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p-complete arc-descent for perfect complexes over integral perfectoid rings

Kazuhiro Ito

We prove p-complete arc-descent results for finite projective modules and perfect complexes over integral perfectoid rings. Using our results, we clarify a reduction argument in the proof of the classification of p-divisible groups over integral perfectoid rings given by Scholze–Weinstein.

1. Introduction

We fix a prime number $p > 0$. In this paper, we will prove p-complete arc-descent results for finite projective modules and perfect complexes over (p-complete integral) perfectoid rings.

Before stating our main results, we first recall the following descent result due to Bhatt–Scholze [6, Theorem 11.2] for the v-topology and Bhatt–Mathew [4, Theorem 5.16] for the arc-topology. Following their work, we will use the ∞-categorical language in this paper. Let Cat_∞ denote the ∞-category of (small) ∞-categories. For a commutative ring A, the ∞-category of perfect complexes over A is denoted by $\mathrm{Perf}(A)$. Recall that, for a homomorphism $A \to B$ of commutative rings, we have a base change functor $\mathrm{Perf}(A) \to \mathrm{Perf}(B)$, $K \mapsto K \otimes^{\mathbb{L}}_A B$.

Theorem 1.1 (Bhatt–Scholze, Bhatt–Mathew). *The functor $A \mapsto \mathrm{Perf}(A)$ from the category of perfect $\mathbb{F}_p$-algebras to* Cat_∞ *satisfies* arc-*hyperdescent.*

The main result of this paper is the following analogous statement for perfectoid rings (in the sense of [5, Definition 3.5]). Here we use the ϖ-complete arc-topology introduced in [7, Section 2.2.1].

Theorem 1.2. *Let R be a perfectoid ring and $\varpi \in R$ an element with $p \in (\varpi^p)$ such that R is ϖ-complete. Then the functors*

$$A \mapsto \mathrm{Perf}(A^\flat), \quad A \mapsto \mathrm{Perf}(W(A^\flat)), \quad \textit{and} \quad A \mapsto \mathrm{Perf}(A)$$

from the category of ϖ-complete perfectoid R-algebras to Cat_∞ *satisfy ϖ-complete* arc*-hyperdescent. Here $A^\flat := \varprojlim_{x \mapsto x^p} A/pA$ is the tilt of A and $W(A^\flat)$ is the ring of Witt vectors of $A^\flat$.*

We will make precise what we mean by "ϖ-complete arc-hyperdescent" in Definition 2.1. From this theorem, we can deduce ϖ-complete arc-descent results for finite projective modules over perfectoid rings, which can be stated using classical category theory; see Corollary 4.2.

We need an analogue of Theorem 1.1 for derived quotients of perfect rings to prove Theorem 1.2. We formulate it in terms of $\mathbb{E}_\infty$-rings and their modules; see Theorem 3.1. We will present two proofs of Theorem 3.1; it can be proved in the same way as Theorem 1.1, and also can be deduced from Theorem 1.1. Even if one is only interested in descent results for finite projective modules (Corollary 4.2), such an analogue will be essential.

Remark 1.3. A similar statement to Theorem 1.2 has been conjectured by Henkel in [10, Conjecture A]. In fact, the proof of Theorem 1.2 shows that [10, Conjecture A] is true; see Remark 4.3 for additional details.

As an application of Theorem 1.2 (or Corollary 4.2), we will discuss the following classification result for p-divisible groups over perfectoid rings obtained by Lau [12, Theorem 9.8] in the case where $p \geq 3$, and by Scholze–Weinstein [20, Theorem 17.5.2] in general; see Theorem 5.2 for a more precise statement.

Theorem 1.4 (Lau, Scholze–Weinstein). *Let A be a perfectoid ring. The category of p-divisible groups over A is anti-equivalent to the category of minuscule Breuil–Kisin–Fargues modules for A.*

Remark 1.5. The strategy of Scholze–Weinstein is to deduce the general statement from the classification of p-divisible groups over perfectoid valuation rings of rank ≤ 1 with algebraically closed fraction fields, which is proved by Berthelot [2, Corollaire 3.4.3] in the equal characteristic case, and by Scholze–Weinstein [20, Theorem 14.4.1] (based on [19, Theorem B]) in the mixed characteristic case. In their original proof, however, there seems to be a technical issue in this reduction procedure.[1] We will verify it by using Theorem 1.2 (or Corollary 4.2) in Section 5.

This paper is organized as follows. In Section 2, we recall the definitions and some basic properties of perfectoid rings and ϖ-complete arc-covers. We also collect some results from the theory of $\mathbb{E}_\infty$-rings and their modules, which are used in the proof of Theorem 1.2. In Section 3, we state and prove an analogue of Theorem 1.1 for derived quotients of perfect rings (Theorem 3.1). In Section 4, we prove Theorem 1.2 and deduce descent results for finite projective modules (Corollary 4.2) from it. In Section 5, we give a proof of Theorem 1.4, following the approach of Scholze–Weinstein.

2. Preliminaries

2.1. Perfectoid rings and ϖ-complete arc-covers

In this subsection, we recall some basic facts about perfectoid rings and ϖ-complete arc-covers. Our basic references are [5, Section 3] and [7, Section 2].

Let us first recall the notion of a ϖ-complete arc-cover from [7, Section 2.2.1]. Let R be a commutative ring and $\varpi \in R$ an element. We say that a homomorphism $A \to B$ of R-algebras is a *ϖ-complete* arc-*cover* if, for any homomorphism $A \to V$ with V a ϖ-complete valuation ring of rank ≤ 1, there exist an extension of $V \hookrightarrow W$ of ϖ-complete valuation rings of rank ≤ 1 and a homomorphism $B \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \longrightarrow & W. \end{array}$$

A 0-complete arc-cover is just an arc-cover in the sense of [4, Definition 1.2]. The reduction modulo ϖ of a ϖ-complete arc-cover is an arc-cover.

[1]More precisely, we should replace the v-cover S over $R^\flat$ given in the last paragraph of the proof of [20, Theorem 17.5.2] with its ξ_0-completion to conclude that $W(S)/(\xi)$ is a perfectoid ring over R whose tilt is isomorphic to S (see also Proposition 2.3). Here $\xi = (\xi_0, \xi_1, \dots) \in W(R^\flat)$ is a generator of the kernel of the usual map $\theta\colon W(R^\flat) \to R$. Accordingly, we need to use Corollary 4.2 (or its "ϖ-complete v-descent" analogue, which, to the best of our knowledge, has not been proved before in the literature, either) instead of Theorem 1.1 or [6, Theorem 4.1]. We thank P. Scholze for e-mail correspondence on the proof of [20, Theorem 17.5.2].

Let R be a *perfectoid ring* in the sense of [5, Definition 3.5], i.e. R is ϖ-complete for some element $\varpi \in R$ with $p \in (\varpi^p)$, the Frobenius map $R/pR \to R/pR$ is surjective, and the kernel of $\theta\colon W(R^\flat) \to R$ is principal. Here

$$R^\flat := \varprojlim_{x \mapsto x^p} R/pR$$

is the tilt of R and $\theta\colon W(R^\flat) \to R$ is the unique ring homomorphism whose reduction modulo p is the projection map $R^\flat \to R/pR$, $(x_0, x_1, \dots) \mapsto x_0$. We note that R is p-complete.

Let $\varpi \in R$ be an element as above. Let $\mathcal{C}_{R,\varpi}$ denote the category of ϖ-complete perfectoid R-algebras. Every diagram $B \leftarrow A \to B'$ in $\mathcal{C}_{R,\varpi}$ has a colimit, which is given by the ϖ-completion of $B \otimes_A B'$; see, for instance, [7, Proposition 2.1.11]. Hence we can define, in the usual way, the Čech conerve $A \to B^\bullet$ for a homomorphism $A \to B$ in $\mathcal{C}_{R,\varpi}$, which is an augmented cosimplicial object in $\mathcal{C}_{R,\varpi}$, and the notion of a *ϖ-complete* arc-*hypercover* $A \to B^\bullet$ using ϖ-complete arc-covers.

In this paper, we will check that some functors on $\mathcal{C}_{R,\varpi}$ (e.g. $A \mapsto \mathrm{Perf}(A)$) satisfy ϖ-complete arc-(hyper)descent. For the sake of clarity, let us recall what it means.

Definition 2.1. Let $\mathcal{F}\colon \mathcal{C}_{R,\varpi} \to \mathcal{D}$ be a (covariant) functor to an ∞-category $\mathcal{D}$. We say that the functor $\mathcal{F}$ satisfies *ϖ-complete* arc-*descent* (resp. *ϖ-complete* arc-*hyperdescent*) if $\mathcal{F}$ preserves finite products and if for every ϖ-complete arc-cover $A \to B$ in $\mathcal{C}_{R,\varpi}$ with Čech conerve $A \to B^\bullet$ (resp. every ϖ-complete arc-hypercover $A \to B^\bullet$) we have

$$\mathcal{F}(A) \xrightarrow{\sim} \lim_{\Delta} \mathcal{F}(B^\bullet) \quad \text{in} \quad \mathcal{D},$$

i.e. $\mathcal{F}(A)$ is a limit of the cosimplicial diagram $\mathcal{F}(B^\bullet)\colon \Delta \to \mathcal{D}$. Here Δ denotes the simplex category. (See also [14, Section A.3.3 and Section A.5.7].)

Example 2.2. (1) An $\mathbb{F}_p$-algebra is perfectoid if and only if it is perfect; see [5, Example 3.15]. If R is a perfect $\mathbb{F}_p$-algebra, then ϖ can be any element such that R is ϖ-complete. If $\varpi = 0$, then $\mathcal{C}_{R,0}$ is just the category of perfect R-algebras.

(2) For every perfectoid algebra R, there is an element $\varpi \in R$ such that ϖ^p is a unit multiple of p; see [5, Lemma 3.9]. For such an element ϖ, we see that $\mathcal{C}_{R,\varpi}$ is the category of perfectoid R-algebras and a ϖ-complete arc-cover in $\mathcal{C}_{R,\varpi}$ is simply a p-complete arc-cover.

We set $x^\sharp := \theta([x])$ for an element $x \in R^\flat$, where $[-]$ denotes the Teichmüller lift. By [5, Lemma 3.9], there is an element $\varpi^\flat \in R^\flat$ such that $(\varpi^\flat)^\sharp$ is a unit multiple of ϖ. The perfect ring $R^\flat$ is $\varpi^\flat$-complete.

Proposition 2.3 ([7, Proposition 2.1.9]). *The functor $A \mapsto A^\flat$ induces an equivalence from the category $\mathcal{C}_{R,\varpi}$ of ϖ-complete perfectoid R-algebras to the category $\mathcal{C}_{R^\flat,\varpi^\flat}$ of $\varpi^\flat$-complete perfect $R^\flat$-algebras. The inverse functor is given by $B \mapsto W(B)/(\xi)$, where ξ is a generator of the kernel of $\theta\colon W(R^\flat) \to R$. Moreover, a map $A \to B$ in $\mathcal{C}_{R,\varpi}$ is a ϖ-complete* arc*-cover if and only if the induced map $A^\flat \to B^\flat$ is a $\varpi^\flat$-complete* arc*-cover.*

Proof. This proposition follows from [7, Proposition 2.1.9]. (See also [7, Lemma 2.2.2].) □

2.2. Preliminaries on $\mathbb{E}_\infty$-rings and their modules

In this subsection, we review some terminology and results from the theory of $\mathbb{E}_\infty$-rings and their modules. Our basic references are [13] and [14].

Let CAlg(Sp) denote the ∞-category of $\mathbb{E}_\infty$-rings; see [13, Section 7.1] for the definition and results which we state below. We say that an $\mathbb{E}_\infty$-ring A is *connective* (resp. *discrete*) if for $n < 0$ (resp. for $n \neq 0$) the homotopy group $\pi_n(A)$ is trivial. Let $\mathrm{CAlg(Sp)^{cn}} \subset \mathrm{CAlg(Sp)}$ be the full subcategory spanned by the connective $\mathbb{E}_\infty$-rings. We have a natural fully faithful functor

$$\mathrm{Ring} \to \mathrm{CAlg(Sp)^{cn}}$$

from the category Ring of commutative rings, whose essential image is the full subcategory spanned by the discrete $\mathbb{E}_\infty$-rings. This functor admits a left adjoint which sends a connective $\mathbb{E}_\infty$-ring A to $\pi_0(A)$. We will regard a commutative ring as an $\mathbb{E}_\infty$-ring via this functor.

Example 2.4. An example of an $\mathbb{E}_\infty$-ring is a derived quotient of a commutative ring, that is defined as follows. We first remark that the ∞-category CAlg(Sp) admits colimits and limits. In particular, a diagram $B \leftarrow A \to B'$ in CAlg(Sp) admits a colimit, denoted by $B \otimes^{\mathbb{L}}_A B'$. Let R be a commutative ring and let $x_1, \ldots, x_r \in R$ be elements. Then we define the derived quotient of R by elements $x_1, \ldots, x_r$ as

$$R/^{\mathbb{L}}(x_1, \ldots, x_r) := R \otimes^{\mathbb{L}}_{\mathbb{Z}[X_1,\ldots,X_r]} \mathbb{Z},$$

where $\mathbb{Z}[X_1, \ldots, X_r] \to R$ is defined by $X_i \mapsto x_i$ and $\mathbb{Z}[X_1, \ldots, X_r] \to \mathbb{Z}$ is defined by $X_i \mapsto 0$. The $\mathbb{E}_\infty$-ring $R/^{\mathbb{L}}(x_1, \ldots, x_r)$ is connective.

For an $\mathbb{E}_\infty$-ring A, let $\mathrm{Mod}(A)$ denote the ∞-category of module spectra over A; see [13, Notation 7.1.1.1] for the definition. We simply say "module" instead of "module spectrum" when there is no ambiguity. The ∞-category $\mathrm{Mod}(A)$ is stable and has the structure of a symmetric monoidal ∞-category. Let us write $\otimes^{\mathbb{L}}_A$ for the tensor product.

Definition 2.5 ([13, Definition 7.2.4.1]). Let

$$\mathrm{Perf}(A) \subset \mathrm{Mod}(A)$$

denote the smallest stable subcategory of $\mathrm{Mod}(A)$ which contains A and is closed under retracts. We say that a module $K \in \mathrm{Mod}(A)$ is *perfect* if $K \in \mathrm{Perf}(A)$.

Example 2.6. Let A be a discrete $\mathbb{E}_\infty$-ring. By [13, Theorem 7.1.2.13], we have a natural equivalence

$$\mathrm{Mod}(A) \cong \mathcal{D}(\pi_0(A)).$$

Here $\mathcal{D}(\pi_0(A))$ is the derived ∞-category of $\pi_0(A)$. Via this equivalence, we may identify $\mathrm{Perf}(A)$ with the ∞-category of perfect complexes over $\pi_0(A)$. This follows, for instance, from the fact that (for every $\mathbb{E}_\infty$-ring A) a module $K \in \mathrm{Mod}(A)$ is perfect if and only if K is a compact object of $\mathrm{Mod}(A)$ in the sense of [15, Definition 5.3.4.5]; see [13, Proposition 7.2.4.2]. (Compare [21, Tag 07LT].)

For a map $A \to B$ of $\mathbb{E}_\infty$-rings, we have a forgetful functor $\mathrm{Mod}(B) \to \mathrm{Mod}(A)$. This admits a left adjoint $- \otimes^{\mathbb{L}}_A B\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$, which induces a functor $\mathrm{Perf}(A) \to \mathrm{Perf}(B)$.

We will need the following notion of a perfect module with Tor-amplitude in $[a, b]$. Here we use homological indexing conventions.

Definition 2.7. Let A be a connective $\mathbb{E}_\infty$-ring (and therefore we have a natural map $A \to \pi_0(A)$). Let $K \in \mathrm{Perf}(A)$ and let a, b be integers with $a \leq b$. We say that K has *Tor-amplitude in* $[a, b]$ if the base change $K \otimes^{\mathbb{L}}_A \pi_0(A)$ has Tor-amplitude in $[a, b]$ (or in other words, for every discrete $\pi_0(A)$-module

M, we have $\pi_n(K \otimes_A^{\mathbb{L}} M) = 0$ for every $n \notin [a,b]$). Let

$$\mathrm{Perf}_{[a,b]}(A) \subset \mathrm{Perf}(A)$$

be the full subcategory spanned by the perfect modules with Tor-amplitude in $[a,b]$.

Lemma 2.8. *Let A be a connective $\mathbb{E}_\infty$-ring. Let a, b be integers with $a \leq b$.*

(1) *Assume that there is an integer $r \geq 0$ such that $\pi_m(A) = 0$ for $m \geq r$. We put $n := b - a + r$. Then the ∞-category $\mathrm{Perf}_{[a,b]}(A)$ is equivalent to an n-category (in the sense of [15, Definition 2.3.4.1]).*

(2) *Let $A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that the induced map $\mathrm{Spec}\,\pi_0(B) \to \mathrm{Spec}\,\pi_0(A)$ is surjective. Then a perfect module $K \in \mathrm{Perf}(A)$ has Tor-amplitude in $[a,b]$ if and only if $K \otimes_A^{\mathbb{L}} B \in \mathrm{Perf}(B)$ has Tor-amplitude in $[a,b]$.*

Proof. (1) By [15, Proposition 2.3.4.18], it suffices to prove that the mapping space $\mathrm{Map}(K, L)$ is $(n-1)$-truncated for all $K, L \in \mathrm{Perf}_{[a,b]}(A)$. By [13, Proposition 7.2.4.4], there is a perfect module $K^\vee \in \mathrm{Perf}(A)$ such that the underlying space of $K^\vee \otimes_A^{\mathbb{L}} L$ is equivalent to $\mathrm{Map}(K, L)$ functorially in $L \in \mathrm{Mod}(A)$. (The module $K^\vee$ is a dual of K in the symmetric monoidal ∞-category $\mathrm{Mod}(A)$; see also the proof of [16, Proposition 2.7.28].) We have

$$K^\vee \otimes_A^{\mathbb{L}} \pi_0(A) \cong R\,\mathrm{Hom}_{\pi_0(A)}(K \otimes_A^{\mathbb{L}} \pi_0(A), \pi_0(A)).$$

It follows that $K^\vee \in \mathrm{Perf}_{[-b,-a]}(A)$, and hence $K^\vee \otimes_A^{\mathbb{L}} L \in \mathrm{Perf}_{[-b+a, b-a]}(A)$ for $L \in \mathrm{Perf}_{[a,b]}(A)$. Then we see that the homotopy group $\pi_m(K^\vee \otimes_A^{\mathbb{L}} L)$ is trivial for $m \geq n$ by [13, Proposition 7.2.4.23 (5)] and the assumption on $\pi_m(A)$. Thus $\mathrm{Map}(K, L)$ is $(n-1)$-truncated as desired.

(2) This follows from the following fact. Let $K \in \mathrm{Perf}(R)$ be a perfect complex over a commutative ring R. If $K \otimes_R^{\mathbb{L}} \kappa(x) \in \mathrm{Perf}_{[a,b]}(\kappa(x))$ for every point $x \in \mathrm{Spec}\, R$, where $\kappa(x)$ is the residue field of x, then we have $K \in \mathrm{Perf}_{[a,b]}(R)$; see the proof of [6, Theorem 11.2 (2)]. □

We record some results on perfect modules over "derived complete" $\mathbb{E}_\infty$-rings. Let R be a commutative ring and let $I = (x_1, \ldots, x_r) \subset R$ be a finitely generated ideal. Let $R \to A$ be a map of $\mathbb{E}_\infty$-rings. We say that a module $M \in \mathrm{Mod}(A)$ is *derived I-complete* if M is derived I-complete (in the sense of [21, Tag 091S]) when we regard it as an object of $\mathcal{D}(R)$; see also [14, Definition 7.3.1.1 and Corollary 7.3.3.6].

Proposition 2.9. *Let A be a connective $\mathbb{E}_\infty$-ring over R. We assume that A is derived I-complete. Let a, b be integers with $a \leq b$.*

(1) *We put $S := \pi_0(A)/I\pi_0(A)$. Let $K \in \mathrm{Mod}(A)$ be a module which is almost connective, i.e. there exists an integer n such that the homotopy group $\pi_m(M)$ is trivial for every $m < n$. Then K belongs to $\mathrm{Perf}_{[a,b]}(A)$ if and only if K is derived I-complete and $K \otimes^{\mathbb{L}}_A S$ belongs to $\mathrm{Perf}_{[a,b]}(S)$.*

(2) *We put $A_m := A \otimes^{\mathbb{L}}_R R/^{\mathbb{L}}(x_1^m, \ldots, x_r^m)$ for an integer $m \geq 1$. We have*

$$\mathrm{Perf}(A) \xrightarrow{\sim} \lim_{m\geq 1} \mathrm{Perf}(A_m) \quad \text{in} \quad \mathrm{Cat}_\infty .$$

Similarly, we have

$$\mathrm{Perf}_{[a,b]}(A) \xrightarrow{\sim} \lim_{m\geq 1} \mathrm{Perf}_{[a,b]}(A_m) \quad \text{in} \quad \mathrm{Cat}_\infty .$$

Proof. (1) By [14, Corollary 8.3.5.9], we see that K belongs to $\mathrm{Perf}(A)$ if and only if K is derived I-complete and $K \otimes^{\mathbb{L}}_A S$ belongs to $\mathrm{Perf}(S)$. We shall prove that a perfect module $K \in \mathrm{Perf}(A)$ belongs to $\mathrm{Perf}_{[a,b]}(A)$ if and only if $K \otimes^{\mathbb{L}}_A S$ belongs to $\mathrm{Perf}_{[a,b]}(S)$. Let $K^\vee$ be a dual of K; see the proof of Lemma 2.8 and the references therein. We have $(K^\vee)^\vee \cong K$. It follows that K has Tor-amplitude in $[a, b]$ if and only if K has Tor-amplitude $\leq b$ and $K^\vee$ has Tor-amplitude $\leq -a$ in the sense of [13, Definition 7.2.4.21]. Thus our claim follows from [14, Corollary 8.3.5.8].

(2) The second equivalence follows from the first one by (1). We prove the first assertion. Let

$$\Phi\colon \mathrm{Mod}(A) \to \lim_{m\geq 1} \mathrm{Mod}(A_m)$$

denote the natural functor. An object of $\lim_{m\geq 1} \mathrm{Mod}(A_m)$ can be identified with a family of objects $\{K_m\}_{m\geq 1}$, where $K_m \in \mathrm{Mod}(A_m)$, together with equivalences $K_{m+1} \otimes^{\mathbb{L}}_{A_{m+1}} A_m \xrightarrow{\sim} K_m$ $(m \geq 1)$. The functor Φ admits a right adjoint

$$\Psi\colon \lim_{m\geq 1} \mathrm{Mod}(A_m) \to \mathrm{Mod}(A)$$

which sends a family $\{K_m\}_{m\geq 1}$ as above to $\lim_m K_m$, where we regard K_m as an object of $\mathrm{Mod}(A)$. We claim that if $K_m \in \mathrm{Perf}(A_m)$ for every $m \geq 1$, then $K := \lim_m K_m$ belongs to $\mathrm{Perf}(A)$ and we have

$$K \otimes^{\mathbb{L}}_A A_m \xrightarrow{\sim} K_m. \tag{2.1}$$

In order to prove the claim, we may assume that K_1 is connective. It then follows that K_m is connective for every $m \geq 1$, and hence K is also connective since $\pi_0(K_{m+1}) \to \pi_0(K_m)$ is surjective for every $m \geq 1$. Now, [14, Lemma 8.3.5.4] implies that the map $K \otimes^{\mathbb{L}}_A A_m \to K_m$ is an equivalence. Since each K_m is derived I-complete as an A-module, the limit K is also derived I-complete. Therefore K is perfect by (1).

We now obtain the following adjunction

$$F\colon \mathrm{Perf}(A) \rightleftarrows \lim_{m\geq 1} \mathrm{Perf}(A_m) :G.$$

It suffices to prove that the unit transformation $\mathrm{id} \to G \circ F$ and the counit transformation $F \circ G \to \mathrm{id}$ are equivalences. For a $K \in \mathrm{Perf}(A)$, we see that $(G \circ F)(K)$ is isomorphic to the derived I-completion of K as an object of $\mathrm{Mod}(R)$; see, for instance, [21, Tag 0920]. Since K is derived I-complete by (1), we have $K \xrightarrow{\sim} (G \circ F)(K)$. Finally, we shall prove $F \circ G \xrightarrow{\sim} \mathrm{id}$. We note that a map in $\lim_{m\geq 1} \mathrm{Perf}(A_m)$ is an equivalence if and only if, for every m, its image in $\mathrm{Perf}(A_m)$ is an equivalence[2]. Thus (2.1) implies $F \circ G \xrightarrow{\sim} \mathrm{id}$. □

We conclude this section with the following result on derived quotients of perfectoid rings.

Lemma 2.10. *Let R be a perfectoid ring and $\varpi \in R$ an element with $p \in (\varpi^p)$ such that R is ϖ-complete. Let $\varpi^\flat \in R^\flat$ be an element such that $(\varpi^\flat)^\sharp$ is a unit multiple of ϖ. Let A be a ϖ-complete perfectoid R-algebra. Then we have an isomorphism of $\mathbb{E}_\infty$-rings*

$$A^\flat/^{\mathbb{L}}\varpi^\flat \cong A/^{\mathbb{L}}\varpi$$

which is functorial in A.

Proof. We may assume that $\varpi = (\varpi^\flat)^\sharp$. Then ϖ admits a p-th root, and thus it suffices to prove that $A^\flat/^{\mathbb{L}}(\varpi^\flat)^p \cong A/^{\mathbb{L}}(\varpi)^p$ functorially in A. Since

[2] This can be deduced from the fact that a natural transformation between two functors from a simplicial set to an ∞-category is an equivalence if and only if it is an objectwise equivalence (which can be found, e.g., in [8, Corollary 3.5.12]). One can also check it by considering fiber sequences. We note that $\lim_{m\geq 1} \mathrm{Perf}(A_m)$ is also a limit in the ∞-category of stable ∞-categories ([13, Theorem 1.1.4.4]).

the map $\theta\colon W(R^\flat) \to R$ is surjective and $p \in (\varpi^p)$, there exists an element $x \in W(R^\flat)$ such that $p = -\theta(x)\varpi^p$. Then $\xi := p + x[\varpi^\flat]^p$ is a generator of $\operatorname{Ker}\theta$; see the proof of [5, Lemma 3.10]. We consider two maps

$$W(R^\flat)[S_1, S_2] \to W(A^\flat) \quad \text{defined by} \quad S_1 \mapsto p, \quad S_2 \mapsto [\varpi^\flat]^p$$

and

$$W(R^\flat)[T_1, T_2] \to W(A^\flat) \quad \text{defined by} \quad T_1 \mapsto \xi, \quad T_2 \mapsto [\varpi^\flat]^p.$$

Since both p and ξ are non-zero divisors in $W(A^\flat)$, we have

$$W(A^\flat) \otimes^{\mathbb{L}}_{W(R^\flat)[S_1,S_2]} W(R^\flat) \cong A^\flat/^{\mathbb{L}}(\varpi^\flat)^p,$$
$$W(A^\flat) \otimes^{\mathbb{L}}_{W(R^\flat)[T_1,T_2]} W(R^\flat) \cong A/^{\mathbb{L}}(\varpi)^p.$$

Here $W(R^\flat)[S_1, S_2] \to W(R^\flat)$ is defined by $S_i \mapsto 0$ and similarly for $W(R^\flat)[T_1, T_2] \to W(R^\flat)$. To conclude the proof, it suffices to observe that we have an isomorphism $W(R^\flat)[T_1, T_2] \xrightarrow{\sim} W(R^\flat)[S_1, S_2]$ defined by $T_1 \mapsto S_1 + xS_2$, $T_2 \mapsto S_2$, which is compatible with the above maps. □

3. An analogue for derived quotients of perfect rings

3.1. An analogue for derived quotients of perfect rings

In this section, we prove the following analogue of Theorem 1.1 for derived quotients of perfect rings:

Theorem 3.1. *Let R be a perfect $\mathbb{F}_p$-algebra and $I = (x_1, \dots, x_r) \subset R$ a finitely generated ideal. Let a, b be integers with $a \leq b$. Then the functors*

$$A \mapsto \mathcal{F}(A) := \operatorname{Perf}(A/^{\mathbb{L}}(x_1, \dots, x_r))$$

and

$$A \mapsto \mathcal{F}_{[a,b]}(A) := \operatorname{Perf}_{[a,b]}(A/^{\mathbb{L}}(x_1, \dots, x_r))$$

from the category of perfect R-algebras to Cat_∞ *satisfy* arc*-hyperdescent (see also Definition 2.1).*

We give two proofs of Theorem 3.1. The first proof, given below, goes along the same line as that of Theorem 1.1; the notion of a *descendable* map of $\mathbb{E}_\infty$-rings introduced by Mathew ([17, Definition 3.18]) plays a central role.

A key input is the fact that an *h-cover* of perfect $\mathbb{F}_p$-algebras is descendable, which is proved by Bhatt–Scholze.

The second proof, which we learned from B. Bhatt, is given in the next subsection; we deduce Theorem 3.1 from Theorem 1.1. It may be easier than the first one for the reader who is familiar with some fundamental results on (almost) perfect modules provided in [13, 14, 21].

Proof. The functor $\mathcal{F}$ commutes with finite products (see, for instance, [14, Lemma D.3.5.5]) and filtered colimits (see [13, Lemma 7.3.5.13]). Then we see that the same results hold for $\mathcal{F}_{[a,b]}$. To prove the theorem, it is enough to prove that $\mathcal{F}_{[a,b]}$ satisfies arc-hyperdescent (for all a, b with $a \leq b$). Moreover, it suffices to prove that $\mathcal{F}_{[a,b]}$ satisfies arc-descent since $\mathcal{F}_{[a,b]}(A)$ is equivalent to an N-category for every perfect R-algebra A by Lemma 2.8 (1), where $N := b - a + r + 1$. We want to apply [4, Proposition 4.8]. For this, we need to prove the following assertions.

(i) (v-descent) For every v-cover $A \to B$ of perfect R-algebras (in the sense of [6, Definition 2.1] or equivalently [18, Definition 2.2]) with Čech conerve $A \to B^\bullet$, the functor

$$\mathcal{F}_{[a,b]}(A) \to \lim_{\Delta} \mathcal{F}_{[a,b]}(B^\bullet)$$

is an equivalence in Cat_∞ (or equivalently, in the full subcategory $\mathrm{Cat}_N \subset \mathrm{Cat}_\infty$ consisting of those ∞-categories which are equivalent to N-categories).

(ii) (aic-v-excision) For every valuation ring V over R with algebraically closed fraction field and every prime ideal $\mathfrak{p} \in \mathrm{Spec}(V)$, the square

$$\begin{array}{ccc} \mathcal{F}_{[a,b]}(V) & \longrightarrow & \mathcal{F}_{[a,b]}(V/\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathcal{F}_{[a,b]}(V_\mathfrak{p}) & \longrightarrow & \mathcal{F}_{[a,b]}(\kappa(\mathfrak{p})) \end{array}$$

is a pullback square in Cat_∞ (or equivalently, in Cat_N).

We first prove (i). We can write B as a colimit of a filtered system $\{B_{0,i}\}_{i\in I}$ of finitely presented A-algebras. Let $B_i := (B_{0,i})_{\mathrm{perf}} := \mathrm{colim}_{x\mapsto x^p} B_{0,i}$ be the perfection of $B_{0,i}$. It follows that $A \to B_i$ is an h-cover (in the sense of [6, Definition 11.1]) and hence it is descendable in the sense of [17, Definition 3.18] (or equivalently [6, Definition 11.14]) by [6, Theorem 11.27]. The base change

$$f_i\colon A/^{\mathbb{L}}(x_1,\dots,x_r)\to B_i/^{\mathbb{L}}(x_1,\dots,x_r)$$

is also descendable. Let $A\to B_i^\bullet$ denote the Čech conerve of $A\to B_i$ taken in the category Ring of commutative rings. The (derived) Čech conerve of f_i taken in the ∞-category CAlg(Sp) of $\mathbb{E}_\infty$-algebras is isomorphic to the base change

$$A/^{\mathbb{L}}(x_1,\dots,x_r)\to B_i^\bullet/^{\mathbb{L}}(x_1,\dots,x_r)$$

of $A\to B_i^\bullet$ along $A\to A/^{\mathbb{L}}(x_1,\dots,x_r)$ by [6, Lemma 3.16 or Proposition 11.6]. Therefore, by [17, Proposition 3.22], we have an equivalence of symmetric monoidal ∞-categories for every $i\in I$

$$\begin{aligned}&\mathrm{Mod}(A/^{\mathbb{L}}(x_1,\dots,x_r))\xrightarrow{\sim}\lim_{\Delta}\mathrm{Mod}(B_i^\bullet/^{\mathbb{L}}(x_1,\dots,x_r)),\\ &\text{hence }\ \mathcal{F}(A)\xrightarrow{\sim}\lim_{\Delta}\mathcal{F}(B_i^\bullet);\end{aligned}$$

here, for the second equivalence, we use [13, Proposition 4.6.1.11] and the fact that an object of $\mathrm{Mod}(A/^{\mathbb{L}}(x_1,\dots,x_r))$ is perfect if and only if it is dualizable (see [16, Proposition 2.7.28]). By Lemma 2.8 (2), this implies

$$\mathcal{F}_{[a,b]}(A)\xrightarrow{\sim}\lim_{\Delta}\mathcal{F}_{[a,b]}(B_i^\bullet)$$

for every $i\in I$. Since we have $\mathrm{colim}_I B_i\cong B$ and $\mathcal{F}_{[a,b]}(A)$ is equivalent to an N-category for every A, it follows from [4, Lemma 3.7] that the functor $\mathcal{F}_{[a,b]}(A)\to\lim_\Delta\mathcal{F}_{[a,b]}(B^\bullet)$ is an equivalence.

Next, we prove (ii). As remarked in the proof of [4, Theorem 5.16], the map $V\to W:=V_{\mathfrak{p}}\times V/\mathfrak{p}$ is descendable. The base change $V/^{\mathbb{L}}(x_1,\dots,x_r)\to W/^{\mathbb{L}}(x_1,\dots,x_r)$ is also descendable. Thus, similarly to the proof of (i), we have

$$\mathcal{F}_{[a,b]}(V)\xrightarrow{\sim}\lim_{\Delta}\mathcal{F}_{[a,b]}(W^\bullet)$$

for the Čech conerve $V\to W^\bullet$ of $V\to W$ taken in Ring. As in the proof of [4, Theorem 5.16], one can prove that the right hand side of the above equivalence is a limit of the diagram

$$\begin{array}{ccc} & & \mathcal{F}_{[a,b]}(V/\mathfrak{p})\\ & & \downarrow\\ \mathcal{F}_{[a,b]}(V_{\mathfrak{p}}) & \longrightarrow & \mathcal{F}_{[a,b]}(\kappa(\mathfrak{p})).\end{array}$$

This proves (ii).

With assertions (i) and (ii), we can now apply the argument as in the proof of [4, Proposition 4.8] to $\mathcal{F}_{[a,b]}$ (or rather the functor $A \mapsto \mathcal{F}_{[a,b]}(A_{\mathrm{perf}})$ from the category of R-algebras to Cat_N) to conclude that $\mathcal{F}_{[a,b]}$ satisfies arc-descent. The proof of Theorem 3.1 is complete. □

The following corollary is a key ingredient in the proof of Theorem 1.2.

Corollary 3.2. *Let R be a perfectoid ring and $\varpi \in R$ an element with $p \in (\varpi^p)$ such that R is ϖ-complete. Let $m \geq 1$ be an integer. The following functors from the category $\mathcal{C}_{R,\varpi}$ of ϖ-complete perfectoid R-algebras to* Cat_∞

$$A \mapsto \mathrm{Perf}(A/^{\mathbb{L}}\varpi) \quad \textit{and} \quad A \mapsto \mathrm{Perf}(A^\flat/^{\mathbb{L}}(\varpi^\flat)^m)$$

satisfy ϖ-complete arc-*hyperdescent. The same statement holds for perfect modules with Tor-amplitude in $[a,b]$ for all a,b with $a \leq b$.*

Proof. By Proposition 2.3 and Lemma 2.10, it suffices to prove that the functors

$$A \mapsto \mathrm{Perf}(A/^{\mathbb{L}}(\varpi^\flat)^m) \quad \text{and} \quad A \mapsto \mathrm{Perf}_{[a,b]}(A/^{\mathbb{L}}(\varpi^\flat)^m)$$

from the category $\mathcal{C}_{R^\flat,\varpi^\flat}$ of $\varpi^\flat$-complete perfect $R^\flat$-algebras to Cat_∞ satisfy $\varpi^\flat$-complete arc-hyperdescent. Moreover, it is enough to prove that the second functor satisfies ϖ-complete arc-descent; compare the first paragraph of the (first) proof of Theorem 3.1. This functor preserves finite products. Let $A \to B$ be a $\varpi^\flat$-complete arc-cover in $\mathcal{C}_{R^\flat,\varpi^\flat}$. Since $A \to C := B \times A[1/\varpi^\flat]$ is an arc-cover, we obtain

$$\mathrm{Perf}_{[a,b]}(A/^{\mathbb{L}}(\varpi^\flat)^m) \xrightarrow{\sim} \lim_{\Delta} \mathrm{Perf}_{[a,b]}(C^\bullet/^{\mathbb{L}}(\varpi^\flat)^m)$$

by Theorem 3.1, where $A \to C^\bullet$ is the Čech conerve of $A \to C$ taken in the category of perfect $R^\flat$-algebras. Let $A \to B^\bullet$ be the (completed) Čech conerve of $A \to B$ taken in $\mathcal{C}_{R^\flat,\varpi^\flat}$. We have $C^\bullet/^{\mathbb{L}}(\varpi^\flat)^m \cong B^\bullet/^{\mathbb{L}}(\varpi^\flat)^m$, and thus

$$\mathrm{Perf}_{[a,b]}(C^\bullet/^{\mathbb{L}}(\varpi^\flat)^m) \cong \mathrm{Perf}_{[a,b]}(B^\bullet/^{\mathbb{L}}(\varpi^\flat)^m).$$

In conclusion, we have $\mathrm{Perf}_{[a,b]}(A/^{\mathbb{L}}(\varpi^\flat)^m) \xrightarrow{\sim} \lim_\Delta \mathrm{Perf}_{[a,b]}(B^\bullet/^{\mathbb{L}}(\varpi^\flat)^m)$, which completes the proof. □

3.2. The second proof of Theorem 3.1

In this subsection, we prove the following proposition, by which one can easily see that Theorem 1.1 implies Theorem 3.1.

Proposition 3.3. *Let $A \to B^\bullet$ be an augmented cosimplicial object in* Ring *and $I = (x_1, \dots, x_r) \subset A$ a finitely generated ideal such that* $\operatorname{Spec} B^0/IB^0 \to \operatorname{Spec} A/I$ *is surjective. Let $A/^{\mathbb{L}}(x_1, \dots, x_r) \to B^\bullet/^{\mathbb{L}}(x_1, \dots, x_r)$ be the augmented cosimplicial object in* CAlg(Sp) *obtained by base change. If* $\mathrm{Perf}(A) \xrightarrow{\sim} \lim_\Delta \mathrm{Perf}(B^\bullet)$, *then we have*

$$F\colon \mathrm{Perf}(A/^{\mathbb{L}}(x_1, \dots, x_r)) \xrightarrow{\sim} \lim_{\Delta} \mathrm{Perf}(B^\bullet/^{\mathbb{L}}(x_1, \dots, x_r)).$$

Proof. To simplify the notation, we write

$$\overline{A} := A/^{\mathbb{L}}(x_1, \dots, x_r) \quad \text{and} \quad \overline{B}^n := B^n/^{\mathbb{L}}(x_1, \dots, x_r).$$

Let

$$\Phi\colon \mathrm{Mod}(\overline{A}) \to \lim_{\Delta} \mathrm{Mod}(\overline{B}^\bullet)$$

denote the natural functor. This functor Φ admits a right adjoint

$$\Psi\colon \lim_{\Delta} \mathrm{Mod}(\overline{B}^\bullet) \to \mathrm{Mod}(\overline{A}).$$

An object $\{K^\bullet\}$ of $\lim_\Delta \mathrm{Mod}(\overline{B}^\bullet)$ gives rise to a natural functor $\Delta \to \mathrm{Mod}(\overline{A})$ such that the image K^n of $[n] \in \Delta$ is isomorphic to that of $\{K^\bullet\}$ in $\mathrm{Mod}(\overline{B}^n)$ regarded as an $\overline{A}$-module. The functor Ψ sends $\{K^\bullet\}$ to $\lim_{[n]\in\Delta} K^n$.

We first prove the fully faithfulness of F. Let $\mathcal{D} \subset \mathrm{Mod}(\overline{A})$ be the full subcategory spanned by those objects K such that the unit map $K \to (\Psi \circ \Phi)(K)$ is an isomorphism. It is clear that $\mathcal{D}$ is a stable subcategory which is closed under retracts. Using the fully faithfulness of $\mathrm{Perf}(A) \to \lim_\Delta \mathrm{Perf}(B^\bullet)$ and using the fiber sequences

$$A/^{\mathbb{L}}(x_1, \dots, x_i) \overset{\times x_{i+1}}{\to} A/^{\mathbb{L}}(x_1, \dots, x_i) \to A/^{\mathbb{L}}(x_1, \dots, x_{i+1})$$

for $0 \leq i \leq r-1$ inductively, where $A/^{\mathbb{L}}(x_1, \dots, x_i) := A$ if $i = 0$, we see that $\overline{A} \in \mathcal{D}$. Thus, by the definition of $\mathrm{Perf}(\overline{A})$, we have $\mathrm{Perf}(\overline{A}) \subset \mathcal{D}$. In particular, it follows that F is fully faithful.

It remains to show that F is essentially surjective. Let $\{K^\bullet\}$ be an object of $\lim_\Delta \mathrm{Perf}(\overline{B}^\bullet)$ and let $K^n \in \mathrm{Perf}(\overline{B}^n)$ denote its image. It suffices to prove that $L := \Psi(\{K^\bullet\})$ belongs to $\mathrm{Perf}(\overline{A})$ and the canonical map $L \otimes^{\mathbb{L}}_{\overline{A}} \overline{B}^n \to K^n$ is an isomorphism. (See also the proof of Proposition 2.9 (2).) Since $\overline{B}^n$ is perfect as a B^n-module, it follows that K^n is perfect over B^n as well. Thus, by our assumption (and a similar argument as in the previous paragraph), we see that L is perfect over A and we have $L \otimes^{\mathbb{L}}_{\overline{A}} \overline{B}^n \cong L \otimes^{\mathbb{L}}_{A} B^n \cong K^n$. It remains to show that L is perfect over $\overline{A}$. For this, we remark that, since $\overline{A}$ is perfect as an A-module, the $\overline{A}$-module L is almost perfect in the sense of [13, Definition 7.2.4.10] by Lemma 3.4 below. Therefore, by [21, Tag 068W] and [14, Proposition 2.7.3.2 (d)] (see also [14, Corollary 8.6.4.3]), it suffices to show that $L \otimes^{\mathbb{L}}_{\overline{A}} \pi_0(\overline{A})/\mathfrak{m}$ is perfect over $\pi_0(\overline{A})/\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subset \pi_0(\overline{A})$. This follows from the fact that $L \otimes^{\mathbb{L}}_{\overline{A}} \overline{B}^0 \cong K^0$ is perfect over $\overline{B}^0$ and the assumption that $\operatorname{Spec} B^0/IB^0 \to \operatorname{Spec} A/I$ is surjective. □

The following easy lemma is used in the proof of Proposition 3.3.

Lemma 3.4. *Let $A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that B is almost perfect as an A-module (in the sense of [13, Definition 7.2.4.10]). Then a B-module M is almost perfect if and only if M is almost perfect as an A-module.*

Proof. Let $n \geq 0$ be an integer. We prove that a connective B-module M is perfect to order n in the sense of [14, Definition 2.7.0.1] if and only if M is perfect to order n as an A-module, from which the lemma follows (see [14, Remark 2.7.0.2]). The "only if" direction is proved in [14, Proposition 2.7.3.3]. The "if" direction can be proved in a similar way; we provide the proof for the reader's convenience. (See also [21, Tag 064Z] for the case where A and B are discrete.)

We proceed by induction on n. Since a connective B-module M is perfect to order 0 if and only if $\pi_0(M)$ is finitely generated over $\pi_0(B)$ by [14, Proposition 2.7.2.1], the case where $n = 0$ is clear. Assume $n > 0$. Since $\pi_0(M)$ is finitely generated over $\pi_0(B)$, there exists a map $f\colon B^{\oplus m} \to M$ of B-modules such that the induced map $\pi_0(B^{\oplus m}) \to \pi_0(M)$ is surjective. Let N be a fiber of f, which is a connective B-module. Since both B and M are perfect to n as A-modules, we see that N is perfect to $n-1$ over A by [14, Remark 2.7.0.7], and hence it is perfect to $n-1$ over B by the induction hypothesis. By [14, Remark 2.7.0.7] again, it follows that M is perfect to n over B. □

4. Proofs of main results

4.1. Proof of Theorem 1.2

We prove Theorem 1.2. By Proposition 2.9 (2) and Corollary 3.2, the functor $A \mapsto \mathrm{Perf}(A^\flat)$ satisfies ϖ-complete arc-hyperdescent.

We prove the assertions for the functors $A \mapsto \mathrm{Perf}(W(A^\flat))$ and $A \mapsto \mathrm{Perf}(A)$ simultaneously. For this, let $\mathcal{G}$ denote the functor $A \mapsto W(A^\flat)$ or $A \mapsto A$ from $\mathcal{C}_{R,\varpi}$ to Ring, and let $\alpha \in \mathcal{G}(A)$ be p or ϖ, respectively. Then, for every integer $m \geq 1$, let $\mathcal{G}_m \colon \mathcal{C}_{R,\varpi} \to \mathrm{CAlg}(\mathrm{Sp})$ be the functor defined by $A \mapsto \mathcal{G}(A)/^{\mathbb{L}}\alpha^m$. By Proposition 2.9 (2), it suffices to prove that the functor $A \mapsto \mathrm{Perf}(\mathcal{G}_m(A))$ satisfies ϖ-complete arc-hyperdescent for every integer $m \geq 1$. This functor preserves finite products. We want to show that the functor

$$F\colon \mathrm{Perf}(\mathcal{G}_m(A)) \to \lim_{\Delta} \mathrm{Perf}(\mathcal{G}_m(B^\bullet))$$

is an equivalence for any ϖ-complete arc-hypercover $A \to B^\bullet$. We first prove the fully faithfulness of F. The natural functor

$$\Phi_m\colon \mathrm{Mod}(\mathcal{G}_m(A)) \to \lim_{\Delta} \mathrm{Mod}(\mathcal{G}_m(B^\bullet))$$

admits a right adjoint

$$\Psi_m\colon \lim_{\Delta} \mathrm{Mod}(\mathcal{G}_m(B^\bullet)) \to \mathrm{Mod}(\mathcal{G}_m(A)).$$

Lemma 4.1. *Let $\mathcal{D} \subset \mathrm{Mod}(\mathcal{G}_m(A))$ be the full subcategory spanned by those objects K such that the unit map $K \to (\Psi_m \circ \Phi_m)(K)$ is an isomorphism. Then we have $\mathrm{Perf}(\mathcal{G}_m(A)) \subset \mathcal{D}$. In particular, the functor F is fully faithful.*

Proof. As in the proof of Proposition 3.3, it suffices to show $\mathcal{G}_m(A) \in \mathcal{D}$. We proceed by induction on m. Assume $m = 1$. In the beginning of this section and Corollary 3.2, we have proved that the functor F is an equivalence (and in particular fully faithful), which implies the result. When $m > 1$, the result follows from the fiber sequence

$$\mathcal{G}_{m-1}(A) \overset{\times\alpha}{\to} \mathcal{G}_m(A) \to \mathcal{G}_1(A) \tag{4.1}$$

and the induction hypothesis. □

We shall prove that F is essentially surjective. Again, we proceed by induction on m. The case $m = 1$ has already been established in the beginning of the proof and Corollary 3.2. Assume $m > 1$. Let $\{K_m^\bullet\}$ be an object of $\lim_\Delta \operatorname{Perf}(\mathcal{G}_m(B^\bullet))$ and let $K_m^n \in \operatorname{Perf}(\mathcal{G}_m(B^n))$ denote its image. It suffices to prove that $L_m := \Psi_m(\{K_m^\bullet\})$ belongs to $\operatorname{Perf}(\mathcal{G}_m(A))$ and the canonical map

$$L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_m(B^n) \to K_m^n \tag{4.2}$$

is an isomorphism. (See also the proofs of Proposition 2.9 (2) and Proposition 3.3.)

By the induction hypothesis, the image $\{K_l^\bullet\}$ of $\{K_m^\bullet\}$ in $\lim_\Delta \operatorname{Perf}(\mathcal{G}_l(B^\bullet))$ corresponds to a perfect module $L_l \in \operatorname{Perf}(\mathcal{G}_l(A))$ for every $l \leq m-1$. By Lemma 4.1, we have $L_l \cong \Psi_l(\{K_l^\bullet\})$. We claim that L_m is almost connective and

$$L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_1(A) \xrightarrow{\sim} L_1. \tag{4.3}$$

Our claim together with Proposition 2.9 (1) shows that $L_m \in \operatorname{Perf}(\mathcal{G}_m(A))$. It also implies that the map (4.2) is an isomorphism. Indeed, the base change of (4.2) along $\mathcal{G}_m(B^n) \to \mathcal{G}_1(B^n)$ is an isomorphism since it can be identified with the base change of (4.3) along $\mathcal{G}_1(A) \to \mathcal{G}_1(B^n)$. Then, using (4.1) repeatedly, we conclude that the map (4.2) is also an isomorphism.

We shall prove our claim. Using (4.1) and the functor Ψ_m, we obtain a fiber sequence $L_{m-1} \to L_m \to L_1$. Since L_{m-1} and L_1 are almost connective, it follows that L_m is also almost connective. It remains to show (4.3). We have a natural fiber sequence $\mathcal{G}_1(A) \to \mathcal{G}_m(A) \to \mathcal{G}_{m-1}(A)$. Moreover, from the following fiber sequence

$$\mathcal{G}(A) \xrightarrow{\times\alpha} \mathcal{G}(A) \to \mathcal{G}_1(A), \tag{4.4}$$

we obtain a fiber sequence $\mathcal{G}_m(A) \to \mathcal{G}_m(A) \to \mathcal{G}_m(A) \otimes^{\mathbb{L}}_{\mathcal{G}(A)} \mathcal{G}_1(A)$. Then, (4.1) and these fiber sequences induce a fiber sequence

$$\mathcal{G}_1(A)[1] \to \mathcal{G}_m(A) \otimes^{\mathbb{L}}_{\mathcal{G}(A)} \mathcal{G}_1(A) \to \mathcal{G}_1(A), \tag{4.5}$$

where [1] denotes the shift by 1. From (4.5), we have a natural map of fiber sequences

$$\begin{array}{ccccc}
L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_1(A)[1] & \longrightarrow & L_m \otimes^{\mathbb{L}}_{\mathcal{G}(A)} \mathcal{G}_1(A) & \longrightarrow & L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_1(A) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
L_1[1] & \longrightarrow & L_m \otimes^{\mathbb{L}}_{\mathcal{G}(A)} \mathcal{G}_1(A) & \longrightarrow & L_1.
\end{array}$$

The first line is obtained by applying $L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} -$ to (4.5). The second line is obtained by applying $K^n_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} -$ to (4.5) and then using the functor Ψ_m. Here we use the fact that the functor $- \otimes^{\mathbb{L}}_{\mathcal{G}(A)} \mathcal{G}_1(A) \colon \mathrm{Mod}(\mathcal{G}(A)) \to \mathrm{Mod}(\mathcal{G}_1(A))$ preserves limits (which can be checked using the fiber sequence (4.4)). We shall prove that the map $L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_1(A) \to L_1$ induces an isomorphism on the k-th homotopy groups for every k. Since L_m is almost connective, so is $L_m \otimes^{\mathbb{L}}_{\mathcal{G}_m(A)} \mathcal{G}_1(A)$. In particular, the assertion holds for a sufficiently small k. Then, by induction on k and using the above diagram, we see that the assertion holds for all k. This concludes the proof of (4.3) and hence the proof of the essential surjectivity of F.

The proof of Theorem 1.2 is now complete.

4.2. ϖ-complete arc-descent for finite projective modules

For future reference, we record a ϖ-complete arc-descent result for finite projective modules over perfectoid rings, which is a consequence of Theorem 1.2.

We first fix some notation. Let $A \to B^\bullet$ be an augmented cosimplicial object in Ring. For any $0 \le i \le 1$, we denote by $p_i \colon B^0 \to B^1$ the map corresponding to the injection $[0] \cong \{i\} \hookrightarrow [1]$ in Δ. Similarly, for any $0 \le i < j \le 2$, we denote by $p_{i,j} \colon B^1 \to B^2$ the map corresponding to the injection $[1] \cong \{i,j\} \hookrightarrow [2]$ in Δ. Let $\mathrm{Vect}(A)$ be the category of finite projective modules over A and $\mathrm{DD}(B^\bullet)$ the category of pairs (M, σ) where $M \in \mathrm{Vect}(B^0)$ is a finite projective module over B^0 and $\sigma \colon p_0^*M \xrightarrow{\sim} p_1^*M$ is an isomorphism in $\mathrm{Vect}(B^1)$ satisfying the usual cocycle condition $p_{0,2}^*\sigma = p_{1,2}^*\sigma \circ p_{0,1}^*\sigma$. Here for a map of commutative rings $f \colon R \to S$ and a module M over R, we denote by $f^*M := M \otimes_A B$ the base change of M along f. We have a natural functor

$$\mathrm{Vect}(A) \to \mathrm{DD}(B^\bullet).$$

Corollary 4.2. *Let R be a perfectoid ring and $\varpi \in R$ an element with $p \in (\varpi^p)$ such that R is ϖ-complete. Let $A \to B^\bullet$ be a ϖ-complete* arc*-hypercover in $\mathcal{C}_{R,\varpi}$. Then we have the equivalences of categories*

$$\mathrm{Vect}(A^\flat) \xrightarrow{\sim} \mathrm{DD}((B^\bullet)^\flat), \quad \mathrm{Vect}(W(A^\flat)) \xrightarrow{\sim} \mathrm{DD}(W((B^\bullet)^\flat)),$$
$$\mathrm{Vect}(A) \xrightarrow{\sim} \mathrm{DD}(B^\bullet).$$

Proof. We only prove $\mathrm{Vect}(A) \xrightarrow{\sim} \mathrm{DD}(B^\bullet)$; the other statements can be proved similarly. First, note that $\mathrm{Vect}(A) \cong \mathrm{Perf}_{[0,0]}(A)$. By Theorem 1.2, Lemma 2.8 (2), and Proposition 2.9 (1), we have $\mathrm{Vect}(A) \xrightarrow{\sim} \lim_\Delta \mathrm{Vect}(B^\bullet)$.

Let $\Delta_{s,\leq n}$ be the subcategory of Δ whose objects are $[m] = \{0, 1, \ldots, m\}$ for $0 \leq m \leq n$ and morphisms are given by injective order preserving maps. Since $\mathrm{Vect}(B^n)$ is a 1-category for any n, it follows that $\lim_{\Delta} \mathrm{Vect}(B^\bullet) \cong \lim_{\Delta_{s,\leq 2}} \mathrm{Vect}(B^\bullet)$; see, for instance, [16, Proposition 4.3.5]. (Here we use the dual of [16, Proposition 4.3.5]. Although it is claimed there that the geometric realization of $X_\bullet$ is isomorphic to a colimit of the diagram $X_\bullet|_{\Delta^{op}_{s,\leq n+1}}$, its proof shows that it is also isomorphic to a colimit of the diagram $X_\bullet|_{\Delta^{op}_{s,\leq n}}$; compare [15, Lemma 5.5.6.17].) One can check that $\mathrm{DD}(B^\bullet) \cong \lim_{\Delta_{s,\leq 2}} \mathrm{Vect}(B^\bullet)$, which concludes the proof. □

Remark 4.3. By the proof of Theorem 1.2 and the argument in the proof of Corollary 4.2, we also have an equivalence of categories $\mathrm{Vect}(W_n(A^\flat)) \xrightarrow{\sim} \mathrm{DD}(W_n((B^\bullet)^\flat))$ for every integer $n \geq 1$, where $W_n(A^\flat) := W(A^\flat)/p^n$. This implies that [10, Conjecture A] holds.

In [10, Section 4.3.3], Henkel proved some p-complete arc-descent results for BK_n-modules for perfectoid rings assuming this conjecture. Here BK_n-modules are "truncated" analogues of (minuscule) Breuil–Kisin–Fargues modules; see [10, Chapter 2] for details. For example, he proved that [10, Conjecture A] implies that the functor sending a perfectoid R-algebra A to the groupoid $\mathrm{BK}_n(A)$ of BK_n-modules for A is a stack with respect to the p-complete arc-topology; see [10, Theorem 4.3.15].

5. The classification of p-divisible groups over perfectoid rings

In this section, we discuss the classification of p-divisible groups over perfectoid rings (Theorem 1.4). We follow the approach of Scholze–Weinstein [20, Theorem 17.5.2].

Let A be a perfectoid ring. Let φ be the Frobenius automorphism of $W(A^\flat)$. For a generator $\xi \in \operatorname{Ker} \theta$, we put $\widetilde{\xi} := \varphi(\xi)$. Recall that a *minuscule Breuil–Kisin–Fargues module* for A is a finite projective module M over $W(A^\flat)$ with a $W(A^\flat)$-linear map $F_M \colon \varphi^* M \to M$ such that the cokernel of F_M is killed by $\widetilde{\xi}$. Note that, since $\widetilde{\xi}$ is a non-zero divisor, the condition on F_M is equivalent to the existence of a $W(A^\flat)$-linear map $V_M \colon M \to \varphi^* M$ such that $F_M \circ V_M = \widetilde{\xi}$. Moreover, for a fixed ξ, such a map V_M is uniquely determined.

We begin by recalling the following special case:

Theorem 5.1. *Let A be a perfectoid ring. If A satisfies one of the following conditions, then there exists an anti-equivalence $\mathcal{M}$ from the category of p-divisible groups over A to the category of minuscule Breuil–Kisin–Fargues modules for A.*

(1) *(Berthelot, Gabber, Lau) A is a perfect ring over $\mathbb{F}_p$.*

(2) *(Fargues, Scholze–Weinstein) A is the ring of integers $\mathcal{O}_C$ of an algebraically closed non-archimedean extension C of $\mathbb{Q}_p$.*

Proof. (1) For a p-divisible group $\mathcal{G}$ over A, let $\mathcal{M}(\mathcal{G}) := \mathbb{D}(\mathcal{G})(W(A))$ be the evaluation on the divided power extension $W(A) \to A$ of the contravariant Dieudonné crystal $\mathbb{D}(\mathcal{G})$ defined in [3, Définition 3.3.6]. This construction induces an anti-equivalence from the category of p-divisible groups over A to the category of minuscule Breuil–Kisin–Fargues modules for A; this fact is proved by Berthelot [2, Corollaire 3.4.3] (see also [2, Proposition 4.3.4]) for a perfect valuation ring over $\mathbb{F}_p$, and it is proved by Gabber and Lau [11, Theorem 6.4] independently for a general perfect ring over $\mathbb{F}_p$.

(2) See [20, Theorem 14.4.1], which is based on [19, Theorem B]. In this paper, for a p-divisible group $\mathcal{G}$ over $\mathcal{O}_C$, we define $\mathcal{M}(\mathcal{G})$ to be the $W(\mathcal{O}_C^\flat)$-linear dual of the Breuil–Kisin–Fargues module attached to $\mathcal{G}$ given in [20, Theorem 14.4.1]. □

We now deduce the general case from Theorem 5.1 by using Corollary 4.2:

Theorem 5.2 (Lau, Scholze–Weinstein). *For each perfectoid ring A, there exists an anti-equivalence $\mathcal{M}_A$ of categories*

$$\{p\text{-divisible groups over } A\} \xrightarrow{\sim} \{\text{minuscule Breuil–Kisin–Fargues modules for } A\}$$

satisfying the following properties:

- *$\mathcal{M}_A$ is compatible with base change in A.*
- *If $p = 0$ in A or $A = \mathcal{O}_C$ for an algebraically closed non-archimedean extension $C/\mathbb{Q}_p$, then $\mathcal{M}_A$ coincides with the anti-equivalence given in Theorem 5.1.*

Proof. As in the proof of [20, Theorem 17.5.2], Theorem 5.1 implies that, for each perfectoid ring $A = \prod_i V_i$ which is a product of perfectoid valuation rings V_i of rank ≤ 1 with algebraically closed fraction fields, we have an anti-equivalence $\mathcal{M}_A$ satisfying the above conditions.

Let A be a general perfectoid ring. By [7, Lemma 2.2.3] (and Example 2.2), there exists a p-complete arc-hypercover $A \to B^\bullet$ whose terms B^n are products of perfectoid valuation rings of rank ≤ 1 with algebraically closed fraction fields. By Corollary 4.2 and [9, Proposition 1.1], the category of p-divisible groups over A is equivalent to the category of p-divisible groups over B^0 with descent data (defined in the same way as the category $\mathrm{DD}(B^\bullet)$ in Section 4.2). Applying Corollary 4.2 again, we see that the same statement holds for minuscule Breuil–Kisin–Fargues modules. Thus we can obtain an anti-equivalence $\mathcal{M}_A$ from the category of p-divisible groups over A to the category of minuscule Breuil–Kisin–Fargues modules for A by using anti-equivalences $\mathcal{M}_{B^n}$. One can check that $\mathcal{M}_A$ does not depend (up to canonical equivalence) on the choice of $A \to B^\bullet$, and $\mathcal{M}_A$ is compatible with base change in A. We also note that, if $p = 0$ in A, then $\mathcal{M}_A$ coincides with the anti-equivalence given in Theorem 5.1 since the formation of $\mathbb{D}(\mathcal{G})(W(A))$ is compatible with base change in A.

The proof of Theorem 5.2 is complete. □

Remark 5.3. The conditions in Theorem 5.2 determine the anti-equivalences $\mathcal{M}_A$ uniquely (up to canonical equivalences). Anschütz–Le Bras give a cohomological description of $\mathcal{M}_A$ using the prismatic site of a perfectoid ring A developed by Bhatt–Scholze; see [1] for details.

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Mathematical Institute, Tohoku University
6-3, Aoba, Aramaki, Aoba-Ku, Sendai 980-8578, Japan
E-mail address: kazuhiro.ito.c3@tohoku.ac.jp

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