Math. Res. Lett.
Volume 30, Number 5, 1299–1333, 2023

H^s bounds for the derivative nonlinear Schrödinger equation

Hajer Bahouri, Trevor M. Leslie, and Galina Perelman

We study the derivative nonlinear Schrödinger equation on the real line and obtain global-in-time bounds on high order Sobolev norms.

1. Introduction

We consider the Cauchy problem for the derivative nonlinear Schrödinger equation (DNLS) on the real line $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = -i\partial_x(|u|^2 u), \\ \qquad u\big|_{t=0} = u_0 \in H^s(\mathbb{R}),\ s \geq \frac{1}{2}. \end{cases} \tag{1}$$

We remark right away that the DNLS is L^2 critical, as it is invariant under the scaling

$$u(t,x) \mapsto u_\mu(t,x) := \sqrt{\mu}\,u(\mu^2 t, \mu x), \qquad \mu > 0. \tag{2}$$

The DNLS equation was introduced by Mio-Ogino-Minami-Takeda and Mjølhus [21, 22] as a model for studying magnetohydrodynamics, and it has received a great deal of attention from the mathematics community after being shown to be completely integrable by Kaup-Newell [14]. The infinitely many conserved quantities admitted by the DNLS equation play an important role in the wellposedness theory. The first three—the mass, momentum, and energy—are as follows.

$$M(u) := \int_{\mathbb{R}} |u|^2 \,\mathrm{d}x, \tag{3}$$

$$P(u) := \operatorname{Im} \int_{\mathbb{R}} \overline{u} u_x \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} |u|^4 \,\mathrm{d}x, \tag{4}$$

$$E(u) := \int_{\mathbb{R}} \big(|u_x|^2 - \frac{3}{2} \operatorname{Im}(|u|^2 u \overline{u}_x) + \frac{1}{2}|u|^6\big) \,\mathrm{d}x. \tag{5}$$

Before stating our main result, let us give a very brief review of what is known about the wellposedness of the DNLS equation. More detailed overviews can be found, for example, in the introductions of [2], [15], and [9]. Local wellposedness in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$ was proven by Takaoka [27], improving earlier work [23] by Ozawa. On the other hand, for $s < \frac{1}{2}$, the uniform continuity of the data-to-solution map fails in $H^s(\mathbb{R})$ [3, 28]. One can, however, close the $\frac{1}{2}$-derivative gap between the $H^{\frac{1}{2}}$ threshold and the critical space $L^2(\mathbb{R})$ by working in more general Fourier-Lebesgue spaces, c.f. Grünrock [6] and references therein.

A line of results, due to Hayashi-Ozawa [8], Colliander-Keel-Staffilani-Takaoka-Tao [4], Wu [30], and Guo-Wu [7], establishes global well-posedness of the DNLS equation in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$, for initial data having mass less than 4π. Another line (Pelinovsky-Saalmann-Shimabukuro [24], Pelinovsky-Shimabukuro [25], and Jenkins-Liu-Perry-Sulem [11–13]) uses inverse scattering techniques to establish global wellposedness under stronger regularity and decay assumptions on the initial data, but without a smallness requirement on the mass.

The first and third authors proved in [2] that the DNLS equation is globally well-posed in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$ and that solutions generated from $H^{\frac{1}{2}}$ initial data remain bounded in $H^{\frac{1}{2}}(\mathbb{R})$ for all time. There have also been some recent works below the aforementioned $s = \frac{1}{2}$ threshold of uniform H^s continuity with respect to initial data [3, 28]. Klaus-Schippa [18] gave H^s a priori estimates for $0 < s < \frac{1}{2}$ in the case of small mass, Killip-Ntekoume-Vişan [15] improved the small mass assumption to 4π and furthermore proved a global wellposedness result in $H^s(\mathbb{R})$, $\frac{1}{6} \leq s < \frac{1}{2}$, for initial data with mass less than 4π. Very recently, Harrop-Griffiths, Killip, and Vişan [10] have removed the small mass assumption both from their H^s a priori bounds, $0 < s < \frac{1}{2}$, as well as from their global wellposedness result in $H^s(\mathbb{R})$ with $\frac{1}{6} \leq s < \frac{1}{2}$.

Finally, during the review period of the present paper, Harrop-Griffiths, Killip, Ntekoume, and Vişan [9] have established wellposedness of the DNLS in $L^2(\mathbb{R})$, and consequently in $H^s(\mathbb{R})$ for all $s \geq 0$.

In this paper, we are concerned with the global-in-time boundedness of solutions to the DNLS equation in H^s spaces, $s \geq \frac{1}{2}$, our goal being to propagate the lower regularity bounds mentioned above to higher regularities. Our main result is:

Theorem 1.1. *Suppose u is a solution to the DNLS equation with initial data $u_0 \in H^s(\mathbb{R})$, with $s \geq \frac{1}{2}$. There exists a finite positive constant $C = C(s, \|u_0\|_{H^s(\mathbb{R})})$, such that*[1]

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s(\mathbb{R})} \leq C(s, \|u_0\|_{H^s(\mathbb{R})}).$$

The main idea is to take advantage of the complete integrability of the equation and to exhibit conserved quantities behaving at leading order as $\|u(t)\|^2_{\dot{H}^s(\mathbb{R})}$ which, together with the low regularity bounds of [10], allow to control the higher Sobolev norms. As in [2], [10], the present work relies heavily on the conservation of the transmission coefficient for the spectral problem associated to the DNLS equation. This property has already been used in many other works; of particular relevance to us are the papers of Gérard [5], Killip-Vişan-Zhang [17], Killip-Vişan [16], and Koch-Tataru [19], on the cubic NLS and KdV equations.

Note that by continuity of the flow, and the preservation of the Schwartz class under the flow, we lose nothing by restricting attention to the Schwartz class; we will thus work exclusively with Schwartz functions for the remainder of the manuscript. We will also suppress the time dependence when it does not play a role.

One can easily prove Theorem 1.1 in the special case $s = 1$, using the conserved quantity $E(u)$. Indeed, simply rearranging (5) yields

$$\|u\|^2_{\dot{H}^1(\mathbb{R})} = E(u) - \frac{1}{2}\|u\|^6_{L^6(\mathbb{R})} + \frac{3}{2}\int_{\mathbb{R}} \operatorname{Im}(|u|^2 u\overline{u}_x)\,\mathrm{d}x.$$

Clearly, the last term can be bounded above in absolute value by $\frac{1}{2}\|u\|^2_{\dot{H}^1(\mathbb{R})} + C\|u\|^6_{L^6(\mathbb{R})}$, whence the desired bound follows for example, from the embedding $\dot{H}^{\frac{1}{3}}(\mathbb{R}) \hookrightarrow L^6(\mathbb{R})$ and the estimate $\|u(t)\|_{H^{\frac{1}{3}}(\mathbb{R})} \leq C(\|u_0\|_{H^{\frac{1}{3}}(\mathbb{R})})$ of [10].

[1] Let us mention that in the case of $s = \frac{1}{2}$ this uniform bound can be also deduced from the analysis of [2].

The higher-order Sobolev norms of integer order can be dealt with similarly, once we have a formula for the corresponding higher-order conserved quantities. We will show that for any nonnegative integer ℓ, one of the conserved quantities is equal to a constant multiple of $\|u\|^2_{\dot{H}^\ell(\mathbb{R})}$, plus terms which are of lower order. For noninteger s, we will use a sort of 'generalized energy', comparable to $\|u\|^2_{\dot{H}^s(\mathbb{R})}$, that will be defined in terms of the transmission coefficient of the DNLS spectral problem. We sketch presently the background necessary to define these objects precisely; for more details, see, for example, [1, 11–14, 20, 25, 29].

The DNLS equation can be obtained as a compatibility condition of the following system [14]:

$$\begin{aligned} \partial_x \psi &= \mathcal{U}(\lambda)\psi, \\ \partial_t \psi &= \Upsilon(\lambda)\psi. \end{aligned} \tag{6}$$

Here $\lambda \in \mathbb{C}$ is a spectral parameter, independent of t and x, and $\psi = \psi(t,x,\lambda)$ is $\mathbb{C}^2$-valued. The operators $\mathcal{U}(\lambda)$ and $\Upsilon(\lambda)$ are defined by

$$\begin{aligned} \mathcal{U}(\lambda) &= -i\sigma_3(\lambda^2 + i\lambda U), \\ \Upsilon(\lambda) &= -i(2\lambda^4 - \lambda^2|u|^2)\sigma_3 \\ &\quad + \begin{pmatrix} 0 & 2\lambda^3 u - \lambda|u|^2 u + i\lambda u_x \\ -2\lambda^3\overline{u} + \lambda|u|^2 u + i\lambda\overline{u}_x & 0 \end{pmatrix}, \end{aligned} \tag{7}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}.$$

To be more specific about the sense in which the DNLS is a compatibility condition, we note that u satisfies the DNLS equation if and only if $\mathcal{U}$ and Υ satisfy the so-called 'zero-curvature' representation

$$\frac{\partial \mathcal{U}}{\partial t} - \frac{\partial \Upsilon}{\partial x} + [\mathcal{U}, \Upsilon] = 0.$$

The first equation of (6) can be written in the form

$$L_u(\lambda)\psi := (i\sigma_3\partial_x - \lambda^2 - i\lambda U)\psi = 0, \tag{8}$$

which defines the scattering transform associated to the DNLS. Let us denote

$$\Omega_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda^2 > 0\}.$$

Then given $u \in \mathcal{S}(\mathbb{R})$ and $\lambda \in \overline{\Omega}_+$, there are unique solutions to (8) (the "Jöst solutions") exhibiting the following behavior at $\pm\infty$:

$$
\begin{aligned}
\psi_1^-(x,\lambda) &= e^{-i\lambda^2 x}\left[\begin{pmatrix}1\\0\end{pmatrix} + o(1)\right], \qquad \text{as } x \to -\infty,\\
\psi_2^+(x,\lambda) &= e^{i\lambda^2 x}\left[\begin{pmatrix}0\\1\end{pmatrix} + o(1)\right], \qquad \text{as } x \to +\infty.
\end{aligned} \tag{9}
$$

Finally, we denote by $a_u(\lambda)$ the Wronskian of the Jöst solutions defined above:[2]

$$a_u(\lambda) = \det(\psi_1^-(x,\lambda), \psi_2^+(x,\lambda)). \tag{10}$$

Using the second equation in (6), it can be shown that $a_u(\lambda)$ is time-independent if u is a solution of (1). Furthermore, a_u is a holomorphic function of λ in Ω_+, and one may determine the behavior of a_u at infinity by transforming (8) into a Zakharov-Shabat spectral problem, linear with respect to the spectral parameter, c.f. [14], [24]. The equivalence between the two problems allows us to write

$$\lim_{|\lambda|\to\infty,\, \lambda\in\overline{\Omega}_+} a_u(\lambda) = e^{-\frac{i}{2}\|u\|^2_{L^2(\mathbb{R})}}. \tag{11}$$

For fixed u, we can thus define the logarithm so that

$$\lim_{|\lambda|\to\infty, \lambda\in\overline{\Omega}_+} \ln a_u(\lambda) = -\frac{i}{2}\|u\|^2_{L^2(\mathbb{R})}. \tag{12}$$

This determines a unique branch for $|\lambda| \gg 1$, which is sufficient for our purposes. Moreover, $\ln a_u(\lambda)$ admits an asymptotic expansion of the following form:

$$\ln a_u(\lambda) = \sum_{j=0}^{\infty} \frac{E_j(u)}{\lambda^{2j}} \qquad \text{as } |\lambda| \to \infty,\ \lambda \in \overline{\Omega}_+. \tag{13}$$

Since $a_u(\lambda)$ is time-independent, the quantities $E_j(u)$ are conservation laws. They are all polynomial in u, $\overline{u}$, and their derivatives. Furthermore, the $E_j(u)$'s inherit scaling properties from $a_u(\lambda)$. That is, for $\mu > 0$, the fact that $a_{u_\mu}(\lambda) = a_u(\frac{\lambda}{\sqrt{\mu}})$ implies that $E_j(u_\mu) = \mu^j E_j(u)$, for each $j \in \mathbb{N}$. The

[2] The *transmission coefficient* mentioned earlier is the inverse of $a_u(\lambda)$.

first several of the $E_j(u)$'s are (up to multiplicative constants) the conserved quantities (3)–(5) mentioned earlier:

$$E_0(u) = -\frac{i}{2}\|u\|^2_{L^2(\mathbb{R})} = -\frac{i}{2}M(u), \qquad E_1(u) = \frac{i}{4}P(u), \qquad E_2(u) = -\frac{i}{8}E(u).$$

For each $\ell \in \mathbb{N}^*$, the quantity $E_{2\ell}(u)$ can be used to control $\|u\|^2_{\dot{H}^\ell(\mathbb{R})}$. Let us define, for ρ positive sufficiently large and $L \in \mathbb{N}$,

$$\varphi_L(u,\rho) = \mathrm{Im}\left[\ln a_u(\sqrt{i\rho}) - \sum_{j=0}^{2L+1} \frac{E_j(u)}{(i\rho)^j}\right]. \tag{14}$$

If u is a solution of the DNLS equation, then $\varphi_L(u,\rho)$ is time-independent, being a sum of time-independent quantities.

In order to establish bounds on the H^s norm of u, for $s \geq \frac{1}{2}$, we will show that for large enough $R > 0$, the quantity $\int_R^\infty \rho^{2s-1}\varphi_{[s]}(u,\rho)\mathrm{d}\rho$ controls the $\dot{H}^s$ seminorm of u, in a sense to be made precise later. Here and below, we use $[s]$ to denote the integer part of a real number s.

Our proof of Theorem 1.1 relies on a good understanding of the structure of the remainder associated to the expansion (13). Note that when $\lambda^2 = i\rho$, the imaginary part of this remainder (which is what we really use) is simply $\varphi_L(u,\rho)$. In Section 2, we will introduce a determinant characterization of $a_u(\lambda)$; we use this characterization to formulate a technical statement (Lemma 2.1 below) on the size of the remainder. Assuming the result of Lemma 2.1, we will prove Theorem 1.1 at the end of Section 2. Then, in Section 3, we will prove our technical Lemma, completing the circle of ideas. Most of the work is contained in this last section.

Before moving on, let us establish a few notational conventions that we wish to add to the ones introduced above. First of all, we use the following normalization for the Fourier transform:

$$\widehat{f}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\zeta} f(x)\,\mathrm{d}x.$$

The symbol $\mathbb{N}$ will denote the nonnegative integers, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We will use $\|\cdot\|_2$ to denote the Hilbert-Schmidt norm, and $\|\cdot\|$ will denote the operator norm on $L^2(\mathbb{R})$. And we will use the following shorthand for derivatives:

$$D = -i\partial_x, \qquad \mathcal{L}_0 = i\sigma_3\partial_x.$$

Whenever $2 \leq p < \infty$, we will use $s^*(p)$ to denote the Sobolev exponent $s^*(p) = \frac{1}{2} - \frac{1}{p}$ such that the embedding $H^{s^*(p)}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ holds.

Finally, we set notation for the following subset of Ω_+:

$$\Gamma_\delta = \{\lambda \in \Omega_+ : \delta < \arg(\lambda^2) < \pi - \delta\}.$$

This notation will be useful in some of the intermediate steps we use to prove Theorem 1.1, as our estimates will frequently depend on $\frac{|\lambda|^2}{\operatorname{Im}\lambda^2}$ (which is $\leq C(\delta)$ on Γ_δ). However, the value of $\delta > 0$ will be inconsequential for our final steps, where we will take λ^2 to be pure imaginary. Therefore, for simplicity of presentation, we will *fix* $\delta > 0$ once and for all and suppress dependence on δ in all bounds below.

2. Proof of the main result

2.1. The determinant characterization of $a_u(\lambda)$

An important property of $a_u(\lambda)$ is the fact that it can be realized as a perturbation determinant, c.f. [10] (see also [26]):

$$a_u(\lambda)^2 = \det(I - T_u(\lambda)^2), \tag{15}$$

where

$$T_u(\lambda) = i\lambda(\mathcal{L}_0 - \lambda^2)^{-1}U, \qquad \lambda \in \Omega_+.$$

The operator $T_u(\lambda)$ is Hilbert-Schmidt, with

$$\|T_u(\lambda)\|_2^2 = \frac{|\lambda|^2}{\operatorname{Im}(\lambda^2)}\|u\|_{L^2(\mathbb{R})}^2. \tag{16}$$

As a consequence of (15), we may write[3]

$$\ln a_u(\lambda) = -\sum_{k=1}^{\infty} \frac{\operatorname{Tr}(T_u(\lambda)^{2k})}{2k}, \qquad \text{if } \|T_u(\lambda)\| < 1. \tag{17}$$

This series will converge whenever $\lambda \in \Gamma_\delta$ has large enough modulus; indeed, using the explicit kernel of $(\mathcal{L}_0 - \lambda^2)^{-1}$, it can easily be shown that for

[3]This series expansion of $\ln a_u(\lambda)$ is consistent with the definition (12).

any $p > 2$, we have

$$\|T_u(\lambda)\| \lesssim \frac{|\lambda|\|u\|_{L^p(\mathbb{R})}}{\mathrm{Im}(\lambda^2)^{1-\frac{1}{p}}}, \qquad \lambda \in \Omega_+,\ u \in L^p(\mathbb{R}). \tag{18}$$

In particular[4], we can find $R_0 = R_0(\|u\|_{H^{\frac{1}{3}}(\mathbb{R})})$ such that $\|T_u(\lambda)\| \leq \frac{1}{2}$ for all $\lambda \in \Gamma_\delta$ satisfying $|\lambda|^2 \geq R_0$. We will fix the notation R_0 for use below.

As we shall see later, each term of the series (17) can be expanded in powers of λ^{-2}:

$$-\frac{\mathrm{Tr}(T_u(\lambda)^{2k})}{2k} = \sum_{j=k-1}^{\infty} \frac{\mu_{j,k}(u)}{\lambda^{2j}}. \tag{19}$$

According to (13) and (17), the $E_j(u)$'s should then satisfy

$$E_j(u) = \sum_{k=1}^{j+1} \mu_{j,k}(u). \tag{20}$$

We will use the following notation for the remainders after truncation of the expansions (17) and (19):

$$\ln a_u(\lambda) = -\sum_{k=1}^{2L+2} \frac{\mathrm{Tr}(T_u(\lambda)^{2k})}{2k} + \tau_L^*(u,\lambda), \qquad L \in \mathbb{N}; \tag{21}$$

$$\begin{aligned} -\frac{\mathrm{Tr}(T_u(\lambda)^{2k})}{2k} &= \sum_{j=k-1}^{2L+1} \frac{\mu_{j,k}(u)}{\lambda^{2j}} + \tau_L^k(u,\lambda), \\ k &\in \{1,\dots,2L+2\},\ L \in \mathbb{N}. \end{aligned} \tag{22}$$

The primary difficulty of the proof of Theorem 1.1—and indeed, the subject of Lemma 2.1—is the understanding of the size and structure of the remainder terms $\tau_L^k(u,\lambda)$, and to a lesser extent, the $\mu_{j,k}(u)$'s. On the other hand, for $\lambda \in \Gamma_\delta$ with large enough modulus, it is easy to bound the $\tau_L^*(u,\lambda)$'s. For

[4] The choice of the $H^{\frac{1}{3}}$-norm here is essentially arbitrary; one can use any H^s-norm with $0 < s < \frac{1}{2}$.

example, if $\|T_u(\lambda)\| \le \frac{1}{2}$, then for $2 < p < \infty$ and $s^*(p) = \frac{1}{2} - \frac{1}{p}$, we have

$$
\begin{aligned}
|\tau_L^*(u,\lambda)| &= \left| \ln a_u(\lambda) + \sum_{k=1}^{2L+2} \frac{\operatorname{Tr} T_u^{2k}(\lambda)}{2k} \right| \\
&\le \sum_{k=2L+3}^{\infty} \|T_u(\lambda)\|^{2k-2} \|T_u(\lambda)\|_2^2 \\
&\lesssim \|T_u(\lambda)\|^{4L+4} \|T_u(\lambda)\|_2^2 \lesssim \frac{\|u\|_{H^{s^*(p)}(\mathbb{R})}^{4L+4} \|u\|_{L^2(\mathbb{R})}^2}{|\lambda|^{(4L+4)(1-\frac{2}{p})}}.
\end{aligned}
\tag{23}
$$

The following table summarizes the various relationships among the quantities introduced above and will be helpful to keep track of the numerology. More precise information about the $\mu_{j,k}(u)$'s and $\tau_L^k(u,\lambda)$'s will be provided below.

$-\frac{\operatorname{Tr} T_u^2(\lambda)}{2}$	$-\frac{\operatorname{Tr} T_u^4(\lambda)}{4}$	$-\frac{\operatorname{Tr} T_u^6(\lambda)}{6}$	$\cdots$	$-\frac{\operatorname{Tr} T_u^{4L+2}(\lambda)}{4L+2}$	$-\frac{\operatorname{Tr} T_u^{4L+4}(\lambda)}{4L+4}$	
$\mu_{0,1}(u)$						$E_0(u)$
$\frac{\mu_{1,1}(u)}{\lambda^2}$	$\frac{\mu_{1,2}(u)}{\lambda^2}$					$\frac{E_1(u)}{\lambda^2}$
$\frac{\mu_{2,1}(u)}{\lambda^4}$	$\frac{\mu_{2,2}(u)}{\lambda^4}$	$\frac{\mu_{2,3}(u)}{\lambda^4}$				$\frac{E_2(u)}{\lambda^4}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$			$\vdots$
$\frac{\mu_{2L,1}(u)}{\lambda^{4L}}$	$\frac{\mu_{2L,2}(u)}{\lambda^{4L}}$	$\frac{\mu_{2L,3}(u)}{\lambda^{4L}}$	$\cdots$	$\frac{\mu_{2L,2L+1}(u)}{\lambda^{4L}}$		$\frac{E_{2L}(u)}{\lambda^{4L}}$
$\frac{\mu_{2L+1,1}(u)}{\lambda^{4L+2}}$	$\frac{\mu_{2L+1,2}(u)}{\lambda^{4L+2}}$	$\frac{\mu_{2L+1,3}(u)}{\lambda^{4L+2}}$	$\cdots$	$\frac{\mu_{2L+1,2L+1}(u)}{\lambda^{4L+2}}$	$\frac{\mu_{2L+1,2L+2}(u)}{\lambda^{4L+2}}$	$\frac{E_{2L+1}(u)}{\lambda^{4L+2}}$
$\tau_L^1(u,\lambda)$	$\tau_L^2(u,\lambda)$	$\tau_L^3(u,\lambda)$	$\cdots$	$\tau_L^{2L+1}(u,\lambda)$	$\tau_L^{2L+2}(u,\lambda)$	$\tau_L^*(u,\lambda)$

2.2. Structure of the traces

In this section, we record all the information about the traces that we need in order to prove our main result. We deal first with the easy case of $\operatorname{Tr} T_u(\lambda)^2$, about which we need more explicit information. A straightforward computation gives us

$$\operatorname{Tr} T_u^2(\lambda) = 2i\lambda^2 \int_{\mathbb{R}} \frac{|\widehat{u}(\zeta)|^2}{\zeta + 2\lambda^2} \mathrm{d}\zeta. \tag{24}$$

We determine the expansion of $\operatorname{Tr} T_u^2(\lambda)$ by simply substituting into (24) the identity

$$\frac{2\lambda^2}{\zeta + 2\lambda^2} = \sum_{j=0}^{2L+1} \left(-\frac{\zeta}{2\lambda^2}\right)^j + \frac{\zeta}{\zeta + 2\lambda^2}\left(\frac{\zeta}{2\lambda^2}\right)^{2L+1}, \qquad L \in \mathbb{N},$$

to obtain (for all $L \in \mathbb{N}$)

$$\begin{aligned} -\frac{\operatorname{Tr} T_u^2(\lambda)}{2} = \sum_{j=0}^{2L+1} \frac{1}{\lambda^{2j}} \cdot \underbrace{\frac{i}{(-2)^{j+1}} \int_{\mathbb{R}} \zeta^j |\widehat{u}(\zeta)|^2 \mathrm{d}\zeta}_{=:\mu_{j,1}(u)} \\ - \underbrace{\frac{i}{4^{L+1}\lambda^{4L+2}} \int_{\mathbb{R}} \frac{\zeta^{2L+2}|\widehat{u}(\zeta)|^2}{\zeta + 2\lambda^2} \mathrm{d}\zeta}_{=:\tau_L^1(u,\lambda)}. \end{aligned} \tag{25}$$

Now we state our main Lemma, which describes the structure of the other $\mu_{j,k}(u)$'s and $\tau_L^k(u,\lambda)$'s.

Lemma 2.1. *For any $k \in \mathbb{N}^*$, $L \in \mathbb{N}$, the traces $\operatorname{Tr} T_u^{2k}(\lambda)$ admit the decomposition (22). The $\mu_{j,k}(u)$'s and $\tau_L^k(u,\lambda)$'s satisfy the properties below, where for any $n \in \mathbb{N}$ we denote $\sigma(n) = \max\{n, \frac{1}{3}\}$.*

- *Each $\mu_{j,k}(u)$ is a homogeneous polynomial of degree $2k$ in u, $\overline{u}$, and their derivatives; it is homogeneous with respect to the natural scaling. We have*

$$\begin{aligned} |\mu_{2\ell,2}(u)| &\lesssim \|u\|^3_{H^{\sigma(\ell-1)}(\mathbb{R})} \|u\|_{H^\ell(\mathbb{R})}, && \ell \in \mathbb{N}^*, \\ |\mu_{2\ell,k}(u)| &\lesssim \|u\|^{2k}_{H^{\sigma(\ell-1)}(\mathbb{R})}, && \ell \in \mathbb{N}^*,\ k \in \{3, \dots, 2\ell+1\}, \\ |\mu_{2\ell+1,k}(u)| &\lesssim \|u\|^{2k}_{H^\ell(\mathbb{R})}, && \ell \in \mathbb{N}^*,\ k \in \{2, \dots, 2\ell+2\}. \end{aligned} \tag{26}$$

- *For* $|\lambda|^2 > R_0$, $\lambda \in \Gamma_\delta$, *we have the following bounds:*

$$|\tau_L^2(u,\lambda)| \lesssim_\alpha \frac{\|u\|^3_{H^{\sigma(L)}(\mathbb{R})}\|u\|_{H^{L+\alpha}(\mathbb{R})}}{|\lambda|^{4L+2+2\alpha}}, \qquad L \in \mathbb{N},\ 0 \le \alpha < 1; \tag{27}$$

$$|\tau_L^k(u,\lambda)| \lesssim \frac{\|u\|^{2k}_{H^L(\mathbb{R})}}{|\lambda|^{4L+4}}, \qquad L \in \mathbb{N}^*,\ k \in \{3,\dots,2L+2\}. \tag{28}$$

We postpone the proof of the Lemma until Section 3.

2.3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1, assuming the result of Lemma 2.1. For $s \in \mathbb{N}^*$, the conclusion follows easily from Lemma 2.1, together with (20), (25), and an induction argument; we provide the details presently. Actually, the case $s = 1$ was already proved in the Introduction. Therefore, let us turn to our inductive hypothesis. For $k = 1, \dots, \ell - 1$, we assume that the following bound holds.

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^k(\mathbb{R})} \le C(k, \|u_0\|_{H^k(\mathbb{R})}). \tag{29}$$

We will prove that the same bound holds with $k = \ell \ge 2$.

First of all, for any integer $\ell \ge 2$, and any time t, we have

$$\begin{aligned}
\|u(t)\|^2_{\dot H^\ell(\mathbb{R})} &= C(\ell)\mu_{2\ell,1}(u(t)) && \text{by (25)}\\
&= C(\ell)\left[E_{2\ell}(u(t)) - \sum_{k=2}^{2\ell+1} \mu_{2\ell,k}(u(t))\right] && \text{by (20)}\\
&\le C(\ell)E_{2\ell}(u_0) + \frac{1}{2}\|u(t)\|^2_{\dot H^\ell(\mathbb{R})} + C(\ell, \|u_0\|_{H^{\ell-1}(\mathbb{R})}).
\end{aligned}$$

To pass to the last line, we used time-independence of $E_{2\ell}(u(t))$, the bounds (26), and our inductive hypothesis (29) (with $k = \ell - 1$). Finally, using that

$$E_{2\ell}(u_0) = \sum_{k=1}^{2\ell+1} \mu_{2\ell,k}(u_0) \le C(\ell, \|u_0\|_{H^\ell(\mathbb{R})}),$$

we get

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\dot H^\ell(\mathbb{R})} \le C(\ell, \|u_0\|_{H^\ell(\mathbb{R})}),$$

which finishes the induction argument, and thus the proof of Theorem 1.1 for $s \in \mathbb{N}^*$.

It remains to consider the situation where $s \notin \mathbb{N}^*$. We start by recording the characterization of $\varphi_L(u,\rho)$ in terms of the remainders $\tau_L^k(u,\sqrt{i\rho})$, and we also set notation for the quadratic part of $\varphi_L(u,\rho)$. We also note that the case $L=0$ is included in the definition.

$$\begin{aligned}(30)\qquad \varphi_L(u,\rho) &= \operatorname{Im}\left[\ln a_u(\sqrt{i\rho}) - \sum_{j=0}^{2L+1} \frac{E_j(u)}{(i\rho)^j}\right] \\ &= \operatorname{Im}\left[\sum_{k=1}^{2L+2} \tau_L^k(u,\sqrt{i\rho}) + \tau_L^*(u,\sqrt{i\rho})\right], \qquad L\in\mathbb{N},\end{aligned}$$

$$(31)\quad \varphi_{L,0}(u,\rho) = \operatorname{Im}\tau_L^1(u,\sqrt{i\rho}) = \frac{(-1)^L}{2^{2L+1}\rho^{2L}}\int_{\mathbb{R}} \frac{\zeta^{2L+2}|\widehat{u}(\zeta)|^2}{\zeta^2+4\rho^2}\mathrm{d}\zeta, \quad L\in\mathbb{N}.$$

The conclusion of Theorem 1.1 for noninteger $s\geq \frac{1}{2}$ will be deduced from the following two Lemmas.

Lemma 2.2. *Suppose $u\in\mathcal{S}(\mathbb{R})$, $s>0$, $s\notin\mathbb{N}^*$, and $R>0$. Then the following comparison holds.*

$$\begin{aligned}(32)\quad \int_{\mathbb{R}_+} \rho^{2s-1}|\varphi_{[s],0}(u,\rho)|\mathrm{d}\rho &\lesssim_s \|u\|^2_{\dot H^s(\mathbb{R})} \\ &\lesssim_s \int_R^\infty \rho^{2s-1}|\varphi_{[s],0}(u,\rho)|\mathrm{d}\rho + R^{2(s-[s])}\|u\|^2_{\dot H^{[s]}(\mathbb{R})}.\end{aligned}$$

Proof. Let us define the function $f_\nu:\mathbb{R}\to\mathbb{R}$, for $0<\nu<1$, by $f_\nu(z) = \frac{|z|^{2\nu-1}}{1+z^2}$. Note that $f_\nu\in L^1(\mathbb{R})$ for this range of ν.

We make a direct substitution of the formula (31) into the integral $\int_R^\infty \rho^{2s-1}|\varphi_{[s],0}(u,\rho)|\mathrm{d}\rho$, then we switch the order of integration. Continuing the computation yields

$$\begin{aligned}\int_R^\infty \rho^{2s-1}|\varphi_{[s],0}(u,\rho)|\mathrm{d}\rho &= \frac{1}{2^{2[s]+1}}\int_{\mathbb{R}}\zeta^{2[s]+2}|\widehat{u}(\zeta)|^2\int_R^\infty \frac{\rho^{2(s-[s])-1}}{\zeta^2+4\rho^2}\mathrm{d}\rho\,\mathrm{d}\zeta \\ &= \frac{1}{2^{2s+1}}\int_{\mathbb{R}}|\zeta|^{2s}|\widehat{u}(\zeta)|^2\int_{\frac{2R}{|\zeta|}}^\infty f_{s-[s]}(z)\,\mathrm{d}z\,\mathrm{d}\zeta \\ &= \frac{1}{4^{s+1}}\|f_{s-[s]}\|_{L^1(\mathbb{R})}\|u\|^2_{\dot H^s(\mathbb{R})} - \frac{1}{2}\int_{\mathbb{R}}\left|\frac{\zeta}{2}\right|^{2s}|\widehat{u}(\zeta)|^2\int_0^{\frac{2R}{|\zeta|}} f_{s-[s]}(z)\,\mathrm{d}z\,\mathrm{d}\zeta.\end{aligned}$$

We estimate the second term on the right by means of the trivial replacement $\frac{1}{1+z^2} \le 1$:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} \left|\frac{\zeta}{2}\right|^{2s} |\widehat{u}(\zeta)|^2 \int_0^{\frac{2R}{|\zeta|}} f_{s-[s]}(z)\,\mathrm{d}z\,\mathrm{d}\zeta \\ \le \frac{1}{2}\int_{\mathbb{R}} \left|\frac{\zeta}{2}\right|^{2s} |\widehat{u}(\zeta)|^2 \int_0^{\frac{2R}{|\zeta|}} z^{2(s-[s])-1}\,\mathrm{d}z\,\mathrm{d}\zeta \\ = \frac{R^{2(s-[s])}}{s-[s]} \cdot \frac{\|u\|^2_{\dot{H}^{[s]}(\mathbb{R})}}{4^{[s]+1}}.\end{aligned}$$

The comparison (32) follows. □

Lemma 2.3. *Suppose* $u \in \mathcal{S}(\mathbb{R})$, $s > 0$, $s \notin \mathbb{N}^*$. *Denoting* $\beta = \max\{[s], \frac{s+[s]+1}{4([s]+1)}, \frac{1}{3}\}$, *we have*

$$|\varphi_{[s]}(u,\rho) - \varphi_{[s],0}(u,\rho)| \le \frac{C(s, \|u\|_{H^\beta(\mathbb{R})})}{\rho^{s+[s]+1}}(\|u\|_{H^s(\mathbb{R})} + 1), \quad \forall \rho \ge R_0, \tag{33}$$

where $R_0 = R_0(\|u\|_{H^{\frac{1}{3}}(\mathbb{R})})$ *is defined as in Section 2.1.*

Proof. Choose $p > 2$ to solve $2([s]+1)(1-\frac{2}{p}) = s+[s]+1$. (Note that $s^*(p) = \frac{s+[s]+1}{4([s]+1)}$ for this choice of p.) Then for $\rho > R_0$, we have

$$\begin{aligned}&|\varphi_{[s]}(u,\rho) - \varphi_{[s],0}(u,\rho)| \le \sum_{k=2}^{2[s]+2} |\tau^k_{[s]}(u, \sqrt{i\rho})| + |\tau^*_{[s]}(u, \sqrt{i\rho})| && \text{by } (30), (31)\\ &\le C(s)\left[\frac{\|u\|^3_{H^\beta(\mathbb{R})}\|u\|_{H^s(\mathbb{R})}}{\rho^{s+[s]+1}} + \sum_{k=3}^{2[s]+2} \frac{\|u\|^{2k}_{H^{[s]}(\mathbb{R})}}{\rho^{2[s]+2}} + \frac{\|u\|^{4[s]+4}_{H^{s^*(p)}(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})}}{\rho^{(2[s]+2)(1-\frac{2}{p})}}\right] && \text{by } (27), (28), (23)\\ &\le \frac{C(s, \|u\|_{H^\beta(\mathbb{R})})}{\rho^{s+[s]+1}}(\|u\|_{H^s(\mathbb{R})} + 1).\end{aligned}$$

In the second line, we understand the sum over k to be empty if $[s] = 0$. □

The conclusion of Theorem 1.1 for noninteger $s \ge \frac{1}{2}$ follows from Lemmas 2.2 and 2.3, the time-independence of the quantity $\varphi_{[s]}(u,\rho)$ for solutions of the DNLS equation, and the bound

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^\beta(\mathbb{R})} \le C(\beta, \|u_0\|_{H^\beta(\mathbb{R})}), \tag{34}$$

where $\beta = \max\{[s], \frac{s+[s]+1}{4([s]+1)}, \frac{1}{3}\}$ is as in the statement of Lemma 2.3. The bound (34) follows from our induction argument if $s > 1$ and from the result of Harrop-Griffiths, Killip, and Vişan [10] if $\frac{1}{2} \leq s < 1$.

Let us give the remaining details of the proof of Theorem 1.1 presently. We choose $R_0 = R_0(\|u_0\|_{H^{\frac{1}{3}}(\mathbb{R})})$ as in Section 2.1, so that $\|T_{u(t)}(\lambda)\| \leq \frac{1}{2}$ for all $\lambda \in \Gamma_\delta$ with $|\lambda|^2 \geq R_0$, for all $t \geq 0$. The fact that we may take R_0 to be independent of time is once again a consequence of the a priori bounds from [10] (in particular, the uniform bound on the $H^{\frac{1}{3}}$ norm). Now, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|u(t)\|^2_{\dot{H}^s(\mathbb{R})} &\lesssim_s \int_{R_0}^{\infty} \rho^{2s-1} |\varphi_{[s],0}(u(t), \rho)| \mathrm{d}\rho + R_0^{2(s-[s])} \|u(t)\|^2_{H^{[s]}(\mathbb{R})} \\
&\leq C(s) \int_{R_0}^{\infty} \rho^{2s-1} |\varphi_{[s]}(u(t), \rho)| \mathrm{d}\rho + C(s, R_0, \|u(t)\|_{H^\beta(\mathbb{R})})(\|u(t)\|_{H^s(\mathbb{R})} + 1) \\
&\leq C(s) \int_{R_0}^{\infty} \rho^{2s-1} |\varphi_{[s]}(u_0, \rho)| \mathrm{d}\rho + C(s, \|u_0\|_{H^s(\mathbb{R})})(\|u(t)\|_{H^s(\mathbb{R})} + 1) \\
&\leq \frac{1}{2} \|u(t)\|^2_{H^s(\mathbb{R})} + C(s, \|u_0\|_{H^s(\mathbb{R})}),
\end{aligned}$$

which establishes the desired conclusion. Note that the first line in the calculation above is simply the upper bound in Lemma 2.2. To pass from the first line to the second, we use Lemma 2.3, followed by the lower bound of Lemma 2.2. We use (34) and the time independence of $\varphi_{[s]}(u(t), \rho)$ to pass to the third line. Finally, we justify the last line by noting that

$$\int_{R_0}^{\infty} \rho^{2s-1} |\varphi_{[s]}(u_0, \rho)| \mathrm{d}\rho \lesssim_s C(s, \|u_0\|_{H^s(\mathbb{R})}),$$

which follows from an application of Lemma 2.3, followed by the lower bound in Lemma 2.2.

3. Proof of Lemma 2.1

3.1. Outline of the proof

In this section, we expand each $\mathrm{Tr}(T_u^{2k}(\lambda))$ in powers of λ^{-2}, up to a specified order, and we establish bounds on the remainders, in order to prove our key Lemma 2.1. In Section 3.2, we consider the case $L = 0$, which is easy to treat explicitly but does not fit naturally into our argument for the other cases. When $L \geq 1$, we follow the strategy of [5], deducing the expansions of the traces from the expansion of the resolvent $L_u(\lambda)^{-1}$. The relationship between $T_u(\lambda)$ and $L_u(\lambda)$ is the following:

$$L_u(\lambda) = (\mathcal{L}_0 - \lambda^2)(I - T_u(\lambda)). \tag{35}$$

Therefore,

$$\begin{aligned} L_u(\lambda)^{-1} &= (I - T_u(\lambda))^{-1}(\mathcal{L}_0 - \lambda^2)^{-1} \\ &= \sum_{n=0}^{\infty} \underbrace{T_u(\lambda)^n(\mathcal{L}_0 - \lambda^2)^{-1}}_{=:\mathcal{R}_n}, \qquad \|T_u(\lambda)\| < 1. \end{aligned} \tag{36}$$

The point is that

$$T_u^{2k}(\lambda) = i\lambda\mathcal{R}_{2k-1}U. \tag{37}$$

Thus, the part of $L_u^{-1}(\lambda)$ that is of relevance to us is $\mathcal{R}_{2k-1}$, i.e., the term in the expansion (36) that is homogeneous of degree $2k-1$ in $u, \overline{u}$. In particular, we seek an expansion of $\lambda\mathcal{R}_{2k-1}$ in powers of λ^{-2}, up to order λ^{4L+2} for a given $L \in \mathbb{N}^*$, and a good understanding of the remainder term.

Our strategy will be to examine the symbol $R(x,\zeta)$ of the pseudodifferential operator $L_u(\lambda)^{-1}$. In Section 3.3, we will expand the diagonal and antidiagonal parts $R^d(x,\zeta)$ and $R^a(x,\zeta)$ of $R(x,\zeta)$ in powers of λ^{-2}, determining recursively the form of each term of the expansion. Homogeneity considerations will then give us the desired expansion of $\lambda\mathcal{R}_{2k-1}$ (and thus of $\operatorname{Tr} T_u^{2k}(\lambda)$) in powers of λ^{-2}. In Section 3.4, we identify the $\mu_{j,k}(u)$'s from (22) and separate them from the remainder term. In Section 3.5 we estimate the remainder term, finishing the proof of the Lemma. The final Section 3.6 consists of the proof by induction of a technical result stated in Section 3.3.1, on the form of the terms of the expansions for R^d and R^a.

3.2. Case $L = 0$

Let us note first of all that the desired decomposition in the case $L = 0$ reads

$$\begin{aligned} \ln a_u(\lambda) = {} & \underbrace{[\mu_{0,1}(u) + \lambda^{-2}\mu_{1,1}(u) + \tau_0^1(u,\lambda)]}_{=-\frac{1}{2}\operatorname{Tr} T_u^2(\lambda)} \\ & + \underbrace{[\lambda^{-2}\mu_{1,2}(u) + \tau_0^2(u,\lambda)]}_{=-\frac{1}{4}\operatorname{Tr} T_u^4(\lambda)} + \tau_0^*(u,\lambda). \end{aligned}$$

(See the table in Section 2.1.) As we have already treated $\operatorname{Tr} T_u^2(\lambda)$ and $\tau_0^*(u,\lambda)$, it remains to understand the term $-\frac{1}{4}\operatorname{Tr} T_u^4(\lambda)$. Recall first of all that the formula for $\mu_{1,2}(u)$ can be read off from the known expression for $E_1(u) = \frac{i}{4}P(u)$. In particular, $\mu_{1,2}(u)$ must be equal to $\frac{i}{8}\|u\|^4_{L^4(\mathbb{R})}$, the part of $E_1(u)$ which is quartic in u. Next, we decompose $T_u^4(\lambda)$ more explicitly, in order to identify and estimate $\tau_0^2(u,\lambda)$. A computation (the details of which are contained, for instance, in [2]) tells us that

$$\operatorname{Tr} T_u^4(\lambda) = i(2\lambda^2)^2 \int_{\mathbb{R}} \overline{u}(x)\big((D+2\lambda^2)^{-1}u(x)\big)^2 (D-2\lambda^2)^{-1}\overline{u}(x)\,\mathrm{d}x.$$

Then, making a few simple manipulations, we can bring the right side of the equation above into the following form.

$$\begin{aligned}
\operatorname{Tr} T_u^4(\lambda) &= \frac{i}{-2\lambda^2}\int_{\mathbb{R}} \overline{u}(x)\Big[u(x) - (D+2\lambda^2)^{-1}Du(x)\Big]^2 \\
&\qquad\qquad \times \big[\overline{u}(x) - (D-2\lambda^2)^{-1}D\overline{u}(x)\big]\,\mathrm{d}x \\
&= -\frac{i}{2\lambda^2}\Big[\int_{\mathbb{R}} |u(x)|^4\,\mathrm{d}x - \int_{\mathbb{R}} |u|^2 u(x)(D-2\lambda^2)^{-1}D\overline{u}(x)\,\mathrm{d}x \\
&\qquad - 2\int_{\mathbb{R}} |u|^2\overline{u}(x)(D+2\lambda^2)^{-1}Du(x)\,\mathrm{d}x \\
&\qquad + 2\int_{\mathbb{R}} |u(x)|^2(D+2\lambda^2)^{-1}Du(x)(D-2\lambda^2)^{-1}D\overline{u}(x)\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} ((D+2\lambda^2)^{-1}Du(x))^2\overline{u}(x)^2\,\mathrm{d}x \\
&\qquad - \int_{\mathbb{R}} \overline{u}(x)((D+2\lambda^2)^{-1}Du(x))^2((D-2\lambda^2)^{-1}D\overline{u}(x))\,\mathrm{d}x\Big] \\
&= -\frac{4}{\lambda^2}\mu_{1,2}(u) - 4\tau_0^2(u,\lambda).
\end{aligned}$$

To estimate $\tau_0^2(u,\lambda)$, we use the following simple Lemma, the proof of which we omit.

Lemma 3.1. *The following estimates hold, for $\lambda \in \Gamma_\delta$.*

- *If $0 \le \alpha_1 \le \alpha_2 \le 1$, then*

$$\big\|(D \pm 2\lambda^2)^{-1}Du\big\|_{\dot H^{\alpha_1}(\mathbb{R})} \lesssim_{\alpha_2-\alpha_1} \frac{\|u\|_{\dot H^{\alpha_2}(\mathbb{R})}}{(2\operatorname{Im}(\lambda^2))^{\alpha_2-\alpha_1}}, \quad \forall\, u \in H^{\alpha_2}(\mathbb{R}). \tag{38}$$

- *If* $2 \le p < \infty$, *then*

$$\left\|(D \pm 2\lambda^2)^{-1} Du\right\|_{L^p(\mathbb{R})} \lesssim_p \|u\|_{H^{s^*(p)}(\mathbb{R})}, \quad \forall\, u \in H^{s^*(p)}(\mathbb{R}). \tag{39}$$

We estimate one of the terms defining $\tau_0^2(u,\lambda)$ explicitly; the others can be dealt with in an entirely similar way.

$$\begin{aligned} &\left|\frac{1}{\lambda^2}\int_{\mathbb{R}} \overline{u}(x)((D+2\lambda^2)^{-1}Du(x))^2((D-2\lambda^2)^{-1}D\overline{u}(x))\,\mathrm{d}x\right| \\ &\quad \le \frac{1}{|\lambda|^2}\|u\|_{L^6(\mathbb{R})}\|(D+2\lambda^2)^{-1}Du\|^2_{L^6(\mathbb{R})}\|(D-2\lambda^2)^{-1}D\overline{u}\|_{L^2(\mathbb{R})} \\ &\quad \lesssim_\alpha \frac{\|u\|^3_{H^{\frac{1}{3}}(\mathbb{R})}\|u\|_{H^\alpha(\mathbb{R})}}{|\lambda|^{2+2\alpha}}. \end{aligned}$$

We conclude that $\tau_0^2(u,\lambda)$ satisfies the required bound, finishing the case $L = 0$.

3.3. Expanding the resolvent

3.3.1. Formal expansion of R^a and R^d. As stated above, for $L \ge 1$ we seek an expansion of the symbol of $L_u^{-1}(\lambda)$, in powers of λ^{-2}. That is, we seek to understand $R(x,\zeta)$ in the expression

$$L_u^{-1}(\lambda)f = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\zeta e^{ix\zeta}R(x,\zeta)\widehat{f}(\zeta). \tag{40}$$

The identity $L_u(\lambda)R(x,D) = I$ implies

$$i\sigma_3\partial_x R(x,\zeta) - (\zeta\sigma_3 + \lambda^2)R(x,\zeta) - i\lambda U(x)R(x,\zeta) = I. \tag{41}$$

Introducing the new variable $p = \frac{\zeta}{\lambda^2}$, this reads

$$i\sigma_3\partial_x R(x,\zeta) - \lambda^2(p\sigma_3 + 1)R(x,\zeta) - i\lambda U(x)R(x,\zeta) = I. \tag{42}$$

We split R into its diagonal and antidiagonal parts R^d and R^a, respectively,

$$R(x,\zeta) = R^d(x,\zeta) + R^a(x,\zeta),$$

and we also split equation (42) accordingly:

$$i\sigma_3\partial_x R^d(x,\zeta) - \lambda^2(p\sigma_3 + 1)R^d(x,\zeta) - i\lambda U(x)R^a(x,\zeta) = I; \tag{43}$$

$$i\sigma_3\partial_x R^a(x,\zeta) - \lambda^2(p\sigma_3+1)R^a(x,\zeta) - i\lambda U(x)R^d(x,\zeta) = 0. \tag{44}$$

Next, we expand[5] R, and thus R^d and R^a, in inverse powers of λ^2:

$$R^d(x,\zeta) = \sum_{k\geq 0} \frac{1}{\lambda^{2+2k}} R^d_k(x,p), \qquad R^a(x,\zeta) = \sum_{k\geq 0} \frac{1}{\lambda^{3+2k}} R^a_k(x,p).$$

We rewrite (43) and (44) in expanded form:

$$I = -(p\sigma_3+1)R^d_0 + \sum_{k=1}^{\infty} \frac{i\sigma_3\partial_x R^d_{k-1} - (p\sigma_3+1)R^d_k - iUR^a_{k-1}}{\lambda^{2k}}; \tag{45}$$

$$0 = -(p\sigma_3+1)R^a_0 - iUR^d_0 + \sum_{k=1}^{\infty} \frac{i\sigma_3\partial_x R^a_{k-1} - (p\sigma_3+1)R^a_k - iUR^d_k}{\lambda^{2k}}. \tag{46}$$

We thus obtain the recursive system (47)–(49) below.

$$R^d_0(x,p) = -\frac{p\sigma_3 - 1}{p^2-1}, \qquad R^a_0(x,p) = -\frac{iU}{p^2-1}, \tag{47}$$

$$\begin{aligned} &R^d_k(x,p) = \frac{1}{p^2-1}\big[-iUR^a_{k-1}(x,p) + i\partial_x R^d_{k-1}(x,p)\sigma_3\big](p\sigma_3-1), \\ &k\geq 1, \end{aligned} \tag{48}$$

$$\begin{aligned} R^a_k(x,p) &= \frac{1}{p^2-1}\big[iUR^d_k(x,p) + i\partial_x R^a_{k-1}(x,p)\sigma_3\big](p\sigma_3+1) \\ &= \frac{1}{p^2-1}\big[U^2R^a_{k-1}(x,p) - U\partial_x R^d_{k-1}(x,p)\sigma_3 \qquad\qquad k\geq 1. \\ &\qquad + i\partial_x R^a_{k-1}(x,p)\sigma_3(p\sigma_3+1)\big], \end{aligned} \tag{49}$$

We used the formula for $R^d_k(x,p)$ to pass to the second line in the formula for $R^a_k(x,p)$. We also used several times the fact that $\sigma_3 A = -A\sigma_3$ for any antidiagonal matrix.

We use the computations above to clarify the form of the R^d_k's and R^a_k's; the precise statement is contained in the following Lemma.

[5]We refer to [5] for the semiclassical interpretation of these expansions.

Lemma 3.2. *The R_k^d's and R_k^a's take the following form:*

$$R_k^d(x,p) = \sum_{r=1}^{k} R_{k,r}^d(x,p), \qquad k \geq 1, \tag{50}$$

$$R_k^a(x,p) = \sum_{r=0}^{k} R_{k,r}^a(x,p), \qquad k \geq 0, \tag{51}$$

where the entries of the $R_{k,r}^d$'s and $R_{k,r}^a$'s are homogeneous polynomials of degrees $2r$ and $2r+1$, respectively, in $u, \overline{u}$, and their derivatives. More specifically, setting $Q_\gamma = \partial_x^{\gamma_1} U \cdots \partial_x^{\gamma_n} U$, for $\gamma \in \mathbb{N}^n$, we have

$$R_{k,r}^d(x,p) = \frac{1}{(p^2-1)^{k+1}} \sum_{\substack{\gamma \in \mathbb{N}^{2r} \\ |\gamma| = k-r}} Q_\gamma(x) P_{|\gamma|}(p)(p\sigma_3 - 1), \tag{52}$$

$$R_{k,r}^a(x,p) = \frac{1}{(p^2-1)^{k+1}} \sum_{\substack{\gamma \in \mathbb{N}^{2r+1} \\ |\gamma| = k-r}} Q_\gamma(x) P_{|\gamma|}(p). \tag{53}$$

Here and below we use the notation P_n to denote any diagonal matrix whose diagonal entries are polynomials in p having degree at most n.

We postpone the proof of this Lemma until Section 3.6, so as not to interrupt the flow of ideas.

3.3.2. The truncated expansion, and a formula for $\mathcal{R}_{2m-1}$. For a fixed $N \in \mathbb{N}^*$, we set the following notation. (Later we will set $N = 2L$.)

$$\begin{aligned} R^{(N)}(x,p) &= \underbrace{\sum_{k=0}^{N} \frac{R_k^d(x,p)}{\lambda^{2+2k}}}_{=:R_d^{(N)}(x,p)} + \underbrace{\sum_{k=0}^{N-1} \frac{R_k^a(x,p)}{\lambda^{3+2k}}}_{=:R_a^{(N)}(x,p)} \\ &= \underbrace{\frac{R_0^d(x,p)}{\lambda^2}}_{=:R_{d,0}^{(N)}(x,p)} + \sum_{r=1}^{N} \underbrace{\sum_{k=r}^{N} \frac{R_{k,r}^d(x,p)}{\lambda^{2+2k}}}_{=:R_{d,r}^{(N)}(x,p)} + \sum_{r=0}^{N-1} \underbrace{\sum_{k=r}^{N-1} \frac{R_{k,r}^a(x,p)}{\lambda^{3+2k}}}_{=:R_{a,r}^{(N)}(x,p)}. \end{aligned} \tag{54}$$

The symbol $R^{(N)}(x,p)$ is a truncated expansion of $R(x,p)$ in inverse powers of λ, having diagonal and antidiagonal parts $R_d^{(N)}$, $R_a^{(N)}$, respectively. The point of this definition is that, using Lemma 3.2, we know that $R_{d,r}^{(N)}$

is homogeneous of degree $2r$ in u, $\overline{u}$, and their derivatives, while $R^{(N)}_{a,r}$ is homogeneous of degree $2r+1$ in these quantities. Expanding $R^{(N)}$ according to (54) and applying the recursive identities (45)–(46), we see that $R^{(N)}(x,p)$ satisfies

$$[i\sigma_3\partial_x - \lambda^2(p\sigma_3+1) - i\lambda U(x)]R^{(N)}(x,p) = I + Y^{(N)}(x,p),$$

where $Y^{(N)}(x,p) = Y^{(N)}_d(x,p) + Y^{(N)}_a(x,p)$,

$$\begin{aligned} Y^{(N)}_d(x,p) &= \frac{1}{\lambda^{2+2N}} i\sigma_3(\partial_x R^d_N)(x,p), \\ Y^{(N)}_a(x,p) &= -\frac{1}{\lambda^{1+2N}} R^a_N(x,p)(p\sigma_3 - 1). \end{aligned}$$

This implies

$$L_u^{-1}(\lambda) = R^{(N)}(x,\lambda^{-2}D) - L_u^{-1}(\lambda)Y^{(N)}(x,\lambda^{-2}D). \tag{55}$$

Recall that $\mathcal{R}_{2m-1}$ is the term in the expansion (36) which is homogeneous of order $2m-1$ in u, $\overline{u}$, and their derivatives. On the other hand, the portion of $R^{(N)}$ which is of this homogeneity is precisely $R^{(N)}_{a,m-1}$. Combining these considerations with (55), we see that $\mathcal{R}_{2m-1}$ is the difference between $R^{(N)}_{a,m-1}$ and the part of $L_u^{-1}(\lambda)Y^{(N)}(x,\lambda^{-2}D)$ that is homogeneous of degree $2m-1$ in u, $\overline{u}$, and their derivatives. Using the expansion (36) to isolate this part, we obtain:

$$\begin{aligned} \mathcal{R}_{2m-1} &= R^{(N)}_{a,m-1}(x,\lambda^{-2}D) \\ &\quad - \frac{1}{\lambda^{2+2N}} \sum_{\substack{k+r'=m-1 \\ k\geq 0,\ 1\leq r'\leq N}} T_u(\lambda)^{2k+1}(\mathcal{L}_0-\lambda^2)^{-1}(\mathcal{L}_0 R^d_{N,r'})(x,\lambda^{-2}D) \\ &\quad - \frac{1}{\lambda^{1+2N}} \sum_{\substack{k+r'=m-1 \\ k\geq 0,\ 0\leq r'\leq N}} T_u(\lambda)^{2k}(\mathcal{L}_0-\lambda^2)^{-1} R^a_{N,r'}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1). \end{aligned} \tag{56}$$

3.4. Extracting the $\mu_{j,m}(u)$'s

Combining (56) with (37), (54), and pulling out inverse powers of λ, we easily find the following formula for $\operatorname{Tr} T_u^{2m}(\lambda)$ with $m \geq 2$, truncated at $N = 2L$.

(57)

$$\begin{aligned}
\operatorname{Tr}(T_u^{2m}(\lambda)) &= \sum_{j=m-1}^{2L-1} \frac{1}{\lambda^{2j+2}} \operatorname{Tr}[iUR^a_{j,m-1}(x,\lambda^{-2}D)] \\
&+ \sum_{\substack{k+r=m-1 \\ k\geq 0,\ 1\leq r\leq 2L}} \frac{(-1)^k}{\lambda^{4L+4+2k}} \operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+2}(\mathcal{L}_0 R^d_{2L,r})(x,\lambda^{-2}D)\right] \\
&+ \sum_{\substack{k+r=m-1 \\ k\geq 0,\ 0\leq r\leq 2L}} \frac{i(-1)^{k+1}}{\lambda^{4L+2+2k}} \operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+1} R^a_{2L,r}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)\right].
\end{aligned}$$

We will refer to the three sums above as I, II, and III, respectively.

We now identify the coefficients $\mu_{j,m}(u)$'s and verify that they satisfy the properties claimed in Lemma 2.1. The claimed homogeneity properties will be clear from the formulas that we derive below; we will just need to verify the bounds (26). The latter are also straightforward to verify but will require us to use the structure of the $R^d_{k,r}$'s and $R^a_{k,r}$'s from (52)–(53).

The first sum has the form

$$-2m \sum_{j=m-1}^{2L-1} \frac{\mu_{j,m}(u)}{\lambda^{2j}}$$

with

(58)

$$\begin{aligned}
\mu_{j,m}(u) &= -\frac{i}{2m\lambda^2} \operatorname{Tr}\left[UR^a_{j,m-1}(x,\lambda^{-2}D)\right] \\
&= -\frac{i}{4m\pi} \sum_{\substack{\gamma\in\mathbb{N}^{2m-1} \\ |\gamma|=j-(m-1)}} \operatorname{Tr}\left[\int U(x)Q_\gamma(x)\,\mathrm{d}x \int \frac{P_{|\gamma|}(\frac{\zeta}{\lambda^2})}{((\frac{\zeta}{\lambda^2})^2-1)^{j+1}} \frac{\mathrm{d}\zeta}{\lambda^2}\right].
\end{aligned}$$

Note that 'Tr' denotes an operator trace in the first line, whereas it refers to the 2×2 matrix trace in the second and third lines. We will use the notation 'Tr' similarly in what follows without further comment.

Since λ is presumed to lie in Γ_δ, a comparison of the degrees in the numerator and denominator ensures that the integrals over ζ are finite and their values are independent of λ.

The integrals over x are all of the form

$$\int U \partial_x^{\gamma_1} U \cdots \partial_x^{\gamma_{2m-1}} U \,\mathrm{d}x, \qquad \text{with } |\gamma| = j-(m-1).$$

Integrating by parts repeatedly allows us to bring each of these into a form where as few derivatives as possible fall on any single U, namely

$$\sum_{\substack{\eta\in\mathbb{N}^{2m}\\ |\eta|=j-(m-1)}} c_\eta \int \partial_x^{\eta_1} U \cdots \partial_x^{\eta_{2m}} U \,\mathrm{d}x,$$

where $c_\eta = 0$ unless

$$\max\{\eta_1,\ldots,\eta_{2m}\} \le \begin{cases} \frac{1}{2}(j-(m-1)), & \text{if } j-(m-1) \text{ is even,}\\ \frac{1}{2}(j-(m-1)+1), & \text{otherwise.}\end{cases}$$

Using this form of the integral over x, we now establish bounds on $\mu_{j,m}(u)$ according to m and the parity of j. In each case below, ℓ is a strictly positive integer.

- If $j = 2\ell$ is even and $m = 2$, then there are $2\ell - 1$ derivatives; thus

  $$|\mu_{2\ell,2}(u)| \lesssim \|u\|_{H^\ell(\mathbb{R})} \|u\|^3_{H^{\sigma(\ell-1)}(\mathbb{R})},$$

  where we recall the notation $\sigma(n) = \max\{n, \frac{1}{3}\}$.
- If $j = 2\ell$ is even and $m \ge 3$, then there are at most $2(\ell-1)$ total derivatives. This establishes the following bounds

  $$|\mu_{2,3}(u)| \lesssim \|u\|^6_{H^{\frac{1}{3}}(\mathbb{R})},$$
  $$|\mu_{2\ell,m}(u)| \lesssim \|u\|^{2m}_{H^{\ell-1}(\mathbb{R})}, \qquad \ell \ge 2,\ m \in \{3,\ldots,2\ell+1\}.$$
- If $j = 2\ell+1$ is odd and $m \ge 2$, then there are at most 2ℓ derivatives, so that we have $|\mu_{2\ell+1,m}(u)| \lesssim \|u\|^{2m}_{H^\ell(\mathbb{R})}$.

Let us remark that the formula (58) determines $\mu_{j,m}(u)$ for all $m \ge 2$ and all $j \ge m-1$, not just for those $\mu_{j,m}(u)$'s that appear in the sum I. Therefore, the above bounds on the $\mu_{j,m}(u)$'s complete our proof of the estimates (26). However, in order to determine the remainders $\tau_L^m(u,\lambda)$, we still need to extract $\mu_{2L,m}(u)$ and $\mu_{2L+1,m}(u)$ from the sums II and III. To this end, we remove the parts of II and III which are of order λ^{-4L} and λ^{-4L-2}; these will correspond to $\mu_{2L,m}(u)$ and $\mu_{2L+1,m}(u)$, respectively. We deal first with $\mu_{2L,m}(u)$; the only term expected to be relevant is the $k=0$ term in III, namely

$$-\frac{i}{\lambda^{4L+2}}\operatorname{Tr}\left[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^a_{2L,m-1}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)\right]. \tag{59}$$

To extract the part of this expression that is really of order λ^{-4L}, we commute the operator $(\lambda^{-2}\mathcal{L}_0-1)^{-1}$ with $R^a_{2L,m-1}(x,\lambda^{-2}D)$. We will have to do something similar several times below, so let us pause to write down a more general formula. Let A denote an antidiagonal operator with symbol $A(x,\zeta)$ and similarly let B denote a diagonal operator with symbol $B(x,\zeta)$. Then a simple application of the product rule gives the following operator identities.

$$A(\mathcal{L}_0+\lambda^2)^{-1} = -(\mathcal{L}_0-\lambda^2)^{-1}A+(\mathcal{L}_0-\lambda^2)^{-1}(\mathcal{L}_0A)(\mathcal{L}_0+\lambda^2)^{-1}; \tag{60}$$

$$B(\mathcal{L}_0-\lambda^2)^{-1} = (\mathcal{L}_0-\lambda^2)^{-1}B+(\mathcal{L}_0-\lambda^2)^{-1}(\mathcal{L}_0B)(\mathcal{L}_0-\lambda^2)^{-1}. \tag{61}$$

Using (60) with $A=R^a_{2L,m-1}$, the expression (59) becomes

$$\overbrace{\frac{i}{\lambda^{4L+2}}\operatorname{Tr}\left[UR^a_{2L,m-1}(x,\lambda^{-2}D)\right]}^{-2m\mu_{2L,m}\lambda^{-4L}} - \frac{i}{\lambda^{4L+4}}\operatorname{Tr}\left[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0R^a_{2L,m-1})(x,\lambda^{-2}D)\right]. \tag{62}$$

To extract $\mu_{2L+1,m}(u)$, we need to determine the part of II and III that is of order $2L+1$ in λ^{-2}. There are three quantities we need to consider:

- The second term in (62):

$$-\frac{i}{\lambda^{4L+4}}\operatorname{Tr}\left[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0R^a_{2L,m-1})(x,\lambda^{-2}D)\right] \tag{63}$$

- The $k=0$ term in II:

$$\frac{1}{\lambda^{4L+4}}\operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^2(\mathcal{L}_0R^d_{2L,m-1})(x,\lambda^{-2}D)\right] \tag{64}$$

- The $k=1$ term in III:

$$\frac{i}{\lambda^{4L+4}}\operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^3R^a_{2L,m-2}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)\right]. \tag{65}$$

We deal with each of these in turn, denoting their contributions to $\mu_{2L+1,m}$ by $\mu^{(1)}_{2L+1,m}$, $\mu^{(2)}_{2L+1,m}$, and $\mu^{(3)}_{2L+1,m}$, respectively. To put (63) in the desired form, we simply apply (60) again, this time with $A = \mathcal{L}_0 R^a_{2L,m-1}$. The result is

$$\begin{aligned} &\frac{i}{\lambda^{4L+4}} \operatorname{Tr}\left[U(\mathcal{L}_0 R^a_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right] \\ &- \frac{i}{\lambda^{4L+6}} \operatorname{Tr}\left[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}(D^2 R^a_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right]. \end{aligned} \tag{66}$$

Thus

$$\mu^{(1)}_{2L+1,m} = -\frac{i}{2m\lambda^2} \operatorname{Tr}\left[U(\mathcal{L}_0 R^a_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right].$$

Next, we look at (64). We perform two commutations, using $A = U$ in (60) (with λ^2 replaced by $-\lambda^2$), then $B = U^2$ in (61), to obtain

$$\begin{aligned} [U(\lambda^{-2}\mathcal{L}_0-1)^{-1}]^2 &= -(\lambda^{-4}D^2-1)^{-1}U^2 \\ &+ \lambda^{-2}(\lambda^{-2}\mathcal{L}_0+1)^{-1}(\mathcal{L}_0 U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1} \\ &- \lambda^{-2}(\lambda^{-4}D^2-1)^{-1}(\mathcal{L}_0 U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}. \end{aligned} \tag{67}$$

Substituting this into (64) yields

$$\begin{aligned} &- \frac{1}{\lambda^{4L+4}} \operatorname{Tr}\left[U^2(\mathcal{L}_0 R^d_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\right] \\ &+ \frac{1}{\lambda^{4L+6}} \operatorname{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(\mathcal{L}_0 U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U \right. \\ &\qquad\qquad \left. \times (\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0 R^d_{2L,m-1})(x,\lambda^{-2}D)\right] \\ &- \frac{1}{\lambda^{4L+6}} \operatorname{Tr}\left[(\lambda^{-4}D^2-1)^{-1}(\mathcal{L}_0 U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0 R^d_{2L,m-1})(x,\lambda^{-2}D)\right]. \end{aligned} \tag{68}$$

We take

$$\mu^{(2)}_{2L+1,m} = \frac{1}{2m\lambda^2} \operatorname{Tr}\left[U^2(\mathcal{L}_0 R^d_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\right].$$

Finally, we look at (65). Proceeding as in (67) but commuting one more time, we get

$$\begin{aligned}
&(\lambda^{-2}\mathcal{L}_0+1)[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}]^3\\
&\quad=(\lambda^{-4}D^2-1)^{-1}U^3\\
&\qquad+\lambda^{-2}(\mathcal{L}_0U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1}\\
&\qquad-\lambda^{-2}(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1}\\
&\qquad-\lambda^{-2}(\lambda^{-4}D^2-1)^{-1}(\mathcal{L}_0U^3)(\lambda^{-2}\mathcal{L}_0-1)^{-1}.
\end{aligned}$$

Substituting the above into (65) yields

(69)
$$\begin{aligned}
&\frac{i}{\lambda^{4L+4}}\operatorname{Tr}\big[U^3R^a_{2L,m-2}(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\big]\\
&+\frac{i}{\lambda^{4L+6}}\operatorname{Tr}\big[(\mathcal{L}_0U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1}U\\
&\qquad\qquad\times(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^a_{2L,m-2}(x,\lambda^{-2}D)\big]\\
&-\frac{i}{\lambda^{4L+6}}\operatorname{Tr}\big[(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U\\
&\qquad\qquad\times(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^a_{2L,m-2}(x,\lambda^{-2}D)\big]\\
&-\frac{i}{\lambda^{4L+6}}\operatorname{Tr}\big[(\lambda^{-4}D^2-1)^{-1}(\mathcal{L}_0U^3)(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^a_{2L,m-2}(x,\lambda^{-2}D)\big].
\end{aligned}$$

Thus

$$\mu^{(3)}_{2L+1,m}=-\frac{i}{2m\lambda^2}\operatorname{Tr}\big[U^3R^a_{2L,m-2}(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\big].$$

In view of (49), we have

$$\begin{aligned}
R^a_{2L+1}(x,\lambda^{-2}D)&=(\mathcal{L}_0R^a_{2L})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\\
&\quad+iU(\mathcal{L}_0R^d_{2L})(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\\
&\quad+U^2R^a_{2L}(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1},
\end{aligned}$$

and therefore,

$$\begin{aligned}
R^a_{2L+1,m-1}(x,\lambda^{-2}D)&=(\mathcal{L}_0R^a_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\\
&\quad+iU(\mathcal{L}_0R^d_{2L,m-1})(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1}\\
&\quad+U^2R^a_{2L,m-2}(x,\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-1},
\end{aligned}$$

which confirms that

$$\mu^{(1)}_{2L+1,m} + \mu^{(2)}_{2L+1,m} + \mu^{(3)}_{2L+1,m} = \mu_{2L+1,m}$$

with $\mu_{2L+1,m}$ defined in (58).

3.5. Estimating the remainder

The final step of the proof is to estimate the remainder terms, which we group together into $\tau^m_L(u,\lambda)$. This expression is a sum of the following terms:

- The $k \geq 1$ terms of II and the $k \geq 2$ terms of III, where II and III denote (as above) the second and third sums in the decomposition (57). We refer to these as the 'Type 1' remainder terms. Note that our assumptions on k force m to be at least 3 for all Type 1 terms.
- The terms in (66), (68), and (69) where λ^{-4L-6} appears (six terms total). We refer to these as the 'Type 2' remainder terms.

3.5.1. Type 1 remainder terms. We begin with the two sums. We want to show that the following expression is bounded by $\|u\|^{2m}_{H^L(\mathbb{R})}$:

(70)
$$\sum_{\substack{k+r=m-1\\ k\geq 1,\ 1\leq r\leq 2L}} \frac{(-1)^k}{\lambda^{2k}} \operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+2}(\mathcal{L}_0 R^d_{2L,r})(x,\lambda^{-2}D)\right]$$
$$+ \sum_{\substack{k+r=m-1\\ k\geq 2,\ 0\leq r\leq 2L}} \frac{i(-1)^{k+1}}{\lambda^{2(k-1)}} \operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+1}R^a_{2L,r}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)\right].$$

By virtue of (52), we can write

$$(\mathcal{L}_0 R^d_{2L,r})(x,p) = \frac{1}{(p^2-1)^{2L+1}} \sum_{\substack{\gamma\in\mathbb{N}^{2r}\\ |\gamma|=2L-r+1}} Q_\gamma(x)P_{|\gamma|}(p).$$

Thus, the traces in the first sum in (70) may be written as a sum of terms of the form

$$\operatorname{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+2}Q_\gamma(x)P_{2L-r+1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\right] \tag{71}$$

with $\gamma\in\mathbb{N}^{2r},\ |\gamma| = 2L-r+1 \leq 2L$. Integrating by parts repeatedly in the above expression until no derivative of order larger than L falls on any single U, we rewrite the expression (71) as a sum of terms of the form

$$\mathrm{Tr}\left[(\partial_x^{\eta_1}U)(x)(\lambda^{-2}\mathcal{L}_0-1)^{-1}\dots(\partial_x^{\eta_{2k+2}}U)(x)(\lambda^{-2}\mathcal{L}_0-1)^{-1}Q_\gamma(x)\right.$$
$$\left.\times P_{2L-r+1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\right],$$

with $\eta=(\eta_1,\dots\eta_{2k+2})\in\mathbb{N}^{2k+2}$, $\gamma=(\gamma_1,\dots,\gamma_{2r})\in\mathbb{N}^{2r}$ satisfying $|\eta|+|\gamma|=2L-r+1$ and $\max_{p,q}(\eta_p,\gamma_q)\leq L$. Thus, the expression (71) can be bounded by $|\lambda|^2\|u\|_{H^L(\mathbb{R})}^{2m}$, and therefore the first sum in (70) by $\|u\|_{H^L(\mathbb{R})}^{2m}$.

We deal with the second sum in (70) in essentially the same way. Invoking (53), we may write

$$R^a_{2L,r}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)$$
$$=\sum_{\substack{\gamma\in\mathbb{N}^{2r+1}\\|\gamma|=2L-r}}Q_\gamma(x)P_{|\gamma|+1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}.$$

Thus

$$\mathrm{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+1}R^a_{2L,r}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)\right]$$

is a sum of terms of the form

$$\mathrm{Tr}\left[(U(\lambda^{-2}\mathcal{L}_0-1)^{-1})^{2k+1}Q_\gamma(x)P_{2L-r+1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\right],\tag{72}$$

where $\gamma\in\mathbb{N}^{2r+1}$ with $|\gamma|=2L-r\leq 2L$. As before, we integrate by parts repeatedly to rewrite the expression (72) as a sum of terms of the form

$$\mathrm{Tr}\left[(\partial_x^{\eta_1}U)(x)(\lambda^{-2}\mathcal{L}_0-1)^{-1}\dots(\partial_x^{\eta_{2k+1}}U)(x)(\lambda^{-2}\mathcal{L}_0-1)^{-1}Q_\gamma(x)\right.$$
$$\left.\times P_{2L-r+1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\right],$$

with $\eta=(\eta_1,\dots\eta_{2k+1})\in\mathbb{N}^{2k+1}$, $\gamma=(\gamma_1,\dots,\gamma_{2r+1})\in\mathbb{N}^{2r+1}$ satisfying $|\eta|+|\gamma|=2L-r$ and $\max_{p,q}(\eta_p,\gamma_q)\leq L$. This allows us to bound (72) by $|\lambda|^2\|u\|_{H^L(\mathbb{R})}^{2m}$, thus completing the desired estimates on the Type 1 remainder terms.

3.5.2. Type 2 remainder terms. We now deal with the Type 2 remainder terms (the terms in (66), (68), and (69) where λ^{-4L-6} appears). When $m\geq 3$, the total number of derivatives falling on the U's is $2L$; therefore we can bound all these terms by $|\lambda|^{-4L-4}\|u\|_{H^L(\mathbb{R})}^{2m}$ by arguing exactly as we did for the Type 1 terms. To complete the proof of Lemma 2.1, it thus remains to consider the Type 2 remainder terms with $m=2$. In this case, some of the U's appear to be overloaded with derivatives, and we need an additional estimate. We state the following Lemma in terms of the 'overloaded' part of the Type 2 remainder term from (66), but the same manipulations will yield the bound we need for the other Type 2 remainders.

Lemma 3.3. *The following estimate holds, for any $j \in \mathbb{N}$ and all $\alpha \in [0,1]$.*

$$\|(\mathcal{L}_0+\lambda^2)^{-1}D^{j+1}U(\mathcal{L}_0-\lambda^2)^{-1}\|_2 \le C(\alpha)|\lambda|^{-1-2\alpha}\|u\|_{H^{j+\alpha}(\mathbb{R})}. \tag{73}$$

Proof. Denoting $T=(\mathcal{L}_0+\lambda^2)^{-1}D^{j+1}U(\mathcal{L}_0-\lambda^2)^{-1}$, we readily compute as follows:

$$\begin{aligned}\|T\|_2^2 &= \frac{1}{\pi}\iint \frac{|\widehat{D^{j+1}u}(\zeta_1-\zeta_2)|^2}{|\zeta_1-\lambda^2|^2|\zeta_2-\lambda^2|^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\ &= \frac{1}{\pi}\int|\zeta_1|^2|\widehat{D^j u}(\zeta_1)|^2\left(\int\frac{\mathrm{d}\zeta_2}{|\zeta_1+\zeta_2-\lambda^2|^2|\zeta_2-\lambda^2|^2}\right)\mathrm{d}\zeta_1\\ &= \frac{2}{\operatorname{Im}\lambda^2}\int\frac{|\zeta_1|^2|\widehat{D^j u}(\zeta_1)|^2}{|\zeta_1+2i\operatorname{Im}\lambda^2|^2}\mathrm{d}\zeta_1 \le C(\alpha)|\lambda|^{-2-4\alpha}\|u\|^2_{H^{j+\alpha}(\mathbb{R})}.\end{aligned}$$

This completes the proof. □

With the above Lemma at our disposal, we now return to the estimation of the Type 2 remainder terms for $m=2$; we deal first with the one that appears in (66). Omitting the prefactor $\frac{-i}{\lambda^{4L+6}}$, the quantity under consideration is

$$\operatorname{Tr}\left[U(\lambda^{-2}\mathcal{L}_0-1)^{-1}(D^2R^a_{2L,1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right],$$

which we write as

$$\operatorname{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(D^2U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^a_{2L,1}(x,\lambda^{-2}D)\right]. \tag{74}$$

Proceeding as we did for the Type 1 terms, we rewrite this expression as a sum of terms of the form

$$\begin{aligned}\operatorname{Tr}\big[&(\lambda^{-2}\mathcal{L}_0+1)^{-1}(D^{\eta+2}U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}Q_\gamma(x)\\ &\times P_{2L-1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\big],\end{aligned} \tag{75}$$

with $\eta\in\mathbb{N}$, $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{N}^3$, $\eta+|\gamma|=2L-1$, $\eta\le L-1$, and $\max(\gamma_1,\gamma_2,\gamma_3)\le L$. The expression (75) can be bounded by

$$\begin{aligned}&\|(\lambda^{-2}\mathcal{L}_0+1)^{-1}(D^{\eta+2}U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}\|_2\\ &\qquad\times\|Q_\gamma(x)P_{2L-1}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\|_2.\end{aligned}$$

By virtue of Lemma 3.3, the above can bounded in turn by

$$C(\alpha)|\lambda|^{4-2\alpha}\|u\|_{H^{L+\alpha}(\mathbb{R})}\|u\|^3_{H^L(\mathbb{R})},$$

for any $\alpha \in [0,1]$.

The next quantity we treat is the third term in (68); we want to estimate

$$\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0R^d_{2L,1})(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right].$$

After an integration by parts we are left with the expression

$$-\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0-1)^{-1}(D^2U^2)(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^d_{2L,1}(x,\lambda^{-2}D)(\lambda^{-2}\mathcal{L}_0+1)^{-1}\right].$$

that can be treated in exactly the same way as (74). We note only the modification to Lemma 3.3 that we use, namely

$$\begin{aligned}\|(\mathcal{L}_0-\lambda^2)^{-1}(D^{L+1}U^2)(\mathcal{L}_0-\lambda^2)^{-1}\|_2 &\lesssim_\alpha |\lambda|^{-1-2\alpha}\|U^2\|_{H^{L+\alpha}(\mathbb{R})}\\ &\lesssim_\alpha |\lambda|^{-1-2\alpha}\|u\|_{H^{L+\alpha}(\mathbb{R})}\|u\|_{H^L(\mathbb{R})}.\end{aligned}$$

We next consider the second term in (68):

$$\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(\mathcal{L}_0U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0R^d_{2L,1})(x,\lambda^{-2}D)\right],$$

where as usual we have suppressed the prefactor λ^{-4L-6}. We start by rewriting it as the sum

$$\begin{aligned}&-\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(D^2U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}U\right.\\ &\qquad\left.\times(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^d_{2L,1}(x,\lambda^{-2}D)\right]\\ &+\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(\mathcal{L}_0U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}(\mathcal{L}_0U)\right.\\ &\qquad\left.\times(\lambda^{-2}\mathcal{L}_0-1)^{-1}R^d_{2L,1}(x,\lambda^{-2}D)\right].\end{aligned}\tag{76}$$

For the first term here we proceed exactly as before: substituting (52) and integrating by parts we rewrite it as a sum of expressions of the form

$$\begin{aligned}\mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0+1)^{-1}(D^{\eta_1+2}U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}(D^{\eta_2}U)(\lambda^{-2}\mathcal{L}_0-1)^{-1}Q_\gamma(x)\right.\\ \left.\times P_{2L}(\lambda^{-2}D)(\lambda^{-4}D^2-1)^{-2L-1}\right],\end{aligned}$$

with $\eta = (\eta_1, \eta_2) \in \mathbb{N}^2$, $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$, $|\eta| + |\gamma| = 2L - 1$, $\max(\eta_1, \eta_2) \leq L - 1$, and $\max(\gamma_1, \gamma_2) \leq L$. We estimate the above by

$$\|(\lambda^{-2}\mathcal{L}_0 + 1)^{-1}(D^{\eta_1+2}U)(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}\|_2 \|(D^{\eta_2}U)(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}\| \\ \times \|Q_\gamma(x)P_{2L}(\lambda^{-2}D)(\lambda^{-4}D^2 - 1)^{-2L-1}\|_2,$$

which can in turn be bounded by

$$C(\alpha)|\lambda|^{4-2\alpha}\|u\|_{H^{L+\alpha}(\mathbb{R})}\|u\|^3_{H^L(\mathbb{R})}.$$

To treat the second term in (76) we distinguish the cases $L = 1$ and $L \geq 2$. In the case of $L = 1$ we estimate this expression by

$$\begin{aligned} &\|(\lambda^{-2}\mathcal{L}_0 + 1)^{-1}(\mathcal{L}_0 U)\|_2 \|(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}(\mathcal{L}_0 U)\| \\ &\quad \times \|(\lambda^{-2}\mathcal{L}_0 - 1)^{-1} R^d_{2L,1}(x, \lambda^{-2}D)\|_2 \\ &\quad\quad \lesssim |\lambda|^{2+\frac{2}{p}}\|u\|^3_{H^1(\mathbb{R})}\|Du\|_{L^p(\mathbb{R})}, \quad 2 \leq p \leq \infty. \end{aligned}$$

Putting $\sigma = \frac{\alpha}{2} \in [0, \frac{1}{2}[$ and choosing p such that $\sigma = \frac{1}{2} - \frac{1}{p}$, we get the bound

$$|\lambda|^{3-2\sigma}\|u\|^3_{H^1(\mathbb{R})}\|u\|_{H^{1+\sigma}(\mathbb{R})} \leq |\lambda|^{4-2\alpha}\|u\|^3_{H^1(\mathbb{R})}\|u\|_{H^{1+\alpha}(\mathbb{R})}.$$

If $L \geq 2$, we can proceed as for the Type 1 remainder terms and bound the second term in (76) by $|\lambda|^2\|u\|^4_{H^L(\mathbb{R})}$. This finishes our considerations of Type 2 remainder terms coming from (68).

The last group of Type 2 remainder terms comes from (69). Omitting the common prefactor, the quantity of interest is

$$\begin{aligned} &\mathrm{Tr}\left[(\mathcal{L}_0 U)(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}U(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}U \right. \\ &\qquad \left. \times (\lambda^{-2}\mathcal{L}_0 - 1)^{-1}R^a_{2L,0}(x, \lambda^{-2}D)\right] \\ &- \mathrm{Tr}\left[(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}(\mathcal{L}_0 U^2)(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}U \right. \\ &\qquad \left. \times (\lambda^{-2}\mathcal{L}_0 - 1)^{-1}R^a_{2L,0}(x, \lambda^{-2}D)\right] \\ &- \mathrm{Tr}\left[(\lambda^{-4}D^2 - 1)^{-1}(\mathcal{L}_0 U^3)(\lambda^{-2}\mathcal{L}_0 - 1)^{-1}R^a_{2L,0}(x, \lambda^{-2}D)\right], \end{aligned} \tag{77}$$

where $R^a_{2L,0}(x, p) = \partial_x^{2L}U(x)P_{2L}(p)(p^2 - 1)^{-2L-1}$. No new ideas are involved in the estimation of these terms; we simply integrate by parts $L - 1$ times to keep $L + 1$ derivatives on U coming from $R^a_{2L,0}$ and then apply Lemma 3.3. We omit the remaining details for these terms. Having now established the required bounds on the remainder $|\tau^m_L(u, \lambda)|$, we have completed the proof of Lemma 2.1, modulo the proof of Lemma 3.2 below.

3.6. Proof of Lemma 3.2

We argue by induction. The base cases are easy to verify explicitly:

$$R_0^a(x,p) = \frac{1}{p^2-1}\cdot U(-i) = R_{0,0}^a(x,p)$$
$$R_1^d(x,p) = \frac{1}{(p^2-1)^2}\cdot U^2\cdot(-1)\cdot(p\sigma_3-1) = R_{1,1}^d(x,p).$$

Using (48)–(49) along with our inductive hypothesis, we may write, for $k \geq 1$,

$$R_k^d(x,p) = \frac{1}{(p^2-1)^{k+1}}\Bigg[\sum_{r=0}^{k-1}\sum_{\substack{\gamma\in\mathbb{N}^{2r+1}\\|\gamma|=(k-1)-r}} -iUQ_\gamma(x)P_{|\gamma|}(p)$$
$$+\sum_{r=1}^{k-1}\sum_{\substack{\gamma\in\mathbb{N}^{2r}\\|\gamma|=(k-1)-r}} i\partial_x Q_\gamma(x)P_{|\gamma|}(p)(p\sigma_3-1)\sigma_3\Bigg](p\sigma_3-1).$$

We check that the inner sums (together with the common factors $(p^2-1)^{-k-1}$ and $(p\sigma_3 - 1)$) can be absorbed into $R_{k,r+1}^d(x,p)$ and $R_{k,r}^d(x,p)$, respectively.

- When $\gamma\in\mathbb{N}^{2r+1}$ and $|\gamma| = (k-1)-r$, the term $iUQ_\gamma(x)P_{|\gamma|}(p)$ can be absorbed into $R_{k,r+1}^d(x,p)$:
  - First, $UQ_\gamma = Q_{(0,\gamma)}$, with $(0,\gamma)\in\mathbb{N}^{2(r+1)}$ and $|(0,\gamma)| = |\gamma| = k-(r+1)$.
  - Second, $\deg P_{|\gamma|}(p) \leq (k-1)-r = k-(r+1)$.
- When $\gamma\in\mathbb{N}^{2r}$ and $|\gamma| = (k-1)-r$, the term $\partial_x Q_\gamma(x)P_{|\gamma|}(p)(p\sigma_3-1)\sigma_3$ can be absorbed into $R_{k,r}^d(x,p)$:
  - First, $\partial_x Q_\gamma$ is a sum of $Q_{\gamma'}$'s, with $|\gamma'| = |\gamma|+1 = k-r$;
  - Second, $\deg P_{|\gamma|}(p)(p\sigma_3-1) \leq [(k-1)-r]+1 = k-r$.

The formula for $R_k^a(x,p)$ may be verified in exactly the same way. We write

$$
\begin{aligned}
R_k^a(x,p) = & \frac{1}{(p^2-1)^{k+1}} \\
& \times \Bigg[\sum_{r=0}^{k-1} \sum_{\substack{\gamma\in\mathbb{N}^{2r+1} \\ |\gamma|=(k-1)-r}} U^2 Q_\gamma(x) P_{|\gamma|}(p) + i\partial_x Q_\gamma(x) P_{|\gamma|}(p)\sigma_3(p\sigma_3+1) \\
& - \sum_{r=1}^{k-1} \sum_{\substack{\eta\in\mathbb{N}^{2r} \\ |\eta|=(k-1)-r}} U\partial_x Q_\eta(x) P_{|\eta|}(p)(p\sigma_3-1)\sigma_3 \Bigg].
\end{aligned}
$$

As above, we perform the routine verifications of the numerology as follows.

- When $\gamma \in \mathbb{N}^{2r+1}$ and $|\gamma| = (k-1)-r$, the term $U^2 Q_\gamma(x) P_{|\gamma|}(x)$ can be absorbed into $R^a_{k,r+1}(x,p)$:
  - First,we note that $U^2 Q_\gamma = Q_{(0,0,\gamma)}$, with $(0,0,\gamma) \in \mathbb{N}^{2(r+1)+1}$, $|(0,0,\gamma)| = |\gamma| = k-(r+1)$;
  - Second, $\deg P_{|\gamma|}(p) \le (k-1)-r = k-(r+1)$.
- When $\gamma \in \mathbb{N}^{2r+1}$ and $|\gamma| = (k-1)-r$, the term $i\partial_x Q_\gamma(x) P_{|\gamma|}(p)\sigma_3(p\sigma_3+1)$ can be absorbed into $R^a_{k,r}(x,p)$. When $\eta \in \mathbb{N}^{2r}$ and $|\eta| = (k-1)-r$, the same is true of $U\partial_x Q_\eta(x) P_{|\eta|}(p)(p\sigma_3 - 1)\sigma_3$.
  - First, $\partial_x Q_\gamma$ and $U\partial_x Q_\eta$ are sums of $Q_{\gamma'}$'s, with $\gamma' \in \mathbb{N}^{2r+1}$ and $|\gamma'| = k-r$.
  - Second, $P_{|\gamma|}(p)\sigma_3(p\sigma_3+1)$ and $P_{|\eta|}(p)(p\sigma_3-1)\sigma_3$ have degree at most $k - r$.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

References

[1] Mark J. Ablowitz and Harvey Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics.*

Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981.

[2] Hajer Bahouri and Galina Perelman. Global well-posedness for the derivative nonlinear Schrödinger equation. *Invent. math.*, 229, 639–688 (2022).

[3] H. A. Biagioni and F. Linares. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations. *Trans. Amer. Math. Soc.*, 353(9):3649–3659, 2001.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 34(1):64–86, 2002.

[5] Patrick Gérard. On the Conservation Laws of the Defocusing Cubic NLS Equation. unpublished, 2015.

[6] Axel Grünrock. Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. *Int. Math. Res. Not.*, (41):2525–2558, 2005.

[7] Zihua Guo and Yifei Wu. Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$. *Discrete Contin. Dyn. Syst.*, 37(1):257–264, 2017.

[8] Nakao Hayashi and Tohru Ozawa. On the derivative nonlinear Schrödinger equation. *Phys. D*, 55(1-2):14–36, 1992.

[9] Benjamin Harrop-Griffiths, Rowan Killip, Maria Ntekoume, and Monica Vişan. Global well-posedness for the derivative nonlinear Schrödinger equation in $L^2(\mathbb{R})$. `arXiv:2204.12548`, 2022. To appear in *J. Eur. Math. Soc.*

[10] Benjamin Harrop-Griffiths, Rowan Killip, and Monica Vişan. Large-data equicontinuity for the derivative NLS. *Int. Math. Res. Notices*, 2023(6):4601–4642, 2023.

[11] Robert Jenkins, Jiaqi Liu, Peter Perry, and Catherine Sulem. The derivative nonlinear Schrödinger equation: global well-posedness and soliton resolution. *Quart. Appl. Math.*, 78(1):33–73, 2020.

[12] Robert Jenkins, Jiaqi Liu, Peter Perry, and Catherine Sulem. Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities. *Anal. PDE*, 13(5):1539–1578, 2020.

[13] Robert Jenkins, Jiaqi Liu, Peter A. Perry, and Catherine Sulem. Global well-posedness for the derivative non-linear Schrödinger equation. *Comm. Partial Differential Equations*, 43(8):1151–1195, 2018.

[14] David J. Kaup and Alan C. Newell. An exact solution for a derivative nonlinear Schrödinger equation. *J. Mathematical Phys.*, 19(4):798–801, 1978.

[15] Rowan Killip, Maria Ntekoume, and Monica Vişan. On the well-posedness problem for the derivative nonlinear schrödinger equation. *Analysis & PDE*, 16(5):1245–1270, 2023.

[16] Rowan Killip and Monica Vişan. KdV is well-posed in H^{-1}. *Ann. of Math. (2)*, 190(1):249–305, 2019.

[17] Rowan Killip, Monica Vişan, and Xiaoyi Zhang. Low regularity conservation laws for integrable PDE. *Geom. Funct. Anal.*, 28(4):1062–1090, 2018.

[18] Friedrich Klaus and Robert Schippa. A priori estimates for the derivative nonlinear schrödinger equation, `arXiv:2007.13161`, 2020.

[19] Herbert Koch and Daniel Tataru. Conserved energies for the cubic nonlinear Schrödinger equation in one dimension. *Duke Math. J.*, 167(17):3207–3313, 2018.

[20] Jyh-Hao Lee. Global solvability of the derivative nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.*, 314(1):107–118, 1989.

[21] Koji Mio, Tatsuki Ogino, Kazuo Minami, and Susumu Takeda. Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Japan*, 41(1):265–271, 1976.

[22] Einar Mjølhus. On the modulational instability of hydromagnetic waves parallel to the magnetic field. *Journal of Plasma Physics*, 16(3):321–334, 1976.

[23] T. Ozawa. On the nonlinear Schrödinger equations of derivative type. *Indiana Univ. Math. J.*, 45(1):137–163, 1996.

[24] Dmitry E. Pelinovsky, Aaron Saalmann, and Yusuke Shimabukuro. The derivative NLS equation: global existence with solitons. *Dyn. Partial Differ. Equ.*, 14(3):271–294, 2017.

[25] Dmitry E. Pelinovsky and Yusuke Shimabukuro. Existence of global solutions to the derivative NLS equation with the inverse scattering transform method. *Int. Math. Res. Not. IMRN*, (18):5663–5728, 2018.

[26] Barry Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005.

[27] Hideo Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Adv. Differential Equations*, 4(4):561–580, 1999.

[28] Hideo Takaoka. Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces. *Electron. J. Differential Equations*, pages No. 42, 23, 2001.

[29] Takayuki Tsuchida and Miki Wadati. Complete integrability of derivative nonlinear Schrödinger-type equations. *Inverse Problems*, 15(5):1363–1373, 1999.

[30] Yifei Wu. Global well-posedness on the derivative nonlinear Schrödinger equation. *Anal. PDE*, 8(5):1101–1112, 2015.

Laboratoire Jacques-Louis Lions (LJLL)
CNRS & Sorbonne Université, Place Jussieu, 75005 Paris, France
E-mail address: hajer.bahouri@sorbonne-universite.fr

Illinois Institute of Technology
10 W 32 St., Chicago IL 60616, USA
E-mail address: tleslie@iit.edu

Laboratoire D'Analyse et de Mathématiques Appliquées
Université Paris-Est Créteil, 94010 Créteil Cedex, France
E-mail address: galina.perelman@u-pec.fr

Received July 26, 2021
Accepted November 16, 2022