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Improved discrete restriction for the parabola

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Using ideas from [7] and working over $\mathbb{Q}_p$, we show that the discrete restriction constant for the parabola is $O_\varepsilon((\log M)^{2+\varepsilon})$.

1. Introduction

Let $e(z) := e^{2\pi iz}$ and let $K(M)$ denote the best constant such that

$$\|\sum_{n=1}^{M} a_n e(nx_1 + n^2 x_2)\|_{L^6([0,1]^2)} \leq K(M)(\sum_{n=1}^{M} |a_n|^2)^{1/2} \tag{1.1}$$

for all sequences of complex numbers $\{a_n\}_{n=1}^{M}$. Trivially, $K(M) \leq M^{1/2}$. In 1993, Bourgain in [2] considered, among other things, the size of $K(M)$

since (1.1) is associated to the periodic Strichartz inequality for the nonlinear Schrödinger equation on the torus. He obtained that

$$(\log M)^{1/6} \lesssim K(M) \leq \exp(O(\frac{\log M}{\log\log M})) \tag{1.2}$$

using number theoretic methods, in particular the upper bound follows from the divisor bound and the lower bound follows from Gauss sums on major arcs (see also [1] for a precise asymptotic in the case of $a_n = 1$ of (1.1)). It is natural to ask what is the true size of $K(M)$ and whether the gap between the upper and lower bounds can be closed.

The lower bound has not been improved since [2]. However by improving the upper bound on the decoupling constant for the parabola, Guth-Maldague-Wang recently in [7] improved the upper bound in (1.2) to $\lesssim (\log M)^C$ for some unspecified but large absolute constant C. Our main result is that C can be reduced to 2+. More precisely:

Theorem 1.1. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$K(M) \leq C_\varepsilon (\log M)^{2+\varepsilon}.$$

Our proof of Theorem 1.1 will rely on a decoupling theorem for the parabola in $\mathbb{Q}_p$. Previous work on studying discrete restriction using decoupling relied on proving decoupling theorems over $\mathbb{R}$ (see for example [3, 4, 7, 9]). Here, we will broadly follow the proof in [7] except to efficiently keep track of the number of logs we will prove a decoupling theorem over $\mathbb{Q}_p$ rather than over $\mathbb{R}$. Additionally we will introduce some extra efficiencies to their argument to decrease the number of logs even further.

Working in $\mathbb{Q}_p$ has two benefits. First, the Fourier transform of a compactly supported function is also compactly supported and hence this allows us to rigorously and efficiently apply the uncertainty principle which is just a heuristic in $\mathbb{R}$. Second, since 6 is even, decoupling in $\mathbb{Q}_p$ still implies discrete restriction estimates.

To avoid confusing the p in $\mathbb{Q}_p$ with the p in L^p norm, henceforth we will replace the p in $\mathbb{Q}_p$ with q.

Let q be a fixed odd prime. Let $|\cdot|$ be the q-adic norm associated to $\mathbb{Q}_q$. We omit the dependence of this norm on q. This is a slight abuse of notation as we will use the same notation for the absolute value on $\mathbb{C}$, as well as the length of a q-adic interval. However, the meaning of the symbol will be clear from context. In Section 2, we summarize all relevant facts of $\mathbb{Q}_q$ that we make use of. See Chapters 1 and 2 of [11] and Chapter 1 (in particular Sections 1 and 4) of [12] for a more complete discussion of analysis on $\mathbb{Q}_q$.

For $\delta \in q^{-\mathbb{N}}$, we write

$$\Xi_\delta = \{(\xi, \eta) \in \mathbb{Q}_q^2 \colon \xi \in \mathbb{Z}_q, |\eta - \xi^2| \leq \delta\}.$$

For a Schwartz function $F : \mathbb{Q}_q^2 \to \mathbb{C}$ and an interval $\tau \subset \mathbb{Z}_q$, let F_τ be defined by $\widehat{F_\tau} := \widehat{F}\, 1_{\tau \times \mathbb{Q}_q}$. Our main decoupling theorem is as follows and is the $\mathbb{Q}_q$ analogue of Theorem 1.2 of [7].

Theorem 1.2. *For every odd prime q and every $\varepsilon > 0$, there exists a constant $C_{\varepsilon,q}$, such that whenever $R \in q^{2\mathbb{N}}$ and a Schwartz function $F \colon \mathbb{Q}_q^2 \to \mathbb{C}$ has Fourier support contained in $\Xi_{1/R}$, one has*

(1.3)
$$\int_{\mathbb{Q}_q^2} |F|^6 \leq C_{\varepsilon,q} (\log R)^{12+\varepsilon} \Big(\sum_{|\tau| = R^{-1/2}} \|F_\tau\|_{L^\infty(\mathbb{Q}_q^2)}^2\Big)^2 \Big(\sum_{|\tau| = R^{-1/2}} \|F_\tau\|_{L^2(\mathbb{Q}_q^2)}^2\Big).$$

Here the sums on the right hand side are over all intervals $\tau \subset \mathbb{Z}_q$ with length $R^{-1/2}$.

This theorem is proved in Sections 4-7. We will in fact show this theorem with ε replaced by 10ε. Since 6 is even, Theorem 1.2 once again immediately implies Theorem 1.1 (as we prove in Section 3).

The 12 powers of log in (1.3) can be accounted for as follows. Reducing from (1.3) to the level set estimate (Proposition 6.3) costs 5 logs. They come from: 3 logs from the Whitney decomposition in Section 5, 1 log from the number of scales in deriving (6.7), and 1 log from pigeonholing to derive (6.12). The level set estimate itself costs 7 logs. These come from: 1 log since we decompose $\mathbb{Q}_q^2$ into sets Ω_k and L in Section 7.3 and (7.6), 2 logs to control g_k^2 by $|g_k^h|^2$ on Ω_k in (7.14), and 4 logs from the appearance of λ^2 in (7.15) (also see (7.8)).

In addition to efficiencies introduced by working with the uncertainty principle q-adically, we introduce a Whitney decomposition, much like in [6], which allows us to more efficiently reduce to a bilinear decoupling problem. Additionally compared to [7], the ratio between our successive scales R_{k+1}/R_k is of size $O((\log R)^\varepsilon)$ rather than in $O((\log R)^{12})$ which allows for further reductions (we essentially have $O(\varepsilon^{-1})$ times many more scales than in [7]). Note that (1.3) is not a true $\mathbb{Q}_q$ analogue of a $l^2 L^6$ decoupling theorem for the parabola. At the cost of a few more logs, a similar argument as in Section 5 of [7] would allow us to upgrade to an actual l^2L^6 decoupling theorem, however (1.3) is already enough for discrete restriction for the parabola.

Since p-adic intervals correspond to residue classes it may be possible to rewrite the proof of Theorem 1.2 in the language of congruences and compare it with efficient congruencing [13]. However we do not attempt this here. For more connections between efficient congruencing and decoupling see [5, 6, 9, 10].

In this paper we consider decoupling over $\mathbb{Q}_p$. However one can also consider the restriction and Kakeya conjectures over $\mathbb{Q}_p$ (or alternatively over more general local fields). We refer the interested reader to [8] and the references therein for more discussion.

For the rest of the paper, for two positive expressions X and Y, we write $X \lesssim Y$ if $X \leq C_{\varepsilon,q} Y$ for some constant $C_{\varepsilon,q}$ which is allowed to depend on ε and q. We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Additionally by writing $f(x) = O(g(x))$, we mean $|f(x)| \lesssim g(x)$. Finally, we say that f has Fourier support in Ω if its Fourier transform $\widehat{f}$ is supported in Ω.

2. Some basic properties of $\mathbb{Q}_q$

For convenience we briefly summarize some key relevant facts about $\mathbb{Q}_q$. First, for a prime q, $\mathbb{Q}_q$ is the completion of the field $\mathbb{Q}$ under the q-adic norm, defined by $|0| = 0$ and $|q^a b/c| = q^{-a}$ if $a \in \mathbb{Z}$, $b, c \in \mathbb{Z} \setminus \{0\}$ and q is relatively prime to both b and c. Then $\mathbb{Q}_q$ can be identified (bijectively) with the set of all formal series

$$\mathbb{Q}_q = \left\{ \sum_{j=k}^{\infty} a_j q^j : k \in \mathbb{Z}, a_j \in \{0, 1, \dots, q-1\} \text{ for every } j \geq k \right\},$$

and the q-adic norm on $\mathbb{Q}_q$ satisfies $|\sum_{j=k}^{\infty} a_j q^j| = q^{-k}$ if $a_k \neq 0$.

The q-adic norm obeys the ultrametric inequality $|x + y| \leq \max\{|x|, |y|\}$ with equality when $|x| \neq |y|$. We also define the q-adic norm on $\mathbb{Q}_q^2$ by setting $|(x, y)| = \max\{|x|, |y|\}$ for $(x, y) \in \mathbb{Q}_q^2$.

Write $\mathbb{Z}_q = \{x \in \mathbb{Q}_q : |x| \leq 1\}$ for the ring of integers of $\mathbb{Q}_q$. This is in analogy to the real interval $[-1, 1]$. In analogy to working over $\mathbb{R}$, for $a \in \mathbb{Z}_q$, we will call sets of the form $\{\xi \in \mathbb{Z}_q : |\xi - a| \leq q^{-b}\}$ an interval inside $\mathbb{Z}_q$ of length q^{-b} (so the length of an interval coincides with its diameter, i.e. maximum distance between two points in that interval). Similarly for $(c_1, c_2) \in \mathbb{Q}_q^2$, we will call sets of the form $\{(x, y) \in \mathbb{Q}_q^2 : |x - c_1| \leq q^{-b}, |y -$

$c_2| \le q^{-b}\}$ a square of side length q^{-b}. Note that because the norm on $\mathbb{Q}_q^2$ is the maximum q-adic norm of each coordinate, this square is the same as $\{(x,y) \in \mathbb{Q}_q^2 : |(x,y) - (c_1,c_2)| \le q^{-b}\}$. Thanks to the ultrametric inequality, if two squares intersect, then one is contained inside the other; hence two squares of the same size are either equal or disjoint.

Observe that $\mathbb{Z}_q$ is a subset of $\mathbb{Q}_q$ consisting of elements of the form $\sum_{j\ge 0} a_j q^j$ where $a_j \in \{0, 1, \dots, q-1\}$. Since each positive integer has a base q representation, we may embed $\mathbb{N}$ into $\mathbb{Z}_q$. Identifying -1 with the element $\sum_{j\ge0}(q-1)q^j$ in $\mathbb{Z}_q$ then allows us to embed $\mathbb{Z}$ into $\mathbb{Z}_q$.

Note that if $\ell \in \mathbb{N}$, the intervals $\{\xi \in \mathbb{Z}_q : |\xi - a| \le 1/q^\ell\}$ for $a = 0, 1, \dots, q^\ell - 1$ partition $\mathbb{Z}_q$ into q^ℓ many disjoint intervals which are pairwise disjoint and each pair of intervals are separated by distance at least $q^{-\ell+1}$. To see this, suppose $|\xi_1 - a| \le q^{-\ell}$ and $|\xi_2 - b| \le q^{-\ell}$ for some $a \ne b$. As $|a-b| \ge q^{-\ell+1}$ and $|(\xi_1 - \xi_2) - (a-b)| \le q^{-\ell}$, the equality case of the ultrametric inequality implies that $|\xi_1 - \xi_2| = |a - b| \ge q^{-\ell+1}$.

Next, for fixed $a \in \{0, 1, \dots, q^\ell - 1\}$, the interval $\{\xi \in \mathbb{Z}_q : |\xi - a| \le 1/q^\ell\}$ is exactly the $\xi \in \mathbb{Z}_q$ such that $\xi \equiv a \pmod{q^\ell}$ (meaning $q^{-\ell}(\xi - a) \in \mathbb{Z}_q$). This illustrates the connection between q-adic intervals in $\mathbb{Q}_q$ and residue classes and both point of views are useful throughout; for instance, it follows easily now that $\mathbb{Z}_q$ is the union of these q^ℓ disjoint intervals.

Finally, let χ be the additive character of $\mathbb{Q}_q$ that is equal to 1 on $\mathbb{Z}_q$ and non-trivial on $q^{-1}\mathbb{Z}_q$ (up to isomorphism, there is essentially just one, given by

$$\chi(x) := e\Big(\sum_{j=k}^{-1} a_j q^j\Big) \quad \text{if } x = \sum_{j=k}^{\infty} a_j q^j$$

where $a_j \in \{0, \dots, q-1\}$ for all j). From this, one can define the Fourier transform for $f \in L^1(\mathbb{Q}_q)$ by $\widehat{f}(\xi) := \int_{\mathbb{Q}_q} f(x)\chi(-\xi x)\, dx$ for $\xi \in \mathbb{Q}_q$, where dx is the Haar measure on $\mathbb{Q}_q$, and we have an analogous definition for the Fourier transform in higher dimensions. The theory of the Fourier transform in $\mathbb{Q}_q$ is essentially the same as in $\mathbb{R}$ and we refer the interested reader to [11, 12] for more details. Note that in $\mathbb{Q}_q$ and in higher dimensions, linear combinations of indicator functions of intervals and squares play the analogue of Schwartz functions in the real setting. For $f, g \in L^1(\mathbb{Q}_q^2) \cap L^2(\mathbb{Q}_q^2)$, we have Plancherel's identity $\int_{\mathbb{Q}_q^2} f\,\overline{g} = \int_{\mathbb{Q}_q^2} \hat{f}\,\overline{\hat{g}}$, which allows one to extend the Fourier transform to a unitary operator on $L^2(\mathbb{Q}_q^2)$. We also have $\widehat{f * g} = \hat{f}\hat{g}$ for any integrable f and g on $\mathbb{Q}_q^2$, where $(f*g)(x)$ is the convolution $\int_{\mathbb{Q}_q^2} f(x-y)g(y)dy$. The inverse Fourier transform will be denoted by

$\check{\cdot}$, and we have $f = \check{\hat{f}}$ for Schwartz functions f. Henceforth we will only deal with Schwartz functions on $\mathbb{Q}_q^2$; note F_τ is Schwartz whenever F is Schwartz.

2.1. Basic geometry and the uncertainty principle

The key property about harmonic analysis in $\mathbb{Q}_q$ is that the Fourier transform of an indicator function of an interval is another indicator function of an interval. The key lemma is following, for a proof see p.42 of [12].

Lemma 2.1. *For $\xi \in \mathbb{Q}_q$ and $\gamma \in \mathbb{Z}$,*

$$\widetilde{1_{|x|\le q^\gamma}}(\xi) = \int_{|x|\le q^\gamma} \chi(\xi x)\, dx = q^\gamma (1_{|\xi|\le q^{-\gamma}})(\xi).$$

Another useful geometric fact about $\mathbb{Q}_q^2$ is that curvature disappears entirely if one considers the intersection of $\Xi_{1/R}$ with a vertical strip of width $R^{-1/2}$.

Lemma 2.2. *For any $R \in q^{2\mathbb{Z}}$ and any interval $I \subset \mathbb{Q}_q$ with length $|I| = R^{-1/2}$, the set $\{(\xi,\eta) \in \mathbb{Q}_q^2 \colon \xi \in I, |\eta - \xi^2| \le R^{-1}\}$ coincides with the parallelogram*

$$\{(\xi,\eta) \in \mathbb{Q}_q^2 \colon |\xi - a| \le R^{-1/2}, |\eta - 2a\xi + a^2| \le R^{-1}\}$$

where a is any point in I.

Proof. Let $a \in I$. The ultrametric inequality implies $I = \{\xi \in \mathbb{Q}_q \colon |\xi - a| \le R^{-1/2}\}$. Now $|\eta - \xi^2| = |\eta - a^2 - 2a(\xi - a) - (\xi - a)^2| = |(\eta - 2a\xi + a^2) - (\xi - a)^2|$. It follows that for $\xi \in I$, i.e. if $|\xi - a| \le R^{-1/2}$, then $|\eta - \xi^2| \le R^{-1}$, if and only if $|\eta - 2a\xi + a^2| \le R^{-1}$. ☐

This motivates the following rigorous q-adic uncertainty prinicple, that is just a heuristic in $\mathbb{R}$.

Lemma 2.3 (Uncertainty Principle). *Let $R \in q^{2\mathbb{Z}}$ and $I \subset \mathbb{Q}_q$ be an interval of length $|I| = R^{-1/2}$. Define the parallelogram*

$$P := \{(\xi,\eta) \in \mathbb{Q}_q^2 : \xi \in I, |\eta - \xi^2| \le R^{-1}\} \tag{2.1}$$

and the dual parallelogram

$$T := \{(x,y) \in \mathbb{Q}_q^2 : |x + 2ay| \le R^{1/2}, |y| \le R\} \tag{2.2}$$

where a is any point in I (this is well-defined independent of the choice of a). Let f be Schwartz and Fourier supported in P. Then $|f|$ is constant on each translate of T.

Proof. One only needs to prove this for $I = \mathbb{Z}_q$, $R = 1$ and then invoke affine invariance. Alternatively, and more directly, we have

$$\begin{aligned}\widecheck{1_P}(x,y) &= \int_{|t|\leq R^{-1}} \int_{|s-a|\leq R^{-1/2}} \chi(sx+s^2y)\chi(ty)\,ds\,dt \\ &= \chi(ax+a^2y)\Big(\int_{|s|\leq R^{-1/2}} \chi(s(x+2ay)+s^2y)\,ds\Big)R^{-1}1_{|y|\leq R}\end{aligned}$$

where the last equality is by Lemma 2.1. Since $|y| \leq R$, $|s^2y| \leq 1$ and therefore $s^2y \in \mathbb{Z}_q$. As χ is trivial on $\mathbb{Z}_q$, after another application of Lemma 2.1, the above expression is equal to $R^{-3/2}\chi(ax+a^2y)1_{|x+2ay|\leq R^{1/2}, |y|\leq R} = R^{-3/2}\chi(ax+a^2y)1_T$.

Suppose $(x,y) \in (A,B)+T$ for some $(A,B) \in \mathbb{Q}_q^2$. Write $x = A + x'$ and $y = B + y'$ for some $(x', y') \in T$. Then since $f = f * \widecheck{1_P}$, we have

$$\begin{aligned}f(x,y) &= R^{-3/2}\chi(ax+a^2y) \\ &\quad\times \int_{\mathbb{Q}_q^2} f(z,w)\chi(-az-a^2w)1_T(x'+A-z, y'+B-w)\,dz\,dw\end{aligned} \tag{2.3}$$

Since $|x'+2ay'| \leq R^{1/2}$, using the ultrametric inequality, $|(x'+A-z)+2a(y'+B-w)| \leq R^{1/2}$ if and only if $|(A-z)+2a(B-w)| \leq R^{1/2}$. Similarly, since $|y'| \leq R$, $|y'+B-w| \leq R$ if and only if $|B-w| \leq R$. Therefore (2.3) is equal to

$$R^{-3/2}\chi(ax+a^2y)\int_{\mathbb{Q}_q^2} f(z,w)\chi(-az-a^2w)1_T(A-z, B-w)\,dz\,dw.$$

Thus $|f(x,y)|$ is independent of $(x,y) \in (A,B)+T$ and therefore $|f|$ is constant on each translate of T (with a constant that depends on f, P, I, and the particular translate of T). □

A similar proof as above shows that if f is Fourier supported in a square of side length L, then $|f|$ is constant on any square of side length L^{-1}. Furthermore, if f is Fourier supported in a square centered at the origin of side length L, then f itself is constant on any square of side length L^{-1}.

In analogy with the real setting, we will say that the parallelogram T in (2.2) has direction $(-2a, 1)$. These parallelograms T enjoy the following nice geometric properties.

Lemma 2.4. *If $R \in q^{2\mathbb{N}}$, $I \subset \mathbb{Z}_q$ is an interval with $|I| = R^{-1/2}$, and T is the parallelogram defined by* (2.2) *(with $a \in I$), then*

(a) each translate of T is the union of $R^{1/2}$ many squares of side length $R^{1/2}$;

(b) any two translates of T are either equal or disjoint;

(c) any square of side length R can be partitioned into translates of T.

We write $\mathbb{T}(I)$ for the set of all translates of T. Note that (c) *implies that $\mathbb{Q}_q^2$ can be tiled by translates of T.*

Proof. (a) First, we claim that if $(x, y) \in T$, and $|(x', y') - (x, y)| \le R^{1/2}$, then $(x', y') \in T$ as well. This is because $|x' + 2ay'| = |x + 2ay + (x' - x) + 2a(y' - y)| \le R^{1/2}$ if both $|x + 2ay| \le R^{1/2}$ and $|(x', y') - (x, y)| \le R^{1/2}$ (recall $|2a| \le 1$ when $a \in \mathbb{Z}_q$). Similarly, $|y| \le R$ and $|y' - y| \le R^{1/2}$ implies $|y'| \le R$. This proves the claim. It follows that if (x, y) belongs to a certain translate of T, then the square of side length $R^{1/2}$ containing (x, y) is also contained in the same translate of T.

Now by the ultrametric inequality, two squares of side length $R^{1/2}$ are either equal or disjoint. Thus every translate of T is a union of squares of side lengths $R^{1/2}$, and volume considerations show that each translate of T contains $R^{1/2}$ many such squares.

(b) It suffices to show that if $(x, y) + T$ intersects T, then $(x, y) \in T$ (because then $(x, y) + T = T$). But if $(x, y) + T$ and T both contains a point (x', y'), then both $|(x' - x) + 2a(y' - y)| \le R^{1/2}$ and $|x' + 2ay'| \le R^{1/2}$, which implies $|x + 2ay| \le R^{1/2}$. Similarly, $|y' - y| \le R$ and $|y'| \le R$ implies $|y| \le R$. Thus $(x, y) \in T$, as desired.

(c) Write $R = q^{2A}$ for $A \ge 1$. It suffices to partition $Q = \{(x, y) \in \mathbb{Q}_q^2 : |x| \le R, |y| \le R\}$ into translates of parallelograms $T_a := \{(x, y) \in \mathbb{Q}_q^2 : |x + 2ay| \le R^{1/2}, |y| \le R\}$.

We first consider the $a = 0$ case. Let $S = \{\sum_{-2A \le j < -A} a_j q^j : a_j \in \{0, 1, \dots, q - 1\}\}$. Note that $\#S = R^{1/2}$.

We claim we can tile Q by $\{(s, 0) + T_0 : s \in S\}$. Indeed, for each $(x, y) \in Q$, we can write $x = \sum_{-2A \le j < -A} x_j q^j + \sum_{j \ge -A} x_j q^j$ for some $x_j \in \{0, 1, \dots, q - 1\}$. As $\sum_{-2A \le j < -A} x_j q^j \in S$,

$x \in (\sum_{-2A \le j < -A} x_j q^j, 0) + T_0$. This shows $Q \subset \bigcup_{s \in S} (s,0) + T_0$. The ultrametric inequality implies that $(s,0) + T_0 \subset Q$ for each $s \in S$ and so $Q = \bigcup_{s \in S}(s,0) + T_0$.

Finally, this union is disjoint as if $(x,y) \in ((s_1,0)+T_0) \cap ((s_2,0)+T_0)$, then $|s_1 - s_2| \le R^{1/2}$ but from the definition of S, $|s_1 - s_2| \ge q^{A+1} = R^{1/2}q$. Therefore we have partitioned Q into translates of T_0.

Next we consider the general case. Let $L_a = \left(\begin{smallmatrix} 1 & 2a \\ 0 & 1 \end{smallmatrix}\right)$. The ultrametric inequality gives that $L_a(Q) = Q$ since $|2a| \le 1$ and for $s \in S$, $L_a((s,0) + T_0) = (s,0) + T_a$. Therefore we can also partition Q into translates of T_a. □

Corollary 2.5. *Let $R \in q^{2\mathbb{N}}$, $I \subset \mathbb{Z}_q$ be an interval with $|I| = R^{-1/2}$, and f be a Schwartz function with Fourier support in $\{(\xi,\eta) \in \mathbb{Q}_q^2 \colon \xi \in I, |\eta - \xi^2| \le 1/R\}$. Then there exist constants $\{c_T\}_{T \in \mathbb{T}(I)}$ such that*

$$|f| = \sum_{T \in \mathbb{T}(I)} c_T 1_T. \tag{2.4}$$

As a result, $|f|^2 = \sum_{T \in \mathbb{T}(I)} c_T^2 1_T$, and

$$\int_{\mathbb{Q}_q^2} |f|^2 = \sum_{T \in \mathbb{T}(I)} c_T^2 |T|.$$

Proof. By Lemma 2.3, for every $T \in \mathbb{T}(I)$, there exists a constant c_T so that $|f| = c_T$ on T. By Lemma 2.4(c), $\mathbb{T}(I)$ tiles $\mathbb{Q}_q^2$. Thus (2.4) holds and the rest follows easily. □

Lemma 2.6. *Suppose $R \in q^{2\mathbb{N}}$ and $a, b \in \mathbb{Z}_q$ with $a \ne b$, let*

$$T = \{(x,y) \in \mathbb{Q}_q^2 : |x + 2ay| \le R, |y| \le R^2\}$$

and

$$T' = \{(x,y) \in \mathbb{Q}_q^2 : |x + 2by| \le R, |y| \le R^2\}.$$

Then

$$|T \cap T'| \le \frac{R^2}{|b-a|}.$$

Proof. By redefining x, we may assume that $a = 0$. Then

$$\begin{aligned} T \cap T' &= \{(x,y) \in \mathbb{Q}_q^2 : \max(|x|, |x+2by|) \le R, |y| \le R^2\} \\ &\subset \{(x,y) \in \mathbb{Q}_q^2 : |x| \le R, |y| \le R/|2b|\}. \end{aligned}$$

Since q is an odd prime, the claim then follows since the Haar measure is normalized so that $|\mathbb{Z}_q| = 1$. □

3. Theorem 1.2 implies Theorem 1.1

Since $K(M)$ is trivially increasing, it suffices to show Theorem 1.1 only in the case when $M = q^t$ for some $t \in \mathbb{N}$. By using the trivial bound for $K(M)$, we may also assume that t is sufficiently large (depending only on an absolute constant). By considering real and imaginary parts, we may also assume that a_n is a sequence of real numbers in (1.1).

Let $R = M^2 = q^{2t}$. Choose F such that

$$\widehat{F}(\xi,\eta) = \sum_{n=1}^{q^t} a_n 1_{(n,n^2)+B(0,q^{-10t})}(\xi,\eta) q^{20t}.$$

Here we are using the embedding of $\mathbb{Z}$ into $\mathbb{Z}_q$, and $(n,n^2) + B(0,q^{-10t})$ denotes the square $\{(\xi,\eta) \in \mathbb{Q}_q^2\colon |(\xi,\eta) - (n,n^2)| \le q^{-10t}\}$. Note that $\widehat{F}$ is indeed supported inside $\Xi_{1/R}$ since if $|(\xi,\eta) - (n,n^2)| \le q^{-10t}$ for some $n \in \mathbb{N}$, then $\xi \in \mathbb{Z}_q$ and

$$\begin{aligned} |\xi^2 - \eta| &= |(\xi - n)^2 + 2n(\xi - n) + n^2 - \eta| \\ &\le \max(|\xi - n|^2, |2n||\xi - n|, |n^2 - \eta|). \end{aligned}$$

Since $q \ge 3$ is an odd prime, $|2n| \le 1$ and so the above is $\le q^{-10t} \le q^{-2t}$.

Inverting the Fourier transform gives that

$$F(x) = \left(\sum_{n=1}^{q^t} a_n \chi(x_1 n + x_2 n^2) \right) 1_{B(0,q^{10t})}(x).$$

Similarly, for each τ on the right hand side of (1.3) (with length $R^{-1/2} = M^{-1} = q^{-t}$), $F_\tau(x) = a_n \chi(x_1 n + x_2 n^2) 1_{B(0,q^{10t})}(x)$ where n is the unique element in $\{1, \dots, q^t\} \cap \tau$; then $\|F_\tau\|_{L^\infty(\mathbb{Q}_q^2)}^2 = |a_n|^2$ and $\|F_\tau\|_{L^2(\mathbb{Q}_q^2)}^2 = |a_n|^2 q^{20t}$. The right hand side of (1.3) is then $\lesssim (\log M)^{12+10\varepsilon} q^{20t} (\sum_{n=1}^{q^t} |a_n|^2)^3$.

It now remains to show that

$$\|F\|^6_{L^6(\mathbb{Q}_q^2)} = q^{20t}\|\sum_{n=1}^{q^t} a_n e(nx_1 + n^2 x_2)\|^6_{L^6([0,1]^2)}. \tag{3.1}$$

This relies on that we are working with L^6. Expanding the left hand side gives

$$\begin{aligned}\sum_{n_1,\ldots,n_6=1}^{q^t} a_{n_1}\cdots a_{n_6}\int_{B(0,q^{10t})} \chi\big((n_1+n_2+n_3-n_4-n_5-n_6)x_1 \\ + (n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2)x_2\big)\,dx.\end{aligned} \tag{3.2}$$

Applying Lemma 2.1 gives that the above is equal to

$$\sum_{n_1,\ldots,n_6=1}^{q^t} q^{20t} a_{n_1}\cdots a_{n_6} 1_{|n_1+n_2+n_3-n_4-n_5-n_6|\le q^{-10t}} 1_{|n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2|\le q^{-10t}}.$$

The statement that $(n_1,\ldots,n_6)\in\{1,\ldots,q^t\}^6$ are such that

$$\begin{aligned}|n_1+n_2+n_3-n_4-n_5-n_6| &\le q^{-10t},\\ |n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2| &\le q^{-10t}\end{aligned} \tag{3.3}$$

is equivalent to the statement that $(n_1,\ldots,n_6)\in\{1,\ldots,q^t\}^6$ are such that

$$\begin{aligned}n_1+n_2+n_3-n_4-n_5-n_6 &\equiv 0 \pmod{q^{10t}},\\ n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2 &\equiv 0 \pmod{q^{10t}}.\end{aligned}$$

Since the $1\le n_i\le q^t$, $n_1+n_2+n_3-n_4-n_5-n_6$ is an integer between $-3q^t$ and $3q^t$, while $n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2$ is an integer between $-3q^{2t}$ and $3q^{2t}$. Since the only integer $\equiv 0 \pmod{q^{10t}}$ between $-3q^{2t}$ and $3q^{2t}$ is 0, (3.3) is true for a given $(n_1,\ldots,n_6)\in\{1,\ldots,q^t\}^6$ if and only if

$$n_1+n_2+n_3-n_4-n_5-n_6=0, \quad n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2=0.$$

Thus (3.2) is equal to

$$q^{20t}\sum_{n_1,\ldots,n_6=1}^{q^t} a_{n_1}\cdots a_{n_6} 1_{n_1+n_2+n_3-n_4-n_5-n_6=0} 1_{n_1^2+n_2^2+n_3^2-n_4^2-n_5^2-n_6^2=0}$$

which in turn is equal to the right hand side of (3.1).

4. Setting up many scales for the proof of Theorem 1.2

We now set out to prove Theorem 1.2. Fix $\varepsilon \in (0,1)$. Let A be an integer with

$$\frac{1}{\varepsilon} \le A \le \frac{2}{\varepsilon}.$$

Henceforth all implicit constants may depend on q, ε and A.

Given $R \in q^{2\mathbb{N}}$, choose $r \in 4\mathbb{N}$ so that

$$q^{q^{A(r-4)}} \le R < q^{q^{Ar}}.$$

Then $q^{Ar} \sim \log R$ and $(\log R)^{\varepsilon/2} \lesssim q^r \lesssim (\log R)^{\varepsilon}$, so for R sufficiently large (depending only on q and ε) we have $r \sim \log\log R$. Henceforth we fix a sufficiently large R, and define

$$R_k := q^{kr} \quad \text{for } k = 0, 1, \ldots, N,$$

where $N \in \mathbb{N}$ is defined such that

$$q^{Nr} \le R < q^{(N+1)r}.$$

The choice $r \in 4\mathbb{N}$ ensures that

$$R_k^{-1/2} \in q^{-2\mathbb{N}} \tag{4.1}$$

for every k. Throughout we write τ_k for a generic interval inside $\mathbb{Z}_q$ of length $R_k^{-1/2}$, for $k = 0, 1, \ldots, N$. For instance, $\sum_{\tau_N}$ means sums over all intervals $\tau_N \subset \mathbb{Z}_q$ with $|\tau_N| = R_N^{-1/2}$.

Let $F\colon \mathbb{Q}_q^2 \to \mathbb{C}$ be Fourier supported in $\Xi_{1/R}$ as in the statement of Theorem 1.2. In order to establish (1.3), it suffices to prove

$$\int_{\mathbb{Q}_q^2} |F|^6 \lesssim (\log R)^{12+9\varepsilon} \Big(\sum_{\tau_N} \|F_{\tau_N}\|^2_{L^\infty(\mathbb{Q}_q^2)}\Big)^2 \Big(\sum_{\tau_N} \|F_{\tau_N}\|^2_{L^2(\mathbb{Q}_q^2)}\Big) \tag{4.2}$$

and then trivially decouple from frequency scale $R_N^{-1/2}$ down to $R^{-1/2}$ (note $R_N^{-1/2}/R^{-1/2} \le q^{r/2} \lesssim (\log R)^{\varepsilon/2}$ which implies $\|F_{\tau_N}\|^2_{L^\infty} \lesssim (\log R)^{\varepsilon/2} \sum_{|\tau|=R^{-1/2}} \|F_\tau\|^2_{L^\infty}$ and $\sum_{\tau_N} \|F_{\tau_N}\|^2_{L^2} = \sum_{|\tau|=R^{-1/2}} \|F_\tau\|^2_{L^2}$ by Plancherel).

5. Bilinearization

The proof of Theorem 1.2 relies on the following key bilinear estimate:

Proposition 5.1. *Let F be Fourier supported in $\Xi_{1/R}$. For $k = 0, 1, \dots, N-1$, and for intervals $\tau_k \subset \mathbb{Z}_q$ with $|\tau_k| = R_k^{-1/2}$, we have*

$$\int_{\mathbb{Q}_q^2} \max_{\substack{\tau_{k+1} \neq \tau'_{k+1} \\ \tau_{k+1}, \tau'_{k+1} \subset \tau_k}} |F_{\tau_{k+1}} F_{\tau'_{k+1}}|^3 \lesssim (\log R)^{9+6\varepsilon} \Big(\sum_{\tau_N \subset \tau_k} \|F_{\tau_N}\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \Big(\sum_{\tau_N \subset \tau_k} \|F_{\tau_N}\|^2_{L^2(\mathbb{Q}_q^2)} \Big).$$

We also need the following Whitney decomposition for $\mathbb{Z}_q^2$, which expresses $\mathbb{Z}_q^2$ into a disjoint union of squares of different scales:

$$\mathbb{Z}_q^2 = \mathcal{W}_0 \sqcup \mathcal{W}_1 \sqcup \cdots \sqcup \mathcal{W}_{N-1} \sqcup \mathcal{W}^N$$

where

$$\mathcal{W}_k := \bigsqcup_{\tau_k \subset \mathbb{Z}_q} \bigsqcup_{\substack{\tau_{k+1} \neq \tau'_{k+1} \\ \tau_{k+1}, \tau'_{k+1} \subset \tau_k}} \tau_{k+1} \times \tau'_{k+1} \quad \text{for } k = 0, 1, \dots, N-1$$

and

$$\mathcal{W}^N := \bigsqcup_{\tau_N \subset \mathbb{Z}_q} \tau_N \times \tau_N.$$

The proof of (4.2), and hence Theorem 1.2 can then be given as follows. First,

$$\int_{\mathbb{Q}_q^2} |F|^6 = \int_{\mathbb{Q}_q^2} |F^2|^3 = \int_{\mathbb{Q}_q^2} \Big| \sum_{\tau_N \subset \mathbb{Z}_q} F_{\tau_N}^2 + \sum_{k=0}^{N-1} \sum_{\tau_{k+1} \times \tau'_{k+1} \subset \mathcal{W}_k} F_{\tau_{k+1}} F_{\tau'_{k+1}} \Big|^3$$

which by the Minkowski inequality is

(5.1)
$$\leq \left[\sum_{\tau_N} \left(\int_{\mathbb{Q}_q^2} \left| F_{\tau_N}^2 \right|^3 \right)^{1/3} + \sum_{k=0}^{N-1} \sum_{\tau_{k+1} \times \tau'_{k+1} \subset \mathcal{W}_k} \left(\int_{\mathbb{Q}_q^2} \left| F_{\tau_{k+1}} F_{\tau'_{k+1}} \right|^3 \right)^{1/3} \right]^3 .$$

Hölder's inequality gives

$$\sum_{\tau_N}\left(\int_{\mathbb{Q}_q^2}\left|F_{\tau_N}^2\right|^3\right)^{1/3} = \sum_{\tau_N}\|F_{\tau_N}\|_{L^6(\mathbb{Q}_q^2)}^2 \le \sum_{\tau_N}\|F_{\tau_N}\|_{L^\infty(\mathbb{Q}_q^2)}^{2\cdot\frac{2}{3}}\|F_{\tau_N}\|_{L^2(\mathbb{Q}_q^2)}^{2\cdot\frac{1}{3}}$$
$$\le (\sum_{\tau_N}\|F_{\tau_N}\|_{L^\infty(\mathbb{Q}_q^2)}^2)^{\frac{2}{3}}(\sum_{\tau_N}\|F_{\tau_N}\|_{L^2(\mathbb{Q}_q^2)}^2)^{\frac{1}{3}}.$$

In addition, for each fixed τ_k, the number of (τ_{k+1},τ'_{k+1}) with $\tau_{k+1},\tau'_{k+1}\subset\tau_k$ is $\le (q^{r/2})^2 \lesssim (\log R)^\varepsilon$. Together with Proposition 5.1, this shows that for each $k=0,1,\dots,N-1$,

$$\sum_{\tau_{k+1}\times\tau'_{k+1}\subset\mathcal{W}_k}\left(\int_{\mathbb{Q}_q^2}\left|F_{\tau_{k+1}}F_{\tau'_{k+1}}\right|^3\right)^{1/3}$$
$$=\sum_{\tau_k}\sum_{\substack{\tau_{k+1}\ne\tau'_{k+1}\\ \tau_{k+1},\tau'_{k+1}\subset\tau_k}}\left(\int_{\mathbb{Q}_q^2}\left|F_{\tau_{k+1}}F_{\tau'_{k+1}}\right|^3\right)^{1/3}$$
$$\lesssim (\log R)^{3+2\varepsilon}(\log R)^\varepsilon\sum_{\tau_k}(\sum_{\tau_N\subset\tau_k}\|F_{\tau_N}\|_{L^\infty(\mathbb{Q}_q^2)}^2)^{\frac{2}{3}}(\sum_{\tau_N\subset\tau_k}\|F_{\tau_N}\|_{L^2(\mathbb{Q}_q^2)}^2)^{\frac{1}{3}}$$
$$\le (\log R)^{3+3\varepsilon}(\sum_{\tau_N}\|F_{\tau_N}\|_{L^\infty(\mathbb{Q}_q^2)}^2)^{\frac{2}{3}}(\sum_{\tau_N}\|F_{\tau_N}\|_{L^2(\mathbb{Q}_q^2)}^2)^{\frac{1}{3}}$$

Thus (5.1) is bounded by

$$N^3(\log R)^{9+9\varepsilon}(\sum_{\tau_N}\|F_{\tau_N}\|_{L^\infty(\mathbb{Q}_q^2)}^2)^2(\sum_{\tau_N}\|F_{\tau_N}\|_{L^2(\mathbb{Q}_q^2)}^2)$$

which proves (4.2) because $N\lesssim\log R$.

Proposition 5.1 can be proved by parabolic rescaling and the proposition below. That is, we use the next proposition with $J=N-k$ and

$$f(x) := \chi(-R_k^{1/2}ax_1+R_ka^2x_2)F_{\tau_k}(R_k^{1/2}x_1-2aR_kx_2, R_kx_2) \tag{5.2}$$

where a is an arbitrary point in τ_k. Note that

$$\widehat{f}(\xi,\eta) = R_k^{-3/2}\widehat{F}_{\tau_k}(a+R_k^{-1/2}\xi, a^2+2aR_k^{-1/2}\xi+R_k^{-1}\eta) \tag{5.3}$$

is supported on $\Xi_{R_k/R}\subset\Xi_{1/R_{N-k}}$.

Proposition 5.2. *Let $J = 1, \dots, N$ and let f be Fourier supported in Ξ_{1/R_J}. Then*

$$\int_{\mathbb{Q}_q^2} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1} f_{\tau_1'}|^3 \lesssim (\log R)^{9+6\varepsilon} \Big(\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)}\Big)^2 \Big(\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}\Big).$$

It remains to prove Proposition 5.2.

6. Broad/Narrow decomposition: Proof of Proposition 5.2

The proof of Proposition 5.2 is via a broad/narrow decomposition. Let $J = 1, \dots, N$ and f be Fourier supported in Ξ_{1/R_J}. For $k = 0, 1, \dots, J-1$, and for $\tau_k \subset \mathbb{Z}_q$ with $|\tau_k| = R_k^{-1/2}$, define

(6.1)
$$\mathcal{B}_{\tau_k} = \{x \in \mathbb{Q}_q^2 \colon |f_{\tau_k}(x)| \le (\log R) q^{r/2} \max_{\substack{\tau_{k+1} \neq \tau_{k+1}' \\ \tau_{k+1}, \tau_{k+1}' \subset \tau_k}} |f_{\tau_{k+1}}(x) f_{\tau_{k+1}'}(x)|^{1/2}$$

(6.2)
$$\text{and} \quad \Big(\sum_{\tau_{k+1} \subset \tau_k} |f_{\tau_{k+1}}(x)|^6\Big)^{1/6} \le (\log R) q^{r/2} \max_{\substack{\tau_{k+1} \neq \tau_{k+1}' \\ \tau_{k+1}, \tau_{k+1}' \subset \tau_k}} |f_{\tau_{k+1}}(x) f_{\tau_{k+1}'}(x)|^{1/2}\}.$$

For $x \notin \mathcal{B}_{\tau_0}$, we have

$$\max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^3 \le \frac{q^{-r/2}}{(\log R)^6} \sum_{\tau_1} |f_{\tau_1}(x)|^6. \tag{6.3}$$

This is because if $x \notin \mathcal{B}_{\tau_0}$, then either (6.1) is violated, in which case

$$\begin{aligned} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^3 &\le \frac{q^{-3r}}{(\log R)^6} |f(x)|^6 = \frac{q^{-3r}}{(\log R)^6} \Big|\sum_{\tau_1} f_{\tau_1}(x)\Big|^6 \\ &\le \frac{q^{-3r}}{(\log R)^6} q^{5r/2} \sum_{\tau_1} |f_{\tau_1}(x)|^6, \end{aligned}$$

or (6.2) is violated, in which case

$$\max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^3 \le \frac{q^{-3r}}{(\log R)^6} \sum_{\tau_1} |f_{\tau_1}(x)|^6.$$

Either way (6.3) holds. Upon splitting the integral in Proposition 5.2 according to whether $x \in \mathcal{B}_{\tau_0}$ or not, (6.3) allows us to obtain

$$\int_{\mathbb{Q}_q^2} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1} f_{\tau_1'}|^3 \leq \int_{\mathcal{B}_{\tau_0}} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1} f_{\tau_1'}|^3 + \frac{q^{-r/2}}{(\log R)^6} \sum_{\tau_1} \int_{\mathbb{Q}_q^2} |f_{\tau_1}|^6. \tag{6.4}$$

Now observe that if $k = 1, \ldots, J-1$ and $|\tau_k| = R_k^{-1/2}$, then

(a) for $x \in \mathcal{B}_{\tau_k}$, we have

$$|f_{\tau_k}(x)|^6 \leq (\log R)^6 q^{3r} \max_{\substack{\tau_{k+1}, \tau_{k+1}' \subset \tau_k \\ \tau_{k+1} \neq \tau_{k+1}'}} |f_{\tau_{k+1}}(x) f_{\tau_{k+1}'}(x)|^3; \tag{6.5}$$

(b) for $x \notin \mathcal{B}_{\tau_k}$, we have

$$|f_{\tau_k}(x)|^6 \leq (1 - (\log R)^{-1})^{-6} \sum_{\tau_{k+1} \subset \tau_k} |f_{\tau_{k+1}}(x)|^6. \tag{6.6}$$

The estimate (6.5) holds because of (6.1). The proof of (6.6) proceeds via the Narrow Lemma:

Lemma 6.1 (Narrow Lemma). *Fix $\tau_k \subset \mathbb{Z}_q$ with $|\tau_k| = R_k^{-1/2}$. Suppose x satisfies*

$$|f_{\tau_k}(x)| > (\log R) q^{r/2} \max_{\substack{\tau_{k+1}, \tau_{k+1}' \subset \tau_k \\ \tau_{k+1} \neq \tau_{k+1}'}} |f_{\tau_{k+1}}(x) f_{\tau_{k+1}'}(x)|^{1/2}.$$

Then there exists a $\tau_{k+1} \subset \tau_k$ such that

$$|f_{\tau_k}(x)| \leq (1 - (\log R)^{-1})^{-1} |f_{\tau_{k+1}}(x)|.$$

Indeed, for $x \notin \mathcal{B}_{\tau_k}$, either (6.1) fails, in which case the Narrow Lemma applies, or (6.1) holds but (6.2) fails, in which case

$$|f_{\tau_k}(x)| \leq (\log R) q^{r/2} \max_{\substack{\tau_{k+1}, \tau_{k+1}' \subset \tau_k \\ \tau_{k+1} \neq \tau_{k+1}'}} |f_{\tau_{k+1}}(x) f_{\tau_{k+1}'}(x)|^{1/2} \leq \Big(\sum_{\tau_{k+1} \subset \tau_k} |f_{\tau_{k+1}}(x)|^6 \Big)^{1/6}.$$

Either way (6.6) holds. From (6.5) and (6.6), we see that for $k = 1, \dots, J-1$ and $|\tau_k| = R_k^{-1/2}$,

$$\frac{q^{-r/2}}{(\log R)^6}(1-(\log R)^{-1})^{-6(k-1)}\int_{\mathbb{Q}_q^2}|f_{\tau_k}|^6$$
$$\leq q^{5r/2}(1-(\log R)^{-1})^{-6(k-1)}\int_{\mathcal{B}_{\tau_k}}\max_{\substack{\tau_{k+1}\neq\tau'_{k+1}\\ \tau_{k+1},\tau'_{k+1}\subset\tau_k}}|f_{\tau_{k+1}}f_{\tau'_{k+1}}|^3$$
$$+\frac{q^{-r/2}}{(\log R)^6}(1-(\log R)^{-1})^{-6k}\sum_{\tau_{k+1}\subset\tau_k}\int_{\mathbb{Q}_q^2}|f_{\tau_{k+1}}|^6.$$

Summing over τ_k, we get

$$\frac{q^{-r/2}}{(\log R)^6}(1-(\log R)^{-1})^{-6(k-1)}\sum_{\tau_k}\int_{\mathbb{Q}_q^2}|f_{\tau_k}|^6$$
$$\leq q^{5r/2}(1-(\log R)^{-1})^{-6(k-1)}\sum_{\tau_k}\int_{\mathcal{B}_{\tau_k}}\max_{\substack{\tau_{k+1}\neq\tau'_{k+1}\\ \tau_{k+1},\tau'_{k+1}\subset\tau_k}}|f_{\tau_{k+1}}f_{\tau'_{k+1}}|^3$$
$$+\frac{q^{-r/2}}{(\log R)^6}(1-(\log R)^{-1})^{-6k}\sum_{\tau_{k+1}}\int_{\mathbb{Q}_q^2}|f_{\tau_{k+1}}|^6$$

for $k = 1, \dots, J-1$. We now apply these successively to the right hand side of (6.4), starting with $k = 1$ and going all the way up to $k = J-1$. Then

$$\int_{\mathbb{Q}_q^2}\max_{\tau_1\neq\tau_1'}|f_{\tau_1}f_{\tau_1'}|^3 \leq \int_{\mathcal{B}_{\tau_0}}\max_{\tau_1\neq\tau_1'}|f_{\tau_1}f_{\tau_1'}|^3$$
$$+\sum_{k=1}^{J-1}q^{5r/2}(1-(\log R)^{-1})^{-6(k-1)}\sum_{\tau_k}\int_{\mathcal{B}_{\tau_k}}\max_{\substack{\tau_{k+1}\neq\tau'_{k+1}\\ \tau_{k+1},\tau'_{k+1}\subset\tau_k}}|f_{\tau_{k+1}}f_{\tau'_{k+1}}|^3$$
$$+\frac{q^{-r/2}}{(\log R)^6}(1-(\log R)^{-1})^{-6(J-1)}\sum_{\tau_J}\int_{\mathbb{Q}_q^2}|f_{\tau_J}|^6.$$

Since $J \le N \lesssim \log R$, this gives

$$\int_{\mathbb{Q}_q^2} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1} f_{\tau_1'}|^3 \lesssim q^{5r/2} (\log R) \max_{k=0,\ldots,J-1} \sum_{\tau_k} \int_{\mathcal{B}_{\tau_k}} \max_{\substack{\tau_{k+1} \neq \tau_{k+1}' \\ \tau_{k+1}, \tau_{k+1}' \subset \tau_k}} |f_{\tau_{k+1}} f_{\tau_{k+1}'}|^3 \tag{6.7}$$

$$+ \frac{q^{-r/2}}{(\log R)^6} \sum_{\tau_J} \int_{\mathbb{Q}_q^2} |f_{\tau_J}|^6. \tag{6.8}$$

Observe that

$$(6.8) \lesssim \frac{q^{-r/2}}{(\log R)^6} (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}) \tag{6.9}$$

which is much better than what we needed in the conclusion of Proposition 5.2. Equation (6.7) is controlled by the following proposition:

Proposition 6.2. *Let $J = 1, \ldots, N$ and let f be Fourier supported in Ξ_{1/R_J}. Let $k = 0, 1, \ldots, J-1$ and $\tau_k \subset \mathbb{Z}_q$ with $|\tau_k| = R_k^{-1/2}$. Then*

$$\int_{\mathcal{B}_{\tau_k}} \max_{\substack{\tau_{k+1} \neq \tau_{k+1}' \\ \tau_{k+1}, \tau_{k+1}' \subset \tau_k}} |f_{\tau_{k+1}} f_{\tau_{k+1}'}|^3 \lesssim (\log R)^{8+\frac{7\varepsilon}{2}} (\sum_{\tau_J \subset \tau_k} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 (\sum_{\tau_J \subset \tau_k} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}). \tag{6.10}$$

Assuming this for the moment, we see that (6.7) is bounded by

$$\begin{aligned}(6.7) &\lesssim q^{5r/2} (\log R)^{1+8+\frac{7\varepsilon}{2}} \max_{k=0,\ldots,J-1} \sum_{\tau_k} (\sum_{\tau_J \subset \tau_k} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 (\sum_{\tau_J \subset \tau_k} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}) \\ &\lesssim (\log R)^{9+6\varepsilon} (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}).\end{aligned}$$

(Recall $q^{5r/2} \le (\log R)^{5\varepsilon/2}$.) Together with (6.9) we finish the proof of Proposition 5.2. It remains to prove Lemma 6.1 and Proposition 6.2.

Proof of Lemma 6.1. Let τ_{k+1}^* be the $\tau_{k+1} \subset \tau_k$ such that

$$\max_{\tau_{k+1} \subset \tau_k} |f_{\tau_{k+1}}(x)| = |f_{\tau_{k+1}^*}(x)|.$$

For $\tau_{k+1} \subset \tau_k$ such that $\tau_{k+1} \neq \tau_{k+1}^*$, note that

$$|f_{\tau_{k+1}}(x)| \le |f_{\tau_{k+1}}(x) f_{\tau_{k+1}^*}(x)|^{1/2} < (\log R)^{-1} q^{-r/2} |f_{\tau_k}(x)|.$$

Therefore

$$\begin{aligned}|f_{\tau^*_{k+1}}(x)| &= |f_{\tau_k}(x) - \sum_{\tau_{k+1}\neq\tau^*_{k+1}} f_{\tau_{k+1}}(x)| \\ &\geq (1 - \#\{\tau_{k+1} : \tau_{k+1} \subset \tau_k, \tau_{k+1} \neq \tau^*_{k+1}\}(\log R)^{-1} q^{-r/2})|f_{\tau_k}(x)| \\ &\geq (1 - (\log R)^{-1})|f_{\tau_k}(x)|.\end{aligned}$$

□

To prove Proposition 6.2, we need the following level set estimate.

Proposition 6.3. *Let $J = 1, \dots, N$ and let f be with Fourier support in Ξ_{1/R_J}. For $\alpha > 0$, let*

$$U_\alpha(f) := \{x \in \mathbb{Q}_q^2 : \max_{\tau_1\neq\tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^{1/2} \sim \alpha \quad \text{and } (\sum_{\tau_1} |f_{\tau_1}(x)|^6)^{1/6} \lesssim (\log R) q^{r/2}\alpha\}.$$

Then

$$\alpha^6 |U_\alpha(f)| \lesssim (\log R)^{7+\frac{7\varepsilon}{2}} (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)})$$

where the implied constant is independent of f and α.

Proof of Proposition 6.2. By the same rescaling as in (5.2)-(5.3), it suffices to prove (6.10) for $k = 0$. For a given $J_0 = 1, 2, \dots, N$ and $k_0 = 1, 2, \dots, J_0 - 1$, the case of $(k, J) = (k_0, J_0)$ in (6.10) follows from the case $(k, J) = (0, J_0 - k_0)$. Note also that in this rescaling, it is important that in the definition of $\mathcal{B}_{\tau_k}$ we have the condition $x \in \mathbb{Q}_q^2$ in (6.1) rather than a smaller spatial region.

Now to prove (6.10) for $k = 0$, for each square $Q_{R_J^{1/2}} \subset \mathbb{Q}_q^2$ of side length $R_J^{1/2}$, we estimate

$$\int_{\mathcal{B}\cap Q_{R_J^{1/2}}} \max_{\tau_1\neq\tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^3 \tag{6.11}$$

where we write $\mathcal{B} := \mathcal{B}_{\tau_0}$ for brevity. Let

$$\mathcal{B}_{\text{small}}(Q_{R_J^{1/2}}) := \{x \in \mathcal{B} \cap Q_{R_J^{1/2}} : \max_{\tau_1\neq\tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^{1/2} \leq R^{-1/2} \max_{\tau_J} \|f_{\tau_J}\|_{L^\infty(Q_{R_J^{1/2}})}\}$$

and partition $(\mathcal{B} \cap Q_{R_J^{1/2}}) \setminus \mathcal{B}_{\text{small}}(Q_{R_J^{1/2}})$ into $O(\log R)$ sets where

$$\max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^{1/2} \sim \alpha \quad \text{and}$$
$$R^{-1/2} \max_{\tau_J} \|f_{\tau_J}\|_{L^\infty(Q_{R_J^{1/2}})} \leq \alpha \leq R \max_{\tau_J} \|f_{\tau_J}\|_{L^\infty(Q_{R_J^{1/2}})}.$$

By pigeonholing, there exists an α_* such that

$$(6.11) \lesssim R_J R^{-3} \max_{\tau_J} \|f_{\tau_J}\|^6_{L^\infty(Q_{R_J^{1/2}})} + (\log R) \alpha_*^6 |Q_{R_J^{1/2}} \cap U_{\alpha_*}(f)|. \tag{6.12}$$

But by the uncertainty principle (see discussion after Lemma 2.3), $|f_{\tau_J}|$ is constant on $Q_{R_J^{1/2}}$, so

$$\|f_{\tau_J}\|^2_{L^\infty(Q_{R_J^{1/2}})} = R_J^{-1} \|f_{\tau_J}\|^2_{L^2(Q_{R_J^{1/2}})} \leq \|f_{\tau_J}\|^2_{L^2(Q_{R_J^{1/2}})}.$$

Thus

$$\begin{aligned} \max_{\tau_J} \|f_{\tau_J}\|^6_{L^\infty(Q_{R_J^{1/2}})} &\leq \max_{\tau_J} \|f_{\tau_J}\|^4_{L^\infty(\mathbb{Q}_q^2)} \|f_{\tau_J}\|^2_{L^2(Q_{R_J^{1/2}})} \\ &\leq \Big(\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)}\Big)^2 \sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(Q_{R_J^{1/2}})}. \end{aligned}$$

Plugging this back into (6.12), and summing over $Q_{R_J^{1/2}}$, we obtain

$$\begin{aligned} &\int_{\mathcal{B}} \max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^3 \\ &\quad \lesssim R_J R^{-3} \Big(\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)}\Big)^2 \sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)} + (\log R) \alpha_*^6 |U_{\alpha_*}(f)| \\ &\quad \lesssim (\log R)^{8 + \frac{7\varepsilon}{2}} \Big(\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)}\Big)^2 \sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)} \end{aligned}$$

where the last inequality is a consequence of Proposition 6.3. This finishes our proof. □

The rest of the argument goes into proving Proposition 6.3.

7. High/Low decomposition: Proof of Proposition 6.3

7.1. Square functions and pruning of wave packets

Fix $J = 1, \dots, N$ and fix f with Fourier support in Ξ_{1/R_J}. For $x \in \mathbb{Q}_q^2$ and λ to be chosen later (see (7.8)), define

$$g_J(x) := \sum_{\tau_J} |f_{\tau_J}(x)|^2 = \sum_{\tau_J} \sum_{T_J \in \mathbb{T}(\tau_J)} |(1_{T_J} f_{\tau_J})(x)|^2$$
$$f_J(x) := \sum_{\tau_J} \sum_{\substack{T_J \in \mathbb{T}(\tau_J) \\ \|1_{T_J} f_{\tau_J}\|_{L^\infty(\mathbb{Q}_q^2)} \leq \lambda}} (1_{T_J} f_{\tau_J})(x)$$

and for $k = J-1, J-2, \dots, 1$, define

$$g_k(x) := \sum_{\tau_k} |(f_{k+1,\tau_k})(x)|^2 = \sum_{\tau_k} \sum_{T_k \in \mathbb{T}(\tau_k)} |(1_{T_k} f_{k+1,\tau_k})(x)|^2$$
$$f_k(x) := \sum_{\tau_k} \sum_{\substack{T_k \in \mathbb{T}(\tau_k) \\ \|1_{T_k} f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)} \leq \lambda}} (1_{T_k} f_{k+1,\tau_k})(x).$$

Note that the Fourier support of g_k is contained in a $R_k^{-1/2}$ square centered at the origin and hence g_k is constant on squares of side length $R_k^{1/2}$. Additionally by definition of the f_k,

$$|f_{k,\tau_k}| \leq |f_{k+1,\tau_k}| \tag{7.1}$$

and so

$$\int_{\mathbb{Q}_q^2} |f_k|^2 = \sum_{\tau_k} \int_{\mathbb{Q}_q^2} |f_{k,\tau_k}|^2 \leq \sum_{\tau_k} \int_{\mathbb{Q}_q^2} |f_{k+1,\tau_k}|^2 = \int_{\mathbb{Q}_q^2} |f_{k+1}|^2,$$

where in the last step we applied L^2 orthogonality. Therefore

$$\int_{\mathbb{Q}_q^2} |f_1|^2 \leq \int_{\mathbb{Q}_q^2} |f_2|^2 \leq \dots \leq \int_{\mathbb{Q}_q^2} |f_J|^2 \leq \int_{\mathbb{Q}_q^2} |f|^2. \tag{7.2}$$

This matches the intuition that when passing from f_J to f_1 we are throwing away wave packets and therefore at least at the L^2 level, we have a monotonicity relation as above.

7.2. High and low lemmas

For $k = 1, \dots, J-1$, define

$$g_k^l = g_k * R_{k+1}^{-1} 1_{B(0,R_{k+1}^{1/2})} \quad \text{and} \quad g_k^h = g_k - g_k^l.$$

Note that g_k (and g_k^h) is Fourier supported on the union of $\{|\xi| \le R_k^{-1/2}, |\eta - 2\alpha\xi| \le R_k^{-1}\}$ where $\{\alpha\}$ is a collection of points chosen from $\{\tau_k\}$, with one α for each τ_k. Additionally, observe that since

$$R_{k+1}^{-1} \widehat{1}_{B(0,R_{k+1}^{1/2})} = 1_{B(0,R_{k+1}^{-1/2})} \tag{7.3}$$

we have $\widehat{g_k^l} = \widehat{g}_k 1_{B(0,R_{k+1}^{-1/2})}$ and so g_k^l is just the restriction of g_k to frequencies less than $R_{k+1}^{-1/2}$. By definition of g_k and g_k^l, both are nonnegative functions.

Lemma 7.1 (Low Lemma). *For $k = 1, \dots, J-1$, we have $g_k^l \le g_{k+1}$.*

Proof of Lemma 7.1. We have

$$\begin{aligned} g_k^l &= g_k * R_{k+1}^{-1} 1_{B(0,R_{k+1}^{1/2})} \\ &= \sum_{\tau_k} \sum_{\tau_{k+1}, \tau'_{k+1} \subset \tau_k} (f_{k+1,\tau_{k+1}} \overline{f_{k+1,\tau'_{k+1}}}) * R_{k+1}^{-1} 1_{B(0,R_{k+1}^{1/2})}. \end{aligned} \tag{7.4}$$

Taking a Fourier transform we see that

$$\begin{aligned} &(f_{k+1,\tau_{k+1}} \overline{f_{k+1,\tau'_{k+1}}}) * R_{k+1}^{-1} 1_{B(0,R_{k+1}^{1/2})} \\ &\quad = \begin{cases} |f_{k+1,\tau_{k+1}}|^2 * R_{k+1}^{-1} 1_{B(0,R_{k+1}^{1/2})} & \text{if } \tau_{k+1} = \tau'_{k+1} \\ 0 & \text{otherwise} \end{cases} \\ &\quad = \begin{cases} |f_{k+1,\tau_{k+1}}|^2 & \text{if } \tau_{k+1} = \tau'_{k+1} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where the last equality is because of (7.3) and that $|f_{k+1,\tau_{k+1}}|^2$ is Fourier supported in $B(0, R_{k+1}^{-1/2})$. Thus (7.4) is equal to

$$\sum_{\tau_{k+1}} |f_{k+1,\tau_{k+1}}|^2 \le \sum_{\tau_{k+1}} |f_{k+2,\tau_{k+1}}|^2 = g_{k+1}$$

by (7.1). Here if $k = J-1$, we interpret f_{k+2} to mean f. $\square$

Lemma 7.2 (High Lemma). *For $k = 1, \dots, J-1$,*

$$\int_{\mathbb{Q}_q^2} |g_k^h|^2 \leq q^{r/2} \sum_{\tau_k} \int_{\mathbb{Q}_q^2} |f_{k+1,\tau_k}|^4.$$

Proof of Lemma 7.2. It suffices to partition $\mathbb{Q}_q^2$ into squares with side length R_{k+1} and prove the estimate on each such square. Fix an arbitrary square $B \subset \mathbb{Q}_q^2$ of side length R_{k+1}. We have by Plancherel,

$$\int_B |g_k^h|^2 = \int \overline{\widehat{g_k^h}} (\widehat{g_k^h} * \widehat{1_B}).$$

Since g_k^h is Fourier supported outside $B(0, R_{k+1}^{-1/2})$ and 1_B is Fourier supported in $B(0, R_{k+1}^{-1})$, $\widehat{g_k^h} * \widehat{1_B}$ is supported in $B(0, R_k^{-1/2}) \setminus B(0, R_{k+1}^{-1/2})$ by the ultrametric inequality. Therefore the above is equal to

$$\sum_{\tau_k} \int_{B(0,R_k^{-1/2}) \setminus B(0,R_{k+1}^{-1/2})} \overline{(|f_{k+1,\tau_k}|^2)^\wedge} \sum_{\tau_k'} ((|f_{k+1,\tau_k'}|^2)^\wedge * \widehat{1_B}). \tag{7.5}$$

We claim that for each τ_k, the Fourier support of $|f_{k+1,\tau_k}|^2$ outside $B(0, R_{k+1}^{-1/2})$ only intersects $q^{r/2}$ many Fourier supports of the $|f_{k+1,\tau_k'}|^2$ outside $B(0, R_{k+1}^{-1/2})$.

Indeed, suppose there exists (ξ, η) such that $\max\{|\xi|, |\eta|\} > R_{k+1}^{-1/2}$ and

$$|\xi| \leq R_k^{-1/2}, \qquad |\eta - 2\alpha\xi|, |\eta - 2\alpha'\xi| \leq R_k^{-1}$$

for some $\alpha \in \tau_k$ and $\alpha' \in \tau_k'$. Then

$$|2(\alpha - \alpha')\xi| \leq R_k^{-1},$$

and so if $|\xi| > R_{k+1}^{-1/2}$, then

$$|\alpha - \alpha'| \leq R_k^{-1}/R_{k+1}^{-1/2} = R_k^{-1/2} q^{r/2}.$$

Else $|\xi| < R_{k+1}^{-1/2}$ and $|\eta| > R_{k+1}^{-1/2}$, which implies $|\eta - 2\alpha\xi| = \max\{|\eta|, |2\alpha\xi|\} > R_{k+1}^{-1/2}$, contradicting $|\eta - 2\alpha\xi| \leq R_k^{-1}$ if $k \geq 1$. So $|\alpha - \alpha'| \leq R_k^{-1/2} q^{r/2}$, the number of overlaps is just $q^{r/2}$ times.

Thus we have

$$\sum_{\tau_k} \int_{B(0,R_k^{-1/2})\setminus B(0,R_{k+1}^{-1/2})} \overline{(|f_{k+1,\tau_k}|^2)^\wedge} \sum_{\tau_k':d(\tau_k,\tau_k')\le R_k^{-1/2}q^{r/2}} (|f_{k+1,\tau_k'}|^2)^\wedge * \widehat{1_B}$$

$$= \sum_{\tau_k} \int_B |f_{k+1,\tau_k}|^2 * (\check{1}_{B(0,R_k^{-1/2})} - \check{1}_{B(0,R_k^{-1/2})}) \sum_{\tau_k':d(\tau_k,\tau_k')\le R_k^{-1/2}q^{r/2}} |f_{k+1,\tau_k'}|^2$$

$$\le \sum_{\tau_k} \int_B |f_{k+1,\tau_k}|^2 \sum_{\tau_k':d(\tau_k,\tau_k')\le R_k^{-1/2}q^{r/2}} |f_{k+1,\tau_k'}|^2$$

where in the last inequality we have used that $|f_{k+1,\tau_k}|^2 * \check{1}_{B(0,R_k^{-1/2})} = |f_{k+1,\tau_k}|^2$, $\check{1}_{B(0,R_{k+1}^{-1/2})}$ is nonnegative, and that the convolution of two nonnegative functions is also nonnegative. Applying Cauchy-Schwarz then gives that (7.5) is

$$\le q^{r/2} \sum_{\tau_k} \int_B |f_{k+1,\tau_k}|^4$$

and summing over all $B \subset \mathbb{Q}_q^2$ of side length R_{k+1} then completes the proof. $\square$

7.3. Decomposition into high and low sets

Let

$$\Omega_{J-1} = \{x \in \mathbb{Q}_q^2 \colon g_{J-1}(x) \le (\log R) g_{J-1}^h(x)\}$$

For $k = J-2, J-3, \ldots, 1$, define

$$\Omega_k = \{x \in \mathbb{Q}_q^2 \setminus (\Omega_{k+1} \cup \cdots \cup \Omega_{J-1}) \colon g_k(x) \le (\log R) g_k^h(x)\}$$

Finally,

$$L = \mathbb{Q}_q^2 \setminus (\Omega_1 \cup \cdots \cup \Omega_{J-1}).$$

Note that g_k is constant on squares of size $R_k^{1/2}$. By definition, g_k^l is constant on squares of size $R_{k+1}^{1/2} > R_k^{1/2}$. Therefore g_k^h is also constant on squares of size $R_k^{1/2}$.

One can view the construction of the Ω_k as follows. Partition $\mathbb{Q}_q^2$ first into squares of size $R_{J-1}^{1/2}$. Then Ω_{J-1} is a union of those squares on which $g_{J-1}(x) \le (\log R) g_{J-1}^h(x)$ where here we have used that both g_{J-1} and g_{J-1}^h are constant on each such square of size $R_{J-1}^{1/2}$.

Next, partition each of the remaining squares not chosen to be part of Ω_{J-1} into squares of size $R_{J-2}^{1/2}$. From these squares of size $R_{J-2}^{1/2}$, Ω_{J-2} is the union of those squares on which $g_{J-2}(x) \leq (\log R) g_{J-2}^h(x)$. Repeat this until we have defined Ω_1 after which we call the remaining set L (which can be written as the union of squares of size $R_1^{1/2}$).

To prove Proposition 6.3, note that

$$\alpha^6 |U_\alpha(f)| \leq \alpha^6 |U_\alpha(f) \cap L| + \sum_{k=1}^{J-1} \alpha^6 |U_\alpha(f) \cap \Omega_k|. \tag{7.6}$$

In view of the definition of the set $U_\alpha(f)$, to control the right hand side, we need to understand the size of $\max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|$ on Ω_k (for $k = 1, \dots, J-1$) and on L. We do so in the next section, and then use it to bound the right hand side of (7.6).

7.4. Approximation by pruned wave packets

Lemma 7.3. *Let $k = 1, 2, \dots, J-1$ and $|\tau| \geq R_k^{-1/2}$. Then for $x \in \mathbb{Q}_q^2$,*

$$|\sum_{\tau_k \subset \tau} f_{k+1,\tau_k}(x) - \sum_{\tau_k \subset \tau} f_{k,\tau_k}(x)| \leq \lambda^{-1} g_k(x).$$

Proof of Lemma 7.3. Fix $x \in \mathbb{Q}_q^2$. We have

$$\begin{aligned} |\sum_{\tau_k \subset \tau} f_{k+1,\tau_k}(x) - \sum_{\tau_k \subset \tau} f_{k,\tau_k}(x)| &= |\sum_{\tau_k \subset \tau} \sum_{\substack{T_k \in \mathbb{T}(\tau_k) \\ \|1_{T_k} f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)} > \lambda}} (1_{T_k} f_{k+1,\tau_k})(x)| \\ &\leq \sum_{\tau_k \subset \tau} \sum_{\substack{T_k \in \mathbb{T}(\tau_k) \\ \|1_{T_k} f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)} > \lambda}} |(1_{T_k} f_{k+1,\tau_k})(x)|. \end{aligned} \tag{7.7}$$

For each τ_k, there exists exactly a parallelogram $\mathcal{T}_k(x)$ depending on x in $\mathbb{T}(\tau_k)$ such that $x \in \mathcal{T}_k(x)$. If for this parallelogram, $\|1_{\mathcal{T}_k(x)} f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)} \leq \lambda$, then the inner sum for this particular τ_k in (7.7) is equal to 0. Otherwise,

$$|(1_{T_k} f_{k+1,\tau_k})(x)| \leq \frac{\|1_{\mathcal{T}_k(x)} f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)}^2}{\lambda}$$

and hence

$$\sum_{\substack{T_k\in\mathbb{T}(\tau_k)\\ \|1_{T_k}f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)}>\lambda}} |(1_{T_k}f_{k+1,\tau_k})(x)| \le \lambda^{-1}\|1_{\mathcal{T}_k(x)}f_{k+1,\tau_k}\|^2_{L^\infty(\mathbb{Q}_q^2)}.$$

Since $|f_{k+1,\tau_k}|$ is constant on $\mathcal{T}_k(x)$, $\|1_{\mathcal{T}_k(x)}f_{k+1,\tau_k}\|^2_{L^\infty(\mathbb{Q}_q^2)} = |(1_{\mathcal{T}_k(x)}f_{k+1,\tau_k})(x)|^2$ and so (7.7) is

$$\le \lambda^{-1}\sum_{\tau_k\subset\tau}\sum_{\substack{T_k\in\mathbb{T}(\tau_k)\\ \|1_{T_k}f_{k+1,\tau_k}\|_{L^\infty(\mathbb{Q}_q^2)}>\lambda}} |(1_{T_k}f_{k+1,\tau_k})(x)|^2 \le \lambda^{-1}g_k(x)$$

which completes the proof of the lemma. □

Lemma 7.4. *Let $k = 1, 2, \ldots, J-1$ and $|\tau| \ge R_k^{-1/2}$. Then for $x\in\Omega_k$,*

$$|f_\tau(x) - \sum_{\tau_k\subset\tau} f_{k+1,\tau_k}(x)| \lesssim \lambda^{-1}\frac{\log R}{\log\log R}\|g_J\|_{L^\infty(\mathbb{Q}_q^2)}.$$

Proof of Lemma 7.4. Fix $x\in\Omega_k$. Since $\sum_{\tau_k\subset\tau} f_{\tau_k} = f_\tau = \sum_{\tau_{k-1}\subset\tau} f_{\tau_{k-1}}$, we have

$$\begin{aligned}|f_\tau(x) - \sum_{\tau_k\subset\tau} f_{k+1,\tau_k}(x)| &\le |f_\tau(x) - \sum_{\tau_J\subset\tau} f_{J,\tau_J}(x)| \\ &\quad + \sum_{j=k+1}^{J-1} |\sum_{\tau_j\subset\tau} f_{j+1,\tau_j}(x) - \sum_{\tau_j\subset\tau} f_{j,\tau_j}(x)| \\ &\le \lambda^{-1}\sum_{j=k+1}^{J} g_j(x)\end{aligned}$$

by Lemma 7.3 (by how f_J is defined, the $f_\tau - \sum_{\tau_J\subset\tau} f_{J,\tau_J}$ term is controlled by the same proof as in Lemma 7.3).

To control this sum, we now use the definition of Ω_k. The low lemma gives

$$g_j(x) = g_j^l(x) + g_j^h(x) \le g_{j+1}(x) + g_j^h(x).$$

Since $x\in\Omega_k$, for $j = k+1, \ldots, J-1$, this is then $\le g_{j+1}(x) + (\log R)^{-1}g_j(x)$ and hence

$$g_j(x) \le (1 - (\log R)^{-1})^{-1}g_{j+1}(x).$$

Therefore for $j = k+1, \dots, J-1$,

$$g_j(x) \le (1 - (\log R)^{-1})^{-(J-j)} \|g_J\|_{L^\infty(\mathbb{Q}_q^2)}.$$

Thus

$$\lambda^{-1} \sum_{j=k+1}^{J} g_j(x) \le \lambda^{-1} \|g_J\|_{L^\infty(\mathbb{Q}_q^2)} \sum_{j=k+1}^{J} (1 - (\log R)^{-1})^{-(J-j)}$$
$$\lesssim \lambda^{-1} \frac{\log R}{\log\log R} \|g_J\|_{L^\infty(\mathbb{Q}_q^2)}$$

which completes the proof of Lemma 7.4. □

Note that the above proof also works for $x \in L$ and we obtain the same conclusion.

Now choose

$$\lambda := (\log R)^2 q^{r/2} \frac{\|g_J\|_{L^\infty(\mathbb{Q}_q^2)}}{\alpha}. \tag{7.8}$$

We can write the conclusion of Lemma 7.4 as for $x \in \Omega_k$ and $|\tau| \ge R_k^{-1/2}$, we have

$$f_\tau(x) = f_{k+1,\tau}(x) + O((\log R)^{-1} q^{-r/2} (\log\log R)^{-1} \alpha)$$

and so for $x \in \Omega_k$ and τ_1, τ_1' disjoint intervals of length $R_1^{-1/2}$,

$$|f_{\tau_1}(x) f_{\tau_1'}(x)| = |f_{k+1,\tau_1}(x) f_{k+1,\tau_1'}(x)|$$
$$+ O\left(\frac{\alpha}{(\log R) q^{r/2} \log\log R} (|f_{\tau_1}(x)| + |f_{\tau_1'}(x)|) + \frac{\alpha^2}{(\log R)^2 q^r (\log\log R)^2} \right).$$

Since $x \in U_\alpha(f)$, we control the $|f_{\tau_1}(x)|$ and $|f_{\tau_1'}(x)|$ by the l^6 sum over all such τ_1 caps and thus by $(\log R) q^{r/2} \alpha$. This gives that for $x \in U_\alpha(f) \cap \Omega_k$,

$$|f_{\tau_1}(x) f_{\tau_1'}(x)| = |f_{k+1,\tau_1}(x) f_{k+1,\tau_1'}(x)| + O(\frac{\alpha^2}{\log\log R}).$$

This implies for $x \in U_\alpha(f) \cap \Omega_k$ and R sufficiently large,

$$\max_{\tau_1 \neq \tau_1'} |f_{\tau_1}(x) f_{\tau_1'}(x)|^2 \lesssim \max_{\tau_1 \neq \tau_1'} |f_{k+1,\tau_1}(x) f_{k+1,\tau_1'}(x)|^2$$

which gives

$$\alpha^4 |U_\alpha(f) \cap \Omega_k| \lesssim \| \max_{\tau_1 \neq \tau_1'} |f_{k+1,\tau_1}(x) f_{k+1,\tau_1'}(x)|^{1/2} \|^4_{L^4(U_\alpha(f)\cap\Omega_k)}. \tag{7.9}$$

Similarly, Lemma 7.3 with $k = 1$ implies $|f_{2,\tau_1}(x) - f_{1,\tau_1}(x)| \leq \lambda^{-1} g_1(x)$ and the beginning of the proof of Lemma 7.4 implies $|f_{\tau_1}(x) - f_{2,\tau_1}(x)| \leq \lambda^{-1} \sum_{j=2}^J g_j(x)$. Following the proof of Lemma 7.4 and the choice of λ in (7.8) shows that for $x \in L$,

$$f_{\tau_1}(x) = f_{1,\tau_1}(x) + O((\log R)^{-1} q^{-r/2} (\log\log R)^{-1} \alpha)$$

from which following the same reasoning as in the Ω_k case, we obtain that

$$\alpha^6 |U_\alpha(f) \cap L| \lesssim \| \max_{\tau_1 \neq \tau_1'} |f_{1,\tau_1}(x) f_{1,\tau_1'}(x)|^{1/2} \|^6_{L^6(U_\alpha(f)\cap L)}. \tag{7.10}$$

In light of (7.6), it remains to estimate the right hand sides of (7.9) and (7.10).

7.5. Estimating $\alpha^6 |U_\alpha(f) \cap \Omega_k|$ for $k = 1, \ldots, J-1$

We first recall the following bilinear restriction theorem whose proof we defer to the end of this section.

Lemma 7.5 (Bilinear restriction). *Suppose $\delta \in q^{-2\mathbb{N}}$, and for $i = 1, 2$, f_i is a function on $\mathbb{Q}_q^2$ whose Fourier support is contained in $\{(\xi, \eta) \colon \xi \in I_i, |\eta - \xi^2| \leq \delta\}$, where I_1, I_2 are intervals in $\mathbb{Z}_q$ (not necessarily of the same length) separated by a distance κ. Assume*

$$\kappa \geq \delta^{1/2}. \tag{7.11}$$

Then

$$\int_{\mathbb{Q}_q^2} |f_1 f_2|^2 \leq \frac{\delta^2}{\kappa} \int_{\mathbb{Q}_q^2} |f_1|^2 \int_{\mathbb{Q}_q^2} |f_2|^2. \tag{7.12}$$

Fix $k = 1, 2, \ldots, J-1$ below. Then (7.9) is bounded by

$$\sum_{\tau_1 \neq \tau_1'} \int_{\Omega_k} |f_{k+1,\tau_1} f_{k+1,\tau_1'}|^2. \tag{7.13}$$

Since g_k and g_k^h are constant on squares of side length $R_k^{1/2}$, we may partition Ω_k into squares Q of side length $R_k^{1/2}$, and integrate on each such Q before we sum over Q. If $k \geq 2$, then the Fourier supports of $f_{k+1,\tau_1} 1_Q$ and $f_{k+1,\tau_1'} 1_Q$ are contained in $\Xi_{R_k^{-1/2}}$, while the distance between τ_1 and τ_1' is $> R_1^{-1/2}$. Since $R_1^{-1/2} \geq (R_k^{-1/2})^{1/2}$ and (4.1) holds, the hypothesis of Lemma 7.5 is satisfied with $\kappa = R_1^{-1/2}$ and $\delta = R_k^{-1/2}$. From (7.12), we then obtain

$$\begin{aligned}\int_Q |f_{k+1,\tau_1} f_{k+1,\tau_1'}|^2 &\leq \frac{(R_k^{-1/2})^2}{R_1^{-1/2}} \int_Q |f_{k+1,\tau_1}|^2 \int_Q |f_{k+1,\tau_1'}|^2 \\ &= \frac{q^{r/2}}{|Q|} \int_Q |f_{k+1,\tau_1}|^2 \int_Q |f_{k+1,\tau_1'}|^2.\end{aligned}$$

The same inequality holds for $k = 1$, because then $|f_{k+1,\tau_1}|$ and $|f_{k+1,\tau_1'}|$ are constants on squares of side length $R_1^{1/2}$. Thus in either case, (7.13) is controlled by

$$\begin{aligned}&\sum_{Q \in P_{R_k^{1/2}}(\Omega_k)} \sum_{\tau_1 \neq \tau_1'} \int_Q |f_{k+1,\tau_1} f_{k+1,\tau_1'}|^2 \\ &\quad \leq q^{r/2} \sum_{Q \in P_{R_k^{1/2}}(\Omega_k)} \frac{1}{|Q|} \sum_{\tau_1 \neq \tau_1'} \int_Q |f_{k+1,\tau_1}|^2 \int_Q |f_{k+1,\tau_1'}|^2 \\ &\quad \leq q^{r/2} \sum_{Q \in P_{R_k^{1/2}}(\Omega_k)} \frac{1}{|Q|} (\sum_{\tau_1} \int_Q |f_{k+1,\tau_1}|^2)^2\end{aligned}$$

where here $P_{R_k^{1/2}}(\Omega_k)$ denotes the partition of Ω_k into squares of side length $R_k^{1/2}$. Since Q has side length $R_k^{1/2}$, Plancherel and the definition of g_k then controls this by

$$\begin{aligned}&q^{r/2} \sum_{Q \in P_{R_k^{1/2}}(\Omega_k)} \frac{1}{|Q|} (\sum_{\tau_k} \int_Q |f_{k+1,\tau_k}|^2)^2 \\ &\quad = q^{r/2} \sum_{Q \in P_{R_k^{1/2}}(\Omega_k)} \frac{1}{|Q|} (\int_Q g_k)^2 = q^{r/2} \int_{\Omega_k} g_k^2\end{aligned}$$

where the last equality is because g_k is constant on squares of size $R_k^{1/2}$.

Therefore we have shown that

$$\alpha^4 |U_\alpha(f) \cap \Omega_k| \lesssim q^{r/2} \int_{\Omega_k} g_k^2.$$

Using that we are in Ω_k and applying the high lemma, this is controlled by

$$(\log R)^2 q^{r/2} \int_{\Omega_k} |g_k^h|^2 \leq (\log R)^2 q^r \sum_{\tau_k} \int_{\mathbb{Q}_q^2} |f_{k+1,\tau_k}|^4 \tag{7.14}$$

Write $f_{k+1,\tau_k} = \sum_{\tau_{k+1} \subset \tau_k} f_{k+1,\tau_{k+1}}$. Note that the sum has $R_k^{-1/2}/R_{k+1}^{-1/2}$ terms. Using Hölder's inequality, we further obtain that

$$\begin{aligned}
(7.14) &\leq (\log R)^2 q^r \Big(\frac{R_k^{-1/2}}{R_{k+1}^{-1/2}}\Big)^3 \sum_{\tau_{k+1}} \int_{\mathbb{Q}_q^2} |f_{k+1,\tau_{k+1}}|^4 \\
&= (\log R)^2 q^{5r/2} \sum_{\tau_{k+1}} \int_{\mathbb{Q}_q^2} |f_{k+1,\tau_{k+1}}|^4 \\
&= (\log R)^2 q^{5r/2} \sum_{\tau_{k+1}} \int_{\mathbb{Q}_q^2} \sum_{\substack{T_{k+1} \in \mathbb{T}(\tau_{k+1}) \\ \|1_{T_{k+1}} f_{k+2,\tau_{k+1}}\|_{L^\infty(\mathbb{Q}_q^2)} \leq \lambda}} |1_{T_{k+1}} f_{k+2,\tau_{k+1}}|^4
\end{aligned}$$

where in the last equality we have used that each $x \in \mathbb{Q}_q^2$ is contained in exactly one $T_{k+1} \in \mathbb{T}(\tau_{k+1})$. Here we have also used the convention that if $k = J-1$, then f_{k+2} is just f. Applying the definition of f_{k+1} shows that this is

$$\begin{aligned}
&\leq (\log R)^2 q^{5r/2} \lambda^2 \sum_{\tau_{k+1}} \int_{\mathbb{Q}_q^2} |f_{k+2,\tau_{k+1}}|^2 \qquad (7.15) \\
&= (\log R)^2 q^{5r/2} \lambda^2 \int_{\mathbb{Q}_q^2} |f_{k+2}|^2 \leq (\log R)^2 q^{5r/2} \lambda^2 \int_{\mathbb{Q}_q^2} |f|^2
\end{aligned}$$

where the last inequality is by (7.2). Using (7.8) then shows that we have proved

$$\alpha^4 |U_\alpha(f) \cap \Omega_k| \lesssim (\log R)^6 q^{7r/2} \alpha^{-2} \|g_J\|_{L^\infty(\mathbb{Q}_q^2)}^2 \int_{\mathbb{Q}_q^2} |f|^2.$$

It follows that

$$\alpha^6 |U_\alpha(f) \cap \Omega_k| \lesssim (\log R)^6 q^{7r/2} \Big(\sum_{\tau_J} \|f_{\tau_J}\|_{L^\infty(\mathbb{Q}_q^2)}^2\Big)^2 \sum_{\tau_J} \|f_{\tau_J}\|_{L^2(\mathbb{Q}_q^2)}^2. \tag{7.16}$$

7.6. Estimating $\alpha^6|U_\alpha(f)\cap L|$

The right hand side of (7.10) is

$$\leq \int_L (\sum_{\tau_1} |f_{1,\tau_1}|^2)^3 \leq \int_L (\sum_{\tau_1} |f_{2,\tau_1}|^2)^3 = \int_L g_1^2 \sum_{\tau_1} |f_{2,\tau_1}|^2 \tag{7.17}$$

where the second inequality is by (7.1). For $x \in L$ and $k = 1, \ldots, J-1$, we have

$$g_k(x) \leq (1-(\log R)^{-1})^{-1} g_{k+1}(x)$$

so

$$g_1(x) \lesssim \sum_{\tau_J} |f_{\tau_J}(x)|^2.$$

Therefore this and (7.2) shows that (7.17) is

$$\lesssim (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 \int_{\mathbb{Q}_q^2} |f|^2.$$

It follows that

$$\alpha^6 |U_\alpha(f) \cap L| \lesssim (\sum_{\tau_J} \|f_{\tau_J}\|^2_{L^\infty(\mathbb{Q}_q^2)})^2 \sum_{\tau_J} \|f_{\tau_J}\|^2_{L^2(\mathbb{Q}_q^2)}. \tag{7.18}$$

Finally, we may sum (7.16) over $k = 1, \ldots, J-1$ with (7.18). Since $J \leq N \lesssim \log R$, this concludes the proof of Proposition 6.3, modulo the proof of Lemma 7.5.

7.7. Proof of Lemma 7.5

Decompose

$$f_i = \sum_{\substack{\theta_i \subset I_i \\ |\theta_i| = \delta^{1/2}}} f_{i,\theta_i}.$$

Then by Plancherel,

$$\begin{aligned}\int_{\mathbb{Q}_q^2} |f_1 f_2|^2 &= \sum_{\theta_1,\theta_1',\theta_2,\theta_2'} \int_{\mathbb{Q}_q^2} f_{1,\theta_1} f_{2,\theta_2} \cdot \overline{f_{1,\theta_1'} f_{2,\theta_2'}} \\ &= \sum_{\theta_1,\theta_1',\theta_2,\theta_2'} \int_{\mathbb{Q}_q^2} (\widehat{f_{1,\theta_1}} * \widehat{f_{2,\theta_2}}) \cdot \overline{(\widehat{f_{1,\theta_1'}} * \widehat{f_{2,\theta_2'}})}.\end{aligned}$$

For the last integral to be non-zero, the support of $\widehat{f_{1,\theta_1}} * \widehat{f_{2,\theta_2}}$ must intersect the support of $\widehat{f_{1,\theta_1'}} * \widehat{f_{2,\theta_2'}}$. Thus we can find (ξ_i, η_i), $i = 1, 2, 3, 4$ such that $\xi_1 + \xi_2 = \xi_3 + \xi_4$ and $\eta_1 + \eta_2 = \eta_3 + \eta_4$ where $|\eta_i - \xi_i^2| \leq \delta$ and $\xi_1 \in \theta_1$, $\xi_2 \in \theta_2$, $\xi_3 \in \theta_1'$, and $\xi_4 \in \theta_2'$. Hence by the ultrametric inequality, for this $(\xi_1, \dots, \xi_4)$, we have

$$\xi_1 + \xi_2 - (\xi_3 + \xi_4) = 0 \tag{7.19}$$

$$|\xi_1^2 + \xi_2^2 - (\xi_3^2 + \xi_4^2)| \leq \delta. \tag{7.20}$$

From (7.19), we have $\xi_1 - \xi_4 = -(\xi_2 - \xi_3)$, so we see from (7.20) that

$$|\xi_1 - \xi_4||\xi_1 + \xi_4 - (\xi_2 + \xi_3)| \leq \delta.$$

But (7.19) also implies $\xi_1 + \xi_4 - (\xi_2 + \xi_3) = 2(\xi_1 - \xi_3)$. Since q is an odd prime, we have

$$|\xi_1 - \xi_4||\xi_1 - \xi_3| \leq \delta.$$

Since $|\xi_1 - \xi_4| \geq \kappa$, this shows

$$|\xi_1 - \xi_3| \leq \frac{\delta}{\kappa}.$$

If $\delta/\kappa \leq \delta^{1/2}$, i.e. (7.11) holds, then $|\xi_1 - \xi_3| \leq \delta^{1/2}$. Since θ_1 and θ_1' are intervals of length $\delta^{1/2}$ and two q-adic intervals of the same length are either disjoint or equal, we must have $\theta_1 = \theta_1'$. Using (7.19) again then implies $\theta_2 = \theta_2'$.

This shows

$$\int_{\mathbb{Q}_q^2} |f_1 f_2|^2 = \sum_{\theta_1, \theta_2} \int_{\mathbb{Q}_q^2} |\widehat{f_{1,\theta_1}} * \widehat{f_{2,\theta_2}}|^2 = \sum_{\theta_1, \theta_2} \int_{\mathbb{Q}_q^2} |f_{1,\theta_1}|^2 |f_{2,\theta_2}|^2.$$

Now for $i = 1, 2$, we may expand

$$|f_{i,\theta_i}|^2 = \sum_{T_i \in \mathbb{T}(\theta_i)} |c_{T_i}|^2 1_{T_i}$$

as in Corollary 2.5, so that $\sum_{T_i \in \mathbb{T}(\theta_i)} |c_{T_i}|^2 |T_i| = \int_{\mathbb{Q}_q^2} |f_{i,\theta_i}|^2$. Thus

$$\begin{aligned}\int_{\mathbb{Q}_q^2} |f_{1,\theta_1}|^2 |f_{2,\theta_2}|^2 &= \int_{\mathbb{Q}_q^2} \sum_{T_1 \in \mathbb{T}(\theta_1)} |c_{T_1}|^2 1_{T_1} \sum_{T_2 \in \mathbb{T}(\theta_2)} |c_{T_2}|^2 1_{T_2} \\ &= \sum_{T_1 \in \mathbb{T}(\theta_1)} \sum_{T_2 \in \mathbb{T}(\theta_2)} |c_{T_1}|^2 |c_{T_2}|^2 |T_1 \cap T_2|\end{aligned}$$

Using the definition of κ, and Lemma 2.6, we see that

$$|T_1 \cap T_2| \leq \delta^{-1/2} \cdot \frac{\delta^{-1/2}}{\kappa} = \frac{\delta^2}{\kappa}|T_1||T_2| \quad \text{for all } T_1 \in \mathbb{T}(\theta_1),\, T_2 \in \mathbb{T}(\theta_2),$$

so

$$\int_{\mathbb{Q}_q^2} |f_{1,\theta_1}|^2 |f_{2,\theta_2}|^2 \leq \frac{\delta^2}{\kappa} \int_{\mathbb{Q}_q^2} |f_{1,\theta_1}|^2 \int_{\mathbb{Q}_q^2} |f_{2,\theta_2}|^2.$$

Summing over θ_1 and θ_2 on both sides, we yield

$$\int_{\mathbb{Q}_q^2} |f_1 f_2|^2 \leq \frac{\delta^2}{\kappa} \int_{\mathbb{Q}_q^2} |f_1|^2 \int_{\mathbb{Q}_q^2} |f_2|^2,$$

as desired.

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