Math. Res. Lett.
Volume 30, Number 5, 1463–1483, 2023

# On certain extensions of vector bundles in $p$-adic geometry

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Given two arbitrary vector bundles on the Fargues-Fontaine curve, we give an explicit criterion in terms of Harder-Narasimhan polygons on whether they realize a semistable vector bundle as their extensions. Our argument is largely combinatorial and builds upon the dimension analysis of certain moduli spaces of bundle maps developed in [1].

## 1. Introduction

### 1.1. The main result

Over the past decade, $p$-adic Hodge theory has undergone a remarkable development driven by a series of new geometric ideas. Of particular importance among such ideas are the theory of perfectoid spaces introduced by Scholze [17] and the geometric reformulation of $p$-adic Hodge theory by Fargues and Fontaine [6] using a noetherian one-dimensional $\mathbb{Q}_p$-scheme called the *Fargues-Fontaine curve*. Some notable applications of these ideas are the geometrization of the local Langlands correspondence by Fargues-Scholze [7] and the construction of local Shimura varieties by Scholze-Weinstein [19].

In this article, we address the question of determining whether there exists a short exact sequence among three given vector bundles on the Fargues-Fontaine curve. This question naturally arises in the study of various objects in $p$-adic geometry. For example, a partial answer to this question obtained by the author and his collaborators in [1] leads to the work of Hansen [8] that describes precise closure relations among the Harder-Narasimhan strata on the stack of vector bundles on the Fargues-Fontaine curve. In addition, a general answer to this question can be used to describe the geometry of the $p$-adic flag variety and the $B_{\mathrm{dR}}^+$-Grassmannian in terms of two natural stratifications, namely the Harder-Narasimhan stratification and the Newton stratification, in line with the work of many authors including Caraiani-Scholze [2], Chen-Fargues-Shen [4], Shen [20], Chen [3], Viehmann [21], and Nguyen-Viehmann [16].

In order to state our main result, let us introduce some notations and terminologies. Let $F$ be an algebraically closed perfectoid field of characteristic $p > 0$. Denote by $X = X_F$ the Fargues-Fontaine curve associated to $F$. The Picard group of $X$ turns out to be naturally isomorphic to $\mathbb{Z}$, and consequently yields a good Harder-Narasimhan formalism for vector bundles on $X$. By a result of Fargues-Fontaine [6] (and also Kedlaya [12]), every vector bundle $\mathcal{V}$ on $X$ is uniquely determined up to isomorphism by its Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$.

We can now state our main result as follows:

**Theorem 1.1.1.** *Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be vector bundles on $X$ such that $\mathcal{E}$ is semistable. There exists a short exact sequence of vector bundles on $X$*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

*if and only if the following conditions are satisfied:*

*(i) All slopes in $\mathrm{HN}(\mathcal{D})$ are less than or equal to the slope of $\mathrm{HN}(\mathcal{E})$.*

*(ii) All slopes in $\mathrm{HN}(\mathcal{F})$ are greater than or equal to the slope of $\mathrm{HN}(\mathcal{E})$.*

*(iii) $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ lies above $\mathrm{HN}(\mathcal{E})$ with the same endpoints.*

In the sequel paper [10], we extend Theorem 1.1.1 to the case where $\mathcal{E}$ is not necessarily semistable. Nonetheless, it is our opinion that Theorem 1.1.1 is worthwhile as an independent statement. In fact, the main result of the article [10] involves a somewhat complicated combinatorial condition and is not at all obviously equivalent to Theorem 1.1.1 in the case where $\mathcal{E}$ is semistable.

### 1.2. Outline of the proof

Let us briefly explain our proof of Theorem 1.1.1. The necessity part of Theorem 1.1.1 is a standard consequence of the slope formalism. Hence the main part of our proof is to establish the sufficiency part of Theorem 1.1.1. When either $\mathcal{D}$ or $\mathcal{F}$ is semistable, we consider the moduli space $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ whose $F$-points parametrize exact sequences $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{F} \to 0$ of vector bundles on $X$, and establish its nonemptiness by a dimension analysis. For the general case, we proceed by induction on the number of distinct slopes in $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{F})$ using a combinatorial argument that utilizes concavity of HN polygons.

In order to study of the space $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$, we adapt the strategies developed in the previous paper [1]. We make sense of this space as a *diamond* in the sense of Scholze [18], and establish a simple dimension formula for this space when $\mathcal{F}$ is semistable. For our dimension analysis, we prove some combinatorial lemmas involving the HN polygons of vector bundles on $X$.

We remark that the previous version of this paper had a mistake and falsely claimed a similar statement of Theorem 1.1.1 in the case where either $\mathcal{D}$ or $\mathcal{F}$ is semistable (without $\mathcal{E}$ being necessarily semistable). The mistake was to assert that the space $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ admits an explicit dimension formula for any vector bundles $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$. However, it turns out that such a dimension formula exists only when $\mathcal{F}$ is semistable, as stated in Proposition 3.1.6.

## 2. Preliminaries

### 2.1. The Fargues-Fontaine curve

Throughout this paper, we fix an algebraically closed perfectoid field $F$ of characteristic $p > 0$. We denote by $\mathcal{O}_F$ the ring of integers of $F$ and by $W(\mathcal{O}_F)$ the ring of Witt vectors over $\mathcal{O}_F$. We choose a pseudouniformizer $\varpi$ of $F$ and write $[\varpi]$ for the Teichmüller lift of $\varpi$. The Frobenius map on $W(\mathcal{O}_F)$ induces a properly discontinuous automorphism $\phi$ on the adic space

$$\mathcal{Y} := \mathrm{Spa}(W(\mathcal{O}_F)) \setminus \{|p[\varpi]| = 0\}$$

defined over $\mathrm{Spa}(\mathbb{Q}_p)$.

**Definition 2.1.1.** We define the *adic Fargues-Fontaine curve* (associated to $F$) by

$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}},$$

and the *schematic Fargues-Fontaine curve* by

$$X := \mathrm{Proj} \left( \bigoplus_{n \geq 0} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^{\phi = p^n} \right).$$

**Remark.** More generally, for any finite extension $E$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, we can define the Fargues-Fontaine curve as an adic space or a scheme over $E$ by replacing $W(\mathcal{O}_F)$ in the above construction with $W_{\mathcal{O}_E}(\mathcal{O}_F)$, the ring of ramified Witt vectors over $\mathcal{O}_F$ with coefficients in $\mathcal{O}_E$. There is also an analogous construction of the equal characteristic Fargues-Fontaine curve as an adic space or a scheme over a finite extension of $\mathbb{F}_p((t))$. Our main result equally holds in these settings with identical proofs.

The two incarnations of the Fargues-Fontaine curve are essentially equivalent to us because of the following GAGA type result:

**Theorem 2.1.2 ([14, Theorems 6.3.12 and 8.7.7]).** *There exists a natural map of locally ringed spaces*

$$\mathcal{X} \to X$$

*which induces by pullback an equivalence of the categories of vector bundles.*

In light of Theorem 2.1.2, we will identify vector bundles on $\mathcal{X}$ with vector bundles on $X$. Vector bundles on the Fargues-Fontaine curve turn out to behave pleasantly well, essentially due to the following fact:

**Proposition 2.1.3 ([6, Théoréme 5.2.7]).** *The scheme $X$ is noetherian and regular of Krull dimension 1 over $\mathbb{Q}_p$. Moreover, it is complete in the sense that every principal divisor on $X$ has degree* 0.

In particular, the degree map is well-defined on the Picard group of $X$, thereby allowing us to define the notion of slope for vector bundles on $X$ as follows:

**Definition 2.1.4.** Let $\mathcal{V}$ be a nonzero vector bundle on $X$.

(1) We write $\text{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$ and $\mathcal{V}^\vee$ for the dual of $\mathcal{V}$.

(2) We define the *degree* and *slope* of $\mathcal{V}$ respectively by

$$\deg(\mathcal{V}) := \deg(\wedge^{\text{rk}(\mathcal{V})}\mathcal{V}) \qquad \text{and} \qquad \mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\text{rk}(\mathcal{V})}.$$

Let $k$ be the residue field of $F$, and let $K_0$ be the fraction field of the ring of Witt vectors over $k$. Recall that an *isocrystal* over $k$ is a finite dimensional vector space over $K_0$ with a Frobenius semi-linear automorphism.

**Lemma 2.1.5.** *There exists a functor from the category of isocrystals over $k$ to the category of vector bundles on $X$ which is compatible with direct sums, duals, ranks, degrees, and slopes.*

*Proof.* Let us write

$$B := H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \qquad \text{and} \qquad P := \bigoplus_{n \geq 0} B^{\phi = p^n}.$$

The desired functor is given by associating to each isocrystal $N$ over $k$ the vector bundle $\mathcal{E}(N)$ on $X$ which corresponds to the graded $P$-module

$$\bigoplus_{n \geq 0} \left(N^\vee \otimes_{K_0} B\right)^{\phi = \varpi^n},$$

where $N^\vee$ denotes the dual isocrystal of $N$. □

**Definition 2.1.6.** Given $\lambda \in \mathbb{Q}$, we write $\mathcal{O}(\lambda)$ for the vector bundle on $X$ that corresponds to the unique simple isocrystal over $k$ of slope $\lambda$ under the functor in Lemma 2.1.5.

**Lemma 2.1.7.** *Let $d$ and $r$ be relatively prime integers with $r > 0$.*

*(1) The bundle $\mathcal{O}(d/r)$ has rank $r$, degree $d$, and slope $d/r$.*

*(2) For any relatively prime integers $d'$ and $r'$ with $r' > 0$, we have*

$$\mathcal{O}\left(\frac{d}{r}\right) \otimes \mathcal{O}\left(\frac{d'}{r'}\right) \simeq \mathcal{O}\left(\frac{d}{r} + \frac{d'}{r'}\right)^{\oplus \gcd(rr', dr' + d'r)}.$$

*In particular, the bundle $\mathcal{O}(d/r) \otimes \mathcal{O}(d'/r')$ has rank $rr'$, degree $dr' + d'r$, and slope $d/r + d'/r'$.*

*(3) $\mathcal{O}(d/r)^\vee \simeq \mathcal{O}(-d/r)$.*

*Proof.* By Lemma 2.1.5, all statements follow immediately from the corresponding statements for isocrystals over $k$. □

**Proposition 2.1.8 ([6, Proposition 5.6.23], [12, Proposition 4.1.3]).** *For every $\lambda \in \mathbb{Q}$, we have the following statements:*

*(1) $H^0(X, \mathcal{O}(\lambda)) = 0$ if and only if $\lambda < 0$.*

*(2) $H^1(X, \mathcal{O}(\lambda)) = 0$ if and only if $\lambda \geq 0$.*

**Definition 2.1.9.** A vector bundle $\mathcal{V}$ on $X$ is *semistable* if every subbundle $\mathcal{W}$ of $\mathcal{V}$ satisfies the inequality $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$.

We can now state the classification theorem for vector bundles on $X$ as follows:

**Theorem 2.1.10 ([6, Théoréme 8.2.10]).** *Let $\mathcal{V}$ be a nonzero vector bundle on $X$.*

*(1) $\mathcal{V}$ is semistable of slope $\lambda$ if and only if it is isomorphic to $\mathcal{O}(\lambda)^{\oplus m}$ for some $m$.*

*(2) $\mathcal{V}$ admits a direct sum decomposition*

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i} \tag{2.1}$$

*where the $\lambda_i$'s are rational numbers with $\lambda_1 > \lambda_2 > \cdots > \lambda_l$.*

**Definition 2.1.11.** Let $\mathcal{V}$ be a nonzero vector bundle on $X$.

(1) We refer to the decomposition (2.1) in Theorem 2.1.10 as the *Harder-Narasimhan (HN) decomposition* of $\mathcal{V}$.

(2) We refer to the numbers $\lambda_i$ in the HN decomposition as the *Harder-Narasimhan (HN) slopes* of $\mathcal{V}$, or often simply as the *slopes* of $\mathcal{V}$.

(3) We write $\mu_{\max}(\mathcal{V})$ (resp. $\mu_{\min}(\mathcal{V})$) for the maximum (resp. minimum) HN slope of $\mathcal{V}$; in other words, we set $\mu_{\max}(\mathcal{V}) := \lambda_1$ and $\mu_{\min}(\mathcal{V}) := \lambda_l$.

(4) For every $\mu \in \mathbb{Q}$, we define the direct summands

$$\mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{and} \qquad \mathcal{V}^{\leq \mu} := \bigoplus_{\lambda_i \leq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i},$$

and similarly define $\mathcal{V}^{>\mu}$ and $\mathcal{V}^{<\mu}$.

(5) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{V}$, denoted by $\mathrm{HN}(\mathcal{V})$, as the upper convex hull of the points $(0,0)$ and $\big(\mathrm{rk}(\mathcal{V}^{\geq \lambda_i}), \deg(\mathcal{V}^{\geq \lambda_i})\big)$.

(6) Given a convex polygon $P$ adjoining $(0,0)$ and $(\mathrm{rk}(\mathcal{V}), \deg(\mathcal{V}))$, we write $\mathrm{HN}(\mathcal{V}) \leq P$ if each point on $\mathrm{HN}(\mathcal{V})$ lies on or below $P$.

**Corollary 2.1.12.** *For an arbitrary vector bundle* $\mathcal{V}$ *on* $X$*, its isomorphism class is determined by the HN polygon* $\mathrm{HN}(\mathcal{V})$*, with the slopes of* $\mathcal{V}$ *precisely being the slopes in* $\mathrm{HN}(\mathcal{V})$*.*

We conclude this subsection by extending the construction of the Fargues-Fontaine curve to relative settings. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(F)$, and let $\varpi_R$ be a pseudouniformizer of $R$. We write $W(R^+)$ for the ring of Witt vectors over $R^+$ and $[\varpi_R]$ for the Teichmüller lift of $\varpi_R$. As in the absolute setting, the Frobenius map on $W(R^+)$ induces a properly discontinuous automorphism $\phi$ on the adic space

$$\mathcal{Y}_S := \mathrm{Spa}(W(R^+)) \setminus \{|p[\varpi_R]| = 0\}$$

defined over $\mathrm{Spa}(\mathbb{Q}_p)$.

**Definition 2.1.13.** Given an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathrm{Spa}(F)$, we define the *adic Fargues-Fontaine curve* associated to $S$ by

$$\mathcal{X}_S := \mathcal{Y}_S/\phi^{\mathbb{Z}},$$

and the *schematic Fargues-Fontaine curve* associated to $S$ by

$$X_S := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}_S})^{\phi=p^n}\right).$$

For an arbitrary perfectoid space $S$ over $\mathrm{Spa}(F)$ with an affinoid cover $S = \bigcup S_i$, we define the adic Fargues-Fontaine curve $\mathcal{X}_S$ by gluing the $\mathcal{X}_{S_i}$.

**Remark.** The schematic Fargues-Fontaine curve $X_S$ is defined only for affinoid perfectoid spaces $S$; in fact, for an arbitrary perfectoid space $S$ over $\mathrm{Spa}(F)$ with an affinoid cover $S = \bigcup S_i$, the schematic curves $X_{S_i}$ do not glue in general. In addition, the readers should be aware that the relative Fargues-Fontaine curve $\mathcal{X}_S$ is not related to $\mathcal{X}$ by a base change, as neither $\mathcal{X}$ nor $\mathcal{X}_S$ is defined over $\mathrm{Spa}(F)$.

## 2.2. Diamonds

In this subsection, we collect some basic facts about diamonds following [18].

**Definition 2.2.1.** Let Perfd denote the category of perfectoid spaces in characteristic $p$.

(1) A morphism $Y \to Z$ of affinoid perfectoid spaces is *affinoid pro-étale* if it can be written as a cofiltered limit of étale morphisms $Y_i \to Z$ of affinoid perfectoid spaces.

(2) A morphism $f : Y \to Z$ of perfectoid spaces is *pro-étale* if there exist open affinoid covers $Z = \bigcup U_i$ and $Y = \bigcup V_{i,j}$ such that $f|_{V_{i,j}}$ factors through an affinoid pro-étale morphism $V_{i,j} \to U_i$.

(3) A pro-étale morphism $f : Y \to Z$ of perfectoid spaces is called a *pro-étale cover* if for any quasicompact open subset $U \subset Z$, there exists some quasicompact open subset $V \subset Y$ with $f(V) = U$.

(4) The *big pro-étale site* is the site on Perfd with covers given by pro-étale covers.

(5) A sheaf $Y$ for the big pro-étale site on Perfd is called a *diamond* if $Y$ can be written as a quotient $Z/R$, where $Z$ is representable by a perfectoid space with a pro-étale equivalence relation $R$ on $Z$.

(6) For a diamond $Y \simeq Z/R$ with a perfectoid space $Z$ and a pro-étale equivalence relation $R$, we define its topological space by $|Y| := |Z|/|R|$, where $|Z|$ and $|R|$ respectively denote the topological spaces for $Z$ and $R$.

(7) For a diamond $Y$, we define its *dimension* to be the Krull dimension of $|Y|$.

**Remark.** For a diamond $Y$, its topological space $|Y|$ does not depend on the choice of presentation $Y \simeq Z/R$ as the quotient of a perfectoid space $Z$ by a pro-étale equivalence relation $R$.

**Proposition 2.2.2 ([18, Corollary 8.6]).** *The big pro-étale site is subcanonical. In other words, for every* $Z \in$ Perfd *the functor* $\mathrm{Hom}(-, Z)$ *is a sheaf for the big pro-étale site.*

**Remark.** By Proposition 2.2.2, we will often identify a perfectoid space $Z$ in characteristic $p$ with the functor $\mathrm{Hom}(-, Z)$ on Perfd.

**Definition 2.2.3.** Let $Y$ be a diamond.

(1) We say that $Y$ is *quasicompact* if it admits a presentation $Y \simeq Z/R$ for some quasicompact perfectoid space $Z$ and a pro-étale equivalence relation $R$ on $Z$.

(2) We say that $Y$ is *quasiseparated* if $U \times_Y V$ is quasicompact for any morphisms $U \to Y$ and $V \to Y$ of diamonds with $U, V$ quasicompact.

(3) We say that $Y$ is *partially proper* if it is quasiseparated with the property that for all characteristic $p$ affinoid perfectoid pair $(R, R^+)$ the restriction map

$$Y(R, R^+) \to Y(R, R^\circ)$$

is bijective where $R^\circ$ denotes the ring of power-bounded elements in $R$.

(4) We say that $Y$ is *spatial* if it is quasicompact and quasiseparated with a neighborhood basis of $|Y|$ given by the set

$$\{|U| : U \subset Y \text{ quasicompact open subdiamonds}\}.$$

(5) We say that $Y$ is *locally spatial* if it admits a covering by spatial open subdiamonds.

**Remark.** In general, quasicompactness (resp. quasiseparatedness) of a diamond $Y$ is not equivalent to quasicompactness (resp. quasiseparatedness) of its topological space $|Y|$.

**Proposition 2.2.4 ([1, Lemma 3.2.3 and Lemma 3.3.4]).** *Let $Y$ be a spatial diamond with a free $\underline{G}$-action for some profinite group $G$. Then $Y/\underline{G}$ is a spatial diamond with*

$$\dim Y/\underline{G} = \dim Y.$$

## 3. Semistable vector bundles arising from extensions

### 3.1. Moduli spaces of extensions

In this subsection, we define and study diamonds that parametrize maps or extensions between two given vector bundles on $\mathcal{X}$. Let us denote by $\mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the category of perfectoid spaces over $\mathrm{Spa}(F)$. By construction, the relative Fargues-Fontaine curve $\mathcal{X}_S$ for any $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ comes with a natural map $\mathcal{X}_S \to \mathcal{X}$.

**Definition 3.1.1.** Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on the Fargues-Fontaine curve $\mathcal{X}$. For any $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$, we write $\mathcal{E}_S$ and $\mathcal{F}_S$ for the pullbacks of $\mathcal{E}$ and $\mathcal{F}$ along the map $\mathcal{X}_S \to \mathcal{X}$.

(1) $\mathcal{H}^i(\mathcal{E})$ is the pro-étale sheafification of the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the group $H^i(\mathcal{X}_S, \mathcal{E}_S)$.

(2) $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \to \mathcal{F}_S$.

(3) $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of surjective $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \twoheadrightarrow \mathcal{F}_S$.

(4) $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of surjective $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \twoheadrightarrow \mathcal{F}_S$ whose kernel becomes isomorphic to $\mathcal{D}$ after pulling back along the map $\mathcal{X}_{E,\overline{x}} \to \mathcal{X}_{E,S}$ for any geometric point $\overline{x}$.

(5) $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of $\mathcal{O}_{\mathcal{X}_S}$-module maps $\mathcal{E}_S \to \mathcal{F}_S$ whose pullback along the map $\mathcal{X}_{\overline{x}} \to \mathcal{X}_S$ for any geometric point $\overline{x} \to S$ gives an injective $\mathcal{O}_{\mathcal{X}_{\overline{x}}}$-module map.

(6) $\mathcal{A}\mathrm{ut}(\mathcal{E})$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the group of $\mathcal{O}_{\mathcal{X}_S}$-module automorphisms of $\mathcal{E}_S$.

(7) $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of isomorphism classes of extensions of $\mathcal{F}_S$ by $\mathcal{D}_S$.

(8) $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is the functor which associates to each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ the set of isomorphism classes of extensions of $\mathcal{F}_S$ by $\mathcal{D}_S$ whose pullback along the map $\mathcal{X}_{E,\overline{x}} \to \mathcal{X}_{E,S}$ for any geometric point $\overline{x} \to S$ yields a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

**Remark.** We have canonical identifications

$$\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee \otimes \mathcal{F}) \qquad \text{and} \qquad \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D}) \cong \mathcal{H}^1(\mathcal{F}^\vee \otimes \mathcal{D}).$$

**Proposition 3.1.2 ([1, Propositions 3.3.2, 3.3.5, 3.3.6, 3.3.7, and 3.3.13]).** *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $\mathcal{X}$.*

*(1) If* $\mathcal{E}$ *is semistable of slope* 0*, then there is a natural identification* $\mathcal{H}^0(\mathcal{E}) \cong \underline{\mathbb{Q}_p}^{\mathrm{rk}(\mathcal{E})}$.

*(2)* $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ *is a partially proper and locally spatial diamond over* $F$*, equidimensional of dimension* $\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0}$.

*(3) Every nonempty open subdiamond of* $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ *has an* $F$*-point.*

*(4)* $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ *and* $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ *are both open, partially proper and locally spatial subdiamonds of* $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$.

*(5)* $\mathcal{A}\mathrm{ut}(\mathcal{E})$ *is a partially proper and locally spatial diamond over* $F$*, equidimensional of dimension* $\deg(\mathcal{E}^\vee \otimes \mathcal{E})^{\geq 0}$.

*(6)* $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ *is a partially proper and locally spatial diamond over* $F$.

**Remark.** The work of Le Bras [15] shows that the diamonds $\mathcal{H}^i(\mathcal{E})$ and $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ also have the structure of Banach-Colmez spaces as defined by Colmez [5].

**Proposition 3.1.3 ([1, Theorem 1.1.2]).** *Let* $\mathcal{D}$*,* $\mathcal{E}$*, and* $\mathcal{F}$ *be vector bundles on* $\mathcal{X}$ *such that* $\mathcal{D}$ *and* $\mathcal{F}$ *are semistable with* $\mu(\mathcal{D}) < \mu(\mathcal{F})$*. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

*if and only if we have* $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

**Proposition 3.1.4.** *Let* $\mathcal{E}$ *be a vector bundle on* $\mathcal{X}$ *with* $\mu_{max}(\mathcal{E}) < 0$.

*(1)* $\mathcal{H}^1(\mathcal{E})$ *is a partially proper and locally spatial diamond over* $F$*, equidimensional of dimension* $\deg(\mathcal{E}^\vee)^{\geq 0}$.

*(2) Every nonempty open subdiamond of* $\mathcal{H}^1(\mathcal{E})$ *has an* $F$*-point.*

*Proof.* Let us write the HN decomposition of $\mathcal{E}$ as

$$\mathcal{E} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i}$$

with $\lambda_i < 0$ for each $i = 1, \cdots, l$. We also set

$$r_i := \mathrm{rk}\left(\mathcal{O}(\lambda_i)\right) \qquad \text{and} \qquad d_i := \deg\left(\mathcal{O}(\lambda_i)\right).$$

By Proposition 3.1.3, each $\mathcal{O}(\lambda_i)^{\oplus m_i}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}(\lambda_i)^{\oplus m_i} \longrightarrow \mathcal{O}^{\oplus m_i(r_i - d_i)} \longrightarrow \mathcal{O}(1)^{\oplus - m_i d_i} \longrightarrow 0.$$

We take the direct sum of all such exact sequences to obtain a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}^{\oplus(r-d)} \longrightarrow \mathcal{O}(1)^{\oplus -d} \longrightarrow 0 \tag{3.1}$$

with $r = \mathrm{rk}(\mathcal{E})$ and $d = \deg(\mathcal{E})$, and consequently get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}^0(\mathcal{E}) \longrightarrow \mathcal{H}^0(\mathcal{O}^{\oplus(r-d)}) \longrightarrow \mathcal{H}^0(\mathcal{O}(1)^{\oplus -d}) \\ \longrightarrow \mathcal{H}^1(\mathcal{E}) \longrightarrow \mathcal{H}^1(\mathcal{O}^{\oplus(r-d)}). \end{aligned}$$

Moreover, by Proposition 2.1.8 and Proposition 3.1.2 we have

$$\mathcal{H}^0(\mathcal{E}) = 0, \qquad \mathcal{H}^0(\mathcal{O}^{\oplus(r-d)}) \cong \underline{\mathbb{Q}_p}^{r-d}, \qquad \mathcal{H}^1(\mathcal{O}^{\oplus(r-d)}) = 0.$$

We thus find a presentation

$$\mathcal{H}^1(\mathcal{E}) \simeq \mathcal{H}^0(\mathcal{O}(1)^{\oplus -d})/\underline{\mathbb{Q}_p}^{r-d} \simeq \mathcal{H}\mathrm{om}(\mathcal{O}, \mathcal{O}(1)^{\oplus -d})/\underline{\mathbb{Q}_p}^{r-d},$$

thereby deducing the desired statements by Proposition 2.2.4 and Proposition 3.1.2. □

**Remark.** The above argument is largely inspired by the proof of [1, Proposition 3.3.2]. It is also presented by Hansen at the Montreal workshop for the geometrization of the local Langlands program held in 2019.

It is worthwhile to note that our use of Proposition 3.1.3 is not essential and is only for brevity. For example, we can prove Proposition 3.1.4 based only on some elementary properties of the Fargues-Fontaine curve, as in the work of Fargues-Scholze [7, Proposition II.2.5]. In fact, Fargues-Scholze [7, Theorem II.2.14] uses a special case of this result in an essential way to give a new conceptual proof of Theorem 2.1.10.

**Lemma 3.1.5 ([1, Lemma 3.3.14]).** *Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be vector bundles on $\mathcal{X}$ such that $\mathcal{F}$ is semistable. Then $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ is either empty or equidimensional with*

$$\dim \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}} = \deg(\mathcal{D}^\vee \otimes \mathcal{E})^{\geq 0} - \deg(\mathcal{D}^\vee \otimes \mathcal{D})^{\geq 0}.$$

**Remark.** The diamond $\mathcal{S}\text{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ is quite obscure if $\mathcal{F}$ is not semistable. For instance, we are highly doubtful that $\mathcal{S}\text{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ admits an explicit dimension formula when $\mathcal{F}$ is not semistable.

**Proposition 3.1.6.** *Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be vector bundles on $\mathcal{X}$ with $\mu_{max}(\mathcal{D}) < \mu_{min}(\mathcal{F})$.*

*(1) $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is a partially proper and locally spatial diamond over $F$.*

*(2) If $\mathcal{E}$ is semistable, then $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is an open subdiamond of $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})$.*

*(3) If $\mathcal{F}$ is semistable, then $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is either empty or equidimensional with*

$$\dim \mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}} = \deg(\mathcal{D}^\vee \otimes \mathcal{E})^{\geq 0} - \deg(\mathcal{E}^\vee \otimes \mathcal{E})^{\geq 0}.$$

*Proof.* By Proposition 2.1.7, we find that all slopes of $\mathcal{F}^\vee \otimes \mathcal{D}$ are negative. Hence Proposition 3.1.4 implies that $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D}) \cong \mathcal{H}^1(\mathcal{F}^\vee \otimes \mathcal{D})$ is a locally spatial diamond over $F$. Let us choose a presentation $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D}) \simeq T/R$ for some perfectoid space $T$ and a pro-étale equivalence relation $R$. Let $\mathcal{V}$ be the vector bundle on $X_T$ which fits into the "universal" exact sequence

$$0 \longrightarrow \mathcal{D}_T \longrightarrow \mathcal{V} \longrightarrow \mathcal{F}_T \longrightarrow 0.$$

We define

$$|T|_{\leq \text{HN}(\mathcal{E})} := \{ \, x \in |\, T| : \text{HN}(\mathcal{V}_x) \leq \text{HN}(\mathcal{E}) \, \} \, ,$$
$$|T|_{\geq \text{HN}(\mathcal{E})} := \{ \, x \in |\, T| : \text{HN}(\mathcal{V}_x) \geq \text{HN}(\mathcal{E}) \, \} \, .$$

By the result of Kedlaya-Liu [14, Theorem 7.4.5], the set $|T|_{\leq \text{HN}(\mathcal{E})}$ (resp. $|T|_{\geq \text{HN}(\mathcal{E})}$) is open (resp. closed) in $|T|$. Moreover, both $|T|_{\leq \text{HN}(\mathcal{E})}$ and $|T|_{\geq \text{HN}(\mathcal{E})}$ are stable under generalizations. Hence the image of $|T|_{\leq \text{HN}(\mathcal{E})} \cap |T|_{\geq \text{HN}(\mathcal{E})}$ under the quotient map $|T| \to |\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})|$ is a locally closed and generalizing subset $|\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})|_{\text{HN}(\mathcal{E})}$ of $|\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})|$. Adapting the argument of Scholze [18, Proposition 11.20], we find that $|\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})|_{\text{HN}(\mathcal{E})}$ gives rise to a locally spatial subdiamond $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\text{HN}(\mathcal{E})}$ of $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})$ with an identification

$$\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\text{HN}(\mathcal{E})} \cong \mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$$

as a functor on $\text{Perfd}_{/\text{Spa}(F)}$. Therefore we deduce that $\mathcal{E}\text{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is a locally spatial diamond over $F$, and also obtain its partial properness from the result of Kedlaya-Liu [14, Theorem 8.7.7]. Moreover, if $\mathcal{E}$ is semistable, then we have $|T|_{\geq \mathrm{HN}(\mathcal{E})} = |T|$ and consequently find that $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}} \cong \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathrm{HN}(\mathcal{E})}$ is an open subdiamond of $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})$.

For the last statement, let us now assume that $\mathcal{F}$ is semistable. Let $\widetilde{\mathcal{E}\mathrm{xt}}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ be the functor which associates each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$ to the set of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{D}_S \longrightarrow \mathcal{E}_S \longrightarrow \mathcal{F}_S \longrightarrow 0.$$

We may identify $\widetilde{\mathcal{E}\mathrm{xt}}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ as an $\mathcal{A}\mathrm{ut}(\mathcal{E})$-torsor over $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ by rigidifying an extension of $\mathcal{F}_S$ by $\mathcal{D}_S$ for each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$. Similarly, we may identify $\widetilde{\mathcal{E}\mathrm{xt}}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ as an $\mathcal{A}\mathrm{ut}(\mathcal{D})$-torsor over $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}}$ by rigidifying a surjective map $\mathcal{E}_S \twoheadrightarrow \mathcal{F}_S$ (and its kernel) for each $S \in \mathrm{Perfd}_{/\mathrm{Spa}(F)}$. Therefore, if $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}}$ is not empty, we find

$$\begin{aligned} \dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_{\mathcal{E}} &= \dim \widetilde{\mathcal{E}\mathrm{xt}}(\mathcal{F}, \mathcal{D})_{\mathcal{E}} - \dim \mathcal{A}\mathrm{ut}(\mathcal{E}) \\ &= \dim \mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})_{\mathcal{D}} + \dim \mathcal{A}\mathrm{ut}(\mathcal{D}) - \dim \mathcal{A}\mathrm{ut}(\mathcal{E}) \\ &= \deg(\mathcal{D}^{\vee} \otimes \mathcal{E})^{\geq 0} - \deg(\mathcal{E}^{\vee} \otimes \mathcal{E})^{\geq 0} \end{aligned}$$

by Proposition 2.2.4, Proposition 3.1.2, and Lemma 3.1.5. □

### 3.2. Main theorem

We now aim to establish our main result classifying all pairs vector bundles on $X$ which realize a given semistable vector bundle as their extension.

**Definition 3.2.1.** Let $\mathcal{V}$ be a vector bundle on $X$ with HN decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i}$$

where the $\lambda_i$'s are in strictly descending order. We define the *HN vectors* of $\mathcal{V}$ by

$$\overrightarrow{\mathrm{HN}}(\mathcal{V}) := (v_i)_{1 \leq i \leq l}$$

where $v_i := \big(\mathrm{rk}(\mathcal{O}(\lambda_i)^{\oplus m_i}), \deg(\mathcal{O}(\lambda_i)^{\oplus m_i})\big)$ is the vector that represents the $i$-th line segment in $\mathrm{HN}(\mathcal{V})$, and write $\mu(v_i) := \lambda_i$ for the slope of $v_i$.

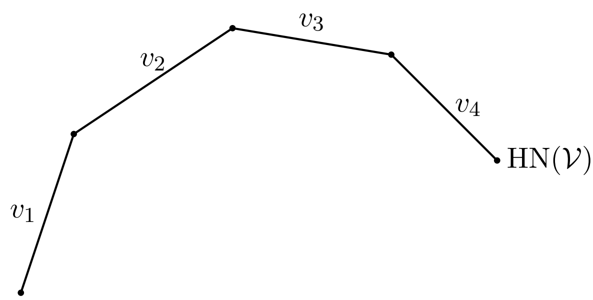


Figure 1: Vector representation of HN($\mathcal{V}$).

**Lemma 3.2.2 ([1, Lemma 2.3.4]).** *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$ with HN vectors $\overrightarrow{\mathrm{HN}}(\mathcal{E}) = (e_i)$ and $\overrightarrow{\mathrm{HN}}(\mathcal{F}) = (f_j)$. Then we have an identity*

$$\deg(\mathcal{E}^\vee \otimes \mathcal{F})^{\geq 0} = \sum_{\mu(e_i) \leq \mu(f_j)} e_i \times f_j$$

*where $e_i \times f_j$ denotes the two-dimensional cross product of the vectors $e_i$ and $f_j$.*

*Proof.* The assertion is straightforward to verify using Lemma 2.1.7. □

**Remark.** Recall that the two-dimensional cross product of two vectors $v = (x_1, y_1)$ and $w = (x_2, y_2)$ is defined by $v \times w := x_1 y_2 - x_2 y_1$.

**Lemma 3.2.3.** *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$ such that* HN($\mathcal{E}$) *lies on or below* HN($\mathcal{F}$)*. For every vector bundle $\mathcal{Q}$ on $X$, we have*

$$\deg(\mathcal{Q}^\vee \otimes \mathcal{E})^{\geq 0} \leq \deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0}.$$

*Proof.* It suffices to consider the case where we have $\mathcal{Q} = \mathcal{O}(\lambda)$ for some $\lambda \in \mathbb{Q}$, as the general case will follow from this special case using the HN decomposition of $\mathcal{Q}$. Then we note by Lemma 2.1.7 that HN($\mathcal{Q}^\vee \otimes \mathcal{E}$) and HN($\mathcal{Q}^\vee \otimes \mathcal{F}$) are respectively obtained from HN($\mathcal{E}$) and HN($\mathcal{F}$) via the composition of the following transformations:

- a shear transformation that makes each slope decrease by $\lambda$, and
- a dilation by the factor $\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{O}(\lambda))$.

Hence we find that HN($\mathcal{Q}^\vee \otimes \mathcal{E}$) lies on or below HN($\mathcal{Q}^\vee \otimes \mathcal{F}$), and in turn deduce the desired inequality by observing that $\deg(\mathcal{Q}^\vee \otimes \mathcal{E})^{\geq 0}$

and $\deg(\mathcal{Q}^\vee \otimes \mathcal{F})^{\geq 0}$ respectively represent the maximum $y$-coordinates of $\mathrm{HN}(\mathcal{Q}^\vee \otimes \mathcal{E})$ and $\mathrm{HN}(\mathcal{Q}^\vee \otimes \mathcal{F})$. □

**Proposition 3.2.4.** *Let $\mathcal{D}, \mathcal{E}, \mathcal{F}$, and $\mathcal{V}$ be vector bundles on $X$ with the following properties:*

*(i)* $\mathrm{HN}(\mathcal{D})$ *lies on or below* $\mathrm{HN}(\mathcal{V})$.

*(ii)* $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{V}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ *with $\mathcal{E}$ and $\mathcal{F}$ being semistable.*

*(iii)* $\mu_{max}(\mathcal{D}) \leq \mu(\mathcal{E}) \leq \mu(\mathcal{F})$.

*Then we have an inequality*

$$\deg(\mathcal{D}^\vee \otimes \mathcal{V})^{\geq 0} - \deg(\mathcal{V}^\vee \otimes \mathcal{V})^{\geq 0} \leq \deg(\mathcal{D}^\vee \otimes \mathcal{F})^{\geq 0}$$

*with equality if and only if $\mathcal{V}$ is isomorphic to $\mathcal{E}$.*

*Proof.* The desired inequality can be written as

$$\deg(\mathcal{D}^\vee \otimes \mathcal{V})^{\geq 0} - \deg(\mathcal{D}^\vee \otimes \mathcal{F})^{\geq 0} \leq \deg(\mathcal{V}^\vee \otimes \mathcal{V})^{\geq 0}.$$

In addition, we have

$$\begin{aligned}\deg\left((\mathcal{V}^{\leq \mu(\mathcal{E})})^\vee \otimes \mathcal{V}\right)^{\geq 0} &\leq \deg\left((\mathcal{V}^{\leq \mu(\mathcal{E})})^\vee \otimes \mathcal{V}\right)^{\geq 0} \\ &\quad + \deg\left((\mathcal{V}^{> \mu(\mathcal{E})})^\vee \otimes \mathcal{V}\right)^{\geq 0} \\ &= \deg(\mathcal{V}^\vee \otimes \mathcal{V})^{\geq 0}\end{aligned} \tag{3.2}$$

where equality holds if and only if $\mathcal{V}^{>\mu(\mathcal{E})}$ is zero, which occurs precisely when $\mathcal{V}$ and $\mathcal{E}$ are isomorphic by the condition (ii). Hence it suffices to show

$$\deg(\mathcal{D}^\vee \otimes \mathcal{V})^{\geq 0} - \deg(\mathcal{D}^\vee \otimes \mathcal{F})^{\geq 0} \leq \deg\left((\mathcal{V}^{< \mu(\mathcal{E})})^\vee \otimes \mathcal{V}\right)^{\geq 0}. \tag{3.3}$$

Let us write $\overrightarrow{\mathrm{HN}}(\mathcal{D}) := (d_i), \overrightarrow{\mathrm{HN}}(\mathcal{F}) := (f)$, and $\overrightarrow{\mathrm{HN}}(\mathcal{V}) := (v_j)$. By the condition (ii), we have $f = \sum v_j - \sum d_i$. Then by Lemma 3.2.2 and the condition (iii) we find

$$\begin{aligned}\deg(\mathcal{D}^\vee \otimes \mathcal{V})^{\geq 0} - \deg(\mathcal{D}^\vee \otimes \mathcal{F})^{\geq 0} \\ &= \sum_{\mu(d_i)\leq\mu(v_j)} d_i \times v_j - \sum d_i \times (\sum v_j - \sum d_i) \\ &= \sum_{\mu(d_i)\leq\mu(v_j)} d_i \times v_j - \sum d_i \times \sum v_j \\ &= - \sum_{\mu(d_i)>\mu(v_j)} d_i \times v_j = \sum_{\mu(v_j)<\mu(d_i)} v_j \times d_i \\ &= \deg\left((\mathcal{V}^{<\mu(\mathcal{E})})^\vee \otimes \mathcal{D}\right)^{\geq 0}.\end{aligned}$$

Moreover, by Lemma 3.2.3 and the condition (i) we have

$$\deg\left((\mathcal{V}^{<\mu(\mathcal{E})})^\vee \otimes \mathcal{D}\right)^{\geq 0} \leq \deg\left((\mathcal{V}^{<\mu(\mathcal{E})})^\vee \otimes \mathcal{V}\right)^{\geq 0}.$$

We thus deduce the desired inequality (3.3), thereby completing the proof. □

**Lemma 3.2.5.** *Let $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ be vector bundles on $X$ which fit into a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

*(1) We have $\mu_{max}(\mathcal{D}) \leq \mu_{max}(\mathcal{E})$, $\mu_{min}(\mathcal{E}) \leq \mu_{min}(\mathcal{F})$, and $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.*

*(2) $\mathrm{HN}(\mathcal{D})$ lies below or on $\mathrm{HN}(\mathcal{E})$.*

*Proof.* The nonzero maps $\mathcal{D} \hookrightarrow \mathcal{E}$ and $\mathcal{E} \twoheadrightarrow \mathcal{F}$ yield the first two inequalities by Lemma 2.1.7 and Proposition 2.1.8. The remaining assertions are standard consequences of the Harder-Narasimhan formalism, as noted by Kedlaya [13, Lemma 3.4.15 and Lemma 3.4.17]. □

**Remark.** In fact, the nonzero maps $\mathcal{D} \hookrightarrow \mathcal{E}$ and $\mathcal{E} \twoheadrightarrow \mathcal{F}$ yield much stronger conditions than the first two inequalities in Lemma 3.2.5, as noted by the author in the previous works [11, Theorem 1.1.2] and [9, Theorem 1.2.1].

**Theorem 3.2.6.** *Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be vector bundles on $X$ such that $\mathcal{E}$ is semistable. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{3.4}$$

*if and only if we have $\mu_{max}(\mathcal{D}) \leq \mu(\mathcal{E}) \leq \mu_{min}(\mathcal{F})$ and $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.*

*Proof.* The necessity part immediately follows from Lemma 3.2.5. For the sufficiency part, we henceforth assume the inequalities $\mu_{\max}(\mathcal{D}) \leq \mu(\mathcal{E}) \leq \mu_{\min}(\mathcal{F})$ and $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$. Let us write $r$ for the number of distinct slopes in $\mathrm{HN}(\mathcal{F})$ and proceed by induction on $r$.

We first consider the base case where $\mathrm{HN}(\mathcal{F})$ is a line segment, which means by Theorem 2.1.10 that $\mathcal{F}$ is semistable. Since all slopes of $\mathcal{F}^\vee \otimes \mathcal{D}$ are negative by Lemma 2.1.7, we know by Proposition 3.1.4 that $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D}) \cong \mathcal{H}^1(\mathcal{F}^\vee \otimes \mathcal{D})$ is a locally spatial diamond over $F$ with

$$\dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D}) = \deg(\mathcal{D}^\vee \otimes \mathcal{F})^{\geq 0}.$$

Let $T$ be the set of all isomorphism classes of vector bundles $\mathcal{V}$ on $X$ which fit into a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{V} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Proposition 3.1.6, Proposition 3.2.4 and Lemma 3.2.5 together yield

$$\dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_\mathcal{V} \leq \dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D}) \qquad \text{for each } \mathcal{V} \in T$$

where equality may hold only for $\mathcal{V} = \mathcal{E}$. Moreover, we have a decomposition

$$|\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})| = \bigsqcup_{\mathcal{V} \in T} |\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_\mathcal{V}|.$$

Since $T$ is a finite set by Lemma 3.2.5, we find

$$\dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D}) = \max_{\mathcal{V} \in T} \dim \mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{D})_\mathcal{V}$$

and consequently deduce that $T$ must contain the isomorphism class of $\mathcal{E}$ as desired.

We now assume $r > 1$ for the induction step. Let us write $\overrightarrow{\mathrm{HN}}(\mathcal{D}) := (d_i)$, $\overrightarrow{\mathrm{HN}}(\mathcal{E}) := (e)$ and $\overrightarrow{\mathrm{HN}}(\mathcal{F}) := (f_j)$. We find $\mu(f_r) \geq \mu(e) > \mu(f_r - \sum d_i)$ by our assumption on the HN polygons. Take $s$ to be the largest integer with $\mu\left(f_r + \sum_{i\leq s} d_i\right) \geq \mu(e)$, and set $e' := e - f_r - \sum_{i\leq s} d_i$. Define the vector bundles $\overline{\mathcal{D}}$, $\mathcal{D}'$, $\mathcal{E}'$, $\overline{\mathcal{F}}$, and $\mathcal{F}'$ by

$$\overrightarrow{\mathrm{HN}}(\mathcal{D}') = (d_i)_{i\leq s},\ \overrightarrow{\mathrm{HN}}(\overline{\mathcal{D}}) = (d_i)_{i>s},\ \overrightarrow{\mathrm{HN}}(\mathcal{E}') = (e'),$$
$$\overrightarrow{\mathrm{HN}}(\overline{\mathcal{F}}) = (f_j)_{j<r},\ \overrightarrow{\mathrm{HN}}(\mathcal{F}') = (f_r)$$

as illustrated in Figure 2.

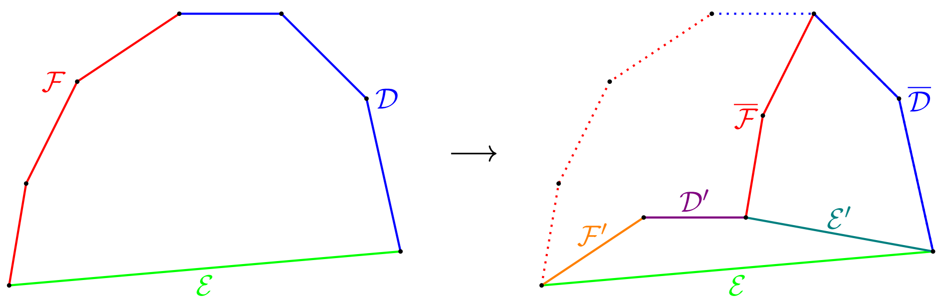


Figure 2: Construction of $\overline{\mathcal{D}}$, $\mathcal{D}'$, $\mathcal{E}'$, $\overline{\mathcal{F}}$, and $\mathcal{F}'$

By construction, we have

$$\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D}' \oplus \mathcal{E}' \oplus \mathcal{F}') \quad \text{and} \quad \mu_{\max}(\overline{\mathcal{D}}) < \mu(\mathcal{E}') \leq \mu(\mathcal{E}) \leq \mu_{\min}(\overline{\mathcal{F}}).$$

Then by the induction hypothesis, we obtain short exact sequences

$$0 \longrightarrow \mathcal{D}' \oplus \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \overline{\mathcal{D}} \longrightarrow \mathcal{E}' \longrightarrow \overline{\mathcal{F}} \longrightarrow 0.$$

These sequences together yield a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}' \oplus \overline{\mathcal{D}} & \xrightarrow{\sim} & \mathcal{D} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \alpha & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{D}' \oplus \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \end{array}$$

which, by the snake lemma, gives rise to a short exact sequence

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \mathrm{coker}(\alpha) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Since we have $\mu_{\min}(\overline{\mathcal{F}}) > \mu_{\max}(\mathcal{F}')$ by construction, this sequence is split by Proposition 2.1.8. Hence we obtain a desired exact sequence (3.4). $\square$

## Acknowledgments

The author would like to sincerely thank David Hansen, Miaofen Chen and the anonymous referee for pointing out a mistake in the previous version of this paper.

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Received August 23, 2021
Accepted May 17, 2022