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Uniqueness of equivariant harmonic maps to symmetric spaces and buildings

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We prove uniqueness of equivariant harmonic maps into irreducible symmetric spaces of non-compact type and Bruhat-Tits buildings associated to isometric actions by Zariski dense subgroups.

1. Introduction

Assume that M and N are Riemannian manifolds, M has finite volume and N has non-positive sectional curvature. Hartman [Ha] proved the following uniqueness result for harmonic maps: *Let $u : M \to N$ be a finite energy harmonic map of rank greater at 1 at some point $p \in M$. If N has negative sectional curvature at $u(p)$, then u is the only harmonic map in its homotopy class* (cf. [Ha, Corollary following (H)]). The second author [Me] generalized Hartman's uniqueness result to the case when the target space is a geodesic metric space $\tilde{X}$ with curvature < 0 in the sense of Alexandrov. On the other hand, if there exists a 2-plane in $T_{u(p)}N$ with sectional curvature 0 for all $p \in M$, then uniqueness fails. For example in the extreme case, when N is a flat torus, then there exists a family of harmonic maps obtained by translations of a given harmonic map.

Analogous uniqueness statements hold for equivariant harmonic maps. More precisely, let $\rho : \pi_1(M) \to \mathsf{Isom}(\tilde{X})$ be a homomorphism into the isometry group of an NPC space $\tilde{X}$ and $\tilde{f}$ be a ρ-equivariant map (cf. Definition 2.10). Using the same principle as in the homotopy problem, a finite energy ρ-equivariant harmonic map $\tilde{u} : \tilde{M} \to \tilde{X}$ is unique provided $\tilde{u}$ has rank greater than 1 at some point $p \in \tilde{M}$ and $\tilde{X}$ has negative curvature at $\tilde{u}(p)$.

In this note, we study uniqueness for equivariant harmonic maps into irreducible symmetric spaces of non-compact type and Bruhat-Tits buildings. Bruhat-Tits buildings are locally finite simplicial complexes. However, we

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conjecture that a similar uniqueness result holds in the case of non-locally finite thick Euclidean buildings with transitive isometry groups (cf. Remark 3.3). The importance of the latter case is that limits of symmetric spaces of non-compact type are such Euclidean buildings (cf. [KL1, Theorem 5.2.1]). This is important in the study of the compactification of character varieties and higher Teichmüller theory.

Symmetric spaces of non-compact type (resp. Euclidean buildings) are examples of Riemannian manifolds of non-positive sectional curvature (resp. NPC spaces or complete CAT(0) spaces). Harmonic maps into Riemannian manifolds of non-positive sectional curvature and NPC spaces have been important in the study of geometric rigidity problems (e.g. [Si], [Co1], [GS], [JY], [MSY], [DMV] among many others). The uniqueness of harmonic maps into symmetric spaces (resp. Euclidean buildings) does not follow from [Ha] (resp. [Me]) unless $\tilde{X}$ has rank 1 (resp. $\tilde{X}$ is a $\mathbb{R}$-tree). Indeed, every point P in a rank n symmetric space $\tilde{X}$ (resp. n-dimensional Euclidean building) is contained in a convex, isometric embedding of $\mathbb{R}^n$. The novelty of this paper is that the uniqueness is proven, not with the assumption on the curvature bound as in [Ha] and [Me], but with an assumption on the homomorphism $\rho : \pi_1(M) \to \mathsf{Isom}(\tilde{X})$.

The main theorem of this paper is the following:

Theorem 1.1 (Existence and Uniqueness). *Let M be a Riemannian manifold with finite volume, $\tilde{X}$ be an irreducible symmetric space of non-compact type, and $\rho : \pi_1(M) \to \mathsf{Isom}(\tilde{X})$ a homomorphism. Assume:*

(i) *The subgroup $\rho(\pi_1(M))$ does not fix a point at infinity.*

(ii) *There exists a finite energy ρ-equivariant map $\tilde{f} : \tilde{M} \to \tilde{X}$.*

Then there exists a unique finite energy ρ-equivariant harmonic map $\tilde{u} : \tilde{M} \to \tilde{X}$.

The same conclusion holds if $\tilde{X}$ is an irreducible Bruhat-Tits building with the additional assumption that the action of $\rho(\pi_1(M))$ does not fix a non-empty closed convex strict subset of $\tilde{X}$.

The existence results for harmonic maps is contained in (e.g. [L], [Do], [Co1], [GS], [J], [KS2], [KS3]). Thus, the goal of this paper is to prove the uniqueness assertion in Theorem 1.1.

The assumptions on the subgroup $\rho(\pi_1(M))$ in Theorem 1.1 are related to the notion of Zariski dense. Indeed, in either the case when $\tilde{X}$ is a symmetric space of non-compact type or a Bruhat-Tits building, if the action of the subgroup Γ of $\mathsf{Isom}(\tilde{X})$ neither fixes a point at infinity nor a non-empty closed convex strict subset, then Γ is Zariski dense (cf. [CaMo, Proposition 2.8]). The converse also holds if $\tilde{X}$ is a symmetric spaces of non-compact type and $\mathsf{rank}(\tilde{X}) \geq 2$ (cf. [KL2, Theorem 4.1]), but there exist Zariski dense subgroups that fix a non-empty closed convex strict subset if $\mathsf{rank}(\tilde{X}) = 1$ (cf. [Ca, Section 4]).

Remark 1.2. *For the case when $\tilde{X} = G/K$ is a symmetric space, Theorem 1.1 may be deduced from the gauge theoretic approach due to Donaldson [Do] and Corlette [Co2]. Indeed, harmonic maps to symmetric spaces can be thought of as a solution to Hitchin's equations and uniqueness follows along the lines of [Co2, Proposition 2.3]. The point of this paper is to provide a simple geometric proof of the uniqueness of harmonic maps that works for Bruhat-Tits buildings as well.*

2. Preliminaries

We start with some definitions. We will assume that $\tilde{X}$ is a complete metric space.

Definition 2.1. *A geodesic $\sigma : I \to \tilde{X}$ is a map from an interval $I \subset \mathbb{R}$ such that $d(\sigma(s), \sigma(s+t)) = |t|$ for all $s, t \in I$. A geodesic line, geodesic ray and geodesic segment are geodesics with domain $\mathbb{R}$, $[0, \infty)$ and closed interval $[a, b]$ respectively.*

Definition 2.2. *Geodesics $\sigma : I \to \tilde{X}$ and $\hat{\sigma} : I \to \tilde{X}$ are said to be parallel if there exists a constant $C > 0$ such that*

$$d(\sigma(s), \hat{\sigma}(s)) = C, \ \forall s \in I.$$

Remark 2.3. *Two geodesic rays $\sigma : [0, \infty) \to \tilde{X}$ and $\bar{\sigma} : [0, \infty) \to \tilde{X}$ are asymptotic if there exists a constant $C > 0$ such that*

$$d(\sigma(s), \bar{\sigma}(s)) \leq C, \ \forall s \in \mathbb{R} \ \ (\textit{resp. } \forall s \in [0, \infty)).$$

By [BH, II.2.13], the terms parallel geodesic rays and asymptotic geodesic rays are equivalent.

Definition 2.4. *A point at infinity is an asymptotic class of geodesic rays. We denote by $[\sigma]$ the asymptotic class containing the geodesic ray σ.*

Definition 2.5. *A symmetric space $\tilde{X}$ is a Riemannian manifold such that, for any $P \in \tilde{X}$, there exists $S_P \in \mathsf{Isom}(\tilde{X})$ such that P is an isolated fixed point of S_P and $S_P \circ S_P$ is the identity map. The isometry S_P is called an inversion symmetry at P.*

Definition 2.6. *Given a geodesic line $\sigma : \mathbb{R} \to \tilde{X}$ and $s \in \mathbb{R}$, the composition*

$$T_s = S_{\sigma(\frac{s}{2})} \circ S_{\sigma(0)}$$

is called a transvection along σ. We have that

$$T_{s+s'} = T_s \circ T_{s'}$$

and $\{T_s\}$ forms a one-parameter subgroup of $\mathsf{Isom}(\tilde{X})$ that act as parallel transports along σ (cf. [Eb, 2.1.1]).

Definition 2.7 (cf. [BH] Definition 10A.1). *A Euclidean building of dimension n is a piecewise Euclidean simplicial complex $\tilde{X}$ such that:*

(1) *$\tilde{X}$ is the union of a collection $\mathcal{A}$ of subcomplexes A, called apartments, such that the intrinsic metric d_A on A makes (A, d_A) isometric to the Euclidean space $\mathbb{R}^n$ and induces the given Euclidean metric on each simplex.*

(2) *Any two simplices B and B' of X are contained in at least one apartment.*

(3) *Given two apartments A and A' containing both simplices B and B', there is a simplicial isometry from (A, d_A) to $(A', d_{A'})$ which leaves both B and B' pointwise fixed.*

Furthermore, will assume

(4) *$\tilde{X}$ is an irreducible Bruhat-Tits building.*

Definition 2.8. *A symmetric space of non-compact type $\tilde{X}$ (resp. a Euclidean building) is said to be irreducible if it is not isometric to a non-trivial product $\tilde{X}_1 \times \tilde{X}_2$ of two symmetric spaces of non-compact type (resp. Euclidean buildings).*

Notation 2.9. *Given $P, Q \in \tilde{X}$ and $s \in \mathbb{R}$, we denote*

$$(1-s)P + sQ$$

to be the geodesic interpolation between P and Q; i.e. $(1-s)P + sQ = \bar{\sigma}(\delta s)$ where $\delta = d(P,Q)$ and $\bar{\sigma} : [0,\delta] \to \tilde{X}$ is a geodesic segment with $\bar{\sigma}(0) = P$ and $\bar{\sigma}(\delta) = Q$.

Definition 2.10. *Let $\mathsf{Isom}(\tilde{X})$ be the group of isometries of $\tilde{X}$ and $\rho : \pi_1(M) \to \mathsf{Isom}(\tilde{X})$ be a homomorphism from the fundamental group of a Riemannian manifold M. Let $\pi_1(M)$ act on the universal cover $\tilde{M}$ of M by deck transformations. A map $\tilde{f} : \tilde{M} \to \tilde{X}$ is said to be ρ-equivariant if*

$$\tilde{f}(\gamma p) = \rho(\gamma)\tilde{f}(p), \quad \forall \gamma \in \pi_1(M),\ p \in \tilde{M}$$

where we write gP for $g \in \mathsf{Is}(\tilde{X})$ and $P \in \tilde{X}$ instead of $g(P)$ for simplicity.

If $\tilde{X}$ is a Riemannian manifold, then $|d\tilde{f}|^2$ is the norm of the differential $d\tilde{f} : T\tilde{M} \to T\tilde{X}$. If $\tilde{X}$ is a NPC space, then $|d\tilde{f}|^2$ is the energy density function in the sense of [KS1]. Either way, if $\tilde{f}$ is ρ-equivariant, then $|d\tilde{f}|^2$ is invariant under the action of $\rho(\gamma)$ for any $\gamma \in \pi_1(M)$, and the energy of $\tilde{f}$ is defined to be

$$E^{\tilde{f}} = \int_M |d\tilde{f}|^2 d\mathrm{vol}_M.$$

3. Proof of Theorem 1.1

The existence results for harmonic maps is contained in (e.g. [L], [Do], [Co1], [GS], [J], [KS2], [KS3]). Thus, we need to only prove the uniqueness assertion.

3.1. Geodesic interpolation

We assume on the contrary that there exist two distinct ρ-equivariant harmonic maps

$$\tilde{u}_0 : \tilde{M} \to \tilde{X} \ \text{ and } \tilde{u}_1 : \tilde{M} \to \tilde{X}.$$

Using Notation 2.9, define the geodesic interpolation of $\tilde{u}_0$ and $\tilde{u}_1$; i.e.

$$\tilde{u}_s : \tilde{M} \to \tilde{X}, \quad \tilde{u}_s(q) = (1-s)\tilde{u}_0(q) + s\tilde{u}_1(q).$$

Since $\tilde{u}_0$ and $\tilde{u}_1$ are ρ-equivariant, $\tilde{u}_s$ is also ρ-equivariant. By the convexity of energy (cf. [KS1, (2.2vi)]),

$$E^{\tilde{u}_s} \leq (1-s)E^{\tilde{u}_0} + sE^{\tilde{u}_1} - s(1-s)\int_M |\nabla d(\tilde{u}_0, \tilde{u}_1)|^2 d\mathrm{vol}_M$$

Lemma 3.1. *Scaling if necessary, assume* $d(\tilde{u}_0(p_0), \tilde{u}_1(p_0)) = 1$ *for some point* $p_0 \in \tilde{M}$*. Then, for* $\tilde{u}_s$ *defined above, we have the following:*

- $d(\tilde{u}_s(p), \tilde{u}_1(p)) = 1 - s, \ \forall p \in \tilde{M}$
- $|(\tilde{u}_s)_*(V)|^2(p) = |(\tilde{u}_0)_*(V)|^2(p),$ *for* $s \in [0,1]$*, a.e.* $p \in M$*, a.e.* $V \in T_p\tilde{M}$*.*

Proof. Since $\tilde{u}_0$ and $\tilde{u}_1$ are energy minimizing, we conclude

$$0 = \int_M |\nabla d(\tilde{u}_0, \tilde{u}_1)|^2 d\mathrm{vol}_M \tag{3.1}$$

$$E^{\tilde{u}_s} = E^{\tilde{u}_0}, \quad \forall s \in [0,1] \tag{3.2}$$

First, (3.1) implies that $\nabla d(\tilde{u}_0, \tilde{u}_1) = 0$ a.e. in $\tilde{M}$. Hence, $d(\tilde{u}_0, \tilde{u}_1)$ is constant; i.e.

$$d(\tilde{u}_0, \tilde{u}_1) \equiv 1. \tag{3.3}$$

Note that equality (3.2) implies that

$$|(\tilde{u}_s)_*(V)|^2(p) = |(\tilde{u}_0)_*(V)|^2(p), \text{for } s \in [0,1], \text{a.e. } p \in \tilde{M}, \text{a.e. } V \in T_p\tilde{M}. \tag{3.4}$$

Indeed, for $\{P, Q, R, S\} \subset \tilde{X}$, the quadrilateral comparison for NPC spaces implies

$$d^2(P_s, Q_s) \leq (1-s)d^2(P,Q) + sd^2(R,S) - s(1-s)(d(P,Q) - d(R,S))^2$$

where $P_s = (1-s)P + sS$ and $Q_s = (1-s)Q + sR$. Applying the above inequality with $P = \tilde{u}_0(p)$, $S = \tilde{u}_1(p)$, $R = \tilde{u}_1(\exp_p(tV))$ and $Q = \tilde{u}_0(\exp_p(tV))$ where $t > 0$ and $V \in T_p\tilde{M}$, dividing by t^2 and letting $t \to 0$, we obtain (cf. [KS1, Theorem 1.9.6])

$$\begin{aligned} |(\tilde{u}_s)_*(V)|^2(p) \leq\ & (1-s)|(\tilde{u}_0)_*(V)|^2(p) + s|(\tilde{u}_1)_*(V)|^2(p) \\ & - s(1-s)(|(\tilde{u}_0)_*(V)|(p) - |(\tilde{u}_1)_*(V)|(p))^2, \\ & \text{a.e. } p \in \tilde{M}, V \in T_p\tilde{M}. \end{aligned}$$

Combining this with (3.2), we conclude

$$|(\tilde{u}_s)_*(V)|^2 = (1-s)|(\tilde{u}_0)_*(V)|^2 + s|(\tilde{u}_1)_*(V)|^2$$

which in turn implies (3.4). □

For each $q \in \tilde{M}$, define the geodesic segment

$$\bar{\sigma}_q : [0,1] \to \tilde{X}, \quad \bar{\sigma}_q(s) = \tilde{u}_s(q). \tag{3.5}$$

Note that up to this point, we have only used the fact that $\tilde{X}$ is an NPC space. We will now specialize to the two cases: (i) $\tilde{X}$ is an irreducible symmetric space of non-compact type and (ii) $\tilde{X}$ is an irreducible Bruhat-Tits building.

3.2. Symmetric spaces

Throughout this subsection $\tilde{X}$ is an irreducible symmetric space of non-compact type. For each $q \in \tilde{M}$, extend the geodesic segment $\bar{\sigma}_q$ of (3.5) to a geodesic line

$$\sigma_q : \mathbb{R} \to \tilde{X}. \tag{3.6}$$

Let

$$F : \tilde{M} \times \mathbb{R} \to \tilde{X}, \quad F(q,s) = \sigma_q(s).$$

Let $\nabla^{F^{-1}}$ be the induced connection on the vector bundle

$$(T^*(M \times \mathbb{R}))^{\otimes k} \otimes F^{-1}T\tilde{X} \to \tilde{M} \times \mathbb{R}. \tag{3.7}$$

For each $s \in [0,1]$, let $\nabla^{\tilde{u}_s^{-1}}$ be the induced connection on the vector bundle

$$(T^*M)^{\otimes k} \otimes \tilde{u}_s^{-1}T\tilde{X} \to \tilde{M}. \tag{3.8}$$

Use the inclusion $\tilde{M} \to \tilde{M} \times \{s\}$ and the identity $F(\cdot, s) = \tilde{u}_s(\cdot)$ to identify (3.8) as a subbundle of (3.7).

Let s, ∂_s denote the standard coordinate and coordinate vector on $\mathbb{R}$. Let $(E_1,\dots,E_n)$ denote a local orthonormal frame of $\tilde{M}$. Set

$$V = dF\left(\partial_s\right), \quad X_\alpha = dF\left(E_\alpha\right) \text{ for } \alpha = 1,\dots,n$$

as sections of $F^{-1}T\tilde{X}$. Applying the usual second variation formula of the energy (e.g. [ES], [Sc]), we obtain

$$\frac{d^2}{ds^2}\Big|_{s=t} E^{\tilde{u}_s}(r) = 2\int_{\tilde{M}} \sum_{\alpha=1}^{n} \left(\|\nabla^{F^{-1}}_{E_\alpha} V\|^2 - \left\langle R^{\tilde{X}}\left(V, X_\alpha\right)V, X_\alpha \right\rangle \right)\Big|_{s=t} d\mathrm{vol}_{\tilde{M}}$$

for any $t \in [0,1]$ where $R^{\tilde{X}}$ is the Riemannian curvature operator of $\tilde{X}$. By (3.2), the left hand side of the above equality is equal to 0. The integrand on the right hand side is non-positive by the assumption of non-positive curvature. Thus, we conclude that for any $\alpha = 1,\dots,n$,

$$\nabla^{F^{-1}}_{E_\alpha} V \equiv 0 \tag{3.9}$$

$$\left\langle R^{\tilde{X}}\left(V, X_\alpha\right)V, X_\alpha \right\rangle \equiv 0. \tag{3.10}$$

From the above, we conclude

$$\nabla^{F^{-1}}_{\partial_s}(d\tilde{u}_s(E_\alpha)) = \nabla^{F^{-1}}_{\partial_s} X_\alpha = 0 \tag{3.11}$$

and

$$\nabla^{\tilde{u}_s^{-1}}_{E_\beta} \nabla^{F^{-1}}_{\partial_s} = \nabla^{F^{-1}}_{\partial_s} \nabla^{\tilde{u}_s^{-1}}_{E_\beta}, \ \forall \beta = 1,\dots,n. \tag{3.12}$$

Since $\nabla_{\partial_s} E_\alpha = \nabla_{\partial_s} E_\beta = 0$, we have

$$\begin{aligned}
&\left(\nabla^{F^{-1}}_{\partial_s} \nabla^{\tilde{u}_s^{-1}} d\tilde{u}_s\right)(E_\alpha, E_\beta) \\
&= \nabla^{F^{-1}}_{\partial_s}\left(\nabla^{\tilde{u}_s^{-1}} d\tilde{u}_s(E_\alpha, E_\beta)\right) - \nabla^{\tilde{u}_s^{-1}} d\tilde{u}_s(\nabla_{\partial_s} E_\alpha, E_\beta) - \nabla^{\tilde{u}_s^{-1}} d\tilde{u}_s(E_\alpha, \nabla_{\partial_s} E_\beta) \\
&= \nabla^{F^{-1}}_{\partial_s}\left(\nabla^{\tilde{u}_s^{-1}}_{E_\alpha}(d\tilde{u}_s(E_\beta)) - d\tilde{u}_s(\nabla_{E_\alpha} E_\beta)\right) \\
&= \nabla^{\tilde{u}_s^{-1}}_{E_\alpha}\left(\nabla^{F^{-1}}_{\partial_s}(d\tilde{u}_s(E_\beta))\right) - \nabla^{F^{-1}}_{\partial_s} d\tilde{u}_s(\nabla_{E_\alpha} E_\beta) \quad \text{(by (3.12))} \\
&= 0 \quad \text{(by (3.11))}.
\end{aligned}$$

More generally, we can inductively use (3.12) multiple times to switch the order of differentiation and apply (3.11) to conclude

$$\nabla^{F^{-1}}_{\partial_s}\left(\nabla^{\tilde{u}_s^{-1}} \cdots \nabla^{\tilde{u}_s^{-1}} d\tilde{u}_s\right) = 0. \tag{3.13}$$

Claim 1. *Fix a point $p \in \tilde{M}$ and let $T_s \in Isom(\tilde{X})$ be the transvection along σ_p as in Definition 2.6. Then*

$$\tilde{u}_s = T_s \tilde{u}_0, \quad \forall s \in [0,1].$$

Proof. For $s \in [0,1]$, define a harmonic map

$$\tilde{v}_s : \tilde{M} \to \tilde{X}, \quad \tilde{v}_s = T_s \tilde{u}_0.$$

Define

$$\Phi : \tilde{M} \times [0,1] \to \tilde{X}, \quad \Phi(q,s) = \tilde{v}_s(q).$$

Since T_s is a transvection along the geodesic σ_p,

$$F(p,s) = \tilde{u}_s(p) = \sigma_p(s) = T_s \sigma_p(0) = T_s \tilde{u}_0(p) = \tilde{v}_s(p) = \Phi(p,s).$$

Furthermore, T_s defines a parallel transport along $\bar{\sigma}_p$, and thus

$$\nabla^{\Phi^{-1}}_{\partial_s} \big(d\tilde{v}_s(E_\alpha)\big) = \nabla^{\Phi^{-1}}_{\partial_s} \big(dT_s(d\tilde{u}_0(E_\alpha))\big) = 0 \text{ at } (p,s), \ \forall s \in (0,1). \tag{3.14}$$

By (3.11) and (3.14), the vector fields $d\tilde{u}_s(E_\alpha)$ and $d\tilde{v}_s(E_\alpha)$ are both parallel along $\sigma_p(s)$. Since $d\tilde{u}_0(E_\alpha) = d\tilde{v}_0(E_\alpha)$ at p, we conclude

$$d\tilde{u}_s(E_\alpha) = d\tilde{v}_s(E_\alpha) \text{ at } p \in \tilde{M}, \ \forall s \in [0,1].$$

Next, since T_s is an isometry,

$$\begin{aligned}
\nabla^{v_s^{-1}}_{E_\alpha} \big(d\tilde{v}_s(E_\beta)\big) &= \nabla^{v_s^{-1}}_{E_\alpha} \big(dT_s \circ d\tilde{u}_0(E_\beta)\big) \\
&= \nabla^{\tilde{X}}_{dT_s \circ d\tilde{u}_0(E_\alpha)} \big(dT_s \circ d\tilde{u}_0(E_\beta)\big) \\
&= dT_s \left(\nabla^{\tilde{X}}_{d\tilde{u}_0(E_\alpha)} d\tilde{u}_0(E_\beta)\right) \\
&= dT_s \left(\nabla^{u_0^{-1}}_{E_\alpha} d\tilde{u}_0(E_\beta)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\nabla^{v_s^{-1}} d\tilde{v}_s(E_\alpha, E_\beta) &= \nabla^{v_s^{-1}}_{E_\alpha} \big(d\tilde{v}_s(E_\beta)\big) - d\tilde{v}_s \left(\nabla^{\tilde{M}}_{E_\alpha} E_\beta\right) \\
&= dT_s \left(\nabla^{u_0^{-1}}_{E_\alpha} d\tilde{u}_0(E_\beta)\right) - dT_s \left(d\tilde{u}_0(\nabla^{\tilde{M}}_{E_\alpha} E_\beta)\right).
\end{aligned}$$

Since T_s defines a parallel transport along $\bar{\sigma}_p(s)$, both vector fields on the right hand side are parallel along $\sigma_p(s)$. Thus,

$$\nabla^{\Phi^{-1}}_{\partial_s} \left(\nabla^{v_s^{-1}} d\tilde{v}_s(E_\alpha, E_\beta)\right) = 0.$$

Continuing inductively, we can prove

$$\nabla^{\Phi^{-1}}_{\partial_s}\left(\nabla^{\tilde{v}_s^{-1}}\cdots\nabla^{\tilde{v}_s^{-1}}d\tilde{v}_s\right)=0.$$

Combined with (3.13) and the fact that $\tilde{u}_0=\tilde{v}_0$, we conclude

$$\nabla^{\tilde{v}_s^{-1}}\cdots\nabla^{\tilde{v}_s^{-1}}d\tilde{v}_s=\nabla^{\tilde{u}_s^{-1}}\cdots\nabla^{\tilde{u}_s^{-1}}d\tilde{u}_s \text{ at } p,\ \forall s\in[0,1].$$

In other words, $\tilde{u}_s$ and $\tilde{v}_s$ agree up to infinitely high order at p which in turn implies that $\tilde{u}_s=\tilde{v}_s=T_s\tilde{u}_0$ by [Sa, Theorem 1]. □

Claim 2. *Let p be the point fixed in Claim 1 and let the geodesic ray $\sigma_q : [0,\infty)\to\tilde{X}$ be the restriction of the geodesic line defined in (3.6). Then*

$$d(\sigma_q(s),\sigma_p(s))=\delta_{p,q},\quad \forall q\in\tilde{M},\ s\in[0,\infty)$$

where $\delta_{p,q}:=d(\sigma_q(0),\sigma_p(0))$. In particular, σ_q is the unique geodesic ray parallel to σ_p with value at $s=0$ equal to $\tilde{u}_0(q)$.

Proof. As above, let T_s be the transvection along σ_p. By Claim 1, $\tilde{u}_s(q)=T_s\tilde{u}_0(q)$ for $s\in[0,1]$. Since

$$T_s\tilde{u}_0(q)=(T_{\frac{1}{2}}\circ T_{s-\frac{1}{2}})\tilde{u}_0(q)=T_{\frac{1}{2}}\sigma_q(s-\frac{1}{2}),\quad \forall s\in[\frac{1}{2},\frac{3}{2}],$$

the restriction of $s\mapsto T_s\tilde{u}_0(q)$ to $[\frac{1}{2},\frac{3}{2}]$ is a geodesic segment. Using an analogous argument, we can inductively show that for any $n\in\mathbb{N}$, the restriction to $[\frac{n}{2},\frac{n}{2}+\frac{1}{2}]$ of the map $s\mapsto T_s\tilde{u}_0(q)=(T_{\frac{1}{2}}\circ T_{s-\frac{1}{2}})\tilde{u}_0(q)$ is a geodesic segment. Thus, we conclude that $s\mapsto T_s\tilde{u}_0(q)$ is a geodesic ray with $T_s\tilde{u}_0(q)=\sigma_q(s)$ for $s\in[0,1]$. Since the two geodesic rays $s\mapsto T_s\tilde{u}_0(q)$ and $s\mapsto\sigma_q(s)$ agree on $[0,1]$, they are the same geodesic ray. Since T_s is an isometry,

$$d(\sigma_q(s),\sigma_p(s))=d(T_s\tilde{u}_0(q),T_s\tilde{u}_0(p))=d(\tilde{u}_0(q),\tilde{u}_0(p))=d(\sigma_0(q),\sigma_0(p)).$$

□

For $Q=\tilde{u}_0(q)$, let $\sigma^Q=\sigma_q$. By Claim 2, there exists map from $\tilde{u}_0(M)$ to a family of pairwise parallel geodesic lines given by

$$Q\mapsto\sigma^Q.$$

Since $\sigma^Q=\sigma_q$ and $\sigma^{\rho(\gamma)Q}=\sigma_{\gamma q}$ are extensions of $\bar{\sigma}_q$ and $\bar{\sigma}_{\gamma q}$ and

$$\rho(\gamma)\bar{\sigma}_q(s)=\rho(\gamma)\tilde{u}_s(q)=\tilde{u}_s(\gamma q)=\bar{\sigma}_{\gamma q}(s),\ \forall s\in[0,1],$$

we have

$$\rho(\gamma)\sigma^Q = \sigma^{\rho(\gamma)Q}, \quad \forall \gamma \in \pi_1(M).$$

Since σ^Q and $\sigma^{\rho(\gamma)Q}$ are parallel geodesic rays, we conclude that that

$$\rho(\gamma)[\sigma_q] = [\sigma_q], \quad \gamma \in \pi_1(M).$$

In other words, $\rho(\pi_1(M))$ fixes a point at infinity, contradicting assumption (i).

3.3. Euclidean buildings

Throughout this subsection, $\tilde{X}$ is an irreducible Bruhat-Tits building of dimension n. An open n-dimensional simplex of $\tilde{X}$ will be referred to as a *chamber*. An apartment of $\tilde{X}$ is a convex isometric embedding of $\mathbb{R}^n$ in $\tilde{X}$.

Let $\tilde{u}_s$ be the geodesic interpolation maps defined in §3.1. The *regular set* $\mathcal{R}(\tilde{u}_s)$ is the set of all points $q \in \tilde{M}$ with the following property: There exists a neighborhood $\mathcal{U}_q$ of q such that $\tilde{u}_s(\mathcal{U}_q)$ is contained in an apartment A_q of $\tilde{X}$.

Theorem 3.2 ([GS] Theorem 6.4). *The singular set $\mathcal{S}(\tilde{u}_s)$, i.e. the complement of $\mathcal{R}(\tilde{u}_s)$, is a closed set of Hausdorff codimension at least 2.*

Let $\mathcal{R}^*(\tilde{u}_s)$ be the set of points q in $\mathcal{R}(\tilde{u}_s)$ such that

$$\exists\ \epsilon > 0 \text{ and a chamber } C^* \text{ such that } \tilde{u}_s(B_q(\epsilon)) \subset \bar{C}^*. \tag{3.15}$$

After identifying $A \simeq \mathbb{R}^n$, $\tilde{u}_s|_{\mathcal{U}_q}$ is a harmonic map into Euclidean space, and it follows that the set $\mathcal{U}_q \backslash \mathcal{R}^*(\tilde{u}_s)$ is a closed set of dimension at most 1.

For each $q \in \tilde{M}$, let $\bar{\sigma}_q(s) = \tilde{u}_s(q)$ (cf. (3.5)) and denote by R_q the set of all points $s \in [0,1]$ such that

$$\exists \epsilon > 0 \text{ and a chamber } C \text{ such that } \bar{\sigma}_q((s-\epsilon, s+\epsilon)) \subset \bar{C}. \tag{3.16}$$

The complement of R_q in $[0,1]$ is a finite set. Thus, the complement of $\mathcal{R}^{**} = \{(q,s) : q \in \mathcal{R}^*(\tilde{u}_s), s \in R_q\}$ in $\tilde{M} \times [0,1]$ is an closed set of measure 0.

Fix $(q,s) \in \mathcal{R}^{**}$ and let C, C^* be the chamber as in (3.15), (3.16) respectively. Let $(x^1, \dots, x^n)$ be local coordinates in a neighborhood $\mathcal{U}_q$ of $q \in \mathcal{R}^*$ with coordinate vector fields $(\partial_1, \dots, \partial_n)$. Let A be the apartment containing $\tilde{u}_s(\mathcal{U}_q)$ for all $s \in [0,1]$. After isometrically identifying A with $\mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$

be the usual inner product defined on $A \simeq \mathbb{R}^n$. Thus, (3.4) with $V = \partial_\alpha$, implies at q_0

$$s \mapsto \left\langle \frac{\partial \tilde{u}_s}{\partial x^\alpha}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle = \text{constant in } [0,1].$$

We can differentiate this twice with respect to s to obtain

$$0 = \frac{\partial^2}{\partial s^2} \left\langle \frac{\partial \tilde{u}_s}{\partial x^\alpha}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle = 2 \left\langle \frac{\partial}{\partial x^\alpha} \frac{\partial^2 \tilde{u}_s}{\partial s^2}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle + 2 \left\langle \frac{\partial}{\partial x^\alpha} \frac{\partial \tilde{u}_s}{\partial s}, \frac{\partial}{\partial x^\alpha} \frac{\partial \tilde{u}_s}{\partial s} \right\rangle.$$

Since $\bar{\sigma}_{q_0}$ is a geodesic, $\frac{\partial^2 \tilde{u}_s}{\partial s^2}(q_0) = \bar{\sigma}''_{q_0}(s) = 0$. Thus,

$$\left. \frac{\partial \bar{\sigma}'_q(s)}{\partial x^\alpha} \right|_{q=q_0} = \frac{\partial}{\partial x^\alpha} \frac{\partial \tilde{u}_s}{\partial s}(q_0) = 0.$$

Since the choice of $\alpha \in \{1, \ldots, n\}$ is arbitrary and $\mathcal{R}^*$ is of full measure in $\tilde{M} \times [0,1]$, we conclude that the geodesic segments $\bar{\sigma}_p$ and $\bar{\sigma}_q$ are parallel for any $p, q \in \tilde{M}$; i.e.

$$d(\bar{\sigma}_p(s), \bar{\sigma}_q(s)) =: \delta_{p,q}, \quad \forall s \in [0,1] \tag{3.17}$$

where $\delta_{p,q} := d(\bar{\sigma}_q(0), \bar{\sigma}_p(0)) = d(\tilde{u}_0(q), \tilde{u}_0(p))$.

Next note the following:

- (Existence of a geodesic extension) Given a geodesic segment, property (2) of Definition 2.7 implies that there exists an apartment containing its endpoints and hence its image. We can thus extend the geodesic segment to a geodesic line in this apartment.
- (Non-uniqueness of a geodesic extension) Unlike symmetric spaces, the geodesic extensions are not necessarily unique in a Euclidean building. Indeed, there may be many apartments containing the endpoints of a given geodesic segment.

Because of the non-uniqueness of geodesic extensions, the proof for the building case is slightly different from the symmetric space case as we see below.

We define the sets

$$C_0, C_1, \ldots, C_n \tag{3.18}$$

inductively follows. First, let $C_0 = \tilde{u}_0(\tilde{M})$, and then let C_n be the union of the images of all geodesic segments connecting points of C_{n-1}. The $\rho(\pi_1(M))$-invariance of C_0 implies the $\rho(\pi_1(M))$-invariance of C_n.

To each $Q = \tilde{u}_0(q) \in C_0$, we assign a geodesic segment $\bar{\sigma}^Q = \bar{\sigma}_q$ (cf. (3.5)). By (3.17), $\{\bar{\sigma}^Q\}_{Q\in C_0}$ is a family of pairwise parallel geodesic segments. Since $\tilde{u}_s$ is ρ-equivariant, the assignment $Q \mapsto \bar{\sigma}^Q$ is $\rho(\pi_1(M))$-equivariant; i.e. $\rho(\gamma)\bar{\sigma}^Q = \bar{\sigma}^{\rho(\gamma)Q}$ for any $Q \in C_0$ and $\gamma \in \rho(\pi_1(M))$.

For $n \in \mathbb{N}$, we inductively define a $\rho(\pi_1(M))$-equivariant map from C_n to a family of pairwise parallel geodesic segments as follows: For any pair of points $Q_0, Q_1 \in C_{n-1}$, apply the Flat Quadrilateral Theorem (cf. [BH, 2.11]) with vertices $Q_0, Q_1, P_1 := \bar{\sigma}^{Q_1}(1), P_0 := \bar{\sigma}^{Q_0}(1)$ to define a one-parameter family of parallel geodesic segments $\bar{\sigma}^{Q_t} : [0,1] \to \tilde{X}$ with initial point $Q_t = (1-t)Q_0 + tQ_1$ and terminal point $P_t = (1-t)P_0 + tP_1$ (cf. (2.9)). The inductive hypothesis implies that the map $Q \mapsto \bar{\sigma}^Q$ defined on C_n is also $\rho(\pi_1(M))$-equivariant. The above construction defines a $\rho(\pi_1(M))$-equivariant map

$$Q \mapsto \bar{\sigma}^Q$$

from $\tilde{X}$ to a family of pairwise geodesic segments. Indeed, we are assuming that the action of $\rho(\pi_1(M))$ does not fix a non-empty closed convex strict subset of $\tilde{X}$. Thus,

$$\tilde{X} = \bigcup_{n=0}^{\infty} C_n \tag{3.19}$$

since the right hand side is the convex hull of $C_0 = \tilde{u}_0(\tilde{M})$ and each C_n is invariant under the action of $\rho(\pi_1(M))$.

Claim 3. *There exists a $\rho(\pi_1(M))$-equivariant map*

$$Q \mapsto \quad \sigma^Q : [0,\infty) \to \tilde{X}$$

from $\tilde{X}$ into a family of pairwise parallel rays; i.e. $\rho(\gamma)\sigma^Q = \sigma^{\rho(\gamma)Q}$ for all $Q \in \tilde{X}$, $\gamma \in \pi_1(M)$ and $d(\sigma_p(s), \sigma_q(s)) = \delta_{p,q}$ for all $s \in [0,\infty)$.

Proof. For $Q \in \tilde{X}$, we inductively construct a sequence $\{Q_i\}$ of points in $\tilde{X}$ by first setting $Q_0 = Q$ and then defining $Q_i = \bar{\sigma}^{Q_{i-1}}(\frac{3}{4})$. Next, let

$$L^Q = \bigcup_{i=0}^{\infty} I^{Q_i}$$

where $I^{Q_i} = \bar{\sigma}^{Q_i}([0,1])$. Therefore, L^Q is a union of pairwise parallel geodesic segments. Thus, $\{L^Q\}_{Q\in\tilde{X}}$ is a family of pairwise parallel geodesic rays. Moreover, the $\rho(\pi_1(M))$-equivariance of the map $Q \mapsto \bar{\sigma}^Q$ implies $\rho(\gamma)\bar{\sigma}^{Q_{i-1}}(\frac{3}{4}) = \bar{\sigma}^{\rho(\gamma)Q_{i-1}}(\frac{3}{4})$. Thus, if $\{Q_i\}$ is the sequence constructed starting with $Q_0 = Q$, then $\{\rho(\gamma)Q_i\}$ is the sequence constructed starting with $\rho(\gamma)Q_0 = \rho(\gamma)Q$. We thus conclude

$$\rho(\gamma)L^Q = \bigcup_{i=0}^{\infty} \rho(\gamma)I^{Q_i} = \bigcup_{i=0}^{\infty} I^{\rho(\gamma)Q_i} = L^{\rho(\gamma)Q}.$$

We are done by letting the geodesic ray $\sigma^Q : [0,\infty) \to \tilde{X}$ be the extension of the geodesic segment $\bar{\sigma}^Q : [0,1] \to \tilde{X}$ parameterizing L^Q. $\square$

Claim 3 implies that $\rho(\pi_1(M))$ fixes the point $[\sigma^Q]$ at infinity. This contradicts assumption (i) and completes the proof.

Remark 3.3 (Generalization to thick Euclidean buildings with transitive isometry groups). *In a forthcoming paper [BDM] by Breiner, Dees and the second author, we generalize Theorem 3.2 to the cases when $\tilde{X}$ is a Euclidean building, not necessarily locally finite. From this, we conjecture that we can generalize Theorem 1.1.*

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References

[BDM] C. Breiner, B. K. Dees, C. Mese. *Harmonic maps into Euclidean buildings and non-Archimedean superrigidity.* arxiv

[BH] M. R. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature. Springer-Verlag, Berlin 1999.

[BT] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Publ. Math. IHES **41** (1972), 5–251.

[Ca] P.-E. Caprace, *Lectures on proper CAT(0) spaces and their isometry groups*, Geometric Group Theory, IAS/Park City Math. Ser., vol. 21, Amer. Math. Soc., Providence, RI, 2014, pp. 91–125.

[CaMo] P.-E. Caprace and N. Monod, *Isometry groups of non-positively curved spaces: discrete subgroups*, Journal of Topology **2** (2009), 701–746.

[Co1] K. Corlette, *Archimedian superrigidity and hyperbolic geometry*, Ann. Math. **135** (1990), 165–182.

[Co2] K. Corlette, *Flat G-bundles with anonical metrics*, J. Differential Geom. **28** (1988), 361–382.

[DMV] G. Daskalopoulos, C. Mese and A. Vdovina, *Superrigidity of hyperbolic buildings*, GAFA **21** (2011), 1–15.

[Do] S. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. **55** (1987), 127–131.

[Eb] P. Eberlein, Geometry of Nonpositively Curved Manifolds, Chicago Lecture Series in Mathematics, Chicago, 1997.

[ES] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, American Journal of Mathematics **86** (1964), 109–160.

[GS] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Publ. Math. IHES **76** (1992), 165–246.

[Ha] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967), 673–687.

[He] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Graduate Studies in Mathematics Vol. 34, American Mathematical Society, Providence, R.I., 2001.

[J] J. Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Lectures in Mathematics. ETH Zürich, Birkhäuser Verlag 1997.

[JY] J. Jost and S. T. Yau, *Harmonic maps and superrigidity*, Differential Geometry: Partial Differential Equations on Manifolds, Proc. Sympos. Pure Math. **54**, part 1 (1993) 245–280.

[KL1] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Publications Mathématiques de IHES **86** (1997), 115–197.

[KL2] B. Kleiner and B. Leeb, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. **163** (2006), 657–676.

[KS1] N. Korevaar and R. Schoen, *Global existence theorems for harmonic maps to non-locally compact spaces*, Comm. Anal. Geom. **5** (1997), 213–266.

[KS2] N. Korevaar and R. Schoen, *Global existence theorem for harmonic maps to non-locally compact spaces*, Comm. Anal. Geom. **5** (1997), 333–387.

[KS3] N. Korevaar and R. Schoen, *Global existence theorem for harmonic maps: finite rank spaces and an approach to rigidity for smooth actions*, unpublished manuscript.

[L] F. Labourie, *Existence d'applications harmoniques tordues a valeurs dans lesvarietes a courbure negative*, Proc. Amer. Math. Soc. **111** (1991), 877–882.

[Me] C. Mese, *Uniqueness theorems for harmonic maps into metric spaces*, Communications in Contemporary Mathematics **4** (2002), 725–750.

[MSY] N. Mok, Y.-T. Siu, and S. K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1993), 57–83.

[Sa] J. H. Sampson, *Some properties and applications of harmonic mappings*, Annales scientifiques de l'École Normale Supérieure, Series **4** (1978), 211–228.

[Sc] R. Schoen. *Analytic aspects of the harmonic map problem*, Seminar on Non-linear Partial Differential Equations (S.S. Chern, ed.), MSRI Publ. 2, Springer-Verlag, New York 1983, 321—358.

[Si] Y.-T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kahler manifolds*, Ann. Math. **112** (1980), 73–112.

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