}_0 \otimes_k A \rightarrow \mathcal{F}_A$ be a morphism over $\mathcal{X}_A = \mathcal{X} \times_{\text{Spec } k} \text{Spec } A$ extending φ , where \mathcal{F}_A is a coherent sheaf flat over A . Let $\mathbf{I}_A^\bullet = \{\mathcal{F}_0 \otimes_k A \xrightarrow{\check{\varphi}_A} \mathcal{F}_A\}$ be the complex concentrated on degree 0 and 1. Then for a given square zero extension $0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$, there is an element*

$$\omega(\mathbf{I}_A^\bullet) \in \text{Ext}^2(\mathbf{I}_A^\bullet, \mathbf{I}_A^\bullet \otimes I)$$

such that there exists a flat extension of $\check{\varphi}_A$ over $\mathcal{X}_{A'}$ if and only if $\omega(\mathbf{I}_A^\bullet) = 0$. If $\omega(\mathbf{I}_A^\bullet) = 0$, then the space of extensions form a torsor under $\text{Ext}^1(\mathbf{I}_A^\bullet, \mathbf{I}_A^\bullet \otimes I)$.

Proof. Notice that since $I^2 = 0$, we have a quasi-isomorphism $V^\bullet \rightarrow \mathbf{I}_A^\bullet$ such that $(V^\bullet \otimes I \rightarrow \mathbf{I}_A^\bullet \otimes I, V^\bullet)$ satisfies conditions (i) and (ii) in Lemma 5.3 (with B replaced by A), which will be used to replace the case of the restriction to the closed fiber in the proof of [27, Proposition 2.3] in the following argument. By Remark 5.4, we have $H^i(\text{Hom}^\bullet(V^\bullet, V^\bullet \otimes I)) \cong \text{Ext}^i(\mathbf{I}_A^\bullet, \mathbf{I}_A^\bullet \otimes I)$ for $i \geq 1$. Let $\tilde{d}_{V^\bullet}^i : L_i \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_i) \rightarrow L_{i+1} \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_{i+1})$ be a lift of morphism $d_{V^\bullet}^i : L_i \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_i) \rightarrow L_{i+1} \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_A, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_{i+1})$. Since $V^\bullet = (V^\bullet, d_{V^\bullet}^i)$ is a complex and $I^2 = 0$, the image of $\varrho^i := \tilde{d}_{V^\bullet}^{i+1} \circ \tilde{d}_{V^\bullet}^i$.

is in $(L_{i+2} \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \mathcal{E} \otimes p_{\mathcal{X}_{A'}, \mathcal{X}}^* \pi^* \mathcal{O}_X(-m_{i+2})) \otimes I$. This yields an element $\omega(\mathbf{I}_A^\bullet) \in H^2(\text{Hom}^\bullet(V^\bullet, V^\bullet \otimes I)) \cong \text{Ext}^2(\mathbf{I}_A^\bullet, \mathbf{I}_A^\bullet \otimes I)$. Again, the proof is completed by following the similar argument in the proof of [27, Proposition 2.3] for the pseudo-torsor result and then using the one of [25, Corollary 3.4] or [35, Corollary 3.2.12] for the torsor result. \square

Remark 5.7. *The first order or infinitesimal deformation theory of the complex \mathbf{I}^\bullet is governed by $\text{Hom}(\mathbf{I}^\bullet, \mathcal{F})$ and $\text{Ext}^1(\mathbf{I}^\bullet, \mathcal{F})$ by Theorem 5.1, or $\text{Ext}^1(\mathbf{I}^\bullet, \mathbf{I}^\bullet)$ and $\text{Ext}^2(\mathbf{I}^\bullet, \mathbf{I}^\bullet)$ by Theorem 5.5. As in [53, Section 2.1], there is a map for $i = 0, 1$*

$$\text{Ext}^i(\mathbf{I}^\bullet, \mathcal{F}) \rightarrow \text{Ext}^{i+1}(\mathbf{I}^\bullet, \mathbf{I}^\bullet)$$

obtained by applying the functor $\text{Hom}(\mathbf{I}^\bullet, \cdot)$ to the following distinguished triangle

$$\mathcal{F}[-1] \rightarrow \mathbf{I}^\bullet \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}.$$

5.2. Virtual fundamental classes

In this subsection, assume that \mathcal{F}_0 is torsion free and polynomials δ, P satisfy $\deg \delta \geq \deg P = 1$. Assume that $k = \mathbb{C}$. In order to prove the existence of virtual fundamental classes, it suffices to provide a perfect obstruction theory in the sense of [9, 34]. We will consider some cases of dimension two and three. We begin with the following result.

Lemma 5.8. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 2 over \mathbb{C} . Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{F}_0, P, \delta)$ and $\mathbf{I}^\bullet = \{\mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}\}$ be the complex concentrated in degree 0 and 1. Then we have*

$$\text{Ext}^i(\mathbf{I}^\bullet, \mathcal{F}) = 0, \quad \text{if } i \neq 0, 1.$$

Proof. Follow the similar argument in the proof of [36, Lemma 6.1] and apply Serre duality in [12, Theorem B.7]. \square

Let $\widehat{M}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{F}_0, P, \delta)$ be the moduli space of δ -stable pairs. By Theorem 4.24, there is a universal δ -stable pair which determines a universal

complex

$$\mathbb{I}^\bullet = \{ \mathcal{O}_{\mathcal{X} \times \hat{M}^s} \rightarrow \mathbb{F} \} \in D^b(\mathcal{X} \times \hat{M}^s)$$

where \mathbb{F} is flat over \hat{M}^s . Let $\pi_{\hat{M}^s} : \mathcal{X} \times \hat{M}^s \rightarrow \hat{M}^s$ and $\hat{\pi}_{\mathcal{X}} : \mathcal{X} \times \hat{M}^s \rightarrow \mathcal{X}$ be the natural projections. As in [36, Section 6], due to Theorem 5.1, we will consider computing the following complex

$$R\pi_{\hat{M}^s*} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{F})$$

to obtain the deformation sheaf $\mathcal{D}ef$ and the obstruction sheaf $\mathcal{O}bs$. To resolve \mathbb{I}^\bullet by a finite complex of locally free sheaves, we need the following generalization of [12, Lemma B.3].

Lemma 5.9. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension d over \mathbb{C} with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. For any bounded complex \mathcal{N}^\bullet of coherent sheaves on \mathcal{X} , there is a bounded complex W^\bullet of locally free sheaves of finite rank, which is of the form $W^i = W_i \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_i)$, such that $W^\bullet \rightarrow \mathcal{N}^\bullet$ is a quasi-isomorphism where W_i is a vector space and m_i is an integer sufficiently large for any $i \in \mathbb{Z}$.*

Proof. We combine the argument in the proof of [27, Proposition 1.1] with the finite resolution property in [12, Lemma B.3]. Let l_1 and l_2 be integers such that $\mathcal{N}^i = 0$ for $i > l_1$ and $i < l_2$ where $l_1 \geq l_2$. Choose an integer m_{l_1} sufficiently large such that $\widehat{W}^{l_1} := \widehat{W}_{l_1} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_{l_1}) \rightarrow \mathcal{N}^{l_1}$ is surjective where $\widehat{W}_{l_1} := H^0(X, \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{N}^{l_1})(m_{l_1}))$. Set $K^{l_1} := \mathcal{N}^{l_1}$ and $K^{l_1-1} := \ker(\mathcal{N}^{l_1-1} \oplus \widehat{W}^{l_1} \rightarrow \mathcal{N}^{l_1})$, then we have the following quasi-isomorphism of complexes

$$\begin{aligned} \{ \mathcal{N}^{l_2} \rightarrow \dots \rightarrow \mathcal{N}^{l_1-1} \rightarrow K^{l_1-1} \rightarrow \widehat{W}^{l_1} \} \\ \xrightarrow{\sim} \{ \mathcal{N}^{l_2} \rightarrow \dots \rightarrow \mathcal{N}^{l_1-1} \rightarrow \mathcal{N}^{l_1-1} \rightarrow \mathcal{N}^{l_1} \} \end{aligned}$$

Suppose that K^i and \widehat{W}^{i+1} are defined for $i \geq c$. For K^c , we choose a sufficiently large integer m_c such that $\widehat{W}^c := \widehat{W}_c \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_c) \rightarrow K^c$ is surjective where $\widehat{W}_c := H^0(X, \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, K^c)(m_c))$. Set $K^{c-1} := \ker(\mathcal{N}^{c-1} \oplus \widehat{W}^c \rightarrow K^c)$. By induction, we have the complex $\widehat{W}^\bullet = (\widehat{W}^i, d_{\widehat{W}}^i)$ defined by $\widehat{W}^i = \widehat{W}_i \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_i)$ for $i \leq l_1$ and $\widehat{W}^i = 0$ for $i > l_1$ with $d_{\widehat{W}}^i : \widehat{W}^i \rightarrow K^i \rightarrow \widehat{W}^{i+1}$. Then we have a quasi-isomorphism $\widehat{W}^\bullet \rightarrow \mathcal{N}^\bullet$.

Since $\mathcal{N}^i = 0$ for $i < l_2$, we have $K^i = \ker(\widehat{W}^{i+1} \rightarrow K^{i+1})$ and then there is a locally free resolution for K^{l_2-1} as follows

$$\dots \rightarrow \widehat{W}^{l_2-2} \rightarrow \widehat{W}^{l_2-1} \rightarrow K^{l_2-1} \rightarrow 0.$$

Since K^{l_2-1} is a coherent sheaf on the smooth projective Deligne-Mumford stack \mathcal{X} , then there exists an integer $l_3 \leq l_2 - 1$ such that $K^{l_3-1} := \ker(\widehat{W}^{l_3} \xrightarrow{d_{\widehat{W}}^{l_3}} \widehat{W}^{l_3+1})$ is locally free of finite rank by [12, Lemma B.3] or [17, Theorem 5.1.30]. Then we have a quasi-isomorphism

$$\{\dots 0 \rightarrow K^{l_3-1} \rightarrow \widehat{W}^{l_3} \rightarrow \dots \rightarrow \widehat{W}^{l_1} \rightarrow 0 \dots\} \xrightarrow{\sim} \mathcal{N}^\bullet$$

Since the morphism $K^{l_3-1} \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(m_{l_3}) \rightarrow \widehat{W}^{l_3} \otimes \mathcal{O}_X$ of locally free sheaves is injective, one can choose a vector space W_{l_3-1} such that $K^{l_3-1} \cong W_{l_3-1} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_{l_3})$. The proof is completed by setting

$$\begin{aligned} W^{l_3-1} &:= W_{l_3-1} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m_{l_3}); \\ W^i &:= \widehat{W}^i, \text{ if } l_3 \leq i \leq l_1; \quad W^i := 0, \text{ otherwise.} \end{aligned}$$

□

The locally free resolution for any bounded complex of coherent sheaves in Lemma 5.9 is called very negative if each m_i is chosen very large. We show the following results generalizing [12, Lemmas B.4, B.5 and Proposition B.6] and Serre duality in [12, Theorem B.7], which are useful in our arguments.

Lemma 5.10. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension d over \mathbb{C} . Let $\mathcal{M}^\bullet, \mathcal{N}^\bullet$ and \mathcal{H}^\bullet be bounded complexes of coherent sheaves on \mathcal{X} . Then we have the following functorial isomorphisms*

$$\begin{aligned} R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \otimes^{\mathbf{L}} \mathcal{H}^\bullet &\cong R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{M}^\bullet, \mathcal{N}^\bullet \otimes^{\mathbf{L}} \mathcal{H}^\bullet) \\ R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{M}^\bullet \otimes^{\mathbf{L}} \mathcal{H}^\bullet, \mathcal{N}^\bullet) &\cong R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{M}^\bullet, R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{H}^\bullet, \mathcal{N}^\bullet)) \\ \mathrm{Hom}_{\mathrm{D}^b(\mathcal{X})}(\mathcal{M}^\bullet \otimes^{\mathbf{L}} (\mathcal{H}^\bullet)^\vee, \mathcal{N}^\bullet) &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{X})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet \otimes \mathcal{H}^\bullet) \\ \mathrm{Hom}_{\mathrm{D}^b(\mathcal{X})}(\mathcal{M}^\bullet \otimes^{\mathbf{L}} \mathcal{H}^\bullet, \mathcal{N}^\bullet) &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{X})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet \otimes (\mathcal{H}^\bullet)^\vee) \end{aligned}$$

where $(\mathcal{H}^\bullet)^\vee = R\mathcal{H}om_{\mathcal{X}}^\bullet(\mathcal{H}^\bullet, \mathcal{O}_{\mathcal{X}})$ satisfying $(\mathcal{H}^\bullet)^{\vee\vee} \cong \mathcal{H}^\bullet$.

Proof. The proof follows from Lemma 5.9 and the techniques used in the proofs of [6, Propositions A.86, A.87, A.88]. □

Lemma 5.11. *Let $p : \mathcal{X} \rightarrow \text{Spec } \mathbb{C}$ be a smooth projective Deligne-Mumford stack of dimension d . Let \mathcal{M}^\bullet and \mathcal{N}^\bullet be bounded complexes of coherent sheaves on \mathcal{X} . Then we have*

$$\text{Ext}^i(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \cong \text{Ext}^{d-i}(\mathcal{N}^\bullet, \mathcal{M}^\bullet \otimes \omega_{\mathcal{X}})^\vee.$$

where $\omega_{\mathcal{X}}$ is the canonical line bundle of \mathcal{X} .

Proof. It follows from the similar argument in the proof of [12, Theorem B.7] by using Serre duality for Deligne-Mumford stacks in [43, Corollary 2.10 and Theorem 2.22] and Lemma 5.10. \square

Since $\hat{M}^s = M_{\mathcal{X}/\mathbb{C}}^{ss}(\mathcal{F}_0, P, \delta)$ is a projective scheme by Theorem 4.24 and Lemma 2.24, one can resolve \mathbb{F} and \mathbb{I}^\bullet by finite complexes P^\bullet and J^\bullet of locally free sheaves respectively by Lemma 5.9, and hence $G^\bullet := R\mathcal{H}om(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{F}), \mathcal{O}_{\mathcal{X} \times \hat{M}^s})$ is a bounded complex of coherent sheaves. Again we take a finite complex Q^\bullet of very negative locally free sheaves resolving G^\bullet by Lemma 5.9, then by Lemma 5.10 we have

$$R\pi_{\hat{M}^{s*}} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{F}) \cong R\pi_{\hat{M}^{s*}}(Q^\bullet)^\vee \cong \pi_{\hat{M}^{s*}}(Q^\bullet)^\vee$$

which is a finite complex of locally free sheaves since Q^\bullet is very negative. Denote this complex by D^\bullet . Together with Lemma 5.8, following the similar argument in [36, Section 6], one has a short exact sequence on \hat{M}^s

$$0 \rightarrow \text{Def} \rightarrow \widetilde{D}^0 \rightarrow \widetilde{D}^1 \rightarrow \text{Obs} \rightarrow 0.$$

where \widetilde{D}^0 and \widetilde{D}^1 are locally free sheaves. This implies the obstruction theory is perfect in the sense of [9, 34]. Then we have the following stacky version of [36, Theorem 1.3].

Theorem 5.12. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 2 over \mathbb{C} . Let \mathcal{F}_0 be a torsion free sheaf and $\deg \delta \geq \deg P = 1$. Then the moduli space $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{F}_0, P, \delta)$ of δ -stable pairs admits a virtual fundamental class.*

Next, we will concentrate on investigating the existence of virtual fundamental classes for some special case of dimension three. We start with the following

Lemma 5.13. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 3 over \mathbb{C} . Let $[(\mathcal{F}, \varphi)]$ be a point in the moduli space $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{F}_0, P, \delta)$*

and $\mathbf{I}^\bullet = \{\mathcal{F}_0 \xrightarrow{\varphi} \mathcal{F}\}$ be the complex concentrated in degree 0 and 1. Then we have

$$\text{Ext}^i(\mathbf{I}^\bullet, \mathcal{F}) = 0, \quad \text{if } i \neq 0, 1, 2.$$

Proof. Follow the similar argument in the proof of Lemma 5.8. □

Remark 5.14. The Ext group $\text{Ext}^2(\mathbf{I}^\bullet, \mathcal{F})$ fits into the following short exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}^1(\ker \varphi, \mathcal{F}) &\rightarrow \text{Ext}^3(\text{coker} \varphi, \mathcal{F}) \\ &\rightarrow \text{Ext}^2(\mathbf{I}^\bullet, \mathcal{F}) \rightarrow \text{Ext}^2(\ker \varphi, \mathcal{F}) \rightarrow \text{Ext}^4(\text{coker} \varphi, \mathcal{F}) = 0. \end{aligned}$$

It seems difficult to prove the vanishness of $\text{Ext}^2(\mathbf{I}^\bullet, \mathcal{F})$ even when $\mathcal{F}_0 = \mathcal{O}_X$.

Due to Remark 5.14, in order to define a perfect (two-term) obstruction theory for the moduli space of δ -stable pairs, we will take the deformation and obstruction theories obtained in Theorem 5.5 and Theorem 5.6 as an alternative approach. Assume in addition $\mathcal{F}_0 = \mathcal{O}_X$. In this case, the moduli space of δ -stable pairs parametrizes the stacky version of PT stable pairs by Remark 2.27.

Lemma 5.15. Let \mathcal{X} be a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} . Assume that $[(\mathcal{F}, \varphi)]$ is a point in the moduli space $\overline{M}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_X, P, \delta)$ and $\overline{\mathbf{I}}^\bullet := \{\mathcal{O}_X \xrightarrow{\varphi} \mathcal{F}\}$ is a complex concentrated in degree 0 and 1. Then we have

$$\begin{aligned} \mathcal{E}xt^{\leq -1}(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet) &= 0 = \mathcal{E}xt^{\leq -1}(\overline{\mathbf{I}}^\bullet, \mathcal{O}_X) \quad \text{and} \\ \mathcal{H}om(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet) &= \mathcal{O}_X = \mathcal{H}om(\overline{\mathbf{I}}^\bullet, \mathcal{O}_X). \end{aligned}$$

Proof. Notice that for 1-dimensional pure sheaf \mathcal{F} , we have

$$\begin{aligned} \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \otimes \omega_X &= \mathcal{H}om(\mathcal{F}, \omega_X) = 0 \quad \text{and} \\ \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_X) \otimes \omega_X &= \mathcal{E}xt^1(\mathcal{F}, \omega_X) = 0 \end{aligned}$$

by [12, Proposition C.1 and Lemma B.4]. Then $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = 0 = \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_X)$. And the proof is completed by following the similar argument in the proofs of [53, Lemma 1.15 and Lemma 1.20]. □

The similar argument for the proof of [53, Proposition 1.21] shows that a point $[(\mathcal{F}, \varphi)]$ in \overline{M}^s can be taken as an object $\overline{\mathbf{I}}^\bullet = \{\mathcal{O}_{\mathcal{X}} \xrightarrow{\varphi} \mathcal{F}\}$ in $D^b(\mathcal{X})$. It follows from Lemma 5.15 and the local-to-global spectral sequence that

$$\text{Ext}^{\leq -1}(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet) = 0 \quad \text{and} \quad \text{Hom}(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet) = \mathbb{C}.$$

By Remark 5.7, the first order or infinitesimal deformation theory of the complex $\overline{\mathbf{I}}^\bullet$ is governed by $\text{Ext}^1(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)$ and $\text{Ext}^2(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)$. Now, we have the following result on traceless Ext groups $\text{Ext}^\bullet(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)_0$ (see [63, Section 3] for the definition of the trace map).

Lemma 5.16. *Let \mathcal{X} be a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} . Assume that $[(\mathcal{F}, \varphi)]$ is a point in the moduli space \overline{M}^s and $\overline{\mathbf{I}}^\bullet = \{\mathcal{O}_{\mathcal{X}} \xrightarrow{\varphi} \mathcal{F}\}$ is a complex concentrated in degree 0 and 1. Then we have*

$$\text{Ext}^i(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)_0 = 0 \quad \text{if } i \neq 1, 2.$$

Proof. It remains to show that $\text{Ext}^3(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)_0 = 0$. By Lemma 5.15 and Lemma 5.10, we have

$$\mathcal{E}xt^{\leq -1}(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet \otimes \omega_{\mathcal{X}}) = 0; \quad \mathcal{H}om(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet \otimes \omega_{\mathcal{X}}) \cong \omega_{\mathcal{X}}$$

Using the local-to-global spectral sequence, we have

$$\text{Hom}(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet \otimes \omega_{\mathcal{X}}) \cong H^0(\omega_{\mathcal{X}})$$

By Lemma 5.11, we have

$$\text{Ext}^3(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet) \cong H^3(\mathcal{O}_{\mathcal{X}})$$

which implies $\text{Ext}^3(\overline{\mathbf{I}}^\bullet, \overline{\mathbf{I}}^\bullet)_0 = 0$. □

Next, we follow the similar argument in [20, 25, 53, 68] to construct a perfect obstruction theory with fixed determinant for \overline{M}^s . Since \overline{M}^s is a fine moduli space, there is a universal complex

$$\overline{\mathbf{I}}^\bullet = \{\mathcal{O}_{\mathcal{X} \times \overline{M}^s} \rightarrow \mathbb{F}\} \in D^b(\mathcal{X} \times \overline{M}^s).$$

where \mathbb{F} is flat over \overline{M}^s . Let $\overline{\pi}_{\overline{M}^s} : \mathcal{X} \times \overline{M}^s \rightarrow \overline{M}^s$ and $\overline{\pi}_{\mathcal{X}} : \mathcal{X} \times \overline{M}^s \rightarrow \mathcal{X}$ be the projections. Since \overline{M}^s is a projective scheme by Theorem 4.24 and

Lemma 2.24, we take a finite complex of locally free sheaves A^\bullet resolving $\bar{\mathbb{I}}^\bullet$ such that

$$R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet) \cong (A^\bullet)^\vee \otimes A^\bullet \cong \mathcal{O}_{\mathcal{X} \times \bar{M}^s} \oplus ((A^\bullet)^\vee \otimes A^\bullet)_0.$$

As in [53, Section 2.3] or [25, Section 4.2], we have $R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet) \cong \mathcal{O}_{\mathcal{X} \times \bar{M}^s} \oplus R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0$ and $R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \cong ((A^\bullet)^\vee \otimes A^\bullet)_0$. Let $\mathbb{L}_{\mathcal{X} \times \bar{M}^s}$ be the truncated cotangent complex (two-term complex quasi-isomorphic to the truncation $\tau^{\geq -1}L_{\mathcal{X} \times \bar{M}^s}^\bullet$ of Illusie’s cotangent complex $L_{\mathcal{X} \times \bar{M}^s}^\bullet$, see also [25, Definition 2.1]), we have the truncated Atiyah class of $\bar{\mathbb{I}}^\bullet$

$$\text{At}(\bar{\mathbb{I}}^\bullet) \in \text{Ext}^1(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet \otimes \mathbb{L}_{\mathcal{X} \times \bar{M}^s})$$

Composing the map $\bar{\mathbb{I}}^\bullet \rightarrow \bar{\mathbb{I}}^\bullet \otimes \mathbb{L}_{\mathcal{X} \times \bar{M}^s}[1]$ with the projection $\mathbb{L}_{\mathcal{X} \times \bar{M}^s} \rightarrow \bar{\pi}_{\bar{M}^s}^* \mathbb{L}_{\bar{M}^s}$ and restricting to the traceless part of $R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)$, we have a class in

$$\text{Ext}^1(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0, \bar{\pi}_{\bar{M}^s}^* \mathbb{L}_{\bar{M}^s}).$$

Tensoring the map $R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \rightarrow \bar{\pi}_{\bar{M}^s}^* \mathbb{L}_{\bar{M}^s}[1]$ with $\bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}}$, together with smooth Serre duality in [43, Theorem 2.22], we have

$$R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}} \rightarrow \bar{\pi}_{\bar{M}^s}^* \mathbb{L}_{\bar{M}^s} \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}}[1] = \bar{\pi}_{\bar{M}^s}^! \mathbb{L}_{\bar{M}^s}[-2].$$

By Serre duality for Deligne-Mumford stacks in [43, Corollary 2.10], we have

$$\Phi : E^\bullet := R\bar{\pi}_{\bar{M}^s,*}(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2] \rightarrow \mathbb{L}_{\bar{M}^s}.$$

Theorem 5.17. *In the notation of [9], the map Φ is a perfect obstruction theory for \bar{M}^s .*

Proof. We combine the argument in the proof of [25, Theorem 4.1] and the one in [68, Theorem 6.2]. Let $T \rightarrow \bar{T}$ be an extension of k -schemes with ideal sheaf J such that $J^2 = 0$ and $g : T \rightarrow \bar{M}^s$ be a morphism of k -schemes. An element $\omega(g) \in \text{Ext}^1(g^* \mathbb{L}_{\bar{M}^s}, J)$ is given by composing the natural map $g^* \mathbb{L}_{\bar{M}^s} \rightarrow \mathbb{L}_T$ with the truncated Kodaira-Spencer class of $T \subset \bar{T}$

$$\kappa(T/\bar{T}) \in \text{Ext}^1(\mathbb{L}_T, J).$$

The composition of $\omega(g)$ with $g^* \Phi$ gives an element

$$\Phi^* \omega(g) \in \text{Ext}^1(g^* E^\bullet, J).$$

By the argument in the proof of [9, Theorem 4.5], to show that Φ is an obstruction theory, we should prove that the obstruction $\Phi^*\omega(g)$ vanishes if and only if an extension \bar{g} of g to \bar{T} exists, and when $\Phi^*\omega(g) = 0$ the space of extensions form a torsor under $\text{Ext}^0(g^*E^\bullet, J)$.

Set $\hat{g} := \text{id}_{\mathcal{X}} \times g : \mathcal{X} \times T \rightarrow \mathcal{X} \times \bar{M}^s$ and let $p : \mathcal{X} \times T \rightarrow \mathcal{X}$ and $q : \mathcal{X} \times T \rightarrow T$ be the natural projections. Then we have

$$g^*(R\pi_{\bar{M}^s*}(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \pi_{\mathcal{X}}^*\omega_{\mathcal{X}})) \cong Rq_*(R\mathcal{H}om(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet)_0 \otimes p^*\omega_{\mathcal{X}}).$$

Using Serre duality in [43, Corollary 2.10] for the map q , we have

$$\begin{aligned} \Phi^*\omega(g) &\in \text{Ext}^1(Rq_*(R\mathcal{H}om(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet)_0 \otimes p^*\omega_{\mathcal{X}})[2], J) \\ &\cong \text{Ext}^2(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet \otimes q^*J)_0 \end{aligned}$$

and

$$\begin{aligned} \text{Ext}^0(g^*E^\bullet, J) &\cong \text{Ext}^0(Rq_*(R\mathcal{H}om(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet)_0 \otimes p^*\omega_{\mathcal{X}})[2], J) \\ &\cong \text{Ext}^1(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet \otimes q^*J)_0. \end{aligned}$$

Then the obstruction $\Phi^*\omega(g)$ is the traceless part of the obstruction class which is a product of a truncated Atiyah class $\overline{\text{At}}(\hat{g}^*\bar{\mathbb{I}}^\bullet)$ in $\text{Ext}^1(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet \otimes q^*\mathbb{L}_T)$ and a truncated Kodaira-Spencer class $q^*\kappa(T/\bar{T})$ in $\text{Ext}^1(q^*\mathbb{L}_T, q^*J)$, that is,

$$\Phi^*\omega(g) = ((\text{id}_{\hat{g}^*\bar{\mathbb{I}}^\bullet} \otimes q^*\kappa(T/\bar{T})) \circ \overline{\text{At}}(\hat{g}^*\bar{\mathbb{I}}^\bullet))_0$$

As in the argument of the proof in [9, Theorem 4.5], to prove that Φ is an obstruction theory we only need to consider the case when both T and \bar{T} are affine. The similar argument in [25, Section 3] shows that the obstruction class to extending $\hat{g}^*\bar{\mathbb{I}}^\bullet$ from $\mathcal{X} \times T$ to $\mathcal{X} \times \bar{T}$ defined as a square of the differential in the proof of Theorem 5.6 (see also [35, Section 3.3]) up to a sign is exactly the obstruction class which is the product of a truncated Atiyah class and a truncated Kodaira-Spencer class. Using Lemma 5.9 to resolve the extension of $\hat{g}^*\bar{\mathbb{I}}^\bullet$ on $\mathcal{X} \times \bar{T}$ (flat over \bar{T}) and its restriction $\hat{g}^*\bar{\mathbb{I}}^\bullet$ on $\mathcal{X} \times T$, the similar argument in the proof of [63, Theorem 3.23] shows that the traceless part is the deformation and obstruction with fixed determinant $\det(\hat{g}^*\bar{\mathbb{I}}^\bullet)$. By Theorem 5.6, $\Phi^*\omega(g) = 0$ if and only if there is an extension of the complex $\hat{g}^*\bar{\mathbb{I}}^\bullet$ from $\mathcal{X} \times T$ to $\mathcal{X} \times \bar{T}$, and in that case the space of extensions with fixed determinant forms a torsor under $\text{Ext}^1(\hat{g}^*\bar{\mathbb{I}}^\bullet, \hat{g}^*\bar{\mathbb{I}}^\bullet \otimes q^*J)_0$. Since \bar{M}^s is a fine moduli space and $\bar{\mathbb{I}}^\bullet$ is a universal complex, then an extension \bar{g} of g

to \bar{T} exists if and only if $\hat{g}^*\bar{\mathbb{I}}^\bullet$ extends from $\mathcal{X} \times T$ to $\mathcal{X} \times \bar{T}$. Thus Φ is an obstruction theory by [9, Theorem 4.5].

Next, we will show that the complex $R\bar{\pi}_{\bar{M}^s}^* R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0$ is quasi-isomorphic to a perfect 2-term complex of locally free sheaves with amplitude contained in $[1, 2]$. Since $R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \cong ((A^\bullet)^\vee \otimes A^\bullet)_0$ is a bounded complex of coherent sheaves, as in the 2-dimensional case, one can take a finite very negative locally free resolution B^\bullet of the bounded complex $R\mathcal{H}om(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0, \mathcal{O}_{\mathcal{X} \times \bar{M}^s})$. Then

$$R\bar{\pi}_{\bar{M}^s}^* R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \cong R\bar{\pi}_{\bar{M}^s}^*(B^\bullet)^\vee \cong \bar{\pi}_{\bar{M}^s}^*(B^\bullet)^\vee$$

is a finite complex of locally free sheaves. Denoted this complex by N^\bullet . By cohomology and base change theorem (cf. [21]) and Lemma 5.16, the complex N^\bullet has cohomology only in degree 1 and 2. By the standard argument in the proof of [53, Lemma 2.1] or [25, Lemma 4.2] for trimming the complex N^\bullet , one can show that $R\bar{\pi}_{\bar{M}^s}^* R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0$ is quasi-isomorphic to 2-term complex of locally free sheaves concentrated only in degree 1 and 2. This implies that $R\bar{\pi}_{\bar{M}^s}^*(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2]$ is perfect of perfect amplitude contained in $[-1, 0]$ in the notation of [9]. Then Φ is perfect. □

Corollary 5.18. *Let \mathcal{X} be a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} . Assume that polynomials δ and P satisfy $\deg \delta \geq \deg P = 1$. Then there exists a virtual fundamental class $[\bar{M}^s]^\text{vir} \in A_{\text{vdim}}(\bar{M}^s)$ of virtual dimension $\text{vdim} = \text{rk}(E^\bullet)$ where $\bar{M}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)$ and $E^\bullet := R\bar{\pi}_{\bar{M}^s}^*(R\mathcal{H}om(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0 \otimes \bar{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2]$.*

Remark 5.19. *Since \bar{M}^s is a projective scheme, then it is proper and $[\bar{M}^s]^\text{vir}$ can be integrated. By Remark 2.27, the intersection theory on $[\bar{M}^s]^\text{vir}$ actually produces a stacky version of Pandharipande-Thomas theory.*

5.3. Orbifold Pandharipande-Thomas theory

In [7], the authors give a definition of orbifold Pandharipande-Thomas invariants for Calabi-Yau 3-orbifolds, which are smooth projective Deligne-Mumford stacks \mathcal{Y} with generically trivial stabilizer groups satisfying $\omega_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{Y}}$ and $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 0$. Compared with their definition employing Behrend’s weighted Euler characteristic [8], our approach here is to take integrations against virtual fundamental classes for more general cases. Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 3 over \mathbb{C} with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. We first

recall some notation in [7, Section 2]. Let $\text{Perf}(\mathcal{X})$ be the subcategory of $D^b(\mathcal{X})$ with objects being perfect complexes, that is, those locally isomorphic to a bounded complex of locally free sheaves. By Lemma 5.9, we have $\text{Perf}(\mathcal{X}) = D^b(\mathcal{X})$. Define the Euler pairing

$$\chi(F^\bullet, G^\bullet) = \sum_i (-1)^i \dim \text{Hom}(F^\bullet, G^\bullet[i])$$

for any $F^\bullet \in \text{Perf}(\mathcal{X})$ and $G^\bullet \in D^b(\mathcal{X})$. The complex G^\bullet is called numerically trivial if $\chi(F^\bullet, G^\bullet) = 0$ for all $F^\bullet \in \text{Perf}(\mathcal{X})$. Denote by $K(\mathcal{X}) = K(D^b(\mathcal{X})) = K(\text{Coh}(\mathcal{X}))$ the Grothendieck group of \mathcal{X} . The numerical Grothendieck group $N(\mathcal{X})$ is define to be the quotient of $K(\mathcal{X})$ by the subgroup generated by numerically trivial complexes. Let $\text{Coh}_{\leq d}(\mathcal{X}) \subset \text{Coh}(\mathcal{X})$ be the subcategory of sheaves supported in dimension at most d . Define $N_{\leq d}(\mathcal{X}) \subset N(\mathcal{X})$ as the subgroup generated by classes of sheaves in $\text{Coh}_{\leq d}(\mathcal{X})$. Set $N_d(\mathcal{X}) = N_{\leq d}(\mathcal{X})/N_{\leq d-1}(\mathcal{X})$. One can choose a splitting of $N_{\leq 1}(\mathcal{X})$ as follows

$$N_{\leq 1}(\mathcal{X}) \cong N_1(\mathcal{X}) \oplus N_0(\mathcal{X}).$$

Given a class $\beta = (\beta_1, \beta_0) \in N_{\leq 1}(\mathcal{X})$ with a sheaf $\mathcal{F} \in \text{Coh}_{\leq 1}(\mathcal{X})$ satisfying $[\mathcal{F}] = \beta$ where $\beta_i \in N_i(\mathcal{X})$ for $i = 0, 1$, then the modified Hilbert polynomial of \mathcal{F} is

$$P_{\mathcal{E}}(\mathcal{F})(m) = \chi(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^* \mathcal{O}_X(m)) := l(\mathcal{F}) \cdot m + \text{deg}(\mathcal{F}).$$

where $\text{deg } \mathcal{F} = \chi(F_{\mathcal{E}}(\mathcal{F}))$. By the definition of numerical Grothendieck group, $P_{\mathcal{E}}(\mathcal{F})(m)$ is independent of the choice of representative \mathcal{F} in $[\mathcal{F}]$. Thus $P_{\mathcal{E}}(\beta)(m)$, $l(\beta)$ and $\text{deg}(\beta)$ (or $\chi(F_{\mathcal{E}}(\beta))$) are well defined for any class $\beta \in N_{\leq 1}(\mathcal{X})$ and we have $P_{\mathcal{E}}(\beta)(m) = l(\beta) \cdot m + \text{deg}(\beta)$.

Notice that the sheaf \mathcal{F} underlying a δ -stable pair (\mathcal{F}, φ) is pure of dimension one. Consider any fixed nonzero class $\beta \in N_{\leq 1}(\mathcal{X})$, we have the corresponding degree one polynomial $P_{\mathcal{E}}(\beta)(m) = P(m)$. For such a polynomial P , and let δ be a rational polynomial with positive leading coefficient and $\text{deg } \delta \geq 1 = \text{deg } P$, we have the subfunctor $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^{(s)s}(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ of $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^{(s)s}(\mathcal{O}_{\mathcal{X}}, P, \delta)$ defined as follows. For any \mathbb{C} -scheme S of finite type, $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^{(s)s}(\mathcal{O}_{\mathcal{X}}, \beta, \delta)(S)$ is the set of isomorphism classes of flat families of nondegenerate δ -(semi)stable pairs (\mathcal{F}, φ) with the fixed numerical class β parametrized by a scheme S , that is, such a flat family (\mathcal{F}, φ) satisfies that for each point $s \in S$, the pair $(\mathcal{F}_s, \varphi|_{(\pi_{\mathcal{X}}^* \mathcal{O}_{\mathcal{X}})_s})$ is a δ -(semi)stable pair with

$[\mathcal{F}_s] = \beta$ (hence with the modified Hilbert polynomial $P_{\tilde{\pi}_s^* \mathcal{E}}(\mathcal{F}_s) = P$ where $\tilde{\pi}_s : \mathcal{X} \times \text{Spec}(k(s)) \rightarrow \mathcal{X}$ is the projection). See Definition 4.8 for more details.

Following the same argument in Section 3 and Section 4 for the functor $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^{(s)s}(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ and the corresponding δ -(semi)stable pairs with fixed numerical class β , and noticing Lemma 2.24, one has

Theorem 5.20. *Let \mathcal{X} be a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} with a moduli scheme $\pi : \mathcal{X} \rightarrow X$ and a polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Assume $0 \neq \beta \in N_{\leq 1}(\mathcal{X})$ and the rational polynomial δ satisfy $\deg \delta \geq 1$. Then there is a projective scheme $\overline{M}_{\beta}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$, which is a fine moduli space for the moduli functor $\mathcal{M}_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$.*

Let $\tilde{\pi}_{\overline{M}_{\beta}^s} : \mathcal{X} \times \overline{M}_{\beta}^s \rightarrow \overline{M}_{\beta}^s$ and $\tilde{\pi}_{\mathcal{X}} : \mathcal{X} \times \overline{M}_{\beta}^s \rightarrow \mathcal{X}$ be the projections. Denote by $\overline{\mathbb{I}}_{\beta}^{\bullet} := \{\mathcal{O}_{\mathcal{X} \times \overline{M}_{\beta}^s} \rightarrow \mathbb{F}\} \in \mathbf{D}^b(\mathcal{X} \times \overline{M}_{\beta}^s)$ the universal complex where \mathbb{F} is the universal sheaf on $\mathcal{X} \times \overline{M}_{\beta}^s$. It follows from the same argument in Section 5.1 and Section 5.2 for the moduli space \overline{M}_{β}^s that

Theorem 5.21. *In the sense of [9], the map*

$$\Phi_{\beta} : E_{\beta}^{\bullet} := R\tilde{\pi}_{\overline{M}_{\beta}^s}^*(R\mathcal{H}om(\overline{\mathbb{I}}_{\beta}^{\bullet}, \overline{\mathbb{I}}_{\beta}^{\bullet})_0 \otimes \tilde{\pi}_{\mathcal{X}}^* \omega_{\mathcal{X}})[2] \rightarrow \mathbb{L}_{\overline{M}_{\beta}^s}.$$

is a perfect obstruction theory for $\overline{M}_{\beta}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$. And there exists a virtual fundamental class $[\overline{M}_{\beta}^s]^{\text{vir}} \in A_{\text{vdim}}(\overline{M}_{\beta}^s)$ of virtual dimension $\text{vdim} = \text{rk}(E_{\beta}^{\bullet})$.

Set

$$S_{P,\delta} = \{\beta = [\mathcal{F}] : [(\mathcal{F}, \varphi)] \in M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)\}.$$

Then we have the following decomposition as a disjoint union

$$M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta) = \coprod_{\beta \in S_{P,\delta}} M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta).$$

Next, we will follow the similar definition of Donaldson-Thomas invariants in [68, Section 6] to give a stacky version of Pandharipande-Thomas invariants. We briefly recall some notation and definitions in [65, Section 2 and Appendix A] to define certain Chern character. Let $I\mathcal{X}$ be the inertia stack, which is defined to be the fiber product $I\mathcal{X} := \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$

where $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is the diagonal map. Its underlying category has objects of the form $\text{Ob}(I\mathcal{X}) = \{(x, g) | x \in \text{Ob}(\mathcal{X}), g \in \text{Aut}_{\mathcal{X}}(x)\}$. Then there is a natural projection $\pi_0 : I\mathcal{X} \rightarrow \mathcal{X}$ with the map $\pi_0((x, g)) = x$ on the level of objects and we have a decomposition of $I\mathcal{X}$ as a disjoint union $I\mathcal{X} := \coprod_{i \in \mathcal{J}} \mathcal{X}_i$ for some index set \mathcal{J} . We also have a canonical involution $\iota : I\mathcal{X} \rightarrow I\mathcal{X}$ with the map $\iota((x, g)) = (x, g^{-1})$ on objects. For any $(x, g) \in \mathcal{X}_i$, one has a decomposition of tangent space $T_x \mathcal{X} = \bigoplus_{0 \leq t < r_i} U^{(t)}$ where $U^{(t)}$ is an eigenspace with eigenvalue $\zeta_{r_i}^t, 0 \leq t < r_i$ and $\zeta_{r_i} = \exp(\frac{2\pi i}{r_i})$. Define $\text{age}_i := \frac{1}{r_i} \sum_{0 \leq t < r_i} t \cdot \dim_{\mathbb{C}} U^{(t)}$. Similarly, for any vector bundle W on $I\mathcal{X}$, there exists a decomposition of W as follows

$$W = \bigoplus_{\zeta} W^{(\zeta)}$$

where $W^{(\zeta)}$ is an eigenbundle with the eigenvalue ζ .

Definition 5.22. *The map $\rho : K(I\mathcal{X}) \rightarrow K(I\mathcal{X})_{\mathbb{C}}$ is defined to be*

$$\rho(W) := \sum_{\zeta} \zeta W^{(\zeta)} \in K(I\mathcal{X})_{\mathbb{C}}.$$

Define $\tilde{\text{ch}} : K(\mathcal{X}) \rightarrow A^*(I\mathcal{X})_{\mathbb{C}}$ to be

$$\tilde{\text{ch}}(V) := \text{ch}(\rho(\pi_0^* V))$$

where ch is the usual Chern character.

The orbifold or Chen-Ruan cohomology of \mathcal{X} (cf. [2, Section 7.3]) is defined as

$$A^*_{\text{orb}}(\mathcal{X}) := \bigoplus_i A^{*- \text{age}_i}(\mathcal{X}_i)$$

where age_i is the degree shift number. Define orbifold Chern character $\tilde{\text{ch}}^{\text{orb}} : K(\mathcal{X}) \rightarrow A^*_{\text{orb}}(\mathcal{X})$ as

$$\tilde{\text{ch}}^{\text{orb}}_k \Big|_{\mathcal{X}_i} := \tilde{\text{ch}}_{k - \text{age}_i} \Big|_{\mathcal{X}_i}.$$

For any $\gamma \in A^l_{\text{orb}}(\mathcal{X})$, define the operators

$$\tilde{\text{ch}}^{\text{orb}}_{k+2}(\gamma) : A_*(\overline{M}^s_{\beta}) \rightarrow A_{*-k+1-l}(\overline{M}^s_{\beta})$$

to be

$$\tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma)(\xi) := \pi_{2*} \left(\tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{F}) \cdot \iota^* \pi_1^* \gamma \cap \pi_2^* \xi \right)$$

where \mathbb{F} is the universal complex on $\mathcal{X} \times \overline{M}_\beta^s$ and the maps π_1, π_2 are the natural projections from $I\mathcal{X} \times \overline{M}_\beta^s$ to the first and second factor respectively. The operator $\tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma)$ has the degree changed as above due to the identity $\text{age}_i + \text{age}_{\iota(i)} = \dim_{\mathbb{C}} \mathcal{X} - \dim_{\mathbb{C}} \mathcal{X}_i$ (cf. [16, Lemma 3.2.1]). To follow the similar definition of Pandharipande-Thomas invariants on nonsingular projective 3-folds in [54, Section 0.5], we use the notation $\overline{M}_{n,\beta}^s := M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, n, \beta, \delta)$ to denote \overline{M}_β^s , which is the moduli space of orbifold PT stable pairs with $[\mathcal{F}] = \beta = (\beta_1, \beta_0) \in N_{\leq 1}(\mathcal{X})$ and $\chi(F_{\mathcal{E}}(\mathcal{F})) = n$ by Remark 2.27.

Definition 5.23. Given $\gamma_i \in A_{\text{orb}}^*(\mathcal{X})$, $1 \leq i \leq r$, define the Pandharipande-Thomas invariants with descendents as

$$\begin{aligned} \left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{n,\beta}^{\mathcal{X}} &:= \int_{[M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, n, \beta, \delta)]^{\text{vir}}} \prod_{i=1}^r \tilde{\text{ch}}_{k_i+2}^{\text{orb}}(\gamma_i) \\ &= \int_{M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, n, \beta, \delta)} \prod_{i=1}^r \tilde{\text{ch}}_{k_i+2}^{\text{orb}}(\gamma_i) ([M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, n, \beta, \delta)]^{\text{vir}}) \end{aligned}$$

The partition function is defined to be

$$Z_{\text{PT},\beta_1} \left(\prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) := \sum_n \left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{n,\beta}^{\mathcal{X}} q^n$$

Remark 5.24. In the proof of Lemma 3.2, we have a short exact sequence $0 \rightarrow F_{\mathcal{E}}(\text{im}\varphi) \rightarrow F_{\mathcal{E}}(\mathcal{F}) \rightarrow F_{\mathcal{E}}(\text{coker}\varphi) \rightarrow 0$, where $\text{coker}\varphi$ and $F_{\mathcal{E}}(\text{coker}\varphi)$ are 0-dimensional. Then we have

$$n = \chi(F_{\mathcal{E}}(\mathcal{F})) = \chi(F_{\mathcal{E}}(\text{im}\varphi)) + \chi(F_{\mathcal{E}}(\text{coker}\varphi)) \geq \chi(F_{\mathcal{E}}(\text{im}\varphi)).$$

For a fixed β_1 , we have no idea wheather $\overline{M}_{n,\beta}^s$ is empty for n very negative. If one can show the boundedness of $\chi(F_{\mathcal{E}}(\text{im}\varphi))$ from below, then $Z_{\text{PT},\beta_1} \left(\prod_{i=1}^r \tau_{k_i}(\gamma_i) \right)$ is a Laurent series in q and hence one can further study its rationality as the one of PT stable pair invariants (e.g., [10, 48–50, 64]). Alternatively, one can define the partition function as in [7].

To conclude this section, we consider some special cases as follows. If \mathcal{X} is a 3-dimensional smooth projective Deligne-Mumford stack over \mathbb{C} satisfying

$\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$. In this case, using Serre duality for Deligne-Mumford stacks, we have two isomorphisms

$$E^{\bullet} \xrightarrow{\theta} E^{\bullet \vee}[1]; \quad E_{\beta}^{\bullet} \xrightarrow{\theta_{\beta}} E_{\beta}^{\bullet \vee}[1]$$

satisfying $\theta^{\vee}[1] = \theta$ and $\theta_{\beta}^{\vee}[1] = \theta_{\beta}$. This shows that two perfect obstruction theories are symmetric in the sense of [8]. Then we have $\text{rk}(E^{\bullet}) = \text{rk}(E_{\beta}^{\bullet}) = 0$. Then $[M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)]^{\text{vir}}$ and $[M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)]^{\text{vir}}$ are 0-cycles. Let $\nu_{\overline{M}^s}$ and $\nu_{\overline{M}_{\beta}^s}$ be the Behrend’s constructible functions in [8] on $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)$ and $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ respectively. Since $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)$ and $M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ are proper, by [8, Theorem 4.18], we have the following

Definition 5.25. *Let \mathcal{X} be a smooth projective Deligne-Mumford stack of dimension 3 satisfying $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$. We define Pandharipande-Thomas invariants of \mathcal{X} corresponding to P and β as follows*

$$\begin{aligned} \text{PT}(\mathcal{O}_{\mathcal{X}}, P, \delta) &:= \chi(M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta), \nu_{\overline{M}^s}) = \text{deg}([M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, P, \delta)]^{\text{vir}}), \\ \text{PT}(\mathcal{O}_{\mathcal{X}}, \beta, \delta) &:= \chi(M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta), \nu_{\overline{M}_{\beta}^s}) = \text{deg}([M_{\mathcal{X}/\mathbb{C}}^s(\mathcal{O}_{\mathcal{X}}, \beta, \delta)]^{\text{vir}}). \end{aligned}$$

Remark 5.26. *When \mathcal{X} is a 3-dimensional Calabi-Yau orbifold, the invariant $\text{PT}(\mathcal{O}_{\mathcal{X}}, \beta, \delta)$ in the above definition is corresponding to $\text{PT}(\mathcal{X})_{\beta}$ defined in [7, Section 1 (1.4)].*

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