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Quillen metric for singular families of Riemann surfaces with cusps and compact perturbation theorem

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We study the behavior of the Quillen metric for families of Riemann surfaces with hyperbolic cusps when the additional cusps are created by degeneration.

More precisely, in our previous paper, we've shown that the renormalization of the Quillen metric associated with a family of Riemann surfaces with cusps extends continuously over the locus of singular curves. Here we show that modulo some explicit universal constant, this continuous extension coincides with the Quillen metric of the normalization of singular curves.

As a consequence, we get an explicit relation in terms of the Bott-Chern classes between the Quillen metric associated with a metric with cusps and the Quillen metric associated with a metric on the compactified Riemann surface. We also prove compatibility between our version of the analytic torsion and the version of Takhtajan-Zograf, defined through lengths of closed geodesics.

1. Introduction

In this paper, we study the behavior of the Quillen metric for families of Riemann surfaces with cusps when the additional cusps are created by degeneration.

More precisely, let $\overline{M}$ be a compact Riemann surface, and let $D_M = \{P_1^M, \ldots, P_m^M\}$ be a finite set of distinct points in $\overline{M}$. Let g^{TM} be a Kähler metric on the punctured Riemann surface $M = \overline{M} \setminus D_M$. We say that g^{TM} is *Poincaré-compatible* with holomorphic coordinates $z_1^M, \ldots, z_m^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon)$, $\epsilon \in]0,1[$, if there is $\epsilon > 0$ such that $g^{TM}|_{V_i^M(\epsilon)}$ is induced by the Kähler form

$$\frac{\sqrt{-1} dz_i^M d\overline{z}_i^M}{\left|z_i^M \log |z_i^M|\right|^2}, \tag{1.1}$$

where $V_i^M(\epsilon)$ are defined as follows

$$V_i^M(\epsilon) = \{x \in M : |z_i^M(x)| < \epsilon\}. \tag{1.2}$$

A triple $(\overline{M}, D_M, g^{TM})$ of a Riemann surface $\overline{M}$, a set of punctures D_M and a metric g^{TM}, which is Poincaré-compatible with some holomorphic coordinates of D_M, is called a *surface with cusps* (cf. [22]). A metric with cusps has scalar curvature equal to -1 away from a compact subset of M.

We fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, $X_t := \pi^{-1}(t)$ is a complex curve with at most double point singularities. We denote by $\Sigma_{X/S} \subset X$ the submanifold of singular points of the fibers (see Corollary 2.6), and by $\Delta = \pi_*(\Sigma_{X/S})$ the divisor formed by the locus of the singular fibers π.

In this article we only consider π for which the associated divisor Δ has normal crossings.

The construction of Grothendieck-Knudsen-Mumford [21] (cf. also [6, §3]) associates for every holomorphic vector bundle ξ over X the "determinant of the direct image of ξ". It is a holomorphic line bundle over S, which we denote (cf. (2.13))

$$\lambda(j^*\xi)^{-1} := \det(R^\bullet \pi_* \xi), \tag{1.3}$$

such that the restriction of $\lambda(j^*\xi)$ at $t \in S \setminus |\Delta|$, is canonically isomorphic to

$$\lambda(j^*\xi)|_t \simeq \left(\Lambda^{\max} H^0(X_t, \xi|_{X_t})\right)^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t}), \tag{1.4}$$

where the letter j designates the injection $j : X_t \to X$ of the fiber in the total space.

Let $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ be disjoint holomorphic sections of π. We introduce a divisor

$$D_{X/S} = \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m). \tag{1.5}$$

Let $\|\cdot\|^{\omega}_{X/S}$ be a $\mathscr{C}^\infty$ Hermitian norm on the relative canonical line bundle $\omega_{X/S} := \omega_X \otimes \pi^* \omega_S^{-1}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ such that its restriction over each nonsingular fiber $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ of π induces the Kähler metric g^{TX_t} on $X_t \setminus \{\sigma_1(t), \dots, \sigma_m(t)\}$ such that the triple $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ is a surface with cusps. In particular, the sections $\sigma_1, \dots, \sigma_m$ parameterize the positions of the hyperbolic cusps in the fibers.

For a complex manifold Y and a divisor $D_0 \subset Y$, we denote by $\|\cdot\|^{\mathrm{div}}_{D_0}$ the singular norm on $\mathscr{O}_Y(D_0)$ such that for the canonical section s_{D_0} of the divisor D_0 with $\mathrm{div}(s_{D_0}) = D_0$, we have

$$\|s_{D_0}\|^{\mathrm{div}}_{D_0}(x) = 1, \quad \text{for any } x \in Y \setminus D_0. \tag{1.6}$$

We endow the *twisted canonical line bundle*

$$\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S}) \tag{1.7}$$

with the canonical norm $\|\cdot\|_{X/S}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, induced by $\|\cdot\|^{\omega}_{X/S}$ and $\|\cdot\|^{\mathrm{div}}_{D_{X/S}}$. The norm $\|\cdot\|_{X/S}$ has logarithmic singularities in the neighborhood of $|D_{X/S}|$.

The metric g^{TX_t} on the regular fiber X_t, $t \in S \setminus |\Delta|$, is not compact, and the spectrum of the associated Kodaira Laplacian is not discrete. This entails technical difficulties for giving a definition of the analytic torsion as the "zeta-regularized" determinant of the Laplacian, as it was done by Ray-Singer in [26]. Nevertheless, in [13, Definition 2.16], for a Hermitian vector bundle (ξ, h^ξ) over X, we defined the analytic torsion $T(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S})$ of the fiber (X_t, g^{TX_t}) associated with a singular Hermitian vector bundle $(\xi \otimes \omega_{X/S}(D)^n, h^\xi \otimes \|\cdot\|^{2n}_{X/S})$ through the regularized trace of the heat kernel, obtained by subtracting a universal contribution from the cusp, see Section 2.1.

In [13], [14] (cf. Section 2.1), we've defined the *Quillen norm* $\|\cdot\|_Q\,(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S})$ on the determinant line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$, $n \leq 0$, over $S \setminus |\Delta|$ as the product of the square root of the analytic torsion of the fiber $T(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n})$ and the L^2-norm of the fiber, defined analogously to the compact case, see (2.3). This gives a non-compact 1-dimensional version of the Quillen metric of Bismut-Gillet-Soulé [6] and it generalizes the definition of Quillen [25], which was given for $n, m = 0$ and trivial π.

Let's denote by $\|\cdot\|_{X/S}^{W}$ the Wolpert norm on $\otimes_{i=1}^{m} \sigma_i^* (\omega_{X/S})$ induced by $\|\cdot\|_{X/S}^{\omega}$, see Wolpert [30, Definition 1] (see Definition 2.3). This norm tracks the variation of the Poincaré-compatible coordinates near the cusp, see (1.1).

The necessary definitions for the following passage are given in Definitions 2.8, 2.9, 2.11.

(1.8) We suppose that the Hermitian norm $\|\cdot\|_{X/S}$ on $\omega_{X/S}(D)$ extends smoothly over $X \setminus (\Sigma_{X/S} \cup |D_{X/S}|)$, has log-log growth with singularities along $\Sigma_{X/S} \cup |D_{X/S}|$, and is good in the sense of Mumford on X with singularities along $\pi^{-1}(\Delta) \cup D_{X/S}$.

We denote by $h^{\det \xi}$ the induced Hermitian metric on $\det \xi := \Lambda^{\max} \xi$. In [14, Theorem C3] (cf. Theorem 2.7), we proved that under the assumption (1.8), the norm

$$\begin{aligned} \|\cdot\|_{\mathscr{L}_n}^{X/S} := & \big(\|\cdot\|_Q \, (g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}) \big)^{12} \\ & \otimes \big(\|\cdot\|_{X/S}^{W} \big)^{-\mathrm{rk}(\xi)} \otimes \big(\|\cdot\|_{\Delta}^{\mathrm{div}} \big)^{\mathrm{rk}(\xi)} \otimes (\otimes_{i=1}^{m} \sigma_i^* h^{\det \xi})^3 \end{aligned} \tag{1.9}$$

on the line bundle

$$\begin{aligned} \mathscr{L}_n^{X/S} := & \lambda \big(j^* (\xi \otimes \omega_{X/S}(D)^n) \big)^{12} \otimes (\otimes_{i=1}^{m} \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)} \\ & \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes (\otimes_{i=1}^{m} \sigma_i^* \det \xi)^6 \end{aligned} \tag{1.10}$$

extends continuously over S. The main goal of this paper is to understand the value of this continuous extension, and to give a geometric interpretation of it as the Quillen norm of the normalization of a singular fiber.

More precisely, as Δ has normal crossings, by shrinking the base S, we may always assume that for any $t \in S$, there is $l \in \mathbb{N}$, so that the divisor Δ decomposes near t as

$$\Delta = k \cdot \Delta_0 + k_1 \cdot \Delta_1 + \cdots + k_l \cdot \Delta_l, \tag{1.11}$$

where Δ_i, $i = 0, \ldots, l$ are divisors induced by the submanifolds $|\Delta_i|$ and $k, k_j \in \mathbb{N}^*$, $j = 1, \ldots, l$. Let $\Delta_j^0 := \Delta_j \cap \Delta_0$ be the induced divisor on $S' :=$

$|\Delta_0|$, and let Δ' be the divisor on S' given by

$$\Delta' := k_1 \cdot \Delta_1^0 + \cdots + k_l \cdot \Delta_l^0. \tag{1.12}$$

Let $\iota : S' \to S$ be the obvious inclusion. We denote $Z := \pi^{-1}(S')$, $Z_t := \pi^{-1}(t)$, $t \in S'$, and let $\rho : Y \to Z$ be the normalization of Z. We denote by $\pi' : Y \to S'$ the family of surfaces, induced by the following commutative square.

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\iota} & S \end{array} \tag{1.13}$$

The restriction of the holomorphic sections $\sigma_1, \ldots, \sigma_m$ on S' induce the holomorphic sections of Y, which we denote by $\sigma'_1, \ldots, \sigma'_m : S' \to Y$. See Figure 1 for an example.

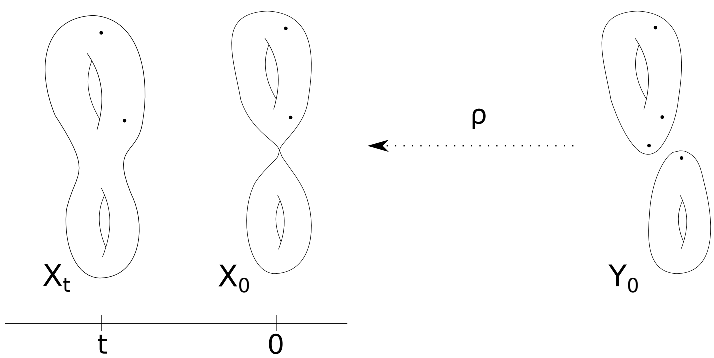

Figure 1: A degenerating family. From left to right, points on the surfaces represent the elements in $D_{X/S}|_{X_t}$, $D_{X/S}|_{X_0}$ and $D_{Y/S'}|_{Y_0}$.

Let $\Sigma_{Z/S'}$ be the locus of points, normalized in ρ. The manifold $\Sigma_{Z/S'}$ is a union of some connected components of $\Sigma_{X/S}$. In particular, $\Sigma_{X/S}$ has codimension 2 in X (see Corollary 2.6). Let

$$\kappa : \Sigma_{Z/S'} \hookrightarrow X \tag{1.14}$$

the obvious inclusion. Then the restriction of π' to $\rho^{-1}(\kappa(\Sigma_{Z/S'}))$ is a covering map of degree $2k$, see (1.11) and Figure 3. By shrinking the base, we may always assume that it is a trivial cover, so there are holomoprhic sections $\sigma'_{m+1}, \ldots, \sigma'_{m+2k} : S' \to Y$ such that $\rho^{-1}(\Sigma_{Z/S'}) = \cup_{i=1}^{2k} \operatorname{Im}(\sigma'_{m+i})$ and $\rho \circ \sigma'_{m+2i-1} = \rho \circ \sigma'_{m+2i}$, $i = 1, \ldots, k$. We define the divisor $D_{Y/S'}$ over Y by

$$D_{Y/S'} := \operatorname{Im}(\sigma'_1) + \cdots + \operatorname{Im}(\sigma'_{m+2k}). \tag{1.15}$$

We also define the *twisted canonical line bundle* of π' as follows

$$\omega_{Y/S'}(D) := \omega_{Y/S'} \otimes \mathscr{O}_Y(D_{Y/S'}). \tag{1.16}$$

We have the canonical isomorphism (cf. Section 2.1)

$$\rho^*(\omega_{X/S}(D)) \simeq \omega_{Y/S'}(D). \tag{1.17}$$

Under assumptions (1.8), more precisely, the smoothness assumption, the isomorphism (1.17) induces the Hermitian norm $\|\cdot\|_{Y/S'}$ on $\omega_{Y/S'}(D)$ over $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$ as follows

$$\|\cdot\|_{Y/S'} := \rho^*(\|\cdot\|_{X/S}). \tag{1.18}$$

Let $\|\cdot\|^{\omega}_{Y/S'}$ be the norm on $\omega_{Y/S'}$, induced by $\|\cdot\|_{Y/S'}$ using (1.6).

(1.19) We assume that the restriction of the norm $\|\cdot\|^{\omega}_{Y/S'}$ over each non-singular fiber $Y_t := \pi^{-1}(t)$, $t \in S' \setminus |\Delta'|$ of π' induces the Kähler metric g^{TY_t}, for which the triple $(Y_t, \{\sigma'_1(t), \ldots, \sigma'_{m+2k}(t)\}, g^{TY_t})$ is a surface with cusps in the sense of Section 2.1.

Essentially the assumption (1.19) says that the hyperbolic cusps on the normalization of the singular fibers are produced either by the extension of the existing cusps or by degeneration. To motivate, in a very important special case, when the marked family of curves $\pi : X \to S$, $\sigma_1, \ldots, \sigma_m : S \to X$ corresponds to a family of stable curves, and the norm $\|\cdot\|^{\omega}_{X/S}$ induces the complete metric of constant scalar curvature -1 over the fibers, both assumptions (1.8) and (1.19) are satisfied by the results of Wolpert, [29] (in the compact case, $m = 0$) and Freixas, [15] (in the non-compact case, $m > 0$), cf. [14, Proposition 5.7].

We denote by $\|\cdot\|^{W}_{Y/S'}$ the Wolpert norm on $\otimes_{i=1}^{m+2k} (\sigma'_i)^* \omega_{Y/S'}$, induced by $\|\cdot\|^{\omega}_{Y/S'}$ (it is well-defined by the assumption (1.19)). Now, by (1.19), similarly to (1.9), (1.10), we define the norm $\|\cdot\|_{\mathscr{L}_n}^{Y/S'}$ on the line bundle

$$\begin{aligned}(1.20)\quad \mathscr{L}_n^{Y/S'} := \lambda\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big)^{12}\otimes(\otimes_{i=1}^{m+2k}(\sigma_i')^*\omega_{Y/S'})^{-\mathrm{rk}(\xi)}\\ \otimes\mathscr{O}_{S'}(\Delta')^{\mathrm{rk}(\xi)}\otimes\big(\otimes_{i=1}^{m+2k}(\sigma_i'\circ\rho)^*\det\xi\big)^6.\end{aligned}$$

The main result of this paper relates the restriction of the Hermitian norm $\|\cdot\|_{\mathscr{L}_n}^{X/S}$ to S' with the Hermitian norm $\|\cdot\|_{\mathscr{L}_n}^{Y/S'}$. To state it, we need to first relate the restriction of the line bundle $\mathscr{L}_n^{X/S}$ to the line bundle S' with $\mathscr{L}_n^{Y/S'}$. By using the residue morphism and the additivity of the determinant, in Section 3.1, we construct the canonical isomorphism

$$(1.21)\qquad \mathscr{L}_n^{X/S}|_{S'}\to\mathscr{L}_n^{Y/S'}.$$

Now, for $k\in\mathbb{N}^*$, we define

$$(1.22)\qquad \begin{aligned}C_0 &= -6\log(\pi),\\ C_k &= -6(1+k)\log(2)-6(1+2k)\log(\pi)-6\log((2k)!).\end{aligned}$$

Theorem 1.1 (Restriction theorem). *Under the assumptions (1.8) and (1.19), the norm $\|\cdot\|_{\mathscr{L}_n}^{X/S}$ extends continuously over S, and under (1.21), we have*

$$(1.23)\qquad \|\cdot\|_{\mathscr{L}_n}^{X/S}|_{S'}=\exp(k\cdot\mathrm{rk}(\xi)\cdot C_{-n})\cdot\|\cdot\|_{\mathscr{L}_n}^{Y/S'}.$$

Remark 1.2. A related theorem was proven by Bismut in [3, Theorems 0.2, 0.3] (cf. Theorem 3.2), but the geometric situation here is different from [3]. In particular, in our case even when the general fiber has no cusps, the metric on the normalization of the singular fiber acquires at least two cusps, which is not the case in [3]. Due to the fact that at this stage there is no precise relation between the Quillen metrics for surfaces with cusps and without cusps, we cannot obtain Theorem 1.1 directly from [3, Theorems 0.2, 0.3] and the anomaly formula of Bismut-Gillet-Soulé [6].

Now, let's describe our second result. We fix a compact Riemann surface $\overline{M}$ and a set of points $D_M\subset\overline{M}$, $\#D_M=m$, $m<+\infty$. We denote $M:=\overline{M}\setminus D_M$. Suppose that a pointed Riemann surface $(\overline{M},D_M)$ is stable, i.e.

the genus $g(\overline{M})$ of $\overline{M}$ satisfies

$$2g(\overline{M}) - 2 + m > 0, \tag{1.24}$$

then, by the uniformization theorem (cf. [12, Chapter IV], [2, Lemma 6.2]), there is exactly one complete metric g^{TM}_{hyp} of constant scalar curvature -1 on M with cusps at D_M. We call this metric the *canonical hyperbolic metric*. We denote by $\|\cdot\|^{\mathrm{hyp}}_M$ the norm induced by g^{TM}_{hyp} on $\omega_M(D)$ over M. Then, as we explain in [13, §2.1], the triple $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ is a surface with cusps (see Section 2.1), in particular, the analytic torsion $T(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n})$ is well-defined by the results from [13].

Alternatively, we denote by $Z_{(\overline{M},D_M)}(s)$, $s \in \mathbb{C}$, the Selberg zeta-function, which is given for $\mathrm{Re}(s) > 1$ by the following absolutely converging product

$$Z_{(\overline{M},D_M)}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})^2, \tag{1.25}$$

where γ runs over the set of all primitive non-oriented closed geodesics on $(M, g^{TM}_{\mathrm{hyp}})$, and $l(\gamma)$ is the length of γ. The function $Z_{(\overline{M},D_M)}(s)$ admits a meromorphic extension to the whole complex s-plane with a simple zero at $s = 1$ (see for example [10, (5.3)]).

Let $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$ be the Riemann zeta function. For $k \in \mathbb{N}^*$, we put

$$\begin{aligned} c_0 &= 4\zeta'(-1) - \frac{1}{2} + \log(2\pi), \\ c_k &= \sum_{l=0}^{k-1} (2k - 2l - 1)\big(\log(2k + 2kl - l^2 - l) - \log(2)\big) \\ &\quad + (\frac{1}{3} + k + k^2)\log(2) + (2k+1)\log(2\pi) + 4\zeta'(-1) - 2(k + \tfrac{1}{2})^2 \\ &\quad - 4\sum_{l=1}^{k-1} \log(l!) - 2\log(k!). \end{aligned} \tag{1.26}$$

For $k \in \mathbb{N}$, we denote by $B_k : \mathbb{N}^2 \to \mathbb{R}$, $E : \mathbb{N}^2 \to \mathbb{R}$ the following functions

$$\begin{aligned} B_k(g,m) &= \exp\Big((2 - 2g(\overline{M}) - m)\frac{c_k}{2}\Big), \\ E(g,m) &= \exp\Big((g(\overline{M}) + 2 - m)\frac{\log(2)}{3}\Big). \end{aligned} \tag{1.27}$$

For surfaces of constant scalar curvature -1 and (ξ, h^{ξ}) trivial, and $l \in \mathbb{Z}$, $l < 0$, Takhtajan-Zograf in [28, (6)] proposed[1] the analogue of the analytic torsion defined via Selbreg zeta function:

$$T_{TZ}(g^{TM}_{\text{hyp}}, 1) = E(g(\overline{M}), m) \cdot B_0(g(\overline{M}), m) \cdot Z'_{(\overline{M}, D_M)}(1),$$
$$T_{TZ}(g^{TM}_{\text{hyp}}, (\|\cdot\|^{\text{hyp}}_M)^{2l}) = B_{-l}(g(\overline{M}), m) \cdot Z_{(\overline{M}, D_M)}(-l+1). \tag{1.28}$$

Theorem 1.3 (Compatibility theorem). *For any surface with cusps $(\overline{M}, D_M, g^{TM}_{\text{hyp}})$, for which g^{TM}_{hyp} has constant scalar curvature -1, the following identity holds*

$$T(g^{TM}_{\text{hyp}}, (\|\cdot\|^{\text{hyp}}_M)^{2n}) = T_{TZ}(g^{TM}_{\text{hyp}}, (\|\cdot\|^{\text{hyp}}_M)^{2n}). \tag{1.29}$$

Remark 1.4. When the surface has no cusps, Theorem 1.3 was proven by D'Hoker-Phong [10, (7.30)], [11, (3.6)] (see also [27], [7, (50)] and [24, (9)]). Our proof is based on their result.

Now let's describe some applications of Theorems 1.1, 1.3 to the moduli space $\mathscr{M}_{g,m}$ of m-pointed Riemann surfaces of genus $g \in \mathbb{N}$, $2g - 2 + m > 0$. We denote by $\overline{\mathscr{M}}_{g,m}$ the *Deligne-Mumford compactification* of $\mathscr{M}_{g,m}$, by $\partial \mathscr{M}_{g,m} := \overline{\mathscr{M}}_{g,m} \setminus \mathscr{M}_{g,m}$ the *compactifying divisor*, by $\mathscr{C}_{g,m}$ and $\overline{\mathscr{C}}_{g,m}$ the universal curves over $\mathscr{M}_{g,m}$ and $\overline{\mathscr{M}}_{g,m}$ respectively. We denote by $\Pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$ the *universal projection*, and by $D_{g,m}$ the divisor on $\overline{\mathscr{C}}_{g,m}$, formed by m fixed points. We denote by $\omega_{g,m}$ the relative canonical line bundle of Π, by $\otimes_{i=1}^m \sigma_i^* \omega_{g,m}$ the determinant of the restriction of $\omega_{g,m}$ to the divisor $D_{g,m}$, and by $\omega_{g,m}(D)$ the twisted relative canonical line bundle

$$\omega_{g,m}(D) := \omega_{g,m} \otimes \mathscr{O}_{\overline{\mathscr{C}}_{g,m}}(D_{g,m}). \tag{1.30}$$

We endow $\omega_{g,m}(D)$ with the Hermitian norm $\|\cdot\|^{\text{hyp}}_{g,m}$ induced by the canonical hyperbolic metric of constant scalar curvature -1 on the fibers. We endow the determinant line bundle $\lambda(j^*(\omega_{g,m}(D)^n))$, $n \leq 0$, with the induced Quillen norm $\|\cdot\|^{Q,n}_{g,m}$ from [13] and $\otimes_{i=1}^m \sigma_i^* \omega_{g,m}$ with the Wolpert norm $\|\cdot\|^{W}_{g,m}$, cf. Definition 2.3.

We denote by $\lambda^n_{g,m}$, $\|\cdot\|^n_{g,m}$ the specifications of $\mathscr{L}^{X/S}_n$, $\|\cdot\|^{X/S}_{\mathscr{L}_n}$ to $X := \overline{\mathscr{C}}_{g,m}$, $S := \overline{\mathscr{M}}_{g,m}$ and ξ trivial. In [14, Corollary 1.11] (cf. Theorem 2.7), we proved that $\|\cdot\|^n_{g,m}$ extends continuously over $\overline{\mathscr{M}}_{g,m}$.

[1] The constant in front of Selberg zeta function didn't appear in [28], as the result of the authors of [28] is independent of it. This normalization was introduced by Freixas [15], [16], [17].

For the definition of the clutching morphisms

$$\begin{aligned} \alpha &: \overline{\mathscr{M}}_{g-1,m+2} \to \overline{\mathscr{M}}_{g,m}, \\ \beta &: \overline{\mathscr{M}}_{g_1,m_1+1} \times \overline{\mathscr{M}}_{g_2,m_2+1} \to \overline{\mathscr{M}}_{g,m}, \end{aligned} \tag{1.31}$$

see Knudsen [20].

For line bundles L_X, L_Y over manifolds X, Y and natural projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, we denote $L_X \boxtimes L_Y := \pi_X^* L_X \otimes \pi_Y^* L_Y$. The isomorphism (1.21) specifies to

$$\alpha^* \lambda_{g,m}^n \simeq \lambda_{g-1,m+2}^n, \tag{1.32}$$

$$\beta^* \lambda_{g,m}^n \simeq \lambda_{g_1,m_1+1}^n \boxtimes \lambda_{g_2,m_2+1}^n. \tag{1.33}$$

Corollary 1.5. a) The isomorphism (1.32) is an isometry if the left-hand side is endowed with $\|\cdot\|_{g,m}^n$, and the right-hand side with $\exp(m \cdot C_{-n}) \cdot \|\cdot\|_{g-1,m+2}^n$.

b) Similarly, the isomorphism (1.33) is an isometry if the left-hand side is endowed with $\|\cdot\|_{g,m}^n$, and the right-hand side is endowed with $\exp(m \cdot C_{-n}) \cdot (\|\cdot\|_{g_1,m_1+1}^n \boxtimes \|\cdot\|_{g_2,m_2+1}^n)$.

Remark 1.6. For a special family of curves from Section 2.3, Freixas proved a version of Corollary 1.5 for $n = 0$ in [16, Corollary 5.8] and for $n > 0$ in [16, Theorem 5.3], cf. Theorem 2.13. There, Freixas used a version of the Quillen norm, defined as a product of (1.28) and the L^2-norm. By Theorem 1.3, our results are compatible. We use the result of Freixas to calculate C_{-n} from (1.22).

Finally, let's state our last result which describes an explicit relation between the Quillen metric associated with a metric with cusps and the Quillen metric associated with a metric on the compactified Riemann surface. It should be regarded as a refinement of [13, Theorem A].

To state it precisely, recall that the Bott-Chern classes $\widetilde{\mathrm{Td}}(\xi, h_1^\xi, h_2^\xi)$, $\widetilde{\mathrm{ch}}(\xi, h_1^\xi, h_2^\xi)$ of a vector bundle ξ with Hermitian metrics h_1^ξ, h_2^ξ were defined in [5, Theorem 1.27], cf. (3.9), (3.10), (3.11).

In (3.43), for a singular $(1,1)$-differential form α with some specified growth near the cusps (see (3.42)), we define a notion of the regularized integral $\int_M^{\mathbf{r}} \alpha \in \mathbb{R}$ by taking out the divergent part of the integral on a truncated surface.

Recall that C_k, $k \in \mathbb{N}$ were defined in (1.22). Now, we define

$$E_k = 4\zeta'(-1) - \log(2\pi) + \frac{1 - C_k}{6}. \tag{1.34}$$

Theorem 1.7 (Compact perturbation theorem). *Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps. We denote by $\|\cdot\|_M$ the induced norm on $\omega_M(D)$ over M as in (1.6). We denote by $\|\cdot\|^W$ the Wolpert norm on $\otimes_{P\in D_M}\omega_{\overline{M}}|_P$ induced by g^{TM}.*

Let $g^{T\overline{M}}$ be a Kähler metric over $\overline{M}$, and let $\|\cdot\|_{\overline{M}}$ be some Hermitian norm on $\omega_M(D)$ over $\overline{M}$. We denote by $\|\cdot\|_{\overline{M}}^{D_M}$ the norm on $\otimes_{P\in D_M}\omega_{\overline{M}}|_P$ induced by $g^{T\overline{M}}$. Let ξ be a holomorphic vector bundle over $\overline{M}$, and let h^ξ and h_0^ξ be two metrics on ξ over $\overline{M}$. Then the following identity holds

$$
\begin{aligned}
&2\log\Big(\|\cdot\|_Q\left(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}\right)/\|\cdot\|_Q\left(g^{T\overline{M}}, h_0^\xi\otimes\|\cdot\|_{\overline{M}}^{2n}\right)\Big)\\
&\quad=\int_M^r\Big[\widetilde{\mathrm{Td}}\big(\omega_{\overline{M}}^{-1}, g^{T\overline{M}}, g^{TM}\big)\mathrm{ch}\big(\xi, h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big)\\
&\qquad\qquad+\mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_0^\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big)\\
&\qquad\qquad+\mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\mathrm{ch}\big(\xi, h^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}, \|\cdot\|_M^{2n}\big)\Big]^{[2]}\\
&\qquad+\frac{\mathrm{rk}(\xi)}{6}\log\left(\|\cdot\|^W/\|\cdot\|_{\overline{M}}^{D_M}\right)-\frac{1}{2}\sum_{P\in D_M}\log\Big(\det(h_0^\xi/h^\xi)|_P\Big)\\
&\qquad+\#(D_M)\cdot\mathrm{rk}(\xi)\cdot E_{-n}.
\end{aligned}
\tag{1.35}
$$

Remark 1.8. Other authors use terminology "compactly supported perturbations" to designate the conformal change with a factor of compact support in M. Such perturbations, unlike ours, do not change the metric at the cusp. The name "compact perturbation theorem" stems from the idea that g^{TM}, viewed as a perturbation of $g^{T\overline{M}}$, "perturbs the compactness" of the metric.

This theorem has some applications to Arakelov geometry. In fact, once it is generalized to metrics with conical singularities, it can be used to calculate some special values of the Selberg zeta function of some modular curves without a direct appeal to the Arithmetic Riemann-Roch theorem as in Freixas [15, Corollary 8.2.2], Freixas-von Pippich [18, Theorem 10.2].

This paper is organized as follows. In Section 2, we recall the definition of the Quillen norm on the family of Riemann surfaces with cusps, the definition of Wolpert norm, and some results from [13], [14], which study those norms. Then we recall an analogue of Theorem 1.1 in the constant scalar curvature case due to Freixas. In Section 3 we extend a result of Bismut [3, Theorem 0.3] to non-Kähler metrics and give a proof of Theorems 1.1, 1.3, 1.7 and Corollary 1.5.

Notation. For a complex manifold X, we denote by ω_X the canonical line bundle $\det(T^{*(1,0)}X)$ of X. For a divisor D in X, we denote by s_D the canonical meromorphic section of $\mathscr{O}_X(D)$. For $\epsilon > 0$, we define $D(\epsilon) = \{u \in \mathbb{C} : |u| < \epsilon\}$, $D^*(\epsilon) = \{u \in \mathbb{C} : 0 < |u| < \epsilon\}$.

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2. Families of nodal curves and hyperbolic cusps

In this section we recall the relevant notations and some preliminary results. More precisely, in Section 2.1, we recall the definition of the Quillen norm from [25], [6], [13] and some notions related to families of Riemann surfaces with cusps from [4], [13], [14], which appeared in the formulation of conditions (1.8) and (1.19). In Section 2.2, we recall several notions of singularities of Hermitian metrics on line bundles and a regularity result from [14] for a push-forward of a differential form in a family of curves with double-point singularities. In Section 2.3, we recall the results of Freixas, [16], [17], related to the degeneration of the Takhtajan-Zograf version of the Quillen metric. In Section 2.4, we recall the results related to the study of the hyperbolic metric near the singular fibers due to Wolpert [29].

2.1. Determinant line bundles and Quillen norms

In this section we recall some basic facts about families of curves due to Bismut-Bost [4] and the Quillen metrics associated with them basing on [13].

We fix a surface with cusps $(\overline{M}, D_M, g^{TM})$ and a Hermitian vector bundle (ξ, h^ξ) over it. We denote by $\omega_{\overline{M}} := T^{*(1,0)}\overline{M}$ the *canonical line bundle* over $\overline{M}$. We denote by $\|\cdot\|_M^\omega$ the norm on $\omega_{\overline{M}}$ induced by g^{TM} over M by the natural identification $TM \ni X \mapsto \frac{1}{2}(X - \sqrt{-1}JX) \in T^{(1,0)}M$, where J is the complex structure of M. Let $\mathscr{O}_{\overline{M}}(D_M)$ be the line bundle associated with the divisor D_M. The *twisted canonical line bundle* is defined as

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{2.1}$$

Using (1.6), the metric g^{TM} endows the line bundle $\omega_M(D)$ with the induced norm $\|\cdot\|_M$ over M.

We recall now briefly the definition of the *analytic torsion* $T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ for $m \in \mathbb{N}$, $n \leq 0$. Assume first $m = 0$, i.e. the surface has no cusps. Then $T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ was defined by Ray-Singer [26, Definition 1.2] as the zeta-regularized determinant of the Kodaira Laplacian $\square^{\xi \otimes \omega_M(D)^n}$ associated with (M, g^{TM}) and $(\xi \otimes \omega_M(D)^n, h^\xi \otimes \|\cdot\|_M^{2n})$.

Now, let $m > 0$. Then M is non-compact, and the spectrum of $\square^{\xi \otimes \omega_M(D)^n}$ is not discrete. Also, the heat operator associated with $\square^{\xi \otimes \omega_M(D)^n}$ is no longer of trace class. Thus, the classical definition of Ray-Singer is not applicable. In this case, in [13, Definition 2.10], for $n \leq 0$, we defined the *regularized heat trace* $\mathrm{Tr}^{\mathbf{r}}[\exp^{\perp}(-t\square^{\xi \otimes \omega_M(D)^n})]$ as a difference of the heat trace of $\square^{\xi \otimes \omega_M(D)^n}$ and the heat trace of the Kodaira Laplacian $\square^{\omega_P(D)^n}$ corresponding to the 3-punctured projective plane $P := \overline{P} \setminus \{0, 1, \infty\}$, $\overline{P} := \mathbb{C}P^1$, endowed with the complete metric g^{TP} of constant scalar curvature -1 and the induced norm $\|\cdot\|_P$ on $\omega_P(D) := \omega_{\overline{P}} \otimes \mathscr{O}_{\overline{P}}(0 + 1 + \infty)$. Then in [13, Definition 2.16], we defined the *regularized spectral zeta function* $\zeta_M(s)$ for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, as the Mellin transform of $\mathrm{Tr}^{\mathbf{r}}[\exp^{\perp}(-t\square^{\xi \otimes \omega_M(D)^n})]$, and we concluded in [13, p. 17] that similarly to the case $m = 0$, the function $\zeta_M(s)$ extends meromorphically to $\mathbb{C}$ and 0 is a holomorphic point. In [13, Definition 2.17], we defined the regularized analytic torsion as

$$T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)/2) \cdot T_{TZ}(g^{TP}, \|\cdot\|_P^{2n})^{m \cdot \mathrm{rk}(\xi)/3}, \tag{2.2}$$

where $T_{TZ}(\cdot, \cdot)$ was defined in (1.28). In other words, our version of the analytic torsion is defined by subtracting the universal contribution of the cusp from the heat trace and by normalizing it in such a way that it coincides with the version of the analytic torsion of Takhtajan-Zograf for $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$, endowed with the complete metric of constant scalar curvature -1.

Then for $n \leq 0$, in [13, §2.1], we explained how the usual definition of the L^2-scalar product extends to the singular setting. More precisely, for $\alpha, \alpha' \in \mathscr{C}^\infty(\overline{M}, \xi \otimes \omega_M(D)^n)$ or $\alpha, \alpha' \in \mathscr{C}^\infty(\overline{M}, \overline{\omega}_{\overline{M}} \otimes \xi \otimes \omega_M(D)^n)$, in [13, §2.1], we've shown that for $n \leq 0$, despite the singularities of the metric, the following L^2-scalar product is well-defined

$$\langle \alpha, \alpha' \rangle_{L^2} := \frac{1}{2\pi} \int_M \langle \alpha(x), \alpha'(x) \rangle_h dv_M(x), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle_h$ is the pointwise scalar product on $\xi \otimes \omega_M(D)^n$ and dv_M is the Riemannian volume form on (M, g^{TM}). This endows the complex line

$$\begin{aligned}(2.4)\quad &\big(\det H^\bullet(\overline{M}, \xi\otimes\omega_M(D)^n)\big)^{-1}\\ &:= \big(\Lambda^{\max} H^0(\overline{M}, \xi\otimes\omega_M(D)^n)\big)^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, \xi\otimes\omega_M(D)^n),\end{aligned}$$

with the induced L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$.

The Quillen norm on the complex line (2.4) is then defined by

$$\begin{aligned}(2.5)\quad &\|\cdot\|_Q(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n})\\ &= T(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n})^{1/2}\cdot\|\cdot\|_{L^2}(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}).\end{aligned}$$

To motivate, when $m = 0$, this coincides (up to a normalization by 2π in (2.3)) with the usual definition of the Quillen norm from Quillen [25] and Bismut-Gillet-Soulé [5, (1.64)].

Following [13], we say that a (smooth) metric g_{f}^{TM} over $\overline{M}$ is a *flattening* of g^{TM} if there is $\nu > 0$ such that g^{TM} is induced by (1.1) over $V_i^M(\nu)$, and

$$(2.6)\qquad g_{\mathrm{f}}^{TM}|_{M\setminus(\cup_i V_i^M(\nu))} = g^{TM}|_{M\setminus(\cup_i V_i^M(\nu))}.$$

Similarly, we define a flattening $\|\cdot\|_M^{\mathrm{f}}$ of the norm $\|\cdot\|_M$.

One of the main results of [13] compares the Quillen metric associated with a metric with cusps and a flattening. Another result gives a formula for the variation of the Quillen metric induced by the variation of the metrics on M and ξ. We recall those results for the convenience of the reader.

Theorem 2.1 ([13, Theorem A]). *Let g_{f}^{TM}, $\|\cdot\|_M^{\mathrm{f}}$ be flattenings of g^{TM}, $\|\cdot\|_M$. Then*

$$\begin{aligned}(2.7)\quad &2\mathrm{rk}(\xi)^{-1}\log\Big(\|\cdot\|_Q\big(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f}}^{TM}, h^\xi\otimes(\|\cdot\|_M^{\mathrm{f}})^{2n}\big)\Big)\\ &- \mathrm{rk}(\xi)^{-1}\int_M c_1(\xi, h^\xi)\Big(2n\log(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M) + \log(g_{\mathrm{f}}^{TM}/g^{TM})\Big)\end{aligned}$$

depends only on the integers $n \in \mathbb{Z}$, $n \leq 0$, and the functions $(g_{\mathrm{f}}^{TM}/g^{TM})|_{V_i^M(1)} \circ (z_i^M)^{-1} : \mathbb{D}^* \to \mathbb{R}$, $(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)|_{V_i^M(1)} \circ (z_i^M)^{-1} : \mathbb{D}^* \to \mathbb{R}$, for $i = 1, \ldots, m$.

Remark 2.2. Directly from the proof of Theorem 2.1, it is easy to see that Theorem 2.1 continues to hold if instead of the flattening g_{f}^{TM} of g^{TM}, we consider a *partial flattening* $g_{\mathrm{f},0}^{TM}$ of g^{TM}, i.e. instead of the condition (2.6), we demand

$$g_{\mathrm{f},0}^{TM}|_{M\setminus(\cup_{i\in I}V_i^M(\nu))} = g^{TM}|_{M\setminus(\cup_{i\in I}V_i^M(\nu))}, \tag{2.8}$$

where I is a subset of the index set parametrizing the cusps.

Definition 2.3 ([13, Definition 1.5]). For a surface with cusps $(\overline{M}, D_M, g^{TM})$, we define the norms $\|\cdot\|_M^{W,i}$ on the complex lines $\omega_{\overline{M}}|_{P_i^M}$, $i = 1, \dots, m$, by requiring $\|dz_i^M\|_M^{W,i} = 1$. They induce the *Wolpert norm* $\|\cdot\|_M^W$ on the complex line $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i^M}$.

The Wolpert norm has been introduced by Wolpert in [30] for metrics of constant scalar curvature -1, and the name "Wolpert norm" was coined up by Freixas in [15], [16]. In [13], we extended the definition of Wolpert to the case of non-constant scalar curvature.

Theorem 2.4 ([13, Theorem B]). *Suppose that for the metric g_0^{TM}, the triple $(\overline{M}, D_M, g_0^{TM})$ is a surface with cusps. We denote by $\|\cdot\|_M, \|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$, and by $\|\cdot\|_M^W$, $\|\cdot\|_M^{W,0}$ the associated Wolpert norms. Let h_0^ξ be a Hermitian metric on ξ over $\overline{M}$. Then the right-hand side of the following equation is finite, and*

$$\begin{aligned}
&2\log\Big(\|\cdot\|_Q\big(g_0^{TM}, h_0^\xi \otimes (\|\cdot\|_M^0)^{2n}\big)/\|\cdot\|_Q\big(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\big)\Big) \\
&= \int_M \Big[\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big) \\
&\qquad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h^\xi, h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big) \\
&\qquad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h_0^\xi\big)\cdot \\
&\qquad\qquad \cdot\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}, (\|\cdot\|_M^0)^{2n}\big)\Big]^{[2]} \\
&\quad - \frac{\mathrm{rk}(\xi)}{6}\log\Big(\|\cdot\|_M^W/\|\cdot\|_M^{W,0}\Big) + \frac{1}{2}\sum_{P\in D_M}\log\Big(\det(h^\xi/h_0^\xi)|_P\Big).
\end{aligned} \tag{2.9}$$

Now let's pass to the study of curves in *families*. By a *curve* we mean, cf. [1, p. 79], an analytic space such that every one of its points is either smooth or is locally complex-analytically isomorphic to a neighborhood of the origin in $\{(z_0, z_1) \in \mathbb{C}^2 : z_0 z_1 = 0\}$.

We fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a curve (in the terminology of [4], [14], π is a f.s.o).

Proposition 2.5 ([4, Proposition 3.1]). For every $x \in X$, there are local holomorphic coordinates $(z_0, \dots, z_q)$ of $x \in X$ and $(w_1, \dots, w_q)$ of $\pi(x) \in S$, such that π is locally defined by

$$w_i = z_i, \qquad \text{for} \quad i = 1, \dots, q, \tag{2.10}$$

$$\text{or} \qquad w_1 = z_0 z_1; \quad w_i = z_i, \qquad \text{for} \quad i = 2, \dots, q. \tag{2.11}$$

Corollary 2.6 ([4, §3(a)]). Let $\Sigma_{X/S} \subset X$ be the locus of double points of the fibers of π. Then $\Sigma_{X/S}$ is a submanifold of X of codimension 2; $\pi|_{\Sigma_{X/S}} : \Sigma_{X/S} \to S$ is a closed immersion and $\pi|_{X \setminus \Sigma_{X/S}} : X \setminus \Sigma_{X/S} \to S$ is a submersion. In particular, $\Delta = \pi_*(\Sigma_{X/S})$ is a divisor in S.

Let $s_0 := \pi(x) \in \Delta$, $x \in \Sigma_{X/S}$, and let $\rho : Y_{s_0} \to X_{s_0}$ be the normalization of X_{s_0} at x. Then for the relative canonical line bundle, $\omega_{X/S} := \omega_X \otimes \pi^* \omega_S^{-1}$, there is a canonical isomorphism

$$\rho^* \omega_{X/S} = \omega_{Y_{s_0}} \otimes \mathscr{O}_{Y_{s_0}}(\rho^{-1}(x)), \tag{2.12}$$

which induces the isomorphism (1.17). Let's fix $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ and $D_{X/S} \subset X$ as in (1.5). Fix a Hermitian norm $\|\cdot\|_{X/S}^{\omega}$ on $\omega_{X/S}$ over $\pi^{-1}(S \setminus |\Delta|) \setminus (\cup_i \operatorname{Im}(\sigma_i))$, satisfying the assumptions described below (1.5).

Now, let (ξ, h^ξ) be a Hermitian vector bundle over X. For $t \in S$, we denote

$$\begin{aligned} \det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))_t := \det H^0(X_t, \xi \otimes \omega_{X/S}(D)^n) \\ \otimes (\det H^1(X_t, \xi \otimes \omega_{X/S}(D)^n))^{-1}. \end{aligned} \tag{2.13}$$

Even though individually $H^0(X_t, \xi \otimes \omega_{X/S}(D)^n)$ and $H^1(X_t, \xi \otimes \omega_{X/S}(D)^n)$ do not necessarily form vector bundles over S, by a result of Grothendick-Knudsen-Mumford [21] (cf. [4, Proposition 4.1]), the family of complex lines $(\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))_t)_{t \in S}$ can be endowed with a natural structure of a holomorphic line bundle, denoted here by $\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))$. We denote

$$\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n)) := \big(\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n)) \big)^{-1}. \tag{2.14}$$

Following [14], by gluing the pointwise Quillen norms, cf. (2.5), we induce the Quillen norm $\|\cdot\|_Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|_{X/S}^{2n} \right)$ on the line bundle $\lambda(j^*(\xi \otimes$

$\omega_{X/S}(D)^n)$). Similarly, the pointwise Wolpert norms, cf. Definition 2.3, glue into the Wolpert norm $\|\cdot\|^{W}_{X/S}$ on $\otimes_{i=1}^{m}\sigma_i^*\omega_{X/S}$. We make no claim about the regularity of the metrics $\|\cdot\|_{Q}\left(g^{TX_t}, h^{\xi}\otimes\|\cdot\|^{2n}_{X/S}\right)$ and $\|\cdot\|^{W}_{X/S}$. This is due to the fact that it is not clear if there is any relation between the coordinates as in (1.1) for nearby fibers.

One of the motivations for introducing Quillen metric is a theorem of Bismut-Gillet-Soulé [6, Theorem 1.6], which says that despite the singularities in the L^2-norm caused by the jumps in the dimension of the cohomology of the fibers, which might occur when one moves over the base space of the family of compact smooth manifolds, the Quillen metric is a smooth metric. Similarly, one can motivate the definition of the Hermitian norm (1.9). More precisely, we have

Theorem 2.7 ([14, Theorem C3]). *Suppose that assumption (1.8) holds. Then the Hermitian norm (1.9) on the line bundle (1.10) extends continuously over S.*

2.2. Singular Hermitian vector bundles

Here we recall several notions of singularities for Hermitian vector bundles, which we already used in conditions (1.8) and (1.19). We fix a complex manifold Y of dimension $q+1$, a *normal crossing divisor* $D_0 \subset Y$ and a submanifold $F \subset Y$.

A triple $(U; z_0, \ldots, z_q; l)$ of an open set $U \subset Y$, coordinates $z_0, \ldots, z_q : U \to \mathbb{C}$ and $l \in \mathbb{N}$ is called an *adapted chart* for D_0 (resp. F) at $x \in D_0$ (resp. $x \in F$) if $U = \{(z_0, \ldots, z_q) \in \mathbb{C}^{q+1} : |z_i| < 1, \text{ for all } i = 0, \ldots, q\}$ and $D_0 \cap U$ (resp. $F \cap U$) is defined by $\{z_0 \cdots z_l = 0\}$ (resp. $\{z_0 = 0, \ldots, z_l = 0\}$). Let $(U; z_0, \ldots, z_q; l)$ be an *adapted chart* for D_0. We denote

$$d\zeta_k = \begin{cases} dz_k/(z_k \log|z_k|), & \text{if } \quad 0 \le k \le l, \\ dz_k, & \text{if } \quad l+1 \le k \le q. \end{cases} \tag{2.15}$$

Definition 2.8. a) [15, Definition 2.1] A function $f : Y \setminus F \to \mathbb{C}$ has *log-log growth on Y, with singularities along F* if for any $x \in Y$, for some adapted chart $(U; z_0, \ldots, z_q; l)$ of F at x, and for some $C > 0$, $p \in \mathbb{N}$, we have

$$|f(z_0, \ldots, z_q)| \le C\Big(\log\big|\log\big(\max_{k=0}^{l}\{|z_k|\}\big)\big|\Big)^p + C. \tag{2.16}$$

b) [23, p. 240] A differential form over $Y \setminus D_0$ has *Poincaré growth on Y, with singularities along D_0*, if it can be expressed as a linear combination of monomials constructed using $d\zeta_k, \overline{d\zeta_k}$, $k = 0, \ldots, q$, with coefficients $f \in \mathscr{C}^\infty(Y \setminus D_0) \cap L^\infty(Y \setminus D_0)$.

c) [15, Definition 2.14] A smooth function $f : Y \setminus D_0 \to \mathbb{C}$ is *P-singular, with singularities along* D_0, if ∂f, $\overline{\partial} f$, $\partial\overline{\partial} f$ have Poincaré growth on Y, with singularities along D_0.

Definition 2.9 ([23, p. 242]). Let L be a holomorphic line bundle over Y and let h^L be a smooth Hermitian metric on L over $Y \setminus D_0$. We say h^L is good with singularities along D_0 if for a local holomorphic frame υ of L, $\log h^L(\upsilon, \upsilon)$ is P-singular, with singularities along D_0.

Remark 2.10. The original definition of Mumford is equivalent to this one, see [15, Proposition 3.2].

Definition 2.11 ([8]). Let L be a holomorphic line bundle over Y and let h^L be a smooth Hermitian metric on L over $Y \setminus F$. We say h^L has log-log growth with singularities along F if for a local holomorphic frame υ of L, $\log h^L(\upsilon, \upsilon)$ has log-log growth with singularities along F.

Now, we fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a curve. Suppose that the divisor of singular curves Δ has normal crossings. Let D be a divisor on X such that $\pi|_D : D \to S$ is a local isomorphism.

Proposition 2.12. Let α be a smooth $(1,1)$-form over $X \setminus (\Sigma_{X/S} \cup |D|)$, with Poincaré growth on $X \setminus (|D| \cup |\pi^{-1}(\Delta)|)$ with singularities along $D \cup \pi^{-1}(\Delta)$. Let $f : X \setminus (\Sigma_{X/S} \cup |D|) \to \mathbb{R}$ be a continuous function, with log-log growth along $\Sigma_{X/S} \cup |D|$.

Then for the normalization $\rho : Y_t \to X_t$ of X_t, $t \in |\Delta|$, the form $\rho^*(f\alpha)$ is integrable over Y_t. Moreover, the function $\pi_*[f\alpha]$ extends continuously over S, and the value of this extension is

$$\pi_*[f\alpha](t) = \textstyle\int_{Y_t} \rho^*(f\alpha). \tag{2.17}$$

Proof. The first part of the statement was proven in [14, Proposition 3.1c)], and the second part follows directly from the proof of [14, Proposition 3.1c)]. □

2.3. Families of hyperbolic surfaces and Quillen metric

In this section we recall the results of Freixas from [16] and [17], which describe how the Quillen metric, defined using the Takhtajan-Zograf version of the analytic torsion, (1.28), behaves in a one specific degenerating family of Riemann surfaces. We will use those results in our proof of Theorem 1.1.

Let's describe the family of Riemann surfaces first. We fix a Riemann surface $\overline{M}$ with m fixed points $D_M = \{P_1^M, \dots, P_m^M\} \subset \overline{M}$ and a Riemann surface $\overline{T}$, homeomorphic to a torus with one fixed point $D_T = \{P^T\} \subset \overline{T}$. Take m copies $(\overline{T}_i, P_i^T)$ of $(\overline{T}, P^T)$, $i = 1, \dots, m$. Let $g \in \mathbb{N}$ be the genus of $\overline{M}$. Clutching morphisms β (see (1.31)), applied to the pairs $\{P_1^M, P_1^T\}, \dots, \{P_m^M, P_m^T\}$, realizes the pointed surface $(M, m \cdot T) := (\overline{M}, D_M) \cup (\overline{T}_1, P_1^T) \cup \dots \cup (\overline{T}_m, P_m^T)$ as a point in a compactifying divisor $\partial \mathscr{M}_{g+m,0}$ of $\overline{\mathscr{M}}_{g+m,0}$. The *plumbing family* associated with $(M, m \cdot T)$ is a family of pointed curves representing a transversal direction to $\partial \mathscr{M}_{g+m,0}$ in $\overline{\mathscr{M}}_{g+m,0}$.

More precisely, we consider a neighborhood U_i of $P_i^M \in \overline{M}$, $i = 1, \dots, m$, biholomorphic to an open disc and a holomorphic coordinate mappings $F_i : U_i \to \mathbb{C}$ with $F_i(P_i^M) = 0$; similarly, a neighborhood V of $P^T \in \overline{T}$, and a holomorphic coordinate mapping $G : V \to \mathbb{C}$ satisfying $G(P^T) = 0$; and a small complex parameter $t \in \mathbb{C}$.

We suppose that U_i are pairwise disjoint. Let $c > 0$ be such that $D(c) \subset \mathbb{C}$ is contained in $\mathrm{Im}(F_i)$, $i = 1, \dots, m$ and $\mathrm{Im}(G)$. We take m copies $G_1, \dots, G_m$ of G, and regard them as local functions on $\overline{T}_1, \dots, \overline{T}_m$ respectively. Let $|t| < c^2$. For $d \in D(c)$, we note

$$
\begin{aligned}
R^{d,*} = &\big(\overline{M} \setminus (\cup_{i=1}^m \{|F_i| \leq |d|\})\big) \\
&\cup \big(\overline{T}_1 \setminus \{|G_1| \leq |d|\}\big) \cup \dots \cup \big(\overline{T}_m \setminus \{|G_m| \leq |d|\}\big).
\end{aligned} \tag{2.18}
$$

Consider the equivalence relation on points of $R^{t/c,*}$ generated by:

$$
p \sim q \text{ if } |t|/c \leq |F_i(p)| \leq c,\ |t|/c \leq |G_i(q)| \leq c,\ F_i(p) G_i(q) = t. \tag{2.19}
$$

Form the identification space $X_t = R^{t/c,*} / \sim$. The curve X_t, $t \in D(c^2)$, is called the *plumbing construction* for $(M, m \cdot T)$ associated with the *plumbing data* $(\cup_i U_i, V, \cup_i F_i, G, t)$. Clearly, $X := \cup_{t \in D(c^2)} X_t$ has a structure of a complex manifold, for which $\pi : X \to S := D(c^2)$ is a proper holomorphic map of codimension 1. The divisor of singular curves is given by $\Delta = m \cdot \{0\}$.

Now, suppose that the pointed surface $(\overline{M}, D_M)$ is stable, i.e. it satisfies (1.24). Then one can take the functions F_i, $i = 1, \dots, m$, from the plumbing construction to be Poincaré-compatible coordinates z_i^M of P_i^M (see (1.1)) with respect to the canonical complete hyperbolic metric of constant scalar curvature -1 on $\overline{M} \setminus D_M$ with cusps at D_M. Similarly, we make the choice for $G_i = z^T$. We call the associated plumbing family the *canonical plumbing family.*

From now on, we fix a canonical plumbing family $\pi : X \to S := D(c^2)$. We denote by

$$Y_0 := (\overline{M} \cup \overline{T}_1 \cup \cdots \cup \overline{T}_m), \qquad \rho : Y_0 \to X_0 \tag{2.20}$$

the normalization of the singular fiber and by

$$\Sigma_{X/S} = \{Q_1, \ldots, Q_m\}, \quad Q_i = \rho(P_i^M), \tag{2.21}$$

the set of singular points in X_0. We denote by $Z_{X_t}(s)$ the Selberg zeta-function associated with X_t, given by the formula (1.25). Let $g^{TX_t}_{\mathrm{hyp}}$, $t \neq 0$ be the canonical hyperbolic metric of constant scalar curvature -1 on X_t. Let $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ be the induced Hermitian norm on $\omega_{X/S}$ over $X \setminus \pi^{-1}(|\Delta|)$.

We consider the determinant line bundle $\lambda(j^*(\omega^n_{X/S}))$, $n \leq 0$, (2.14), and endow it over $S \setminus \Delta$ with the Takhtajan-Zograf version of the Quillen norm (cf. [15, §6]), given by

$$\begin{aligned} &\|\cdot\|^{TZ}_{Q} \left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right) \\ &\qquad := T_{TZ}\left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right)^{1/2} \cdot \|\cdot\|_{L^2} \left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right). \end{aligned} \tag{2.22}$$

We construct the norm (compare with (1.9))

$$\|\cdot\|^{TZ}_{\mathscr{L}_n} := \left(\|\cdot\|^{TZ}_{Q} \left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right)\right)^{12} \otimes \|\cdot\|^{\mathrm{div}}_{\Delta} \tag{2.23}$$

on the line bundle (compare with (1.10))

$$\mathscr{L}^{TZ}_n := \lambda\left(j^*(\omega^n_{X/S})\right)^{12} \otimes \mathscr{O}_S(\Delta). \tag{2.24}$$

We denote by $\|\cdot\|^{\mathrm{hyp}}_M, \|\cdot\|^{\mathrm{hyp}}_T$ the norms on $\omega_M(D)$, $\omega_T(D)$ induced by the canonical hyperbolic metrics g^{TM}_{hyp}, g^{TT}_{hyp} of constant scalar curvature -1 on $(\overline{M}, D_M)$, $(\overline{T}, D_T)$. We denote by $\|\cdot\|^{W,\mathrm{hyp}}_M$, $\|\cdot\|^{W,\mathrm{hyp}}_T$ the associated Wolpert norms on the complex lines $\det(\omega_{\overline{M}}|_{D_M})$ and $\det(\omega_{\overline{T}}|_{D_T})$. Now, we define the norm

$$\begin{aligned} \|\cdot\|^{TZ}_{\mathscr{L}'_n} := &\left(\|\cdot\|^{TZ}_{Q} \left(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n}\right) \otimes \left(\|\cdot\|^{TZ}_{Q} \left(g^{TT}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_T)^{2n}\right)\right)^m\right)^{12} \\ &\otimes \left(\|\cdot\|^{W,\mathrm{hyp}}_M \otimes (\|\cdot\|^{W,\mathrm{hyp}}_T)^m\right)^{-1} \end{aligned} \tag{2.25}$$

on the complex line (compare with (1.20))

$$\mathscr{L}_n^{TZ\prime} := \Big(\lambda\big(\omega_M(D)^n\big) \otimes \lambda\big(\omega_T(D)^n\big)^m\Big)^{12} \otimes \Big(\det(\omega_{\overline{M}}|_{D_M}) \otimes (\det(\omega_{\overline{T}}|_{D_T}))^m\Big)^{-1}. \tag{2.26}$$

Then the isomorphism (1.21) specifies in this case to the canonical isomorphism

$$\mathscr{L}_n^{TZ}|_\Delta \to \mathscr{L}_n^{TZ\prime}. \tag{2.27}$$

We recall that the constants C_k, $k \in \mathbb{N}$ were defined in (1.22).

Theorem 2.13 (Freixas, [16, Corollary 5.8] for $n = 0$ and [17, Theorem 5.3] for $n < 0$). *The norm $\|\cdot\|_{\mathscr{L}_n}^{TZ}$ extends continuously over S, and, under the isomorphism (2.27), the following identity holds*

$$\|\cdot\|_{\mathscr{L}_n}^{TZ}|_\Delta = \exp(m \cdot C_{-n}) \cdot \|\cdot\|_{\mathscr{L}_n'}^{TZ}. \tag{2.28}$$

Remark 2.14. Theorem 2.13 corresponds exactly to Theorem 1.1 for a special choice of a family of curves, a special choice of the metric and a different definition of the analytic torsion.

Remark also that Freixas states his theorems for $\lambda(j^*(\omega_{X/S}^{-n+1}))$, $n \leq 0$, but since Serre duality is an isometry, cf. [9, p. 310], his result holds for $\lambda(j^*(\omega_{X/S}^n))$, $n \leq 0$.

2.4. Model grafting and pinching expansion

The goal of this section is to recall the model grafting construction due to Wolpert [29]. For simplicity, we state it only in the setting of Section 2.3. We conserve the notation from Section 2.3.

To be compatible with further notation, we denote

$$z_0^i := z_i^M, \qquad z_1^i := z_i^T. \tag{2.29}$$

By the definition of the plumbing family from Section 2.3, the coordinates (z_0^i, z_1^i) can be regarded as local holomorphic charts in the neighborhood

$Q_i \in X$. We denote

$$U(Q_i, \epsilon) = \Big\{ x \in X : |z_0^i(x)| < \epsilon, |z_1^i(x)| < \epsilon \Big\}. \tag{2.30}$$

By the definition of plumbing family, in t-coordinates on S (see (2.19)), we have

$$\pi(z_0^i, z_1^i) = z_0^i z_1^i. \tag{2.31}$$

The canonical hyperbolic metric on M (resp. T) with cusps at D_M (resp. D_T) induces a metric $g_{\mathrm{hyp}}^{TR^{\epsilon,*}}$ on $R^{\epsilon,*}$ (see (2.18) for the definition of $R^{\epsilon,*}$). Let ϵ be so small, so that $g_{\mathrm{hyp}}^{TR^{\epsilon,*}}$ is induced by (1.1) in coordinate z_j^M over $\{|z_j^M| < 2\epsilon\}$ and by (1.1) in coordinate z_j^T over $\{|z_j^T| < 2\epsilon\}$.

We choose $c = \epsilon^2$ in the plumbing construction from Section 2.3. Now, since the manifold $X \setminus (\cup_{i=1}^k U(Q_i, \epsilon))$ is naturally isomorphic to the product $R^{\epsilon,*} \times D(\epsilon^2)$, the metric $g_{\mathrm{hyp}}^{TR^{\epsilon,*}}$ induces the Kähler metric g^{TX_t} on $X_t \setminus (\cup_{i=1}^k U(Q_i, \epsilon))$.

The *model grafted metric* $g_{\mathrm{gft}}^{TX_t}$ on X_t, is built from the metric g^{TX_t} and the *hyperbolic metric on a cylinder*, (2.34). It models the degeneration of the metric of constant scalar curvature -1.

More precisely, let $\nu : X \to [0, 1]$ be smooth function, satisfying

$$\nu(x) = \begin{cases} 0, & \text{for } x \in X \setminus (\cup_{i=1}^k U(Q_i, 2\epsilon)), \\ 1, & \text{for } x \in \cup_{i=1}^k U(Q_i, \epsilon). \end{cases} \tag{2.32}$$

For $t \in D(\epsilon^2)$, we denote by $g_{i,t}^{\mathrm{Cyl}}$ the metric over the set

$$\big\{ (z_0^i, z_1^i) \in X_t : |t|/(2\epsilon) < |z_0^i| < 2\epsilon \big\}, \tag{2.33}$$

induced by the Kähler form

$$\left(\frac{\pi}{|z_0^i| \log |t|} \left(\sin \frac{\pi \log |z_0^i|}{\log |t|} \right)^{-1} \right)^2 \sqrt{-1} dz_0^i d\overline{z}_0^i. \tag{2.34}$$

We remark that since over X_t, we have $z_0^i z_1^i = t$, the expression (2.34) is symmetric in z_0^i and z_1^i.

Following Wolpert [29], we define the *model grafted metric* $g_{\mathrm{gft}}^{TX_t}$ as follows: over $X_t \setminus (\cup_{i=1}^m U(Q_i, 2\epsilon))$, $g_{\mathrm{gft}}^{TX_t}$ coincides with g^{TX_t}, and over

$U(Q_i, 2\epsilon)$, it is given by

$$g_{\mathrm{gft}}^{TX_t} = \big(g_{i,t}^{\mathrm{Cyl}}\big)^{\nu} \big(g^{TX_t}\big)^{1-\nu}. \tag{2.35}$$

The metric $g_{\mathrm{gft}}^{TX_t}$ is designed to model the metric of constant scalar curvature near the degeneration, according to Wolpert [29, Expansion 4.2]. The advantage of the grafted metric over the hyperbolic one is that its construction is local near the singularities.

Proposition 2.15. The norm $\|\cdot\|_{X/S}^{\mathrm{gft}}$, induced by $g_{\mathrm{gft}}^{TX_t}$ over $X \setminus \pi^{-1}(|\Delta|)$, extends continuously over $X \setminus \Sigma_{X/S}$. Moreover, it is good in the sense of Mumford on $X \setminus \pi^{-1}(|\Delta|)$ with singularities along $\pi^{-1}(\Delta)$ and has log-log growth with singularities along $\Sigma_{X/S}$.

Proof. This follows from an explicit calculation, see Wolpert [29, Lemma 1.5]. □

3. The behavior of the Quillen metric near singular fibers

The main goal of this section is to prove Theorems 1.1, 1.3 and Corollaries 1.5, 1.7. More precisely, this section is organized as follows. In Section 3.1, we use Theorems 2.1, 2.4, 2.7, 2.13 to prove Theorems 1.1, 1.3 modulo a certain universality statement. In Section 3.2, we slightly generalize the result of Bismut [3, Theorem 0.3] about the behavior of the Quillen norm in a smooth Kähler family of degenerating compact Riemann surfaces by dropping out the Kähler assumption on the metric. Finally, in Section 3.3, by using this result, we prove the universality statement, which is used in Section 3.1. From this, we also deduce Theorem 1.7.

3.1. Quillen metric on the singular locus, proof of Theorems 1.1, 1.3

In this section we prove Theorems 1.1, 1.3 modulo a certain universality statement, which will be established in Section 3.3. We then establish Corollary 1.5. We conserve the notation from the statement of Theorem 1.1.

Before describing the proofs, let's explain the construction of the isomorphism (1.21). Recall that $\Sigma_{Z/S'}$, $\Sigma_{X/S}$ and κ were defined in (1.14) and in a paragraph before it. We denote by $N_{\Sigma_{Z/S'}/X}$ (resp. $N_{S'/S}$) the normal vector bundle of $\Sigma_{Z/S'}$ in X (resp. of S' in S). The fibers of X have only double-point singularities, so the projection π induces the following canonical isomorphism (see (2.11), cf. also [3, (2.9)])

$$d\pi^2 : \wedge^2(N_{\Sigma_{Z/S'}/X}) \otimes (\det \rho_*(\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})) \to \kappa^* \pi^* N_{S'/S}. \tag{3.1}$$

The square of $\det \rho_*(\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})$ is canonically trivialized, so from the metrical point of view, it doesn't contribute to our analysis and we omit this bundle from now on. For the relative tangent bundle TY/S' of π' and for any $i = 1, \dots, k$, the normalization map ρ induces the canonical isomorphism

$$(\sigma'_{m+2i-1})^*(TY/S') \otimes (\sigma'_{m+2i})^*(TY/S') \to \wedge^2(N_{\Sigma_{Z/S'}/X}). \tag{3.2}$$

We denote by ω_S and $\omega_{S'}$ the canonical line bundles over S and S'. By combining the duals of the isomorphisms (3.1), (3.2), for $i = 1, \dots, k$, we get the canonical isomorphism

$$(\omega_S \otimes \omega_{S'}^{-1})|_{S'} \to (\sigma'_{m+2i-1})^*(\omega_{Y/S'}) \otimes (\sigma'_{m+2i})^*(\omega_{Y/S'}). \tag{3.3}$$

Poincaré residue morphism (cf. [19, p. 147]) gives a canonical isomorphism

$$(\omega_S \otimes \mathscr{O}_S(\Delta_0))|_{S'} \to \omega_{S'}. \tag{3.4}$$

By combining the isomoprhism (3.3), applied for each $i = 1, \dots, k$, the isomorphism (3.4) and by multiplying by $(\otimes_{i=1}^m \sigma_i^* \omega_{X/S})^{-1} \otimes \mathscr{O}_S(\sum k_i \Delta_i)$, we get the canonical isomorphism

$$\begin{aligned} \Big(\big(\otimes_{i=1}^m \sigma_i^* \omega_{X/S} \big)^{-1} \otimes \mathscr{O}_S(\Delta) \Big)\Big|_{S'} \\ \to \big(\otimes_{i=1}^{m+2k} (\sigma'_i)^* \omega_{Y/S'} \big)^{-1} \otimes \mathscr{O}_{S'}(\Delta'). \end{aligned} \tag{3.5}$$

For $t \in S'$, we have the following exact sequence of sheaves (cf. [3, (5.53)])

$$\begin{aligned} 0 \to \mathscr{O}_{Z_t}\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big) \to \rho_* \mathscr{O}_{Y_t}\big(j^*(\rho^*(\xi) \otimes \omega_{Y/S'}(D)^n)\big) \\ \to \mathscr{O}_{\Sigma_{Z/S'}}\big(\kappa^*(\xi) \otimes \det(\rho_* \mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})\big) \to 0, \end{aligned} \tag{3.6}$$

where the first map is induced by the pull-back and (1.17), and the second map is the difference of the residue morphism at $\rho^{-1}(\Sigma_{Z/S'})$. By the additivity of the determinant, the short exact sequence (3.6) induces the canonical isomorphism (cf. [3, (5.55)])

$$\begin{aligned}(3.7)\quad \lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big)|_{S'} \to \lambda\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big)\\ \otimes \det\big(\pi_*(\kappa^*(\xi))\big)\otimes\det\big((\pi\circ\rho)_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}}\big)^{\mathrm{rk}(\xi)}.\end{aligned}$$

Trivially, we have an isomorphism

$$\begin{aligned}(3.8)\quad \det\big(\pi_*(\kappa^*(\xi))\big)^2\\ \to \big(\otimes_{i=1}^{2k}(\sigma'_{m+i}\circ\rho)^*\det\xi\big)\otimes\big(\det\rho_*(\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})\big)^{\mathrm{rk}(\xi)}.\end{aligned}$$

The composition of the isomorphisms (3.5), (3.7) and (3.8) induce the isomorphism (1.21).

We can now state a crucial theorem in our analysis, which will be proved in Section 3.3.

Theorem 3.1. *For any $n\leq 0$, there exists a universal constant $A_{-n}\in\mathbb{R}$, such that Theorem 1.1 holds for any family of curves $\pi: X\to S$ without cusps (i.e. $m=0$) with A_{-n} in place of C_{-n}.*

Now, by [5, Theorem 1.27], the Bott-Chern classes of a vector bundle ξ with metrics h_1^ξ, h_2^ξ are natural differential forms $\widetilde{\mathrm{Td}}(\xi,h_1^\xi,h_2^\xi)$, $\widetilde{\mathrm{ch}}(\xi,h_1^\xi,h_2^\xi)$, defined modulo $\mathrm{Im}(\partial)+\mathrm{Im}(\overline{\partial})$, so that

$$\begin{aligned}\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{Td}}(\xi,h_1^\xi,h_2^\xi)&=\mathrm{Td}(\xi,h_1^\xi)-\mathrm{Td}(\xi,h_2^\xi),\\ \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{ch}}(\xi,h_1^\xi,h_2^\xi)&=\mathrm{ch}(\xi,h_1^\xi)-\mathrm{ch}(\xi,h_2^\xi),\end{aligned}\tag{3.9}$$

where Td, ch are Todd and Chern forms. By [5, Theorem 1.27], we have the following identities

$$\widetilde{\mathrm{ch}}(\xi,h_1^\xi,h_2^\xi)^{[0]}=2\widetilde{\mathrm{Td}}(\xi,h_1^\xi,h_2^\xi)^{[0]}=\log\big(\det(h_1^\xi/h_2^\xi)\big).\tag{3.10}$$

If, moreover, $\xi:=L$ is a line bundle, we have

$$\begin{aligned}\widetilde{\mathrm{ch}}(L,h_1^L,h_2^L)^{[2]}&=6\widetilde{\mathrm{Td}}(L,h_1^L,h_2^L)^{[2]}\\ &=\log(h_1^L/h_2^L)\Big(c_1(L,h_1^L)+c_1(L,h_2^L)\Big)/2,\end{aligned}\tag{3.11}$$

where c_1 is the first Chern form. In what follows, when we write a Bott-Chern class, one should interpret it as a *differential form*, given by (3.10), (3.11).

Now let's see how Theorem 3.1 can be used to prove Theorems 1.1, 1.3.

Proof of Theorem 1.1. The proof consists of 3 steps. In *Steps 1, 2*, we prove that it is enough to establish Theorem 1.1 for $m = 0$. This reduces Theorem 1.1 by Theorem 3.1 to the proof that A_{-n} from Theorem 3.1 coincides with C_{-n} from Theorem 1.1. We establish this in *Step 3*.

Before describing the proof in details, let us remark that by Theorem 2.7, we may assume $S = D(1)$, $|\Delta| = \{0\}$. Also, by Theorem 2.4, the metrics from Theorem 1.1 depend in the same way on the variation of h^{ξ}. Hence, we may assume that the Hermitian vector bundle (ξ, h^{ξ}) is trivial over a small neighborhood of $|D_{X/S}| \cup \Sigma_{X/S}$. We make those simplifications below.

Step 1. In this step we show that it is enough to establish Theorem 1.1 for families of Riemann surfaces with cusps such that the metric near the cusp is constant in the horizontal direction (with respect to some fixed local holomorphic coordinates near the cusp). For this, let $V_{i,c}$, $i = 1, \dots, m$, $c > 0$ (resp. U) be a neighborhood of $\sigma_i(t_0)$ (resp. t_0) such that for some local coordinates $(z_0, \dots, z_q)$ of $\sigma_i(t_0)$ and $(w_1, \dots, w_q)$ of $t_0 \in S$, satisfying (2.10), we have $V_{i,c} = \{x \in \pi^{-1}(U) : |z_0(x)| < c\}$ and $\{z_0(x) = 0\} = \{\sigma_i(t) : t \in U\}$. For simplicity, we note $V_i := V_{i,1}$. Let $\nu_0 : \mathbb{R}_+ \to [0, 1]$ be a smooth function satisfying

$$\nu_0(u) = \begin{cases} 0, & \text{if} \quad u < 1/2, \\ 1, & \text{if} \quad u > 3/4. \end{cases} \tag{3.12}$$

We denote by $\|\cdot\|_{X/S}^{\omega,0}$ the norm on $\omega_{X/S}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ such that $\|\cdot\|_{X/S}^{\omega,0}$ coincides with $\|\cdot\|_{X/S}^{\omega}$ away from $\cup_{i=1}^m V_i$, and over $(\cup_{i=1}^m V_i) \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, we have

$$\|dz_0\|_{X/S}^{\omega,0} = \big|z_0 \log |z_0|\big|^{1-\nu_0(|z_0|)} \cdot \big(\|dz_0\|_{X/S}^{\omega}\big)^{\nu_0(|z_0|)}. \tag{3.13}$$

Let $\|\cdot\|_{X/S}^{0}$ be the norm on $\omega_{X/S}(D)$ induced from $\|\cdot\|_{X/S}^{\omega,0}$ as in (1.6), and let $g_0^{TX_t}$, $t \in S$ be the induced Kähler metric $X_t \setminus D_{X/S}$ with cusps at $D_{X/S} \cap X_t$. We denote by $\|\cdot\|_{X/S}^{W,0}$ the norm on the line bundle $\otimes_{i=1}^m \sigma_i^* \omega_{X/S}$ associated with $g_0^{TX_t}$ as in Definition 2.3. Then by (3.13), we see that if $\|\cdot\|_{X/S}$ satisfies assumptions (1.8) and (1.19), then $\|\cdot\|_{X/S}^{0}$ satisfies assumptions (1.8) and (1.19) as well. In fact, this property along with the fact that $\|\cdot\|_{X/S}^{\omega,0}$ doesn't vary in the horizontal direction around the cusps (with respect to some fixed local holomorphic coordinates) are the only facts we need from the construction (3.13).

We denote by $g_0^{TY_0}$ the Kähler metric on $Y_0 \setminus D_{Y_0}$, constructed from $\|\cdot\|_{Y_0}^0 := \rho^*(\|\cdot\|_{X/S}^0)$ as in (1.6). We denote by $\|\cdot\|_{Y_0}^{W,0}$ the Wolpert norm on $\otimes_{i=1}^{m+2k}(\sigma_i')^*\omega_{Y_0}$ induced by $g_0^{TY_0}$.

As we assumed that (ξ, h^ξ) is trivial in a neighborhood of $|D_{X/S}|$, and the metrics $\|\cdot\|_{X/S}$, $\|\cdot\|_{X/S}^0$ differ only in the neighborhood of $|D_{X/S}|$, by Theorem 2.4, applied pointwise for the line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))^{12} \otimes (\otimes_{i=1}^m \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)}$, for any $t \in S \setminus |\Delta|$, we have

(3.14)
$$\begin{aligned}
&\frac{1}{6}\log\Big(\|\cdot\|_Q\big(g_0^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S}^0)^{2n}\big)^{12} \otimes \big(\|\cdot\|_{X/S}^{W,0}\big)^{-\mathrm{rk}(\xi)}\Big)\\
&\qquad - \frac{1}{6}\log\Big(\|\cdot\|_Q\big(g^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S})^{2n}\big)^{12} \otimes \big(\|\cdot\|_{X/S}^{W}\big)^{-\mathrm{rk}(\xi)}\Big)\\
&= \mathrm{rk}(\xi)\cdot \int_{X_t} \Big(\widetilde{\mathrm{Td}}\big(\omega_{X/S}(D)^{-1}, \|\cdot\|_{X/S}^{-2}, (\|\cdot\|_{X/S}^0)^{-2}\big)\mathrm{ch}\big(\omega_{X/S}(D)^n, \|\cdot\|_{X/S}^{2n}\big)\\
&\qquad\qquad + \mathrm{Td}\big(\omega_{X/S}(D)^{-1}, (\|\cdot\|_{X/S}^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\omega_{X/S}(D)^n, \|\cdot\|_{X/S}^{2n}, (\|\cdot\|_{X/S}^0)^{2n}\big)\Big).
\end{aligned}$$

We note that the conformal factor corresponding to the change of the norm from $\|\cdot\|_{X/S}^\omega$ to $\|\cdot\|_{X/S}^{\omega,0}$ is non-trivial in the neighborhood of the cusp. Thus, in (3.14) we use a strong version of Theorem 2.4 with the conformal factor of non-compact support in the punctured surface.

By applying Theorem 2.4, we get

$$\begin{aligned}
&\frac{1}{6}\log\Big(\|\cdot\|_Q\big(g_0^{TY_0}, \rho^*(h^\xi) \otimes (\|\cdot\|_{Y_0}^0)^{2n}\big)^{12} \otimes \big(\|\cdot\|_{Y_0}^{W,0}\big)^{-\mathrm{rk}(\xi)}\Big)\\
&\qquad - \frac{1}{6}\log\Big(\|\cdot\|_Q\big(g^{TY_0}, \rho^*(h^\xi) \otimes (\|\cdot\|_{Y_0})^{2n}\big)^{12} \otimes \big(\|\cdot\|_{Y_0}^{W}\big)^{-\mathrm{rk}(\xi)}\Big)\\
&= \mathrm{rk}(\xi)\cdot \int_{Y_0} \Big(\widetilde{\mathrm{Td}}\big(\omega_{Y_0}(D)^{-1}, \|\cdot\|_{Y_0}^{-2}, (\|\cdot\|_{Y_0}^0)^{-2}\big)\mathrm{ch}\big(\omega_{Y_0}(D)^n, \|\cdot\|_{Y_0}^{2n}\big)\\
&\qquad\qquad + \mathrm{Td}\big(\omega_{Y_0}(D)^{-1}, (\|\cdot\|_{Y_0}^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\omega_{Y_0}(D)^n, \|\cdot\|_{Y_0}^{2n}, (\|\cdot\|_{Y_0}^0)^{2n}\big)\Big).
\end{aligned} \tag{3.15}$$

By Proposition 2.12, we see that the right-hand-side of (3.14) extends continuously over S, moreover, as $t \to 0$, by (2.17), the right-hand side of (3.14) converges to the right-hand side of (3.15). Thus, it is enough to prove Theorem 1.1 for the metrics $\|\cdot\|_{X/S}^0$, $\|\cdot\|_{X/S}^{\omega,0}$, $\|\cdot\|_{X/S}^{W,0}$ instead of $\|\cdot\|_{X/S}$, $\|\cdot\|_{X/S}^\omega$, $\|\cdot\|_{X/S}^W$. We also note that by (3.13), for $i = 1, \ldots, m$, the following identity holds

$$\big\| dz_0|_{\sigma_i(t)} \big\|_{X/S}^{W,0,i} = \big\| dz_0|_{\sigma_i'(0)} \big\|_{Y/S'}^{W,0,i} = 1. \tag{3.16}$$

In other words, the Wolpert norms associated with σ_i, $i = 1, \dots, m$, are trivial.

Step 2. In this step we show that by Theorem 2.1, one can delete the cusps from the metric obtained in *Step 1.* We denote $V_i' = V_{i,1/2} \subset V_i$, $i = 1, \dots, m$. Let $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ be the Hermitian norm on $\omega_{X/S}$ over $X \setminus \pi^{-1}(|\Delta|)$ such that $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ coincides with $\|\cdot\|_{X/S}^{\omega,0}$ away from $\cup_{i=1}^m V_i'$, and for $\nu_0 : \mathbb{R} \to [0,1]$ as in (3.12), over V_i', we have

$$\|dz_0\|_{X/S}^{\omega,\mathrm{cmp}} = |z_0 \log|z_0||^{\nu_0(2|z_0|)}. \tag{3.17}$$

We denote by $g_{\mathrm{cmp}}^{TX_t}$ the induced Kähler metric on X_t. By (3.17), we see that if $\|\cdot\|_{X/S}^{\omega,0}$ satisfies the assumptions (1.8) and (1.19), then $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ satisfies the assumptions (1.8) and (1.19) as well, but for $D_{X/S} = \emptyset$, i.e. without the cusps. In fact, this property along with the fact that $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ doesn't vary in the horizontal direction around the cusps (with respect to some fixed local holomorphic coordinates) are the only facts we need from (3.17).

We denote by $g_{\mathrm{cmp}}^{TY_0}$ the Kähler metric over $Y_0 \setminus \rho^{-1}(\Sigma_{X/S})$ induced from $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ as in (1.18) for $D_{X/S} = \emptyset$. We denote by $\|\cdot\|_{X/S}^{\mathrm{cmp}}$ the norm on $\omega_{X/S}(D)$ over $X \setminus \pi^{-1}(|\Delta|)$, such that $\|\cdot\|_{X/S}^{\mathrm{cmp}}$ coincides with $\|\cdot\|_{X/S}^{0}$ away from $\cup_{i=1}^m V_i'$, and over V_i', we have

$$\left\|dz_0 \otimes s_{D_{X/S}}/z_0\right\|_{X/S}^{\mathrm{cmp}} = |\log|z_0||^{\nu_0(2|z_0|)}. \tag{3.18}$$

We denote by $\|\cdot\|_{Y_0}^{\mathrm{cmp}} := \rho^*(\|\cdot\|_{X/S}^{\mathrm{cmp}})$ the induced Hermitian norm on $\omega_{Y_0}(D)$ over $Y_0 \setminus \rho^{-1}(\Sigma_{X/S})$. Remark that $g_{\mathrm{cmp}}^{TY_0}, \|\cdot\|_{Y_0}^{\mathrm{cmp}}$ form a partial flattening of $g_0^{TY_0}$ and $\|\cdot\|_{Y_0}^{0}$ in the sense of Remark 2.2. Now, since in $g_0^{TX_t}$ the Poincaré-compatible coordinates of the cusps are trivialized, we see by Theorem 2.1 and Remark 2.2 that for $t \in S \setminus |\Delta|$ the following holds

$$\begin{aligned} &\log\Big(\|\cdot\|_Q\big(g_{\mathrm{cmp}}^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S}^{\mathrm{cmp}})^{2n}\big) / \|\cdot\|_Q\big(g_0^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S}^{0})^{2n}\big)\Big) \\ &= \log\Big(\|\cdot\|_Q\big(g_{\mathrm{cmp}}^{TY_0}, \rho^*(h^\xi) \otimes (\|\cdot\|_{Y_0}^{\mathrm{cmp}})^{2n}\big) / \|\cdot\|_Q\big(g_0^{TY_0}, \rho^*(h^\xi) \otimes (\|\cdot\|_{Y_0}^{0})^{2n}\big)\Big). \end{aligned} \tag{3.19}$$

In (3.19), we didn't mention the last term of (2.7) since (ξ, h^ξ) is trivial in the neighborhood of $|D_{X/S}|$, and the norms $\|\cdot\|_{X/S}^{\mathrm{cmp}}, \|\cdot\|_{X/S}^{0}$ differ only in the neighborhood of $|D_{X/S}|$.

We denote by $\|\cdot\|_{D_{X/S}}^{\mathrm{cmp}}$ the norm on $\mathscr{O}_X(D_{X/S})$, given by $\|\cdot\|_{X/S}^{\mathrm{cmp}} / \|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$. The norm $\|\cdot\|_{D_{X/S}}^{\mathrm{cmp}}$ is trivial away from $\cup_{i=1}^m V_i$ (with respect to the canonical trivialization of $\mathscr{O}_X(D_{X/S})$), and it is smooth over X.

By (3.5), (3.16) and (3.19), it is enough to prove Theorem 1.1 for the Hermitian vector bundles $(\xi \otimes \mathscr{O}_X(D_{X/S})^n, h^\xi \otimes (\|\cdot\|^{\rm cmp}_{D_{X/S}})^{2n})$, $(\omega_{X/S}, \|\cdot\|^{\omega,{\rm cmp}}_{X/S})$ and $D_{X/S} = \emptyset$, instead of (ξ, h^ξ), $(\omega_{X/S}, \|\cdot\|^{\omega,0}_{X/S})$ and $D_{X/S}$, given by (1.5). Since $h^\xi \otimes (\|\cdot\|^{\rm cmp}_{D_{X/S}})^{2n}$ is smooth over X, such a statement is equivalent to Theorem 1.1 for $m = 0$. Thus, it is enough to prove Theorem 1.1 only for $m = 0$, which we assume from now on.

Step 3. As we established in *Steps 1, 2*, to prove Theorem 1.1, it is enough to do so for surfaces without cusps, by Theorem 3.1, it would be sufficient to establish that the constant A_{-n} from Theorem 3.1 actually coincides with the constant C_{-n} from Theorem 1.1.

For this, we consider a stable pointed Riemann surface $(\overline{M}, D_M)$ and the associated canonical plumbing family $\pi : X \to S$ (see Section 2.3) with the norm $\|\cdot\|^{\rm hyp}_{X/S}$ on $\omega_{X/S}$ induced by the constant scalar curvature -1 metric. The generic fiber of this family has no cusps, hence, in the notations of Section 2.3, by a theorem of D'Hoker-Phong [10], [11], (cf. Remark 1.4), the following identity of norms over $S \setminus |\Delta|$ holds

$$\|\cdot\|_Q \left(g^{TX_t}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{X/S})^{2n}\right) = \|\cdot\|^{TZ}_Q \left(g^{TX_t}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{X/S})^{2n}\right). \tag{3.20}$$

We apply this for $(\overline{M}, D_M) := (\overline{T}, D_T)$, where $(\overline{T}, D_T)$ is a 1-pointed torus, considered in Section 2.3. Then by taking limit $t \to 0$ in (3.20), by Theorems 2.13 and 3.1, we get

$$\begin{aligned}\exp(A_{-n}/2) \cdot \|\cdot\|_Q \left(g^{TT}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{T})^{2n}\right) \\ = \exp(C_{-n}/2) \cdot \|\cdot\|^{TZ}_Q \left(g^{TT}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{T})^{2n}\right).\end{aligned} \tag{3.21}$$

By applying (3.20) again, but now for any $(\overline{M}, D_{\overline{M}})$, and by taking limit $t \to 0$ again, by Theorems 2.13, 3.1 and (3.21), we see that for any $(\overline{M}, D_M)$, $m := \# D_M$, we have

$$\begin{aligned}\exp(m \cdot A_{-n}/2) \cdot \|\cdot\|_Q \left(g^{TM}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{M})^{2n}\right) \\ = \exp(m \cdot C_{-n}/2) \cdot \|\cdot\|^{TZ}_Q \left(g^{TM}_{\rm hyp}, (\|\cdot\|^{\rm hyp}_{M})^{2n}\right).\end{aligned} \tag{3.22}$$

But by (2.2), our definition of the analytic torsion coincides with the definition of Takhtajan-Zograf for the 3-punctured hyperbolic sphere $P := \mathbb{P} \setminus \{0, 1, \infty\}$. From this and (3.22), we get $A_{-n} = C_{-n}$, which finishes the proof of Theorem 1.1 for $m = 0$. Thus, by *Steps 1,2*, Theorem 1.1 holds for any $m \in \mathbb{N}$. $\square$

Proof of Theorem 1.3. In Step 3 of the proof of Theorem 1.1, we proved that $A_{-n} = C_{-n}$ for any $n \in \mathbb{N}$. From this and (3.22), we deduce

$$\|\cdot\|_Q \,(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n}) = \|\cdot\|^{TZ}_Q \,(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n}), \tag{3.23}$$

for any surface with cusps $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$, for which g^{TM}_{hyp} has constant scalar curvature -1. From (2.5), (2.22) and (3.23), we deduce the result. $\square$

Proof of Corollary 1.5. By the results of Wolpert, [29] (in the compact case) and Freixas, [15] (in the non-compact case), (cf. [14, Proposition 5.6]), the norm $\|\cdot\|^{\mathrm{hyp}}_{g,m}$ satisfies assumptions (1.8) and (1.19). Thus, Corollary 1.5 is a direct consequence of Theorem 1.1. The fact that the underlying spaces are orbifolds doesn't pose any problem, as our methods are local. $\square$

3.2. Quillen metric for families of Riemann surfaces with smooth metric

In this section we describe a generalization of the result of Bismut [3, Theorem 0.3] for non-necessarily Kähler metrics. This theorem describes the behavior of the Quillen norm in a family of degenerating Riemann surfaces endowed with a (non-singular) metric, coming from the metric on the total space of the family. It will be used in our proof of Theorem 3.1.

To describe it precisely, let's fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a curve. Let (ξ, h^ξ) be a Hermitian vector bundle over X. Let g^{TX} be a Riemannian metric over X, which is compatible with the complex structure of X. By h^{TX} we note the Hermitian metric on $T^{(1,0)}X$ induced by g^{TX} as in Section 2.1. We denote by g^{TX_t} the restriction of the metric g^{TX} on X_t, $t \in S \setminus |\Delta|$. Since g^{TX} is compatible with the complex structure of X, and X_t is a complex submanifold of dimension 1, the metric g^{TX_t} is Kähler on X_t. We denote by $\|\cdot\|_Q \,(g^{TX_t}, h^\xi)$ the Quillen norm on the line bundle $\lambda(j^*\xi)$ over $S \setminus |\Delta|$ (see (2.5)).

For simplicity, assume that $S = D(1)$ and $|\Delta| = \{0\}$. We write $\Sigma_{X/S} = \{Q_1, \dots, Q_k\}$. Let $\rho : Y_0 \to X_0$ be the normalization of X_0. We denote

$$\rho^{-1}(\Sigma_{X/S}) = \{P_1, \dots, P_{2k}\}, \tag{3.24}$$

where P_i are enumerated in such a way that $\rho(P_{2j-1}) = \rho(P_{2j}) = Q_j$ for $j = 1, \dots, k$. We notice that $g^{TY_0} := \rho^*(g^{TX})$ is the well-defined Riemannian metric on Y_0 and we denote by $\|\cdot\|^{\omega}_{Y_0}$ the induced Hermitian norm on ω_{Y_0}. Since g^{TX} is compatible with the complex structure, g^{TY_0} is Kähler. We denote by $\|\cdot\|_Q\,(g^{TY_0}, \rho^*(h^\xi))$ the induced Quillen norm on the complex line $\lambda(\rho^*\xi)$.

Let $\|\cdot\|^i_{\Sigma_{X/S}/X}$ be the Hermitian norm induced by the natural isomorphism (3.2) on the complex lines $\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}}$, $i = 1, \ldots, k$. More explicitly, let local holomorphic coordinates z_0^i, z_1^i around $Q_i \in X$ and t around $0 \in S$ be as in (2.31). We denote

(3.25)
$$a_i = h^{TX}\Big(\frac{\partial}{\partial z_0^i}, \frac{\partial}{\partial z_0^i}\Big), \quad b_i = h^{TX}\Big(\frac{\partial}{\partial z_0^i}, \frac{\partial}{\partial z_1^i}\Big), \quad c_i = h^{TX}\Big(\frac{\partial}{\partial z_1^i}, \frac{\partial}{\partial z_1^i}\Big).$$

Then, by definition, we have

$$\big\|dz_0^i \otimes dz_1^i\big\|^i_{\Sigma_{X/S}/X} = \big(a_i c_i - |b_i|^2\big)^{-1/2}(Q_i). \tag{3.26}$$

We denote by $\|\cdot\|_{\Sigma_{X/S}/X}$ the induced norm on the complex line $\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}})$.

Over S, we introduce the holomorphic line bundle

$$\mathscr{L} := \lambda(j^*\xi)^{12} \otimes \mathscr{O}_S(\Delta)^{2\cdot \mathrm{rk}(\xi)}. \tag{3.27}$$

We endow it with a norm

$$\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}} := \|\cdot\|_Q\,(g^{TX_t}, h^\xi)^{12} \otimes (\|\cdot\|^{\mathrm{div}}_\Delta)^{2\cdot\mathrm{rk}(\xi)}. \tag{3.28}$$

Notice that the power of the divisor line bundle $\mathscr{O}_S(\Delta)$ in $\mathscr{L}$ is different from (1.10). This discrepancy is motivated by Theorem 3.2, which contrasts with Theorem 2.7. This is due to the fact that the geometric setting in this section is different from Section 1, see Remark 1.2. In fact, it turns out that the appearance of the hyperbolic cusps in the degenerated fiber entails a different singularity in the Quillen metric, compared to the case when the metric comes from a smooth metric on the total space of the fibration.

More precisely, we introduce the complex line

$$\begin{aligned} \mathscr{L}' := \lambda(\rho^*\xi)^{12} &\otimes (\otimes_{i=1}^{2k} \det \rho^*(\xi)|_{P_i})^6 \\ &\otimes (\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}}))^{-2\cdot\mathrm{rk}(\xi)}. \end{aligned} \tag{3.29}$$

We denote by $\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}'}$ the norm on $\mathscr{L}'$ induced by $\|\cdot\|_Q\,(g^{TY_0}, \rho^*(h^\xi))$, h^ξ and $\|\cdot\|_{\Sigma_{X/S}/X}$. Similarly to (1.21), one has the following canonical isomorphism

$$\mathscr{L}|_\Delta \to \mathscr{L}'. \tag{3.30}$$

Now we can state the main result of this section.

Theorem 3.2. *The norm* $\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}}$ *extends continuously over* S*. Moreover, under the isomorphism (3.30), the following identity holds*

$$\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}}|_{\Delta} = \exp\Big(\mathrm{rk}(\xi) \cdot k \cdot \big(24\zeta'(-1) - 6\log(2\pi)\big)\Big) \cdot \|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}'}. \tag{3.31}$$

Proof. First of all, let's assume that g^{TX} is Kähler. Then we argue that Theorem 3.2 is just a restatement of [3, Theorem 0.3] due to Bismut.

To see this, let's fix a holomorphic coordinate t on S such that $\Delta = k \cdot \{t = 0\}$. We denote by $\|\cdot\|_{\Delta}$ the Hermitian norm on $\mathscr{O}_S(\Delta)$, characterized by

$$\|s_{\Delta}/t^k\|_{\Delta} = 1. \tag{3.32}$$

As $\mathrm{div}(s_{\Delta}) = k\{0\}$, we deduce that $\|\cdot\|_{\Delta}$ is smooth over S. By the definition of the singular norm $\|\cdot\|^{\mathrm{div}}_{\Delta}$ from (1.6), by (3.32), we have $\|\cdot\|^{\mathrm{div}}_{\Delta} = |t|^{-k} \cdot \|\cdot\|_{\Delta}$. We denote by $\|d\pi^2\|$ the norm of the isomorphism (3.5), calculated with respect to $\|\cdot\|_{\Delta}$ and $\|\cdot\|_{\Sigma_{X/S}/X}$.

Due to our normalization of the L^2-norm, (2.3), the difference between our definition of the Quillen norm, and the one from [6], [3], which we denote by $\|\cdot\|^{BGS}_{Q}$, is given by

$$\|\cdot\|_{Q}(g^{TX_t}, h^{\xi}) = \exp\big(\log(2\pi) \cdot \chi(X_t, \xi|_{X_t})/2\big) \cdot \|\cdot\|^{BGS}_{Q}(g^{TX_t}, h^{\xi}), \tag{3.33}$$

where $\chi(X_t, \xi|_{X_t}) = \dim H^0(X_t, \xi|_{X_t}) - \dim H^1(X_t, \xi|_{X_t})$ is the Euler characteristic. By Riemann-Roch theorem, the value $\chi(X_t, \xi|_{X_t})$ depends only on the topological invariants of X_t and $\xi|_{X_t}$, and thus, by the flatness of the family, is constant over $S \setminus |\Delta|$.

We denote by $\|\cdot\|^{\xi}_{Q}(g^{TY_0}, \rho^*(h^{\xi}))$ the norm on the complex line $\lambda(j^*\xi) \otimes (\otimes_{i=1}^{2k} \det \xi|_{P_i})^6$ induced by $\|\cdot\|_{Q}(g^{TY_0}, \rho^*(h^{\xi}))$ and h^{ξ}. Similarly, due to our normalization of the L^2-norm, (2.3), the difference between our definition of the norm $\|\cdot\|^{\xi}_{Q}(g^{TY_0}, \rho^*(h^{\xi}))$, and the one from [6], [3], which we denote by $\|\cdot\|^{\xi,BGS}_{Q}(g^{TY_0}, \rho^*(h^{\xi}))$, is

$$\begin{aligned}\|\cdot\|^{\xi}_{Q}&(g^{TY_0}, \rho^*(h^{\xi})) \\ &= \exp\big(\log(2\pi) \cdot \chi(Y_0, \rho^*(\xi)|_{Y_0})/2\big) \cdot \|\cdot\|^{\xi,BGS}_{Q}(g^{TY_0}, \rho^*(h^{\xi})).\end{aligned} \tag{3.34}$$

We fix a smooth frame υ of $\lambda(j^*\xi)$ over S. As g^{TX} is Kähler, we can apply [3, Theorem 0.3, (0.5)], to see that under the isomorphisms (3.7), (3.8), the following identity holds

$$(3.35)\quad \lim_{t\to 0}\Big(\log\big(\|\upsilon(t)\|_Q^{BGS}(g^{TX_t},h^{\xi})\big)-\frac{\mathrm{rk}(\xi)}{6}\log\big(\big\|s_\Delta(t)\big\|_\Delta\big)\Big)$$
$$=\log\big(\|\upsilon(0)\|_Q^{\xi,BGS}(g^{TY_0},\rho^*(h^{\xi}))\big)+\frac{\mathrm{rk}(\xi)}{6}\log\Big\|d\pi^2\Big\|+2\zeta'(-1)\cdot k\cdot\mathrm{rk}(\xi).$$

Now, by (3.6) and the induced long exact sequence, we deduce $\chi(X_t,\xi|_{X_t}) = \chi(Y_0,\rho^*(\xi)) - k\cdot\mathrm{rk}(\xi)$. By this, (2.31) and (3.32), we see that (3.35) is a restatement of (3.31).

Now let's prove (3.31) for non-necessarily Kähler metrics g_0^{TX}. We note that π is locally projective (cf. Bismut-Bost [4, Proposition 3.4]), thus for some small neighborhood U of $0\in S$, we may find a Kähler metric g^{TX} over $\pi^{-1}(U)$. As the statement of Theorem 3.2 is local over the base, without losing the generality, we may suppose that g^{TX} is defined over X. We denote by $\|\cdot\|_{\mathscr{L}}^{\mathrm{cmp},0}$ the norm on $\mathscr{L}$, induced by g_0^{TX}. The idea of the proof is to use the above result and the anomaly formula to relate the norms $\|\cdot\|_{\mathscr{L}}^{\mathrm{cmp},0}$ and $\|\cdot\|_{\mathscr{L}}^{\mathrm{cmp}}$ near the locus of singular curves.

We denote by $\|\cdot\|_{\Sigma_{X/S}/X}^{0}$ the norm on the line bundle $\otimes_{i=1}^{k}(\omega_{Y_0}|_{P_{2i-1}}\otimes\omega_{Y_0}|_{P_{2i}})$, induced by g_0^{TX} as in (3.26). Similarly to the functions a_i, b_i, c_i from (3.25), we define the functions a_i^0, b_i^0, c_i^0 associated with g_0^{TX}.

Let us assume that $a_i^0, c_i^0 = 1, b_i^0 = 0$. This loses no generality, as we can fix a Riemannian metric g_*^{TX} which is compatible with the complex structure satisfying this assumption and then by applying Theorem 3.2 twice for g_*^{TX} and g^{TX} and for g_*^{TX} and g_0^{TX}, we infer the original statement.

Now, by (3.26) and the above assumption, we trivially have

$$(3.36)\qquad 2\log\Big(\|\cdot\|_{\Sigma_{X/S}/X}/\|\cdot\|_{\Sigma_{X/S}/X}^{0}\Big)=-\sum_{i=1}^{k}\log(a_ic_i-|b_i|^2)(Q_i).$$

Let the differential form F on X be given by

$$(3.37)\qquad F=\widetilde{\mathrm{Td}}\big(TX/S,g^{TX/S},g_0^{TX/S}\big)\mathrm{ch}\big(\xi,h^{\xi}\big),$$

where $g^{TX/S}, g_0^{TX/S}$ are the Hermitian norms on TX/S induced by g^{TX}, g_0^{TX}. Now, as the map π is a submersion away from $\Sigma_{X/S}$, and the metrics g^{TX}, g_0^{TX} are smooth over X, by the first part of the proof, (3.36) and the anomaly formula of Bismut-Gillet-Soulé [6] (cf. Theorem 2.4 for $m=0$), to prove Theorem 3.2, it is enough to establish that for any $i=1,\dots,k$, the following holds

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \int_{X_t \cap U(Q_i, \epsilon)} F = \frac{\mathrm{rk}(\xi)}{6} \log(a_i c_i - |b_i|^2)(Q_i). \tag{3.38}$$

For brevity, we fix $1 \leq i \leq k$, and denote $z_0 := z_0^i$, $z_1 := z_1^i$. As $z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}$ is a local holomoprhic frame of TX/S, locally around Q_i, we have

$$\begin{aligned} g^{TX/S}\Big(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}, z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}\Big) \\ = a_i |z_0|^2 + c_i |z_1|^2 - b_i z_0 \overline{z}_1 - \overline{b}_i z_1 \overline{z}_0. \end{aligned} \tag{3.39}$$

By using the fact that $z_0 z_1 = t$ over X_t, we deduce that locally around Q_i, we have

$$\begin{aligned} c_1(TX/S, g^{TX})|_{X_t} &= \frac{\partial \overline{\partial}}{2\pi\sqrt{-1}} \Big(\log \big(a_i |z_0|^2 + c_i |z_1|^2 - b_i z_0 \overline{z}_1 - \overline{b}_i z_1 \overline{z}_0 \big) \Big) \\ &= \frac{4(a_i c_i - |b_i|^2)|z_0|^2 |t|^2}{(a_i |z_0|^4 + c_i |t|^2 - b_i z_0^2 \overline{t} - \overline{b}_i \overline{z}_0^2 t)^2} \frac{dz_0 d\overline{z}_0}{2\pi\sqrt{-1}} + o\bigg(\Big(\frac{|t|^2}{|z_0|^6} + \frac{|z_0|^2}{|t|^2} \Big) dz_0 d\overline{z}_0 \bigg). \end{aligned} \tag{3.40}$$

As we are only interested in the limit (3.38), we may suppose that a_i, b_i, c_i are constants. By this, the change of variables $y_0 = z_0 |t|^{-1/2}$ and (3.10), (3.11), we see that (3.38) now reduces to proving that for any $a, c > 0$, $b \in \mathbb{R}$, $ac - b^2 > 0$, we get

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Big(\log \frac{a(x^2+y^2) + c - 2bx}{x^2 + y^2 + 1} \Big) \\ \times \Big(\frac{4ac - 4b^2}{(a(x^2+y^2) + c - 2bx)^2} + \frac{4}{(x^2+y^2+1)^2} \Big) dx dy \\ = 4\pi \log(ac - b^2). \end{aligned} \tag{3.41}$$

But (3.41) can be verified directly by switching to polar coordinates, changing the integration over the radius by the integration over its square and applying tedious derivation by parts. □

3.3. Univesality in restriction theorem, proofs of Theorems 1.7, 3.1

The goal of this section is to prove Theorems 1.7, 3.1.

Let's first define the notion of *regularized integral* on a surface with cusps $(\overline{M}, D_M, g^{TM})$. Let α be a $(1,1)$-form on M. We suppose that for any $P_i \in D_M$, there are coordinates z_i around $P_i \in D_M$, such that for some $\epsilon > 0$ small enough, there are $C \in \mathbb{C}$, $l \in \mathbb{N}$ such that the following estimate holds

$$\alpha|_{\{|z_i|<\epsilon\}} = -\frac{C \cdot \sqrt{-1} dz_i d\overline{z}_i}{|z_i|^2 |\log|z_i||} + O\Big(\frac{\log|\log|z_i||^l dz_i d\overline{z}_i}{|z_i \log|z_i||^2}\Big). \tag{3.42}$$

We define $\int^{\mathbf{r}}_M \alpha \in \mathbb{C}$ by the following limit

$$\int^{\mathbf{r}}_M \alpha = \lim_{\epsilon \to 0} \Big(\int_{M \setminus (\cup \{|z_i|<\epsilon\})} \alpha + 4\pi \cdot C \cdot (\# D_M) \cdot \log|\log \epsilon| \Big). \tag{3.43}$$

In other words, $\int^{\mathbf{r}}_M \alpha$ is the non-divergent part of $\int_{M \setminus (\cup \{|z_i|<\epsilon\})} \alpha$, as $\epsilon \to 0$. It is an easy verification that $\int^{\mathbf{r}}_M \alpha$ doesn't depend on the choice of the coordinates z_i.

Proof of Theorem 3.1. The main idea of the proof is to give a general construction of a special norm $\|\cdot\|^{\deg}_{X/S}$ on $\omega_{X/S}$ over $X \setminus |\pi^{-1}(\Delta)|$ for any $\pi : X \to S$ using the local coordinates z_i near Q_i, for which the assumptions (1.8), (1.19) hold, and to show that Theorem 3.1 holds for $\|\cdot\|^{\deg}_{X/S}$ (i.e. the value $\|\cdot\|^{X/S}_{\mathscr{L}_n}|_{S'} / \|\cdot\|^{Y/S'}_{\mathscr{L}_n}$ is independent of $\pi : X \to S$, etc., once $\omega_{X/S}$ is endowed with $\|\cdot\|^{\deg}_{X/S}$). Then, by the anomaly formula, Theorem 2.4, and Proposition 2.12, we deduce that Theorem 3.1 holds in its full generality.

More precisely, we proceed in the following way. First, we construct a certain Riemannian metric g^{TX}_{sm} compatible with the complex structure on X. By using Theorem 3.2, we will calculate the asymptotics of the norm $\|\cdot\|^{\mathrm{sm}}_{\mathscr{L}_n}$ induced on the line bundle $\mathscr{L}_n^{X/S}$ by g^{TX}_{sm}. Then by modifying locally this metric in the neighborhood of $\Sigma_{X/S}$, we construct a family of metrics $g^{TX_t}_{\deg}$ on X_t for $t \in S \setminus |\Delta|$, which degenerates to the hyperbolic metric at the singular fiber through the family of degenerating hyperbolic cylinders. The construction of the metric $g^{TX_t}_{\deg}$ is highly motivated by (2.35). By applying Theorem 2.4, and the previous result on the asymptotics of $\|\cdot\|^{\mathrm{sm}}_{\mathscr{L}_n}$, we compute the asymptotics of the norm $\|\cdot\|^{\deg}_{\mathscr{L}_n}$ induced on the line bundle $\mathscr{L}_n^{X/S}$ by $g^{TX_t}_{\deg}$.

Now, let's give a precise construction of the metric g^{TX}_{sm}. Fix the local coordinates z^j_0, z^j_1 (cf. (3.24)) around Q_j as in (2.31). For simplicity, assume $D(1) \subset \mathrm{Im}(z^j_0) \cap \mathrm{Im}(z^j_1)$. Recall that $U(Q_i, \epsilon)$, $\epsilon > 0$ was defined in (2.30).

We specify the function ν from (2.32) as follows

$$\nu(x) = \begin{cases} 0, & \text{for } x \in X \setminus (\cup_{i=1}^k U(Q_i, 1)), \\ 1 - \nu_0(|z_0^i|^2 + |z_1^i|^2), & \text{for } x \in U(Q_i, 1), \end{cases} \tag{3.44}$$

where ν_0 was defined in (3.12). By (3.12), the function (3.44) satisfies (2.32) for $\epsilon = 1/2$.

As π is locally projective (cf. Bismut-Bost [4, Proposition 3.4]), there is a neighborhood U of $0 \in S$ and a Kähler metric g_0^{TX} over $\pi^{-1}(U)$. But Theorem 3.2 is local over the base, so we may suppose from now on that g_0^{TX} is defined over X. We define the Riemannian metric g_{sm}^{TX} over X so that it coincides with g_0^{TX} over $X \setminus (\cup_{i=1}^k U(Q_i, 1))$, and over $U(Q_i, 1)$ it is given by

$$g_{\mathrm{sm}}^{TX} = (1 - \nu) \cdot g_0^{TX} + \nu \cdot (|dz_0^i|^2 + |dz_1^i|^2). \tag{3.45}$$

We denote by $g_{\mathrm{sm}}^{TX_t}$ the induced metric on X_t, $t \in S \setminus |\Delta|$, and define $g_{\mathrm{sm}}^{TY_0} = \rho^*(g_{\mathrm{sm}}^{TX})$, where $\rho : Y_0 \to X_0$ is the normalization map. The metric g_{sm}^{TX} is not necessarily Kähler, but it is compatible with the complex structure of X. In particular, the metrics $g_{\mathrm{sm}}^{TX_t}$, $g_{\mathrm{sm}}^{TY_0}$ are Kähler.

We endow $\omega_{X/S}$ with the Hermitian norm $\|\cdot\|_{X/S}^{\mathrm{sm,ind}}$ induced by g_{sm}^{TX} over $X \setminus \Sigma_{X/S}$. Let $\widetilde{\nu} : X \to [0, 1]$ be defined as ν in (3.44), where in place of $\nu_0(\cdot)$, we put $\nu_0(4\cdot)$. Then $\widetilde{\nu}(x) = 1$ for $x \in X \setminus (\cup_{i=1}^k U(Q_i, 1/2))$. We define the Hermitian norm $\|\cdot\|_{X/S}^{\mathrm{sm}}$ on $\omega_{X/S}$ over X as follows. Over $X \setminus (\cup_{i=1}^k U(Q_i, 1/2))$, we demand it to be equal to $\|\cdot\|_{X/S}^{\mathrm{sm,ind}}$, and over $U(Q_i, 1/2)$, we define it by

$$\left\| dz_0^i / z_0^i \right\|_{X/S}^{\mathrm{sm}} = (1 - \widetilde{\nu}) \cdot \left\| dz_0^i / z_0^i \right\|_{X/S}^{\mathrm{sm,ind}} + \widetilde{\nu}. \tag{3.46}$$

The Hermitian norm $\|\cdot\|_{X/S}^{\mathrm{sm}}$ on $\omega_{X/S}$ is smooth over X. Moreover, it is trivial on $\cup_{i=1}^k U(Q_i, 1/4)$. We also define the norm $\|\cdot\|_{Y_0}^{\mathrm{sm}} := \rho^*(\|\cdot\|_{X/S}^{\mathrm{sm}})$ over $\omega_{Y_0}(D)$.

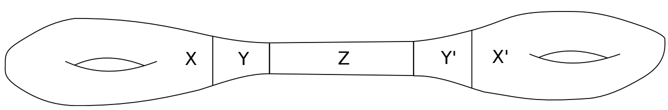

Figure 2: Over X, X', the metric $g_{\mathrm{sm}}^{TX_t}$ is induced by g_0^{TX}. Over Y, Y', it is an interpolation between g_0^{TX} and $|dz_0|^2 + |dz_1|^2$, and over Z, it is given by $|dz_0|^2 + |dz_1|^2$.

We endow $\mathscr{L}_n^{X/S}$ with the norm $\|\cdot\|_{\mathscr{L}_n}^{\mathrm{sm}}$, induced by the Quillen norm $\|\cdot\|_Q\,(g_{\mathrm{sm}}^{TX_t}, h^{\xi} \otimes (\|\cdot\|_{X/S}^{\mathrm{sm}})^{2n})$ and the singular norm (1.6). We endow $\mathscr{L}_n'$ with the norm $\|\cdot\|_{\mathscr{L}_n'}^{\mathrm{sm}}$, induced by the Quillen norm $\|\cdot\|_Q\,(g_{\mathrm{sm}}^{TY_0}, \rho^*(h^{\xi}) \otimes (\|\cdot\|_{Y_0}^{\mathrm{sm}})^{2n})$ and the norm $\|\cdot\|_{\Sigma_{X/S}/X}^{\mathrm{sm}}$ (see (3.26)) on $\otimes_{i=1}^{k}(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}})$.

The metrics $g_{\mathrm{sm}}^{TX_t}, \|\cdot\|_{X/S}^{\mathrm{sm}}$ and $g_{\mathrm{sm}}^{TY_0}, \|\cdot\|_{Y_0}^{\mathrm{sm}}$ satisfy the assumptions of Theorem 3.2. Let

$$A_{-n}' := 24\zeta'(-1) - 6\log(2\pi). \tag{3.47}$$

By Theorem 3.2, (3.26) and (3.45), for a frame υ of $\mathscr{L}_n^{X/S}$, under the isomorphism (1.21), we have

$$\begin{aligned} \lim_{t\to 0}\Big(\log\left(\|\upsilon(t)\|_{\mathscr{L}_n}^{\mathrm{sm}}\right) - k\cdot \mathrm{rk}(\xi)\cdot \log|t|\Big) \\ = \log\left(\|\upsilon(0)\|_{\mathscr{L}_n'}^{\mathrm{sm}}\right) + k\cdot \mathrm{rk}(\xi)\cdot A_{-n}'. \end{aligned} \tag{3.48}$$

Let's now modify the metric $g_{\mathrm{sm}}^{TX_t}$ to $g_{\mathrm{deg}}^{TX_t}$, so that $g_{\mathrm{deg}}^{TX_t}$ satisfies the assumptions of Theorem 1.1. We define $g_{\mathrm{deg}}^{TX_t}$ on X_t, $t \in S \setminus |\Delta|$, as follows: over $X_t \setminus (\cup_{i=1}^{k} U(Q_i, 1/2))$ it coincides with $g_{\mathrm{sm}}^{TX_t}$, and over $U(Q_i, 1/2)$ it is given by

$$g_{\mathrm{deg}}^{TX_t} := (1-\widetilde{\nu})\cdot g_{\mathrm{sm}}^{TX_t} + \widetilde{\nu}\cdot g_{i,t}^{\mathrm{Cyl}}, \tag{3.49}$$

where the metric $g_{j,t}^{\mathrm{Cyl}}$ was defined in (2.34). We also define the metric $g_{\mathrm{deg}}^{TY_0}$ as follows: over $Y_0 \setminus (\cup_{i=1}^{k} U(Q_i, 1/2))$ it coincides with $g_{\mathrm{sm}}^{TY_0}$, and over $U(Q_i, 1/2)$ it is given by

$$g_{\mathrm{deg}}^{TY_0} := (1-(\widetilde{\nu}\circ\rho))\cdot g_{\mathrm{sm}}^{TY_0} + (\widetilde{\nu}\circ\rho)\cdot\left(g_{i,0}^{\mathrm{Poinc}} + g_{i,1}^{\mathrm{Poinc}}\right), \tag{3.50}$$

where the metrics $g_{i,0}^{\mathrm{Poinc}}, g_{i,1}^{\mathrm{Poinc}}$ are the metrics induced by the Poincaré metric (1.1) with respect to the coordinates z_0^i and z_1^i. We denote by $\|\cdot\|_{X/S}^{\mathrm{deg}}$ the Hermitian norm on $\omega_{X/S}$ induced by $g_{\mathrm{deg}}^{TX_t}$. By (3.49) (cf. Proposition 2.15), we see that the Hermitian norm $\|\cdot\|_{X/S}^{\mathrm{deg}}$ extends smoothly over $X \setminus \Sigma_{X/S}$, and the assumptions (1.8) are satisfied. We define the norm $\|\cdot\|_{Y_0}^{\mathrm{deg}}$ on $\omega_{Y_0}(D)$ by

$$\|\cdot\|_{Y_0}^{\mathrm{deg}} = \rho^*(\|\cdot\|_{X/S}^{\mathrm{deg}}). \tag{3.51}$$

Then we see that $\|\cdot\|_{X/S}^{\mathrm{deg}}$ satisfies assumptions (1.19), and by (3.49), (3.50), the associated metric on $Y_0 \setminus D_{Y_0}$, constructed as in Section 1, coincides with $g_{\mathrm{deg}}^{TY_0}$.

Let's pause and explain this construction. By Section 2.4, the metrics $g^{TX_t}_{\mathrm{deg}}$ degenerate near the singular fibers to a metric with cusps in the similar way as the hyperbolic metrics. The advantage of the metrics $g^{TX_t}_{\mathrm{deg}}$ over the hyperbolic one is that over the region $\cup_{i=1}^k U(Q_i, 1/2)$, it is independent of any exterior data (as $\pi : X \to S$), and over $X_t \setminus (\cup_{i=1}^k U(Q_i, 1/2))$, the metric $g^{TX_t}_{\mathrm{deg}}$ coincides with a metric $g^{TX_t}_{\mathrm{sm}}$, for which Theorem 3.2 holds.

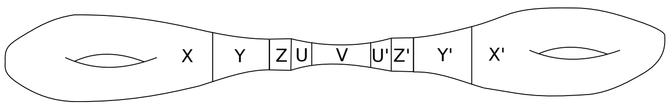

Figure 3: The metric $g^{TX_t}_{\mathrm{deg}}$. Over the regions X, Y, Z, Z', Y', X' it coincides with $g^{TX_t}_{\mathrm{sm}}$. Over the regions U, U', it is an interpolation between $g^{TX_t}_{\mathrm{sm}}$ and the hyperbolic cylinder metric, (2.34), and over the region V, it coincides with the hyperbolic cylinder metric, (2.34).

To get the asymptotic near the singular fibers of the Hermitian norm $\|\cdot\|^{\mathrm{deg}}_{\mathscr{L}_n}$ on the holomorphic line bundle $\mathscr{L}_n^{X/S}$, induced by the Quillen norm $\|\cdot\|_Q\,(g^{TX_t}_{\mathrm{deg}}, h^\xi \otimes (\|\cdot\|^{\mathrm{deg}}_{X/S})^{2n})$ and the singular norm (1.6), it is enough to apply the anomaly formula and use (3.48). More precisely, let $\|\cdot\|^{W,\mathrm{deg}}_{Y_0}$ be the Wolpert norm on $\otimes_{i=1}^k (\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}})$. We endow $\mathscr{L}'_n$ with the Hermitian norm $\|\cdot\|^{\mathrm{deg}}_{\mathscr{L}'_n}$, induced by the Quillen norm $\|\cdot\|_Q\,(g^{TY_0}_{\mathrm{deg}}, \rho^*(h^\xi) \otimes (\|\cdot\|^{\mathrm{deg}}_{Y_0})^{2n})$ and the Wolpert norm $\|\cdot\|^{W,\mathrm{deg}}_{Y_0}$.

Recall that the regularized integral was defined in (3.43). For $n \in \mathbb{N}$ we define $A''_{-n} \in \mathbb{R}$ as follows (from (1.1), (3.10) and (3.11), the regularized integral below is well-defined)

$$
\begin{aligned}
(3.52)\quad A''_{-n} := 1 + 6k^{-1} \cdot \int_{Y_0}^{\mathbf{r}} \Big(& \widetilde{\mathrm{Td}}\big(\omega_{Y_0}^{-1}, g^{TY_0}_{\mathrm{sm}}, g^{TY_0}_{\mathrm{deg}}\big) \mathrm{ch}\big(\omega_{Y_0}(D)^n, (\|\cdot\|^{\mathrm{sm}}_{Y_0})^{2n}\big) \\
& + \mathrm{Td}\big(\omega_{Y_0}^{-1}, g^{TY_0}_{\mathrm{deg}}\big) \widetilde{\mathrm{ch}}\big(\omega_{Y_0}(D)^n, (\|\cdot\|^{\mathrm{sm}}_{Y_0})^{2n}, (\|\cdot\|^{\mathrm{deg}}_{Y_0})^{2n}\big)\Big).
\end{aligned}
$$

By (3.45), (3.46), (3.49), (3.50) and (3.52), we see that the integral in A''_{-n} is a sum of $2k$ identical contributions, one for each cusp, and those contributions do not depend on the global geometry of $\pi : X \to S$ or ξ. In fact, A''_{-n} depends only on $n \in \mathbb{N}$ and ν_0. We would like to prove that the following holds

$$
(3.53)\quad \lim_{t \to 0} \Big(\log \big(\|\cdot\|^{\mathrm{deg}}_{\mathscr{L}_n} / \|\cdot\|^{\mathrm{sm}}_{\mathscr{L}_n}\big)(t) + k \cdot \mathrm{rk}(\xi) \cdot \log|t| \Big) = k \cdot \mathrm{rk}(\xi) \cdot A''_{-n}.
$$

Assume we proved (3.53). Remark that by Theorem 2.1, (3.26), (3.45), (3.50), (3.51) and the similar reasoning as below (3.52), the value

$$A'''_{-n} := (k \cdot \mathrm{rk}(\xi))^{-1} \log \big(\|\cdot\|^{\deg}_{\mathscr{L}'_n} / \|\cdot\|^{\mathrm{sm}}_{\mathscr{L}'_n} \big) \tag{3.54}$$

depends only the choice of $n \in \mathbb{N}$ and ν_0. Thus, by (3.48), (3.53) and (3.54), we deduce that under the isomorphism (1.21), the following identity holds (3.55)

$$\|\cdot\|^{\deg}_{\mathscr{L}_n}|_{|\Delta|} = \exp(k \cdot \mathrm{rk}(\xi) \cdot A_{-n}) \cdot \|\cdot\|^{\deg}_{\mathscr{L}'_n} \quad \text{with} \quad A_{-n} := A'_{-n} + A''_{-n} - A'''_{-n}.$$

So Theorem 3.1 holds for $\|\cdot\|^{\deg}_{X/S}$ and the universal constant A_{-n}, defined above. But then by applying anomaly formula, and by proceeding similarly to *Steps 1, 2* of the proof of Theorem 1.1, we see that Theorem 3.1 holds in its full generality.

Now, to prove (3.53), we apply the anomaly formula of Bismut-Gillet-Soulé [6] (cf. Theorem 2.4 for $m = 0$). By the triviality of (ξ, h^{ξ}) near $\Sigma_{X/S}$, and the fact that $g^{TX_t}_{\deg}$ coincides with $g^{TX_t}_{\mathrm{sm}}$ away from a small neighborhood of $\Sigma_{X/S}$, for any $t \in S \setminus |\Delta|$, we have

$$\log \Big(\|\cdot\|^{\deg}_{\mathscr{L}_n} / \|\cdot\|^{\mathrm{sm}}_{\mathscr{L}_n} \Big)(t) = 6 \cdot \mathrm{rk}(\xi) \cdot \int_{X_t} G, \tag{3.56}$$

where the differential form G is given by

$$\begin{aligned} G = \Big(& \widetilde{\mathrm{Td}}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\mathrm{sm,ind}}_{X/S})^{-2}, (\|\cdot\|^{\deg}_{X/S})^{-2}\big) \mathrm{ch}\big(\omega^n_{X/S}, (\|\cdot\|^{\mathrm{sm}}_{X/S})^{2n}\big) \\ & + \mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\deg}_{X/S})^{-2}\big) \widetilde{\mathrm{ch}}\big(\omega^n_{X/S}, (\|\cdot\|^{\mathrm{sm}}_{X/S})^{2n}, (\|\cdot\|^{\deg}_{X/S})^{2n}\big)\Big)^{[2]}. \end{aligned} \tag{3.57}$$

We decompose $G = G_1 + G_2$, where

$$G_1 = \widetilde{\mathrm{Td}}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\mathrm{sm,ind}}_{X/S})^{-2}, (\|\cdot\|^{\deg}_{X/S})^{-2}\big)^{[2]}, \tag{3.58}$$

$$\begin{aligned} G_2 = \Big(& \widetilde{\mathrm{Td}}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\mathrm{sm,ind}}_{X/S})^{-2}, (\|\cdot\|^{\deg}_{X/S})^{-2}\big)^{[0]} \mathrm{ch}\big(\omega^n_{X/S}, (\|\cdot\|^{\mathrm{sm}}_{X/S})^{2n}\big) \\ & + \mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\deg}_{X/S})^{-2}\big) \widetilde{\mathrm{ch}}\big(\omega^n_{X/S}, (\|\cdot\|^{\mathrm{sm}}_{X/S})^{2n}, (\|\cdot\|^{\deg}_{X/S})^{2n}\big)\Big)^{[2]}. \end{aligned} \tag{3.59}$$

By (3.58), (3.59) and the fact that the norms $\|\cdot\|^{\mathrm{sm,ind}}_{X/S}, \|\cdot\|^{\mathrm{sm}}_{X/S}, \|\cdot\|^{\deg}_{X/S}$ coincide over $X \setminus (\cup_{i=1}^k U(Q_i, 1/2))$, we conclude that G_i, $i = 1, 2$ have support over $\cup_{i=1}^k U(Q_i, 1/2)$.

Now, the norm $\|\cdot\|^{\mathrm{sm}}_{X/S}$ is smooth over X and by Proposition 2.15, the norm $\|\cdot\|^{\deg}_{X/S}$ is good on $X \setminus \pi^{-1}(|\Delta|)$ with singularities along $\pi^{-1}(\Delta)$, and it has log-log growth along $\Sigma_{X/S}$. By this, the first summand in (3.59) has logarithmic singularities in the neighborhood of $\Sigma_{X/S}$ and the integrand satisfies the assumptions of Proposition 2.12. Hence, by Proposition 2.12 and (3.10), (3.11), (3.59), we conclude that

$$\lim_{t\to 0}\int_{X_t} G_2 = \int_{Y_0} \Big(\widetilde{\mathrm{Td}}\big(\omega_{Y_0}^{-1}, g^{TY_0}_{\mathrm{sm}}, g^{TY_0}_{\mathrm{deg}}\big)^{[0]}\mathrm{ch}\big(\omega_{Y_0}(D)^n, (\|\cdot\|^{\mathrm{sm}}_{Y_0})^{2n}\big) + \mathrm{Td}\big(\omega_{Y_0}^{-1}, g^{TY_0}_{\mathrm{deg}}\big)\widetilde{\mathrm{ch}}\big(\omega_{Y_0}(D)^n, (\|\cdot\|^{\mathrm{sm}}_{Y_0})^{2n}, (\|\cdot\|^{\mathrm{deg}}_{Y_0})^{2n}\big)\Big). \tag{3.60}$$

By (3.56) and (3.60), we see that to prove (3.53), it is enough to establish

$$\lim_{t\to 0}\Big(\int_{X_t} G_1 + k\cdot\frac{\log|t|}{6}\Big) = \int^{\mathbf{r}}_{Y_0} \widetilde{\mathrm{Td}}\big(\omega_{Y_0}^{-1}, g^{TY_0}_{\mathrm{sm}}, g^{TY_0}_{\mathrm{deg}}\big) + \frac{k}{6}. \tag{3.61}$$

To show (3.61), we remark that by (3.40) and (3.45), for $t \in S\setminus|\Delta|$, we have

$$c_1(\omega_{X/S}^{-1}, (\|\cdot\|^{\mathrm{sm,ind}}_{X/S})^{-2})|_{X_t\cap U(Q_i,1/2)} = \frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i}{2\pi\sqrt{-1}}. \tag{3.62}$$

By the fact that the norm $\|\cdot\|^{\mathrm{sm,ind}}_{X/S}$ coincides with $\|\cdot\|^{\mathrm{deg}}_{X/S}$ away from $U(Q_i,1/2)$, by Green identities and (3.10), (3.11), (3.45), (3.49), (3.62), we see that the following identity holds

$$\int_{X_t} G_1 = \frac{1}{12}\sum_{i=1}^k \int_{2|t|<|z_0^i|<1/2} \log\big(|z_0^i|^2 + |t/z_0^i|^2\big)\frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i}{2\pi\sqrt{-1}} - \frac{1}{6}\sum_{i=1}^k \int_{2|t|<|z_0^i|<1/2} \log\Big(\frac{1}{\sqrt{2}}\Big\|\frac{dz_0^i}{z_0^i} - \frac{dz_1^i}{z_1^i}\Big\|^{\mathrm{deg}}_{X/S}\Big) c_1(\omega_{X/S}, (\|\cdot\|^{\mathrm{deg}}_{X/S})^2). \tag{3.63}$$

After a change of variables $y := z_0^i\cdot|t|^{-1/2}$, a simple calculation yields

$$\int_{2|t|<|z_0^i|<1/2} \log\big(|z_0^i|^2 + |t/z_0^i|^2\big)\frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i}{2\pi\sqrt{-1}} = -2\log|t| - 2 + o(1). \tag{3.64}$$

Also, we see easily by (3.49) that

$$\begin{aligned}(3.65)\quad \lim_{t\to 0}\int_{2|t|<|z_0^i|<1/2} &\log\Big(\frac{1}{\sqrt{2}}\Big\|\frac{dz_0^i}{z_0^i}-\frac{dz_1^i}{z_1^i}\Big\|^{\deg}_{X/S}\Big)c_1(\omega_{X/S},(\|\cdot\|^{\deg}_{X/S})^2)\\ &= -\int_{0<|z_0^i|<1/2}\log\Big((1-\widetilde{\nu})|z_0^i|^2+\frac{\widetilde{\nu}}{(\log|z_0^i|)^2}\Big)c_1(\omega_{Y_0}(D),(\|\cdot\|^{\deg}_{Y_0})^2).\end{aligned}$$

Thus, by (3.10), (3.11), (3.45), (3.49) and the fact that $c_1(\omega_{Y_0}^{-1}, g_{\mathrm{sm}}^{TY_0}) = 0$ over $\{|z_0^i| < 1/2\}$:

$$\begin{aligned}(3.66)\quad &\sum_{i=1}^{k}\int_{0<|z_0^i|<1/2}\log\Big((1-\widetilde{\nu})|z_j^i|^2+\frac{\widetilde{\nu}}{(\log|z_j^i|)^2}\Big)c_1(\omega_{Y_0}(D),(\|\cdot\|^{\deg}_{Y_0})^2)\\ &= 6\int^{\mathbf{r}}_{Y_0}\widetilde{\mathrm{Td}}\big(\omega_{Y_0}^{-1},g^{TY_0}_{\mathrm{sm}},g^{TY_0}_{\deg}\big)+2k\cdot\int^{\mathbf{r}}_{0<|z_0^i|<1/2}\log|z_j^i|c_1(\omega_{Y_0}(D),(\|\cdot\|^{\deg}_{Y_0})^2).\end{aligned}$$

However, by (3.49) and Green identities, we have

$$\begin{aligned}(3.67)\quad \int_{\epsilon<|z_0^i|<1/2}&\log|z_0^i|\cdot c_1(\omega_{Y_0}(D),(\|\cdot\|^{\deg}_{Y_0})^2)\\ &= \frac{1}{4\pi}\int_{\epsilon<|z_0^i|<1/2}\log|z_0^i|\cdot\Delta\log\Big((1-\widetilde{\nu})+\frac{\widetilde{\nu}}{|z_0^i\log|z_0^i||^2}\Big)dxdy\\ &= 1-\log|\log\epsilon|.\end{aligned}$$

Thus, we deduce by (3.43), (3.67) that

$$(3.68)\qquad \int^{\mathbf{r}}_{0<|z_0^i|<1/2}\log|z_0^i|c_1(\omega_{Y_0}(D),(\|\cdot\|^{\deg}_{Y_0})^2)=1.$$

By (3.63)–(3.68) the proof of (3.61), and thus of (3.53), is complete. □

Proof of Theorem 1.7. First, for fixed $n\in\mathbb{N}$, $(\overline{M}, D_M, g^{TM})$, g^{TM}, $g^{T\overline{M}}$, $\|\cdot\|_{\overline{M}}$, h^ξ, h_0^ξ, we define $E_{-n}\in\mathbb{R}$ by equality (1.35). Let's prove the independence of E_{-n} on g^{TM}, $g^{T\overline{M}}$, $\|\cdot\|_{\overline{M}}$, h^ξ, h_0^ξ.

To be brief, we restrict ourselves to the proof of independence on $g^{T\overline{M}}$. By the anomaly formula of Bismut-Gillet-Soulé [6] (cf. Theorem 2.4 for $m = 0$), it is enough to establish that for any Kähler metrics $g_1^{T\overline{M}}$, $g_2^{T\overline{M}}$ over $\overline{M}$, we have

$$\int_M^{\mathbf{r}} \Big[\Big(\widetilde{\mathrm{Td}}\big(\omega_{\overline{M}}^{-1}, g_1^{T\overline{M}}, g^{TM}\big) - \widetilde{\mathrm{Td}}\big(\omega_{\overline{M}}^{-1}, g_2^{T\overline{M}}, g^{TM}\big) \Big) \times \mathrm{ch}\big(\xi, h_0^{\xi}\big) \mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big) \Big]^{[2]} = \int_{\overline{M}} \Big[\widetilde{\mathrm{Td}}\big(\omega_{\overline{M}}^{-1}, g_1^{T\overline{M}}, g_2^{T\overline{M}}\big) \mathrm{ch}\big(\xi, h_0^{\xi}\big) \mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big) \Big]^{[2]} + \frac{\mathrm{rk}(\xi)}{6} \log \Big(\|\cdot\|_{\overline{M},1}^{D_M} / \|\cdot\|_{\overline{M},2}^{D_M} \Big), \tag{3.69}$$

where the norms $\|\cdot\|_{\overline{M},1}^{D_M}, \|\cdot\|_{\overline{M},2}^{D_M}$ on the complex line $\otimes_{P \in D_M} \omega_{\overline{M}}|_P$ are induced by $g_1^{T\overline{M}}$ and $g_2^{T\overline{M}}$ respectively. But (3.69) follows from (3.10), (3.11) and Green identities.

Now, we proved that E_{-n} is a universal constant, depending only on $n \in \mathbb{N}$. To show that it is indeed equal to (1.34), it is enough to verify this for at least one example. We will do so for $\overline{M} = Y_0$, $g^{T\overline{M}} := g_{\mathrm{sm}}^{TY_0}$, $g^{TM} := g_{\mathrm{deg}}^{TY_0}$, $\xi := \rho^* \xi$, $h^{\xi} = h_0^{\xi} := \rho^*(h^{\xi})$ in the notation from the proof of Theorem 3.1. For this choice of the data, by (3.54), we have

$$2 \log \Big(\|\cdot\|_Q \big(g_{\mathrm{deg}}^{TY_0}, \rho^*(h^{\xi}) \otimes (\|\cdot\|_{Y_0}^{\mathrm{deg}})^{2n}\big) / \|\cdot\|_Q \big(g_{\mathrm{sm}}^{TY_0}, \rho^*(h^{\xi}) \otimes (\|\cdot\|_{Y_0}^{\mathrm{sm}})^{2n}\big) \Big) - \frac{\mathrm{rk}(\xi)}{6} \log \Big(\|\cdot\|_{\mathrm{deg}}^{W} / \|\cdot\|_{\overline{M},\mathrm{sm}}^{D_M} \Big) = \frac{k \cdot \mathrm{rk}(\xi)}{6} A_{-n}''' \tag{3.70}$$

However, by (3.55), we have $A_{-n}''' = A_{-n}' + A_{-n}'' - A_{-n}$. By Theorem 1.1, A_{-n} is equal to C_{-n}. But by (3.47), (3.52), this exactly means that Theorem 1.7 holds for this particular data. This finishes the proof since E_{-n} is a universal constant. □

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