On numerically trivial automorphisms of threefolds of general type

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In this paper, we prove that the group $\mathrm{Aut}_{\mathbb{Q}}(X)$ of numerically trivial automorphisms are uniformly bounded for *smooth* projective threefolds X of general type which either satisfy $q(X) \geq 3$ or have a Gorenstein minimal model. If X is furthermore of maximal Albanese dimension, then $|\mathrm{Aut}_{\mathbb{Q}}(X)| \leq 4$, and the equality can be achieved by an unbounded family of threefolds previously constructed by the third author. Along the way we prove a Noether type inequality for log canonical pairs of general type with the coefficients of the boundary divisor from a given subset $\mathcal{C} \subset (0, 1]$ such that $\mathcal{C} \cup \{1\}$ attains the minimum.

1. Introduction

In studying automorphism groups of projective varieties, it is a basic problem to understand its induced action on the cohomology groups, say, with rational coefficients. More explicitly, for a complex smooth projective variety X, the action of the automorphism group $\mathrm{Aut}(X)$ on the cohomology with rational cohomology is the group homomorphism $\psi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(H^*(X, \mathbb{Q}))$ such that $\psi(\sigma) = (\sigma^{-1})^*$ for any $\sigma \in \mathrm{Aut}(X)$. We denote $\ker(\psi)$ by $\mathrm{Aut}_{\mathbb{Q}}(X)$ and call its elements the *numerically trivial automorphisms* of X. In this paper, we are interested in the size of $\mathrm{Aut}_{\mathbb{Q}}(X)$.

Obviously, the identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is contained in $\mathrm{Aut}_{\mathbb{Q}}(X)$. It is proved by Lieberman [35] that the index $[\mathrm{Aut}_{\mathbb{Q}}(X) : \mathrm{Aut}_0(X)]$ is finite. The question is how large $[\mathrm{Aut}_{\mathbb{Q}}(X) : \mathrm{Aut}_0(X)]$ can be.

If $\dim X = 1$, then X is a curve and $\mathrm{Aut}_{\mathbb{Q}}(X) = \mathrm{Aut}_0(X)$ holds by the explicit description of $\mathrm{Aut}_0(X)$ and an easy application of the Riemann–Hurwitz formula.

From dimension two on, the situation becomes very delicate. It is part of the package of the Torelli theorem that $\mathrm{Aut}(X)$ acts faithfully on $H^*(X, \mathbb{Q})$ (or equivalently, on $H^2(X, \mathbb{Q})$) for K3 surfaces [5, 47].[1] In relation to the Torelli type problem in arbitrary dimension, $\mathrm{Aut}_{\mathbb{Q}}(X)$ is shown to be trivial for the following classes of varieties: generalized Kummer manifolds ([41]), smooth cubic threefolds and fourfolds as well as their Fano varieties of lines ([42]), and complete intersections $X \subset \mathbb{P}^n$ such that $\deg X \geq 3$, $\dim X \geq 2$ and X is not of type $(2, 2)$ ([16]).

However, examples of elliptic surfaces with nontrivial $\mathrm{Aut}_{\mathbb{Q}}(X)$ were found around 1980 by several authors ([2, 18, 40, 45, 46]). Mukai–Namikawa [40] obtained the optimal inequality $|\mathrm{Aut}_{\mathbb{Q}}(X)| \leq 4$ for Enriques surfaces.[2] By a recent work of Catanese and the second author ([12]), $[\mathrm{Aut}_{\mathbb{Q}}(X) : \mathrm{Aut}_0(X)] \leq 12$ holds for surfaces with Kodaira dimension 0, but the index can be arbitrarily large for the class of smooth projective surfaces with Kodaira dimension $-\infty$ and 1.

If X is a surface of general type, as for all smooth projective varieties of general type, the identity component $\mathrm{Aut}_0(X)$ is trvial, and hence $[\mathrm{Aut}_{\mathbb{Q}}(X) : \mathrm{Aut}_0(X)] = |\mathrm{Aut}_{\mathbb{Q}}(X)|$. Some restrictions on the invariants of X with nontrivial $\mathrm{Aut}_{\mathbb{Q}}(X)$ were found by Peters [46] under the condition

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[1]For K3 surfaces in positive characteristics, $\mathrm{Aut}(X)$ acts faithfully on the second l-adic étale cohomology group $H^2_{\text{ét}}(X, \mathbb{Q}_l)$ by [30, Theorem 1.4].

[2]We refer to [19–21] for numerically trivial automorphisms of Enriques surfaces in positive characteristics.

that the canonical linear system $|K_X|$ is base point free. Cai [6] used the Bogomolov–Miyaoka–Yau inequality to show that $|\operatorname{Aut}_{\mathbb{Q}}(X)|$ is uniformly bounded; in fact, $|\operatorname{Aut}(X)| \leq 4$ as soon as $\chi(X, \mathcal{O}_X) \geq 189$. For irregular surfaces of general type, the results are quite complete by now:

1) If $q(X) \geq 3$, then $\operatorname{Aut}_{\mathbb{Q}}(X)$ is trivial ([8, 11]).

2) If $q(X) = 2$, then $|\operatorname{Aut}_{\mathbb{Q}}(X)| \leq 2$, and there is a complete classification of the equality case: they are certain surfaces isogenous to a product of curves of unmixed type ([9, 11, 36]).

3) If $q(X) = 1$ then $|\operatorname{Aut}_{\mathbb{Q}}(X)| \leq 4$, and again equality holds only if X is isogenous to a product of curves of unmixed type. Moreover, there are infinite serie of such surfaces with $\operatorname{Aut}_{\mathbb{Q}}(X) = 4$ ([10]).

For regular surfaces of general type, it follows from [7] that $\operatorname{Aut}_{\mathbb{Q}}(X)$ is trivial if X has a genus 2 fibration and $\chi(X, \mathcal{O}_X) \geq 5$.

Not much is known about $\operatorname{Aut}_{\mathbb{Q}}(X)$ for higher dimensional general type varieties. Unlike the surface case, one easily constructs smooth threefolds of general type with $\operatorname{Aut}_{\mathbb{Q}}(X)$ nontrivial and $q(X)$ arbitrarily large: Just take $X = S \times C$, where S is a smooth projective surface of general type with nontrivial $\operatorname{Aut}_{\mathbb{Q}}(S)$ and C be a smooth projective curve with $g(C) \geq 2$; then $\operatorname{Aut}_{\mathbb{Q}}(X)$, containing $\operatorname{Aut}_{\mathbb{Q}}(S) \times \{\operatorname{id}_C\}$, is nontrivial, and $q(X) = q(S) + g(C)$ can be arbitrarily large.

Nevertheless, we have the following question about the boundedness of $\operatorname{Aut}_{\mathbb{Q}}(X)$ for smooth projective varieties of general type:

Question 1.1. Given a positive integer n, does there exist a constant $M(n)$, depending on n, such that $|\operatorname{Aut}_{\mathbb{Q}}(X)| \leq M(n)$ for any n-dimensional *smooth* projective variety X of general type?

In this paper we give a confirmative partial answer to Question 1.1 in dimension 3.

Theorem 1.2. *There is a positive constant M such that, for any smooth projective threefold of general type satisfying one of the following conditions:*

(i) $q(X) \geq 3$.

(ii) X has a Gorenstein minimal model.

the inequality $|\operatorname{Aut}_{\mathbb{Q}}(X)| \leq M$ holds.

Curiously, the smoothness condition in Theorem 1.2 cannot be weakened to, say, the condition of having terminal singularities: There exist projective threefolds X with terminal singularities and maximal Albanese dimension such that $|\operatorname{Aut}_{\mathbb{Q}}(X)|$ can be arbitrarily large ([52, Example 6.3]). We observe that the numerically trivial automorphism σ appearing there induces a nontrivial translation of the Albanese variety A_X. This cannot happen if X were smooth; see Lemma 2.6.

For smooth threefolds with maximal Albanese dimension, we have a more precise statement.

Theorem 1.3 (= Corollary 3.2). *If X is smooth projective threefold of general type with maximal Albanese dimension, then $|\operatorname{Aut}_{\mathbb{Q}}(X)| \leq 4$, and in the equality case we have $\operatorname{Aut}_{\mathbb{Q}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $q(X) = 3$.*

This generalizes and refines the effective boundedness results of [52]. Thanks to [52, Example 6.1], the bound of Theorem 1.3 is optimal.

By the Hodge decomposition of $H^*(X, \mathbb{C})$, $\operatorname{Aut}_{\mathbb{Q}}(X)$ acts trivially on $H^i(X, \mathcal{O}_X)$ and $H^i(X, \omega_X)$ for each i. It is this condition we are mainly using in the arguments, and it already yields the boundedness we are searching for. Theorems 1.2 and 1.3 are consequences of the following more technical theorem; see also Corollaries 3.2, 3.4, 3.7. Here we allow mild singularities.

Theorem 1.4. *There is a constant $M > 0$ satisfying the following. Let X be a projective threefold of general type with canonical singularities. Let $\operatorname{Aut}_{\mathcal{O}}(X) < \operatorname{Aut}(X)$ be the group of automophisms acting trivially on $H^i(X, \mathcal{O}_X)$ (or, equivalently, $H^i(X, \omega_X)$) for $i \geq 0$, and let $\operatorname{Aut}_A(X) < \operatorname{Aut}(X)$ the group of automorphisms preserving each fiber of the Albanese map $a_X \colon X \to A_X$. If X satisfies one of the following conditions:*

(i) X has maximal Albanese dimension, that is, $\dim a_X(X) = 3$;

(ii) X has Albanese dimension two and either $q(X) \geq 3$ or A_X is a simple abelian surface;

(iii) X has Albanese dimension one and $q(X) \geq 2$;

(iv) X has a Gorenstein minimal model;

then $|\operatorname{Aut}_{\mathcal{O}}(X) \cap \operatorname{Aut}_A(X)| \leq M$. In case (i), we have $|\operatorname{Aut}_{\mathcal{O}}(X) \cap \operatorname{Aut}_A(X)| \leq 4$, and if the equality holds then $\operatorname{Aut}_{\mathbb{Q}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $q(X) = 3$.

The third author [52] proved the inequality $|\operatorname{Aut}_{\mathcal{O}}(X)| \leq 6$ for threefolds of general type satisfying the conditions (i) and (iv) simultaneously; the proof was a straightforward application of the Noether–Severi type inequality and the Miyaoka–Yau inequality bounding the volume of K_X in terms of $\chi(X, \omega_X)$ from below and from above respectively.

In dealing with the case (iv) of Theorem 1.4, we take a similar approach, but the required Noether type inequality in this more general setting is not readily available. For that reason, we prove in the appendix of the paper the existence of a Noether type inequality for log canonical pairs with coefficients from a given subset $\mathcal{C} \subset (0, 1]$ such that $\mathcal{C} \cup \{1\}$ attains the minimum. This result is of independent interest.

In the case (i) of Theorem 1.4, we use the eventual paracanonical maps [1, 25] to obtain the effective bound $|\operatorname{Aut}_{\mathcal{O}}(X) \cap \operatorname{Aut}_A(X)| \leq 4$. The point is that the eventual paracanonical map $\varphi_X \colon X \dashrightarrow Z$ factors through the quotient map $\pi \colon X \to X/G$, where $G = \operatorname{Aut}_{\mathcal{O}}(X) \cap \operatorname{Aut}_A(X)$. It follows that $|G| \leq \deg \varphi_X$. We can then draw on the explicit description of the cases with $\deg \varphi_X \geq 2$ by [25].

In general, there is a factorization of the Albanese map $a_X \colon X \xrightarrow{\pi} Y \xrightarrow{a_Y} A_X$, where $Y = X/G$ with $G = \operatorname{Aut}_{\mathcal{O}}(X) \cap \operatorname{Aut}_A(X)$ and $\pi \colon X \to Y$ is the quotient map. If $\dim a_X(X) < 3 \leq q(X)$, we use the Chen–Jiang decomposition of $a_{X*}\omega_X$ and $a_{Y*}\omega_Y$ to bound the difference between the ranks of $a_{X*}\omega_X$ and $a_{Y*}\omega_Y$, which in turn gives the required bound on $|G|$. The case with $q(X) = 2$ and A_X a simple abelian surface requires a more involved application of the Chen–Jiang decomposition, but the idea is similar.

2. Preliminaries

We work over the complex numbers throughout the paper.

In this section we recall the notion of Albanese morphisms for normal projective varieties X as well as the Chen–Jiang decomposition on abelian varieties, and prove some basic properties of numerically trivial automorphisms in relation to their induced actions on the cohomology groups $H^*(X, \mathcal{O}_X)$, $H^*(X, \omega_X)$ as well as the Albanese variety.

2.1. The Albanese morphism

Let X be a normal projective variety. Then there exists a morphism $a_X \colon X \to A_X$ to an abelian variety A_X such that any morphism from X to an abelian variety factors through a_X ([48, Théorème 5]). This morphism is

unique up to translations on A_X. We call A_X the Albanese variety of X and $a_X : X \to A_X$ the Albanese morphism.

When X is a smooth projective variety, it is well known that $\dim A_X = q(X) := h^1(X, \mathcal{O}_X)$ and A_X is dual to the Picard variety $\mathrm{Pic}^0(X)$. In general, if X is singular, then the Albanese morphism $a_{\widetilde{X}}$ of $\widetilde{X}$ may not factor through ρ, where $\rho : \widetilde{X} \to X$ is a desingularization. But when X has rational singularities, then $a_{\widetilde{X}}$ factors through ρ and $A_X \cong A_{\widetilde{X}}$ (see for instance [29, Lemma 8.1]).

Let X be a normal projective variety, and $a_X : X \to A_X$ the Albanese map with respect to a base point. We call the number $\dim a_X(X)$ the *Albanese dimension* of X; if $\dim a_X(X) = X$ then X is said to be *of maximal Albanese dimension*. Let $X \to S \xrightarrow{a_S} A_X$ be the Stein factorization of a_X. By the universal property of the Albanese morphisms, it is easy to check that $A_S = A_X$, and the morphism $a_S : S \to A_X$ is indeed the Albanese morphism of S. For any $\sigma \in \mathrm{Aut}(X)$, by the universal properties of the Albanese morphisms and of the Stein factorization, there are induced automorphisms $\sigma_S \in \mathrm{Aut}(S)$ and $\sigma_A \in \mathrm{Aut}(A_X)$ such that the following diagram is commutative

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & S & \xrightarrow{\ a_S\ } & A_X \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma_S} & & \downarrow{\scriptstyle\sigma_A} \\
X & \xrightarrow{\ f\ } & S & \xrightarrow{\ a_S\ } & A_X
\end{array}
$$

In this way, we obtain group homomorphisms

$$
\begin{aligned}
(2.1) \qquad & \psi_S \colon \mathrm{Aut}(X) \to \mathrm{Aut}(S), \ \sigma \mapsto \sigma_S, \\
& \text{and} \quad \psi_A \colon \mathrm{Aut}(X) \to \mathrm{Aut}(A_X), \ \sigma \mapsto \sigma_A.
\end{aligned}
$$

In other words, $\mathrm{Aut}(X)$ acts on S and A_X such that the morphisms f and a_S are $\mathrm{Aut}(X)$-equivariant.

2.2. The Chen–Jiang decomposition

An important ingredient of the proofs of the main results on irregular three-folds is the Chen–Jiang decomposition theorem. We briefly recall this result.

Let A be an abelian variety. For a coherent sheaf $\mathcal{F}$ on A, we define the i-th *cohomological support loci* for integer $i \geq 0$ to be the following closed

subset of $\mathrm{Pic}^0(A)$ with the reduced scheme structure:

$$V^i(\mathcal{F}) := \{P \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes P) \neq 0\}.$$

We say that $\mathcal{F}$ is *GV* if $\mathrm{codim}_{\mathrm{Pic}^0(A)} V^i(\mathcal{F}) \geq i$ for each $i \geq 1$ and we say that $\mathcal{F}$ is *M-regular* if $\mathrm{codim}_{\mathrm{Pic}^0(A)} V^i(\mathcal{F}) > i$ for each $i \geq 1$. These two definitions were introduced respectively by Hacon [23] and by Pareschi–Popa [43]. GV sheaves have certain positivity properties. Hacon proved in [23] that if $0 \neq \mathcal{F}$ is GV, then $V^0(\mathcal{F}) \neq \emptyset$, that is, there exists $P \in \mathrm{Pic}^0(A)$ such that $H^0(A, \mathcal{F} \otimes P) \neq 0$. M-regular sheaves have stronger positivity properties. Pareschi and Popa proved in [43] that if $\mathcal{F}$ is M-regular, then $V^0(\mathcal{F}) = \mathrm{Pic}^0(A)$ and, roughly speaking, $\mathcal{F}$ can be generated by the twisted global sections.

Given a morphism $f : X \to A$ from a smooth projective variety to an abelian variety, the main result of [23] states that $R^i f_* \omega_X$ is a GV-sheaf for each $i \geq 0$. The following improvement of Hacon's theorem, which was first proved when f is generically finite in [14] and later proved for f general in [44], is now called the Chen–Jiang decomposition theorem.

Theorem 2.1. *Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. For each integer $k \geq 0$, there exist finitely many quotient maps $p_I : A \to B_I$ between abelian varieties with connected fibers, and, for each index I, an M-regular sheaf $\mathcal{F}_I$ on B_I and a torsion line bundle $Q_I \in \mathrm{Pic}^0(A)$ such that*

$$R^k f_* \omega_X \cong \bigoplus_I p_I^* \mathcal{F}_I \otimes Q_I.$$

Remark 2.2. We remark that under the assumption of Theorem 2.1, for any torsion line bundle $Q \in \mathrm{Pic}^0(X)$, $R^k f_*(\omega_X \otimes Q)$ also has the Chen–Jiang decomposition (see for instance [44, Note on page 2449]). Moreover, any direct summand of $R^k f_*(\omega_X)$ has the Chen–Jiang decomposition (see [37, Proposition 3.6]). Similar decompositions also hold for varieties with klt singularities, see for instance [26] and [38].

2.3. Some basic properties of numerically trivial automorphisms

Let X be a normal projective variety, and $a_X : X \to A_X$ the Albanese morphism. We are interested in various normal subgroups of the automorphism group $\mathrm{Aut}(X)$ of X, defined as follows:

- the group of *numerically trivial automorphisms*

$$\mathrm{Aut}_{\mathbb{Q}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma|_{H^*(X,\mathbb{Q})} = \mathrm{id}_{H^*(X,\mathbb{Q})}\},$$

- the group of $\mathcal{O}$-*cohomologically trivial automorphisms*

$$\mathrm{Aut}_{\mathcal{O}}(X) := \left\{\sigma \in \mathrm{Aut}(X) \mid \sigma|_{H^*(X,\mathcal{O}_X)} = \mathrm{id}_{H^*(X,\mathcal{O}_X)}\right\},$$

- the group of *A-trivial automorphisms*

$$\mathrm{Aut}_A(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma|_{A_X} = \mathrm{id}_{A_X}\},$$

where $\sigma|_{H^*(X,\mathbb{Q})}$, $\sigma|_{H^*(X,\mathcal{O}_X)}$ and $\sigma|_{A_X}$ denote the induced actions of σ on $H^*(X,\mathbb{Q}) := \bigoplus_i H^i(X,\mathbb{Q})$, $H^*(X,\mathcal{O}_X) := \bigoplus_i H^i(X,\mathcal{O}_X)$ and A_X respectively.

If X is Cohen–Macaulay, then by Serre duality, $\mathrm{Aut}_{\mathcal{O}}(X)$ acts trivially on $H^i(X,\omega_X)$ for each $i \geq 0$. This is the case when X has rational singularities. The following Lemma 2.3 implies that $\mathrm{Aut}_{\mathbb{Q}}(X) < \mathrm{Aut}_{\mathcal{O}}(X)$, provided that there is an effective $\mathbb{R}$-divisor Δ such that (X,Δ) is a log canonical pair. We will see in Proposition 2.6 that $\mathrm{Aut}_{\mathbb{Q}}(X) < \mathrm{Aut}_A(X)$ if X is a smooth projective threefold of general type.

Lemma 2.3. *Let X be a normal projective variety.*

(i) *If there is an effective $\mathbb{R}$-divisor Δ on X such that (X,Δ) is log canonical, then $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on $H^i(X,\mathcal{O}_X)$ for all $i \geq 0$.*

(ii) *If X is smooth, then $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on $H^i(X,\Omega_X^j)$ for all $0 \leq i,j \leq \dim X$.*

Proof. Note that $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on $H^*(X,\mathbb{C}) = H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. If (X,Δ) is log canonical then the homomorphism $H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X)$ induced by the inclusion of sheaves $\mathbb{C} \subset \mathcal{O}_X$ is surjective ([33]). It follows that $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on $H^i(X,\mathcal{O}_X)$. If X is smooth, $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on its direct summands $H^i(X,\Omega_X^j)$ of $H^{i+j}(X,\mathbb{C})$ in the Hodge decomposition. $\square$

Lemma 2.4. *Let X be a normal projective variety with rational singularities. Suppose that $\sigma \in \mathrm{Aut}(X)$ induces a trivial action on $H^1(X,\mathcal{O}_X)$. Then the following holds.*

(i) *The induced automorphism $\sigma_A \in \mathrm{Aut}(A_X)$ is a translation.*

(ii) If σ is of finite order and $\sigma_A \neq \mathrm{id}_{A_X}$, then $\chi(X, \omega_X) = (-1)^{\dim X}\chi(X, \mathcal{O}_X) = 0$ and $e(X) = 0$.

Proof. (i) Suppose that $\sigma \in \mathrm{Aut}(X)$ induces the trivial action on $H^1(X, \mathcal{O}_X)$. Since $a_X^* \colon H^1(A, \mathcal{O}_A) \to H^1(X, \mathcal{O}_X)$ is an isomorphism satisfying

$$\sigma^* \circ a_X^* = a_X^* \circ \sigma_A^* \colon H^1(A_X, \mathcal{O}_{A_X}) \to H^1(X, \mathcal{O}_X)$$

and σ^* is the identity map of $H^1(X, \mathbb{C})$, we infer that σ_A^* is the identity map of $H^1(A_X, \mathcal{O}_{A_X})$. It follows that σ_A is a translation of A_X.

(ii) If $\sigma_A \neq \mathrm{id}_{A_X}$, then it is a nontrivial translation by (i) and hence has no fixed points. It follows that σ does not fix any point either. We then conclude by the topological Lefschetz fixed point theorem that $e(X) = 0$.

For the vanishing of $\chi(X, \mathcal{O}_X)$, we take a σ-equivariant resolution $\rho \colon \widetilde{X} \to X$ ([22]). Since X has rational singularities, we have $a_X \circ \rho = a_{\widetilde{X}}$ and $\chi(X, \mathcal{O}_X) = \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Let $\widetilde{\sigma} \in \mathrm{Aut}(\widetilde{X})$ be the lift of σ. Then $\widetilde{\sigma}$ has no fixed points either. By the holomorphic Lefschetz fixed point theorem, applied to $\widetilde{\sigma}$, we obtain $\chi(X, \mathcal{O}_X) = \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. $\square$

We refer to [34] for the notions appearing in Lemma 2.5.

Lemma 2.5. *Let (X, Δ) be a projective log canonical pair, and $\varphi_R \colon X \to Y$ the contraction with respect to a $(K_X + \Delta)$-negative extremal ray R of the Kleiman–Mori cone of (X, Δ). Then any numerically trivial automorphism $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(X)$ induces an automorphism $\sigma_Y \in \mathrm{Aut}(Y)$. Moreover, the following holds.*

  (i) If Y has rational singularities, then σ_Y is $\mathcal{O}$-cohomologically trivial, that is, $\sigma_Y \in \mathrm{Aut}_{\mathcal{O}}(Y)$.

  (ii) If X and Y are smooth, then σ_Y is numerically trivial, that is, $\sigma_Y \in \mathrm{Aut}_{\mathbb{Q}}(Y)$.

Proof. By the contraction theorem, there is a line bundle L on X such that $L = \varphi_R^* L_Y$ for some ample line bundle L_Y on Y. Since σ acts trivially on $H^2(X, \mathbb{Q})$, it preserves the cohomology class of L in $H^2(X, \mathbb{Q})$. Note that homological euqivalence coincides with algebraic equivalence for Cartier divisors, we infer that $\sigma^* L$ is numerically equivalent to L. For any curve C whose numerical class generates the extremal ray R, using the projection

formula,

$$L_Y \cdot (\varphi_R \circ \sigma)_*(C) = L \cdot \sigma_*(C) = (\sigma^* L) \cdot C = 0,$$

we get that C is contracted by $\varphi_R \circ \sigma$. By the rigidity of birational contractions, there exists a morphism $\sigma_Y \colon Y \to Y$ such that the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{\ \sigma\ } & X \\
{\scriptstyle \varphi_R}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_R} \\
Y & \xrightarrow{\ \sigma_Y\ } & Y
\end{array}
$$

Since σ is an isomorphism, we conclude that σ_Y is an isomorphism of Y.

(i) If Y has rational singularities, then we have isomorphisms

$$H^i(X, \mathcal{O}_X) \cong H^i(Y, \mathcal{O}_Y)$$

for each $i \geq 0$ which is compatible with induced actions of σ and σ_Y on $H^i(X, \mathcal{O}_X)$ and $H^i(Y, \mathcal{O}_Y)$ respectively. Since $\sigma \in \mathrm{Aut}_{\mathcal{O}}(X)$, we infer that $\sigma_Y \in \mathrm{Aut}_{\mathcal{O}}(Y)$.

(ii) By the decomposition theorem for resolutions ([17]), $\varphi_R^* \colon H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C})$ is injective. Thus the numerical triviality of σ implies that of σ_Y. $\qquad\square$

Proposition 2.6. *Let X be a smooth projective threefold of general type. Then $\mathrm{Aut}_{\mathbb{Q}}(X)$ acts trivially on A_X.*

Proof. The statement is trivial when $q(X) = 0$. We can thus assume that $q(X) > 0$.

Take $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(X)$ and let $\sigma_A \in \mathrm{Aut}_{\mathbb{Q}}(A_X)$ be the induced automorphism of A_X. Then σ_A is a translation of A_X by Lemmas 2.3 and 2.4. If X has a smooth minimal model, then the Miyaoka–Yau inequality, $\mathrm{vol}(K_X) \leq 72\chi(\mathcal{O}_X)$ implies that $\chi(\mathcal{O}_X) > 0$. By Lemma 2.4, σ_A is trivial.

Now we assume that the minimal models of X are singular. In order to show that $\sigma_A = \mathrm{id}_{A_X}$, it suffices to show that σ_A fixes a point. Consider a minimal model program

$$X = X_0 \ \dashrightarrow^{\rho_0} \ X_1 \ \dashrightarrow^{\rho_1} \ X_2 \ \dashrightarrow^{\rho_2} \ \cdots \ \dashrightarrow^{\rho_{n-1}} \ X_n$$

where for $0 \leq i \leq n$, X_i has terminal singularities, the ρ_i are either divisorial extremal contractions or flips, and K_{X_n} is nef. As we have explained in last section, the induced maps $a_i \colon X_i \dashrightarrow A_X$ are indeed the corresponding Albanese morphisms of X_i and the above minimal model program is over A_X.

By assumption, $X_0 = X$ is smooth and X_n is singular, so we there is an index i_0 such that X_{i_0} is smooth and X_{i_0+1} is singular. By the classification of extremal contractions in dimension three ([39]; see also [34, Theorem 1.32]), the birational maps $\rho_i \colon X_i \dashrightarrow X_{i+1}$ are divisorial contractions for $i \leq i_0$ and, moreover, $\rho_{i_0} \colon X_{i_0} \to X_{i_0+1}$ is a morphism contracting a divisor $E \subset X_{i_0}$ to a point $p \in X_{i_0+1}$. By Lemma 2.5, σ descends to a numerically trivial automorphism $\sigma_{i_0} \in \mathrm{Aut}_{\mathbb{Q}}(X_{i_0})$. In particular, σ_{i_0} preserves the exceptional locus $\mathrm{Exc}(\rho_{i_0})$, and this implies that the induced automorphism σ_A fixes the point $a_{i_0}(E) = a_{i_0+1}(p) \in A_X$. $\qquad\square$

3. Irregular threefolds of general type

In this section, we prove Theorem 1.4, cases (i)-(iii). Let X be an irregular projective threefold of general type with canonical singularities and $a_X \colon X \to A_X$ its Albanese map with respect to a base point.

We proceed according to the Albanese dimension $\dim a_X(X)$.

3.1. The case $\dim a_X(X) = 3$

In this subsection we assume that $\dim a_X(X) = 3$, so X is of maximal Albanese dimension.

Theorem 3.1. *Let X be a projective threefold of general type with canonical singularities and of maximal Albanese dimension. Then $|\mathrm{Aut}_{\mathcal{O}}(X) \cap \mathrm{Aut}_A(X)| \leq 4$, and if the equality holds then $\mathrm{Aut}_{\mathcal{O}}(X) \cap \mathrm{Aut}_A(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $q(X) = 3$.*

Proof. Let $G := \mathrm{Aut}_{\mathcal{O}}(X) \cap \mathrm{Aut}_A(X)$. Then the Albanese morphism a_X factors through the quotient morphism $\pi \colon X \to Y := X/G$.

If $\chi(X, \omega_X) = 0$, then a_X is birationally an $(\mathbb{Z}/2\mathbb{Z})^2$-cover by [13]. Thus G is an (abelian) subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$.

If $\chi(X, \omega_X) > 0$, then $\chi(X, \omega_X) = \chi(Y, \omega_Y) > 0$. By [25], the eventual paracanonical map $\varphi_X \colon X \dashrightarrow Z$ factors through the quotient map as $\varphi_X \colon X \xrightarrow{\pi} Y \dashrightarrow Z$, where $Y \dashrightarrow Z$ is the eventual paracanonical map φ_Y of Y ([25, Proposition 1.5]). Note that, $\deg \varphi_X \leq 8$ by [25, Theorem 1.7], and hence

$$(3.1) \qquad |G| = \deg \pi = \deg \varphi_X / \deg \varphi_Y \leq \frac{8}{\deg \varphi_Y}.$$

If $\deg \varphi_Y \geq 2$, then either $|G| \leq 3$ or $|G| = 4$ and $\deg \varphi_X = 8$. In the latter case, φ_X is birationally a $(\mathbb{Z}/2\mathbb{Z})^3$-cover by [25, Theorem 1.7]. If $\deg \varphi_Y = 1$, then φ_Y is birational and hence

$$\chi(Z, \omega_Z) = \chi(Y, \omega_Y) = \chi(X, \omega_X) > 0.$$

By [25, Theorem 1.7] again, φ_X induces a Galois field extenstion $\mathbb{C}(X)/\mathbb{C}(Z)$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$. In conclusion, we have $|G| \leq 4$, and if the equality holds then $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

In the case $G \cong (\mathbb{Z}/2\mathbb{Z})^2$, we will show that $q(X) = 3$. When $\chi(X, \omega_X) = 0$, we automatically have $q(X) = 3$ by [13]. We can thus assume that $\chi(X, \omega_X) = \chi(Y, \omega_Y) > 0$. If $q(X) > 3$, we are in the situation of [25, Subsection 4.1, Case 1]. By [25, Proposition 4.4], the quotient map π is birationally equivalent to the eventual paracanonical map of X. Let $V := a_X(X)$ be the Albanese image of X with the reduced scheme structure. Since $\dim A_X > 3$, V is fibred by an abelian subvariety K such that the quotient V/K is not fibred by any positive dimensional abelian subvariety of A_X/K and any desingularization of V/K is of general type (see [49, Theorem 10.9]). Let $B := A/K$ and $V_B := V/K$. We have the following picture:

$$
\begin{array}{ccccc}
X & \xrightarrow{\pi} & Y & \longrightarrow & V \hookrightarrow A_X \\
 & \searrow_f & \downarrow^h & & \downarrow \\
 & & V_B & \hookrightarrow & B.
\end{array}
$$

Let $t \in V_B$ be a general point and let X_t and Y_t be the fibers of f and h over t respectively. We remark that X_t and Y_t may be reducible and each component of X_t or Y_t is of general type. Fix a general torsion line bundle $Q \in \operatorname{Pic}^0(A_X)$. Note that A_X is the Albanese variety of both X and Y. We can simply regard Q as a general torsion line bundle on X or on Y. By generic vanishing,

$$
\begin{aligned}
h^0(X, \omega_X \otimes Q) &= \chi(X, \omega_X \otimes Q) = \chi(X, \omega_X) \\
&= \chi(Y, \omega_Y) = \chi(Y, \omega_Y \otimes Q) = h^0(Y, \omega_Y \otimes Q).
\end{aligned}
$$

Moreover, since Q is general, we also have $h^i(X_t, \omega_{X_t} \otimes Q) = h^i(Y_t, \omega_{Y_t} \otimes Q) = 0$. Thus both $R^i f_*(\omega_X \otimes Q)$ and $R^i h_*(\omega_Y \otimes Q)$ are supported on a proper subset of V_B. However, by the main theorem of [31], these sheaves are torsion-free on V_B if they are not zero. Thus, $R^i f_*(\omega_X \otimes Q) = R^i h_*(\omega_Y \otimes$

$Q) = 0$ for $i \geq 1$. Therefore, for $Q' \in \mathrm{Pic}^0(B)$ general,

$$(3.2) \quad h^0(V_B, f_*(\omega_X \otimes Q) \otimes Q') = \chi(X, \omega_X \otimes Q \otimes Q')$$
$$= \chi(Y, \omega_Y \otimes Q \otimes Q') = h^0(V_B, h_*(\omega_Y \otimes Q) \otimes Q').$$

We claim that the inclusion of coherent sheaves $h_*(\omega_Y \otimes Q) \subset f_*(\omega_X \otimes Q)$ is an isomorphism. Assume the contrary, and let $\mathcal{Q}$ be the *nonzero* quotient sheaf. Then $V^0(\mathcal{Q})$ is a proper subset of $\mathrm{Pic}^0(B)$ by (3.2). On the other hand, by [31], these two sheaves are torsion-free and since V_B is not fibred by positive dimensional abelian subvariety, using the Chen–Jiang decomposition for $h_*(\omega_Y \otimes Q)$ and $f_*(\omega_X \otimes Q)$ (see 2.1 and Remark 2.2), both of them are M-regular. It follows that $\mathcal{Q}$ is also M-regular and hence $V^0(\mathcal{Q}) = \mathrm{Pic}^0(B)$, which is a contradiction.

The equality $h_*(\omega_Y \otimes Q) = f_*(\omega_X \otimes Q)$ implies that $\chi(X_t, \omega_{X_t}) = \chi(Y_t, \omega_{Y_t})$. Note that $\pi_t \colon X_t \to Y_t$ is again of degree 4. The case $\dim X_t = 1$ is impossible by Riemann–Roch. The remaining case $\dim X_t = 2$ is impossible by [25, Theorem 1.6] or [1, Proposition 3.3]. $\square$

Corollary 3.2. *Let X be any smooth projective threefold of general type with maximal Albanese dimension. Then $|\mathrm{Aut}_{\mathbb{Q}}(X)| \leq 4$, and if the equality holds then $\mathrm{Aut}_{\mathbb{Q}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $q(X) = 3$.*

Proof. By Lemmas 2.3 and 2.6, we know that $\mathrm{Aut}_{\mathbb{Q}}(X)$ is contained in the group $\mathrm{Aut}_{\mathcal{O}}(X) \cap \mathrm{Aut}_A(X)$ of Theorem 3.1. $\square$

Note that the case $\mathrm{Aut}_{\mathbb{Q}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is realized by an unbounded series of threefolds by [52, Example 6.1].

3.2. The case $\dim a_X(X) < 3$

In this subsection we deal with irregular threefolds of general type which are not of maximal Albanese dimension. We will denote by $G := \mathrm{Aut}_{\mathcal{O}}(X)$ the $\mathcal{O}$-cohomologically trivial automorphism subgroup, and $\pi \colon X \to Y := X/G$ the quotient morphism.

Lemma 3.3. *There is a positive constant M_1 such that the following holds. Let $f \colon X \to B$ be a fibration from a smooth projective threefold of general type onto a smooth projective curve B with $g(B) \geq 2$. Then $|\mathrm{Aut}_{\mathcal{O}}(X)| \leq M_1$.*

Proof. Since $f^*\colon H^1(B, \mathcal{O}_B) \to H^1(X, \mathcal{O}_X)$ is injective and $G := \mathrm{Aut}_{\mathcal{O}}(X)$ acts trivially on $H^1(X, \mathcal{O}_X)$, G induces a trivial action on $H^1(B, \mathcal{O}_B)$. Since $g(B) \geq 2$, G acts trivially on B, that is, $G \subset \mathrm{Aut}_B(X)$, the group of automorphisms preserving each fiber of f.

Let F be a general fiber of f. Then G acts on F via the injective homomorphism $G \hookrightarrow \mathrm{Aut}(F)$, $\sigma \mapsto \sigma|_F$. Let $\pi\colon X \to Y := X/G$ and $\pi_F\colon F \to F' := F/G$ be the quotient morphisms. Since B is of general type, by Theorem 2.1 or [27, Lemma 2.1], both $a_{X*}\omega_X$ and $a_{Y*}\omega_Y$ are M-regular. Moreover, $p_g(X) = p_g(Y)$, thus $a_{Y*}\omega_Y = a_{X*}\omega_X$. Hence $p_g(F) = p_g(F')$.

By [3, Propositions 2.1 and 4.1], if $\chi(F, \mathcal{O}_F) \geq 31$, then the canonical map φ_F of F is either of degree ≤ 9 or is a fibration such that the genus g of a general fiber is ≤ 5. Thus in this case we obtain

$$|G| \leq \begin{cases} \deg \varphi_F \leq 9 & \text{if } \varphi_F \text{ is generically finite} \\ 84(g-1) \leq 336 & \text{if } \varphi_F \text{ is composed with a pencil} \end{cases}$$

where the inequality in second case follows from the Hurwitz bound of the full automorphism group of a curve with genus $g \geq 2$. On the other hand, surfaces F with $\chi(F, \mathcal{O}_F) < 31$ form a birationally bounded family and thus the order of their automorphism groups are bounded by a constant. As a conclusion, there exists a constant M_1 such that $|G| \leq M_1$. $\qquad\square$

Corollary 3.4. *The inequality* $|\mathrm{Aut}_{\mathcal{O}}(X)| \leq M_1$ *holds for any smooth projective threefold* X *of general type with* $q(X) \geq 2$ *and* $\dim a_X(X) = 1$.

Proof. Since $q(X) \geq 2$ and $\dim a_X(X) = 1$, we infer that the Albanese image of X factors as $a_X\colon X \xrightarrow{f} C \hookrightarrow A_X$, where $g(C) = q(X)$ and f is a fibration. Now apply Lemma 3.3. $\qquad\square$

Remark 3.5. We can indeed work out an explicit bound for M_1. Note that in the proof of Lemma 3.3, when $\chi(F, \mathcal{O}_F) < 31$, we have $\mathrm{vol}(K_F) \leq 9\chi(F, \mathcal{O}_F) \leq 180$ by the Bogomolov–Miyaoka–Yau inequality. Thus $|G| \leq 42^2 \mathrm{vol}(K_F) \leq 317520$ by [51], and we may take $M_1 = 317520$.

Proposition 3.6. *Let* X *be a smooth threefold of general type with* $q(X) \geq 3$ *and* $\dim a_X(X) = 2$. *Then* $|\mathrm{Aut}_{\mathcal{O}}(X)| \leq M_1$.

Proof. Since a_X is not surjective, the image Z of the Albanese morphism (or rather its desingularization) has Kodaira dimension ≥ 1.

Let H be the image of $\mathrm{Aut}_{\mathcal{O}}(X)$ in $\mathrm{Aut}(A_X)$; see (2.1). By Lemma 2.4, H acts on A_X by translations. Thus the quotient morphism $A_X \to A_X/H$ is étale.

Consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi\ } & Y \\
& & \\
Z & \longrightarrow & Z/H \\
& & \\
A_X & \longrightarrow & A_X/H
\end{array}
$$

where $Y = X/G$ and $\pi\colon X \to Y$ is the quotient map. Since the morphism $Z \to Z/H$ is étale, the Kodaira dimension of the desingularization of Z/H is the same as the Kodaira dimension of the desingularization of Z.

If Z is of general type, then so is Z/H. Hence $a_{Y*}\omega_Y$ and $a_{Y*}\pi_*\omega_X$ are M-regular sheaves supported on Z/H by Theorem 2.1. By the trace map, we write $\pi_*\omega_X = \omega_Y \oplus \mathcal{M}$. Suppose that G is non-trivial. Then, since the general fibers of a_Y are curves, the rank of $a_{Y*}\pi_*\omega_X$ is strictly larger than that of $a_{Y*}\omega_Y$. Hence $a_{Y*}\mathcal{M}$ is non-zero and, being M-regular, has a non-zero global section. But then $p_g(X) = h^0(A_X/H, a_{Y*}\pi_*\omega_X) > h^0(A_X/H, a_{Y*}\omega_Y) = p_g(Y)$, which is a contradiction.

If Z has Kodaira dimension 1, then it is an elliptic fiber bundle over a curve C with genus $g(C) \geq 2$, by [49, Theorem 10.9]. Therefore, the Stein factorization $X \to \widetilde{C}$ of the natural map $X \to C$ is a fibration with $g(\widetilde{C}) \geq 2$. It follows that $|G| \leq M_1$, where M_1 is as in Lemma 3.3. $\qquad\square$

Using the same argument for Corollary 3.2, we obtain

Corollary 3.7. *Let X be a smooth projective threefold of general type such that $\dim a_X(X) = 2$ and $q(X) \geq 3$. Then $|\mathrm{Aut}_{\mathbb{Q}}(X)| \leq M_1$, where M_1 is the constant appearing in Lemma 3.3.*

The rest of this section is devoted to the proof of the following

Proposition 3.8. *There is a positive constant M_2 such that the following holds. Let X be a smooth threefold of general type with $q(X) = \dim a_X(X) = 2$ such that its Albanese variety A_X is a simple abelian surface. Then $|\mathrm{Aut}_{\mathcal{O}}(X)| \leq M_2$.*

We need some preparations for the proof.

Assume that X is as in Proposition 3.8. Since X is of general type, $V^0(a_{X*}\omega_X)$ generates $\mathrm{Pic}^0(A_X)$ by [28]. By the simplicity of $\mathrm{Pic}^0(A_X)$, we conclude that

$$(3.3) \qquad\qquad V^0(a_{X*}\omega_X) = \mathrm{Pic}^0(A_X)$$

In particular, $p_g(X) > 0$.

Let $f\colon X \to S$ be the Stein factorization of $a_X\colon X \to A_X$. Then S is a normal projective surface which is finite over the simple abelian surface A_X. Consider $G := \mathrm{Aut}_{\mathcal{O}}(X)$ and its images $\psi_S(G) \subset \mathrm{Aut}(S)$ and $\psi_A(G) \subset \mathrm{Aut}(A_X)$. Then we have the following commutative diagram

$$(3.4)$$

$$
\begin{array}{ccccc}
 & & \overset{a_X}{\longrightarrow} & & \\
X & \overset{f}{\longrightarrow} & S & \overset{a_S}{\longrightarrow} & A_X \\
\pi\downarrow & & \pi_S\downarrow & & \pi_A\downarrow \\
Y & \overset{h}{\longrightarrow} & T & \overset{a_T}{\longrightarrow} & B \\
 & & \underset{a_Y}{\longrightarrow} & &
\end{array}
$$

where $Y = X/G$, $T = S/\psi_S(G)$, $B = A_X/\psi_A(G)$ are the quotient varieties, and π, π_S, π_A are the quotient maps. One can check that the morphism $Y \to B$ is indeed the Albanese morphism of Y.

Lemma 3.9. $\psi_A\colon G \to \mathrm{Aut}(A_X)$ *is trivial, so* $B = A_X$ *and the quotient map* π_A *is the identity.*

Proof. Assume on the contrary that $\psi_A(G)$ is nontrivial. Note that $\psi_A(G)$ consists of translations of A_X, so $\pi\colon A_X \to B = A_X/\psi_A(G)$ is étale. Let $Y' \cong X/\ker(\psi_A)$. Then $Y = Y'/\bar{G}$, where $\bar{G} = G/\ker(\psi_A)$, and we have a commutative diagram:

$$
\begin{array}{ccccc}
X & \overset{\pi'}{\longrightarrow} & Y' & \overset{\pi''}{\longrightarrow} & Y \\
a_X\downarrow & & a_{Y'}\downarrow & & a_Y\downarrow \\
A_X & =\!=\!= & A_X & \overset{\pi_A}{\longrightarrow} & B.
\end{array}
$$

where $\pi'\colon X \to Y'$ and $\pi''\colon Y' \to Y$ are the quotient maps. By the universal property of fiber products, one sees easily that $Y' = Y \times_B A_X$, and $\pi'\colon Y' \to Y$ and $a_{Y'}\colon Y' \to A_X$ are the projections under this identification. Since

$\pi_{A*}\mathcal{O}_A = \bigoplus_{P \in \ker \pi_A^*} P$, where $\pi_A^* \colon \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A_X)$ is the pull-back, we have

$$(3.5) \qquad \pi_*''\omega_{Y'} = \bigoplus_{P \in \ker \pi_A^*} (\omega_Y \otimes P).$$

Since G acts trivially on $H^0(X, \omega_X)$, we have

$$H^0(X, \omega_X) \cong H^0(Y', \omega_{Y'}) \cong H^0(Y, \omega_Y).$$

Thus this implies that for P non-trivial in (3.5), $H^0(Y, \omega_Y \otimes P) = 0$. Thus $V^0(a_{Y*}\omega_Y)$ is a proper subset of $\mathrm{Pic}^0(B)$. Since A_X and hence B is simple, so are $\mathrm{Pic}^0(A_X)$ and $\mathrm{Pic}^0(B)$. Thus $V^0(a_{Y*}\omega_Y)$ consists of finitely many points. By Theorem 2.1,

$$(3.6) \qquad a_{Y*}\omega_Y = \bigoplus_i Q_i,$$

where $Q_i \in \mathrm{Pic}^0(B)$ are torsion line bundles on B.

On the other hand, as a consequence of (3.3), we have $p_g(Y) = p_g(X) > 0$. It follows that some Q_i in the decomposition (3.6) must be $\mathcal{O}_B$. By the Leray spectral sequence, $q(Y) - q(B)$ is the number of trivial line bundles in (3.6), and hence

$$q(Y) = q(B) + (q(Y) - q(B)) > 2$$

which contradicts the assumption that $q(Y) = q(X) = 2$. $\qquad \square$

By Lemma 3.9, the commutative diagram (3.4) simplifies to

$$(3.7)$$

Note that every variety in (3.7) have A_X as its Albanese variety.

Lemma 3.10. *Let $\pi_*\omega_X = \omega_Y \oplus \mathcal{M}$ be the splitting induced by the trace map. Then the following holds.*

(i) *$H^i(Y, \mathcal{M}) = 0$ and $H^i(A_X, R^j a_{Y*}\mathcal{M}) = 0$ for all $i, j \geq 0$.*

(ii) *$R^j a_{Y*}\mathcal{M} = \bigoplus_k Q_k^j$, where $Q_k^j \in \mathrm{Pic}^0(A_X)$ are nontrivial torsion line bundles on A_X for $j \geq 0$.*

Proof. Since G acts trivially on $H^i(X, \omega_X)$, we have $H^i(X, \omega_X) \cong H^i(Y, \omega_Y)$, hence $H^i(Y, \mathcal{M}) = 0$ for all $i \geq 0$. Pushing the decomposition $\pi_*\omega_X = \omega_Y \oplus \mathcal{M}$ forward to A_X by a_{Y*}, we obtain

$$(3.8) \qquad a_{X*}\omega_X = a_{Y*}\omega_Y \oplus a_{Y*}\mathcal{M}.$$

Note that $H^0(A, a_{Y*}\mathcal{M}) = H^0(Y, \mathcal{M}) = 0$. By Theorem 2.1 and Remark 2.2, taking the simplicity of A_X into account, we infer that

$$a_{Y*}\mathcal{M} = \bigoplus_k Q_k$$

where $Q_k \in \mathrm{Pic}^0(A)$ are nontrivial torsion line bundles on A_X. This implies that ([23, Corollary 3.2])

$$H^i(A_X, a_{Y*}\mathcal{M}) = \bigoplus_k H^i(A_X, Q_k) = 0 \text{ for all } i \geq 0.$$

Now it follows from the Leray spectral sequences for $H^i(Y, \mathcal{M})$ that

$$H^i(A_X, R^1 a_{Y*}\mathcal{M}) = 0$$

for all $i \geq 0$. Using the Chen–Jiang decomposition of $R^1 a_{Y*}\mathcal{M}$ and the simplicity of A_X, we obtain

$$R^1 a_{Y*}\mathcal{M} = \bigoplus_k Q_k^1$$

where the Q_k^1 are all nontrivial torsion line bundles on A_X.

Finally, we remark that $R^j a_{Y*}\mathcal{M} \subset R^j a_{Y*}\omega_Y = 0$ for $j \geq 2$ by [31, Theorem 2.1]. $\square$

Lemma 3.11. *The homomorphism $\psi_S \colon G \to \mathrm{Aut}(S)$ is trivial, or equivalently, the quotient morphism $\pi_S \colon S \to T = S/\psi_S(G)$ in (3.7) is an isomorphism.*

Proof. If $\kappa(S) = 0$, then S is an abelian surface and $S \to A_X$ is an isomorphism by the universal properties of the Stein factorization and the Albanese morphism. As a consequence, $S = T$.

If $\kappa(S) = 1$ then there is a pencil of (possibly singular) elliptic curves on S, whose image in A_X is necessarily again a pencil of elliptic curves, contradicting the simplicity of A_X.

Suppose now $\kappa(S) = 2$. Since π_S is finite, in order to show that π_S is an isomorphism, it suffices to show that π_S is birational. For this purpose, we may pass to smooth higher birational models X', Y', S', T' of X, Y, S and T respectively, and by abuse of notation, assume that S and T are themselves smooth. We will show that $\chi(S, \omega_S) = \chi(T, \omega_T)$. Grant this for the moment. Then, since the eventual paracanonical map of S factors through $S \to T$ by [25, Proposition 1.5], we can apply [25, Theorem 1.6] to conclude that $\pi_S \colon S \to T$ is birational.

It remains to show that $\chi(S, \omega_S) = \chi(T, \omega_T)$. By [31, Proposition 7.6], we have $R^1 f_* \omega_X = \omega_S$ and $R^1 h_* \omega_Y = \omega_T$. Therefore,

$$
\begin{aligned}
\chi(X, \omega_X) &= \chi(S, f_* \omega_X) - \chi(S, \omega_S) \\
\text{and} \quad \chi(Y, \omega_Y) &= \chi(T, h_* \omega_Y) - \chi(T, \omega_T).
\end{aligned}
\tag{3.9}
$$

As in Lemma 3.10, we have a splitting $\pi_* \omega_X = \omega_Y \oplus \mathcal{M}$ such that

$$
H^i(A_X, a_{Y*}\mathcal{M}) = 0 \text{ for any } i \geq 0
\tag{3.10}
$$

Since $a_{S*} f_* \omega_X = a_{X*} \omega_X = a_{Y*} \omega_Y \oplus a_{Y*} \mathcal{M} = a_{T*} h_* \omega_Y \oplus a_{Y*} \mathcal{M}$, we obtain

$$
\begin{aligned}
\chi(S, f_* \omega_X) &= \chi(A_X, a_{S*} f_* \omega_X) \\
&= \chi(A_X, a_{T*} h_* \omega_Y) + \chi(A_X, a_{Y*} \mathcal{M}) \\
&= \chi(A_X, a_{T*} h_* \omega_Y) = \chi(T, h_* \omega_Y)
\end{aligned}
\tag{3.11}
$$

where the first and the last equalities are due to the vanishing of $R^1 a_{S*}(f_* \omega_X)$ and $R^1 a_{T*}(h_* \omega_Y)$ by [32, Theorem 3.4 (iii)] and the third equality is by (3.10). Moreover, since $h^i(X, \omega_X) = h^i(Y, \omega_Y)$ for each $i \geq 0$, we have $\chi(X, \omega_X) = \chi(Y, \omega_Y)$. Combining this with (3.9) and (3.11), we obtain $\chi(T, \omega_T) = \chi(S, \omega_S)$. $\square$

By Lemma 3.11, the commutative diagram (3.7) further simplifies to

(3.12)

$$
\begin{array}{c}
\xymatrix{
X \ar[r]^{a_X} \ar[d]_{\pi} \ar[dr]^{f} & \\
Y \ar[r]^{h} \ar[rr]_{a_Y} & S \ar[r]^{a_S} & A_X
}
\end{array}
$$

Proof of Proposition 3.8. In the diagram (3.12), the general fiber C of f is preserved by the action of G. Let $D = \pi(C)$ be the fiber of h over the point $f(C) \in S$. Then $D \cong C/G$ and $\pi|_C \colon C \to D$ is the quotient map under the action of G. By the Riemann–Hurwitz formula, we have

(3.13)
$$
2g(C) - 2 = |G|(2g(D) - 2) + \delta
$$

where δ is the degree of the ramification divisor of $\pi|_C \colon C \to D$. Also, we have

$$
\mathrm{rk}(a_{Y*}\mathcal{M}) = \mathrm{rk}(a_{S*}h_*\mathcal{M}) = (\deg a_S) \cdot \mathrm{rk}(h_*\mathcal{M}) = (\deg a_S)(g(C) - g(D)),
$$

where $\mathrm{rk}(\cdot)$ denotes the rank of a sheaf. One has $p_g(Y) = p_g(X) > 0$ by (3.3) and hence $g(D) \geq 1$.

By Lemma 3.10, there is an abelian étale cover $\mu : \widetilde{A}_X \to A_X$ such that $\mu^*(R^1 a_{Y*}\mathcal{M})$ is a direct sum of trivial line bundles. Consider the base change of $X \to Y \to S \to A_X$ by $\mu \colon \widetilde{A}_X \to A_X$,

$$
\begin{array}{c}
\xymatrix{
\widetilde{X} \ar[r]^{\widetilde{\pi}} \ar[d]_{\mu_X} \ar@/^1.2pc/[rr]^{\widetilde{f}} & \widetilde{Y} \ar[r]^{\widetilde{h}} \ar[d]_{\mu_Y} & \widetilde{S} \ar[r]^{\widetilde{a}_S} \ar[d]_{\mu_S} & \widetilde{A}_X \ar[d]^{\mu} \\
X \ar[r]_{\pi} \ar@/_1.2pc/[rr]_{f} & Y \ar[r]_{h} & S \ar[r]_{a_S} & A_X
}
\end{array}
$$

where $\widetilde{X} = X \times_{A_X} \widetilde{A}_X$, $\widetilde{Y} \times_{A_X} \widetilde{A}_X$ and $\widetilde{S} = S \times_{A_X} \widetilde{A}_X$, and the squares are those of fiber products.

By [32, Corollary 3.2], we have

$$H^2(\widetilde{X}, \omega_{\widetilde{X}}) \cong H^1(\widetilde{S}, R^1\widetilde{f}_*\omega_{\widetilde{X}}) \oplus H^2(\widetilde{S}, \widetilde{f}_*\omega_{\widetilde{X}})$$
$$\cong H^1(\widetilde{S}, \omega_{\widetilde{S}}) \oplus H^2(\widetilde{A}_X, \mu^*(a_{X*}\omega_X)),$$

so $q_{\widetilde{f}} := q(\widetilde{X}) - q(\widetilde{S}) = h^2(\widetilde{A}_X, \mu^*(a_{X*}\omega_X))$. Since $a_{Y*}\mathcal{M}$ is a direct summand of $a_{X*}\omega_X$, $\mu^*(a_{Y*}\mathcal{M})$ is a direct sum of $\mu^*(a_{X*}\omega_X)$ and we conclude that

$$h^2(\widetilde{A}_X, \mu^*(a_{X*}\omega_X)) \geq h^2(\widetilde{A}_X, \mu^*(a_{Y*}\mathcal{M}))$$
$$= \mathrm{rk}(a_{Y*}\mathcal{M}) = (\deg a_S)(g(C) - g(D)),$$

where the second equality is because $\mu^*(a_{Y*}\mathcal{M})$ is a direct sum of copies of $\mathcal{O}_{\widetilde{A}_X}$ by construction.

Let B be a general hyperplane section of $\widetilde{S}$ and $\widetilde{X}_B = \widetilde{f}^{-1}(B)$. Let $\widetilde{f}_B \colon \widetilde{X}_B \to B$ be the restriction of $\widetilde{f}$ over B.

Claim 3.12. $h^0(B, R^1\widetilde{f}_{B*}\mathcal{O}_{\widetilde{X}_B}) = h^0(\widetilde{S}, R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}})$

Proof of the claim. Applying $\widetilde{f}_*$ to the short exact sequence of sheaves on $\widetilde{X}$:

$$0 \to \mathcal{O}_{\widetilde{X}}(-\widetilde{f}^*B) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}_B} \to 0$$

we obtain an exact sequence of sheaves on $\widetilde{S}$:

$$\widetilde{f}_*\mathcal{O}_{\widetilde{X}_B} \to (R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}})(-B) \to R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}} \to R^1\widetilde{f}_{B*}\mathcal{O}_{\widetilde{X}_B} \to (R^2\widetilde{f}_*\mathcal{O}_{\widetilde{X}})(-B)$$

For a general hyperplane $B \subset \widetilde{S}$, one deduces

$$0 \to (R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}})(-B) \to R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}} \to R^1\widetilde{f}_{B*}\mathcal{O}_{\widetilde{X}_B} \to 0$$

Since we have $H^0(\widetilde{S}, (R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}})(-B)) = 0$ and $H^1(\widetilde{S}, (R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}})(-B)) = 0$ for a sufficiently ample hypersurface B on $\widetilde{S}$, the induced exact sequence of cohomology groups gives the desired isomorphism:

$$H^0(\widetilde{S}, R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}}) \cong H^0(B, R^1\widetilde{f}_{B*}\mathcal{O}_{\widetilde{X}_B}).$$

$\square$

By Claim 3.12, we obtain

$$(3.14) \qquad q_{\widetilde{f}_B} = h^0(B, R^1\widetilde{f}_{B*}\mathcal{O}_{\widetilde{X}_B}) = h^0(\widetilde{S}, R^1\widetilde{f}_*\mathcal{O}_{\widetilde{X}}) = q_{\widetilde{f}}$$
$$\geq (\deg a_S)(g(C) - g(D)).$$

Suppose that $g(D) = 1$. If $g(C) \geq 8$ then $q_{\widetilde{f}_B} \geq g(C) - 1 \geq \frac{5g(C)+1}{6}$, and hence $\widetilde{f}_B$ is birationally trivial by Xiao's result [50]. Since $B \subset \widetilde{S}$ is a general hypersurface, $\pi_1(B) \to \pi_1(\widetilde{S})$ is surjective and the birational triviality of $\widetilde{f}_B \colon \widetilde{X}_B \to B$ implies that of $\widetilde{f} \colon \widetilde{X} \to \widetilde{S}$. Therefore, $\widetilde{X}$ is birational to $\widetilde{S} \times C$ and $\widetilde{Y}$ is birational to $\widetilde{S} \times D$. By the fact that

$$\chi(\widetilde{X}, \omega_{\widetilde{X}}) = (\deg \mu)\chi(X, \omega_X) = (\deg \mu)\chi(Y, \omega_Y) = \chi(\widetilde{Y}, \omega_{\widetilde{Y}}),$$

we infer that $C = D$, which is absurd. Thus in this case $g(C) \leq 7$ and hence $|G| \leq |\mathrm{Aut}(C)| \leq 84 \times (7 - 1) = 504$, where the second inequality is Hurwitz's bound of the full automorphism group of a curve.

If $g(D) \geq 2$, then by (3.13) and (3.14)

$$q_{\widetilde{f}_B} \geq (\deg a_S)(g(C) - g(D)) = (\deg a_S)\left(\frac{\delta}{2} + (|G| - 1)(g(D) - 1)\right).$$

where δ is as in (3.13). By a simple calculation, if $g(C) \geq 8$ and $|G| \geq 13$, then $q_{\widetilde{f}_B} > \frac{5g(C)+1}{6}$ and by Xiao's result [50], $\widetilde{f}_B$ and hence $\widetilde{f}$ is birationally trivial, and we draw a contradiction by the same argument as in the last paragraph. Thus either $|G| \leq 12$ or $g(C) \leq 7$, in which case $|G| \leq \frac{g(C)-1}{g(D)-1} \leq 6$. $\qquad\square$

4. Threefolds of general type with Gorenstein minimal models

Theorem 4.1. *There is a constant M such that the following holds. Let X be a projective threefold of general type with canonical singularities such that its minimal models are Gorenstein. Then $|\mathrm{Aut}_{\mathcal{O}}(X)| < M$.*

Proof. Since X has a Gorenstein minimal model, the Miyaoka–Yau inequality gives

$$(4.1) \qquad \mathrm{vol}(K_X) \leq 72\chi(X, \omega_X) = 72(p_g(X) - h^1(X, \omega_X) + q(X) - 1)$$
$$\leq 72(p_g(X) + q(X) - 1)$$

In particular, $\chi(X, \omega_X) > 0$.

By Lemma 2.4, $\text{Aut}_{\mathcal{O}}(X) = \text{Aut}_{\mathcal{O}}(X) \cap \text{Aut}_A(X)$. If $q(X) \geq 3$ then $|\text{Aut}_{\mathcal{O}}(X)|$ is uniformly bounded by Theorem 3.1.

In the following we can assume that $q(X) \leq 2$. Let $Y = X/\text{Aut}_{\mathcal{O}}(X)$ be the quotient variety and $\pi\colon X \to Y$ the quotient morphism. Then there is an effective divisor B, which is supported on the branched divisor of π and with coefficients from the set $\mathcal{C}_2 := \{1 - \frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$, such that $K_X = \pi^*(K_Y + \Delta)$. Obviously, $K_Y + \Delta$ is big and, by [34, Proposition 5.20], the pair (Y, Δ) is klt. Since $\text{Aut}_{\mathcal{O}}(X)$ acts trivially on $H^0(X, K_X)$, one has $p_g(X) = p_g(Y, \Delta)$.

By Theorem A.2, there are some positive constants a and b such that

$$(4.2) \qquad \text{vol}(K_X) = |\text{Aut}_{\mathcal{O}}(X)|\,\text{vol}(K_Y + \Delta) \geq |\text{Aut}_{\mathcal{O}}(X)|(ap_g(X) - b).$$

Combining the inequalities (4.1) and (4.2), we obtain

$$|\text{Aut}_{\mathcal{O}}(X)|(ap_g(X) - b) \leq 72(p_g(X) + 1).$$

Now one sees easily that, there is a sufficiently large integer N such that $|\text{Aut}_{\mathcal{O}}(X)| \leq \lceil \frac{72}{a} + 1 \rceil$ holds if $p_g(X) > N$. On the other hand, if $p_g(X) \leq N$ then, by (4.1), we have $\text{vol}(K_X) \leq 72(N + 1)$, which implies that this class of threefolds of general type are bounded. It follows that there is a constant $M' > 0$ such that $|\text{Aut}(X)| \leq M'$, provided $p_g(X) \leq N$.

Then $M = \max\{\lceil \frac{72}{a} + 1 \rceil, M'\}$ is the bound we wanted. $\qquad\square$

Corollary 4.2. *Let X be a projective threefold of general type with canonical singularities such that its minimal models are Gorenstein. Then $|\text{Aut}_{\mathbb{Q}}(X)| \leq M$, where M is as in Theorem 4.1.*

Proof. By Lemma 2.3, $\text{Aut}_{\mathbb{Q}}(X)$ is contained in the group $\text{Aut}_{\mathcal{O}}(X)$. $\qquad\square$

Recall that a smooth projective variety is isogenous to a product of curves if it admits the product of smooth projective curves as an étale cover.

Corollary 4.3. *Let X be a projective threefold of general type isogenous to a product of curves. Then $|\text{Aut}_{\mathbb{Q}}(X)| \leq M$, where M is as in Theorem 4.1.*

Proof. Since X is isogenous to a product of curves, it is smooth with ample K_X. Now apply Corollary 4.2. $\qquad\square$

Appendix A. Noether type inequalities for log canonical pairs of general type

Let $\mathcal{C} \subset (0,1]$ be a subset, possibly empty. Define $\mathfrak{P}_n(\mathcal{C})$ to be the set of n-dimensional projective log canonical pairs (X, Δ) such that $\mathcal{C}_\Delta \subset \mathcal{C}$ and $K_X + \Delta$ is big, where $\mathcal{C}_\Delta$ denotes the set of nonzero coefficients appearing in Δ. Furthermore, set

$$\mathfrak{P}_n^+(\mathcal{C}) := \{(X, \Delta) \in \mathfrak{P}_n(\mathcal{C}) \mid p_g(X, \Delta) > 0\},$$

where $p_g(X, \Delta) := h^0(X, K_X + \lfloor \Delta \rfloor)$, and

$$\mathfrak{V}_n^+(\mathcal{C}) := \{\text{vol}(K_X + \Delta) \mid (X, \Delta) \in \mathfrak{P}_n^+(\mathcal{C})\} \text{ and } v_n^+(\mathcal{C}) := \inf \mathfrak{V}_n^+(\mathcal{C}).$$

Remark A.1. If $\mathcal{C}$ satisfies the descending chain condition, then the set

$$\mathfrak{V}_n(\mathcal{C}) := \{\text{vol}(K_X + \Delta) \mid (X, \Delta) \in \mathfrak{P}_n(\mathcal{C})\}$$

also satisfies the descending chain condition ([24, Theorem 1.3]). In particular, any nonempty subset of $\mathfrak{V}_n(\mathcal{C})$ attains its minimum.

Now we state the main result of this appendix.

Theorem A.2. *Let n be a positive integer and $\mathcal{C} \subset (0,1]$ a subset such that $\mathcal{C} \cup \{1\}$ attains the minimum, denoted by c. Then there exist positive numbers $a_n(\mathcal{C})$ and $b_n(\mathcal{C})$, depending on n and c, such that for any $(X, \Delta) \in \mathfrak{P}_n(\mathcal{C})$ the following inequality holds*

$$\text{vol}(K_X + \Delta) \geq a_n(\mathcal{C})p_g(X, \Delta) - b_n(\mathcal{C})$$

We make some preparation before proving Theorem A.2. First we prove a lemma on the extension of pluri-log-canonical sections:

Lemma A.3. *Let $(X, \Delta + H)$ be a divisorial log terminal pair with $\mathbb{Q}$-coefficients such that $\lfloor \Delta \rfloor = 0$ and H is a reduced smooth divisor. Suppose that $K_X + \Delta + H$ is big and no component of H is contained in the stable base locus of $K_X + \Delta + H$. Then the natural restriction map*

$$r_m \colon H^0(X, m(K_X + \Delta + H)) \to H^0(H, m(K_H + \Delta_H))$$

is surjective for any sufficiently divisible positive integer $m \geq 2$, where $\Delta_H = \Delta|_H$ be restriction of Δ to H.

Proof. The proof is similar to the one sketched in [15, Remark 2.5]. Since $K_X + \Delta + H$ is big and H is not contained in the stable base locus of $K_X + \Delta + H$, there are an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor B such that $K_X + \Delta + H \sim_{\mathbb{Q}} A + B$ and H is not contained in the support of B. For $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, we may choose an effective $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} (1 - \epsilon_1)(\Delta + H) + (\epsilon_1(\Delta + H) + \epsilon_2 A) + \epsilon_2 B$$

so that (X, D) is klt. By [4], there is a minimal model program yielding a birational contraction $\pi \colon (X, D) \dashrightarrow (\bar{X}, \bar{D})$ onto a minimal model $(\bar{X}, \bar{D})$ of (X, D).

Now observe that $K_X + D \sim_{\mathbb{Q}} (1 + \epsilon_2)(K_X + \Delta + H)$, so π yields also a (divisorial log terminal) minimal model $(\bar{X}, \bar{\Delta} + \bar{H})$ of $(X, \Delta + H)$, where $\bar{\Delta} = \pi_* \Delta$ and $\bar{H} = \pi_* H$; see [34, 3.31].

We claim that no component of H is contracted by π. Otherwise, suppose that a component H_i of H is contracted by π. Then the discrepancy of H_i with respect to $(\bar{X}, \bar{\Delta} + \bar{H})$ is larger than -1. It follows that H_i is contained in the stable base locus of $K_X + \Delta + H$, which is a contradiction to the assumption.

The Kawamata–Viehweg vanishing theorem, applied to the big and nef divisor $(m-1)(K_{\bar{X}} + \bar{\Delta} + \bar{H})$ on the Kawamata log terminal pair $(\bar{X}, \bar{\Delta})$, gives the vanishing $H^1(X, m(K_{\bar{X}} + \bar{\Delta} + \bar{H}) - \bar{H}) = 0$ for any positive integer $m \geq 2$ such that $m(K_{\bar{X}} + \bar{\Delta} + \bar{H})$ is Cartier. By the long exact sequence of cohomology groups, the natural restriction map $\bar{r}_m \colon H^0(\bar{X}, m(K_{\bar{X}} + \bar{\Delta} + \bar{H})) \to H^0(\bar{H}, m(K_{\bar{H}} + \bar{\Delta}_{\bar{H}}))$ is surjective, where $\bar{\Delta}_{\bar{H}} := \bar{\Delta}|_{\bar{H}}$.

By going to a common log resolution of $(X, \Delta + H)$ and $(\bar{X}, \bar{\Delta} + \bar{H})$, one sees easily that there is a commutative diagram

$$
\begin{array}{ccc}
H^0(\bar{X}, m(K_{\bar{X}} + \bar{\Delta} + \bar{H})) & \xrightarrow{\bar{r}_m} & H^0(\bar{H}, m(K_{\bar{H}} + \bar{\Delta}_{\bar{H}})) \\
\cong \downarrow & & \downarrow \cong \\
H^0(X, m(K_X + \Delta + H)) & \xrightarrow{r_m} & H^0(H, m(K_H + \Delta_H))
\end{array}
$$

where the vertical arrows are isomorphisms induced by $\pi \colon X \dashrightarrow \bar{X}$ and $\pi|_H \colon H \dashrightarrow \bar{H}$. Now the surjectivity of r_m follows from that of $\bar{r}_m$. $\qquad \square$

Lemma A.4. *Let (X, Δ) be a projective log canonical pair of dimension n such that X is smooth and $K_X + \Delta$ is big. Suppose that the linear system $|K_X + \lfloor \Delta \rfloor|$ has positive dimension and its movable part $\mathrm{Mov}|K_X + \lfloor \Delta \rfloor|$ is base point free. Let $H \in \mathrm{Mov}|K_X + \lfloor \Delta \rfloor|$ be a general element, and $\Delta_H :=$*

$\Delta|_H$ *the restriction of* Δ *to* H*. Then*

$$\mathrm{vol}(K_X + \Delta) \geq 2^{1-n}\,\mathrm{vol}(K_H + \Delta_H).$$

Proof. Since $\mathrm{Mov}|K_X + \lfloor \Delta \rfloor|$ is a positive dimensional base-point-free linear system, its general element H is a (nonzero) smooth divisor by the Bertini theorem.
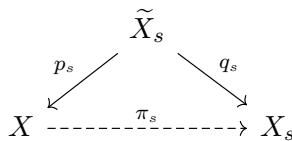
Since (X, Δ) is a projective log canonical pair such that X is smooth and $K_X + \Delta$ is big, we can decrease the coefficients of Δ slightly to obtain a $\mathbb{Q}$-divisor $\Delta^{(s)}$ for each positive integer s such that the following holds.

1) $(K_X, \Delta^{(s)} + H)$ is a divisorial log terminal pair with $\mathbb{Q}$-coefficients such that $\lfloor \Delta^{(s)} \rfloor = 0$.

2) $K_X + \Delta^{(s)}$ is big, and $K_X + \Delta^{(s)} - (1 - \frac{1}{s})H$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

3) $\lim_{s \to \infty} \Delta^{(s)} = \Delta$.

By [4], there is a minimal model program yielding a birational contraction

$$\pi_s \colon (X, \Delta^{(s)} + H) \dashrightarrow (X_s, \Delta_s + H_s),$$

where $\Delta_s = \pi_{s*}\Delta^{(s)}$ and $H_s = \pi_{s*}H$, such that $(X_s, \Delta_s + H_s)$ is a minimal model of $(X, \Delta^{(s)} + H)$ with divisorial log terminal singularities. Resolving the indeterminacy of π_s, we obtain the following commutative diagram of birational maps:

$$
\begin{array}{ccc}
 & \widetilde{X}_s & \\
{\scriptstyle p_s}\swarrow & & \searrow{\scriptstyle q_s} \\
X & \xdashrightarrow{\ \ \pi_s\ \ } & X_s
\end{array}
$$

where p_s and q_s are log resolutions of $(X, \Delta^{(s)} + H)$ and $(X_s, \Delta_s + H_s)$ respectively.

Denote by $\widetilde{H}_s$ be the strict transform of H on $\widetilde{X}_s$. There is a uniquely determined divisor $\widetilde{\Delta}_s$ on $\widetilde{X}_s$ such that $p_s^*(K_X + \Delta^{(s)} + H) = K_{\widetilde{X}_s} + \widetilde{\Delta}_s + \widetilde{H}_s$ and $p_{s*}\widetilde{\Delta}_s = \Delta^{(s)}$. Let $\Delta_{\widetilde{H}_s}$ be the restriction of $\widetilde{\Delta}_s^{>0}$ to $\widetilde{H}_s$, and $\Delta_H^{(s)}$ the restriction of $\Delta^{(s)}$ to H. We claim that

(A.1) $$\mathrm{vol}\left(K_H + \Delta_H^{(s)}\right) = \mathrm{vol}(K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s}).$$

In fact, since $p_{s*}\Delta_{\widetilde{H}_s} = \Delta_H^{(s)}$, one has $\mathrm{vol}(K_H + \Delta_H^{(s)}) \geq \mathrm{vol}(K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s})$. On the other hand,

$$p_s^*(K_H + \Delta_H^{(s)}) = p_s^*(K_X + H + \Delta^{(s)})|_{\widetilde{H}_s} = (K_{\widetilde{X}_s} + \widetilde{H}_s + \widetilde{\Delta}_s)|_{\widetilde{H}_s}$$
$$\leq (K_{\widetilde{X}_s} + \widetilde{H}_s + \widetilde{\Delta}_s^{>0})|_{\widetilde{H}_s} = K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s}$$

and thus $\mathrm{vol}(K_H + \Delta_H^{(s)}) = \mathrm{vol}(p_s^*(K_H + \Delta_H^{(s)})) \leq \mathrm{vol}(K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s})$.

For sufficiently large and divisible integer m, consider the commutative diagram

(A.2)
$$\begin{array}{ccc}
H^0(\widetilde{X}_s, mq_s^*(K_{X_s}+\Delta_s+H_s)) & \xrightarrow{r'_{s,m}} & H^0(\widetilde{H}_s, mq_s^*(K_{X_s}+\Delta_s+H_s)|_{\widetilde{H}_s}) \\
\cong \downarrow \varphi_{s,m} & & \downarrow \lambda_{s,m} \\
H^0(\widetilde{X}_s, m(K_{\widetilde{X}_s}+\widetilde{\Delta}_s^{>0}+\widetilde{H}_s)) & \xrightarrow{r_{s,m}} & H^0(\widetilde{H}_s, m(K_{\widetilde{X}_s}+\widetilde{\Delta}_s^{>0}+\widetilde{H}_s)|_{\widetilde{H}_s})
\end{array}$$

where the horizontal arrows are restriction maps and the vertical maps are induced by inclusions of sheaves and hence injective. Since $(X_s, \Delta_s + H_s)$ is a minimal model of $(\widetilde{X}_s, \widetilde{\Delta}_s^{>0} + \widetilde{H}_s)$, the map $\varphi_{s,m}$ is an isomorphism; the restriction map $r_{s,m}$ is surjective by Lemma A.3. It follows that $\lambda_{s,m}$ is surjective and hence an isomorphism. Letting m go to infinity, we obtain

(A.3)
$$\mathrm{vol}(K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s}) = \mathrm{vol}((K_{\widetilde{X}_s} + \widetilde{\Delta}_s^{>0} + \widetilde{H}_s)|_{\widetilde{H}_s})$$
$$= \mathrm{vol}(q_s^*(K_{X_s} + \Delta_s + H_s)|_{\widetilde{H}_s})$$
$$= q_s^*(K_{X_s} + \Delta_s + H_s)^{n-1} \cdot \widetilde{H}_s$$

where the second equality is due to the isomorphisms $\lambda_{s,m}$ and the last equality is because of the nefness of $q_s^*(K_{X_s} + \Delta_s + H_s)|_{\widetilde{H}_s}$.

Since $K_X + \Delta \geq H$ and $K_X + \Delta \geq K_X + \Delta^{(s)}$ by construction, we have

(A.4)
$$2(K_X + \Delta) \geq K_X + \Delta^{(s)} + H$$

Also, since $K_X + \Delta^{(s)} \geq (1 - \frac{1}{s})H$, we have

(A.5)
$$K_{X_s} + \Delta_s + H_s = \pi_{s*}(K_X + \Delta^{(s)} + H)$$
$$\geq \left(2 - \frac{1}{s}\right)\pi_{s*}H = \left(2 - \frac{1}{s}\right)H_s.$$

Now we can compute the volume

$$
\begin{aligned}
2^n \operatorname{vol}(K_X + \Delta) = \operatorname{vol}(2(K_X + \Delta)) \\
\geq \operatorname{vol}(K_X + \Delta^{(s)} + H) \quad \text{(by (A.4))} \\
= \operatorname{vol}(K_{X_s} + \Delta_s + H_s) \\
= (K_{X_s} + \Delta_s + H_s)^n \quad \text{(since } K_{X_s} + \Delta_s + H_s \text{ is nef)} \\
\geq (K_{X_s} + \Delta_s + H_s)^{n-1} \cdot \left(2 - \frac{1}{s}\right) H_s \quad \text{(by (A.5))} \\
= \left(2 - \frac{1}{s}\right) (q_s^*(K_{X_s} + \Delta_s + H_s))^{n-1} \cdot q_s^* H_s \\
\geq \left(2 - \frac{1}{s}\right) (q_s^*(K_{X_s} + \Delta_s + H_s))^{n-1} \cdot \widetilde{H}_s \quad \text{(since } q_s^* H_s \geq \widetilde{H}_s) \\
= \left(2 - \frac{1}{s}\right) \operatorname{vol}(K_{\widetilde{H}_s} + \Delta_{\widetilde{H}_s}) \quad \text{(by (A.3))} \\
= \left(2 - \frac{1}{s}\right) \operatorname{vol}(K_H + \Delta_H^{(s)}) \quad \text{(by (A.1))}
\end{aligned}
$$

Since $\lim_{s \to \infty} \Delta_H^{(s)} = \Delta_H$, we obtain

$$
\operatorname{vol}(K_X + \Delta) \geq 2^{-n} \lim_{s \to \infty} \left(2 - \frac{1}{s}\right) \operatorname{vol}\left(K_H + \Delta_H^{(s)}\right) \geq 2^{1-n} \operatorname{vol}(K_H + \Delta_H).
$$

$\square$

Proof of Theorem A.2. By ensuring that $b_n(\mathcal{C}) \geq a_n(\mathcal{C})$, the inequality of the theorem is trivially true in the case $p_g(X, \Delta) \leq 1$. We can thus assume that $p_g(X, \Delta) \geq 2$.

We proceed by induction on the dimension n.

If $n = 1$ then X is a smooth projective curve, and, using the Riemann–Roch theorem,

$$
\operatorname{vol}(K_X + \Delta) = \deg(K_X + \Delta) \geq \deg(K_X + \lfloor \Delta \rfloor) \geq h^0(X, K_X + \lfloor \Delta \rfloor) - 1
$$

where the last inequality is an equality if and only if $g(X) = 0$. Thus we can take $a_1(\mathcal{C}) = b_1(\mathcal{C}) = 1$ for any $\mathcal{C} \subset (0, 1]$.

From now on, we assume that $n \geq 2$ and that the theorem has been proven in lower dimensions. Let (X, Δ) be an n-dimensional projective log canonical pair of general type with coefficients of Δ belonging to $\mathcal{C}$.

First we assume that $1 \in \mathcal{C}$. Let $\rho \colon \widetilde{X} \to X$ be a log resolution of (X, Δ) such that the movable part $\operatorname{Mov}|K_{\widetilde{X}} + \lfloor \widetilde{\Delta}^{>0} \rfloor|$ of the linear system $|K_{\widetilde{X}} +$

$\lfloor \widetilde{\Delta}^{>0} \rfloor |$ is base point free, where $\widetilde{\Delta}$ is the divisor on $\widetilde{X}$ satisfying $K_{\widetilde{X}} + \widetilde{\Delta} = \rho^*(K_X + \Delta)$ and $\rho_* \widetilde{\Delta} = \Delta$, and $\widetilde{\Delta}^{>0}$ denotes the effective part of $\widetilde{\Delta}$. Let $\widetilde{\Delta}'$ be the divisor on $\widetilde{X}$, obtained from $\widetilde{\Delta}^{>0}$ by raising the coefficients not in $\mathcal{C}$ to 1 and then decreasing those coefficients in $\mathcal{C} \cap (0,1)$ (if any) to $c = \min(\mathcal{C} \cup \{1\})$. Then $(\widetilde{X}, \widetilde{\Delta}') \in \mathfrak{P}_n(\{c,1\})$, and one sees easily that

$$0 < \operatorname{vol}(K_{\widetilde{X}} + \widetilde{\Delta}') \le \operatorname{vol}(K_X + \Delta) \text{ and } p_g(\widetilde{X}, \widetilde{\Delta}') = p_g(X, \Delta).$$

Replacing (X, Δ) with $(\widetilde{X}, \widetilde{\Delta}')$, we can assume that X is smooth, Δ is a simple normal crossing divisor with coefficients lying in $\{c, 1\}$, and $\operatorname{Mov}|K_X + \lfloor \Delta \rfloor|$ is base point free.

Let $H \in \operatorname{Mov}|K_X + \lfloor \Delta \rfloor|$ be a general member. Then H is smooth and $H + \Delta$ is simple normal crossing by the Bertini theorem. Let Δ_H be the restriction of Δ to H.

Let $f : X \to \mathbb{P}^{p_g - 1}$ denote the morphism defined by $|K_X + \lfloor \Delta \rfloor|$, where $p_g = p_g(X, \Delta)$. In order to bound $\operatorname{vol}(K_X + \Delta)$ from below in terms of $p_g(X, \Delta)$, we have to distinguish two cases depending on $\dim f(X)$.

Case $\dim f(X) \ge 2$. In this case, H is connected. Then $(H, \Delta_H) \in \mathfrak{P}_{n-1}(\{c,1\}) \subset \mathfrak{P}_{n-1}(\mathcal{C})$, and hence by induction there are constants $a_{n-1}(\mathcal{C})$ and $b_{n-1}(\mathcal{C})$ such that

$$(A.6) \qquad \operatorname{vol}(K_H + \Delta_H) \ge a_{n-1}(\mathcal{C}) p_g(H, \Delta_H) - b_{n-1}(\mathcal{C})$$

Since $p_g(H, \Delta_H) \ge p_g(X, \Delta) - 1$, we have by Lemma A.4 and (A.6):

$$(A.7) \quad \begin{aligned} \operatorname{vol}(K_X + \Delta) &\ge 2^{1-n} \operatorname{vol}(K_H + \Delta_H) \\ &\ge 2^{1-n} \left(a_{n-1}(\mathcal{C})(p_g(X, \Delta) - 1) - b_{n-1}(\mathcal{C}) \right) \\ &= 2^{1-n} a_{n-1}(\mathcal{C}) p_g(X, \Delta) - 2^{1-n} \left(a_{n-1}(\mathcal{C}) + b_{n-1}(\mathcal{C}) \right) \end{aligned}$$

Case $\dim f(X) = 1$. In this case, the number k of connected components of H is $\deg f(X) \ge p_g(X, \Delta) - 1$. Let $H_1, \ldots, H_k$ be the connected components of H. For each $1 \le i \le k$, let Δ_{H_i} be the restriction of Δ to H_i. Then $(H_i, \Delta_{H_i}) \in \mathfrak{P}_{n-1}^+(\{c,1\})$, and $\operatorname{vol}(K_{H_i} + \Delta_{H_i})$ is independent of i. We have thus

$$(A.8) \quad \operatorname{vol}(K_H + \Delta_H) = k \operatorname{vol}(K_{H_i} + \Delta_{H_i}) \ge v_{n-1}^+(\{c,1\})(p_g(X, \Delta) - 1).$$

Combining this with Lemma A.4, we obtain

$$(A.9) \qquad \operatorname{vol}(K_X + \Delta) \ge 2^{1-n} v_{n-1}^+(\{c,1\})(p_g(X, \Delta) - 1).$$

By (A.7) and (A.9), if we set

$$a_n(\mathcal{C}) = 2^{1-n} \min\{a_{n-1}(\mathcal{C}), v_{n-1}^+(\{c, 1\})\},$$
$$b_n(\mathcal{C}) = 2^{1-n} \max\{a_{n-1}(\mathcal{C}) + b_{n-1}(\mathcal{C}), v_{n-1}^+(\{c, 1\})\}.$$

then the inequality $\mathrm{vol}(K_X + \Delta) \geq a_n(\mathcal{C})p_g(X, \Delta) - b_n(\mathcal{C})$ is valid in both cases.

In general, if $1 \notin \mathcal{C}$, since $\mathfrak{P}_n(\mathcal{C}) \subset \mathfrak{P}_n(\mathcal{C} \cup \{1\})$, we can simply set $a_n(\mathcal{C}) := a_n(\mathcal{C} \cup \{1\})$ and $b_n(\mathcal{C}) := b_n(\mathcal{C} \cup \{1\})$. $\qquad\square$

Remark A.5. The choices of $a_n(\mathcal{C})$ and $b_n(\mathcal{C})$ in the proof of Theorem A.2 are not optimal, but are computable. If the following Question A.6 admits a positive answer, then one can see by induction that, for $n \geq 2$,

$$a_n(\mathcal{C}) = \frac{1}{2^{n-1}} v_{n-1}^+(\{c, 1\})$$

and

$$b_n = a_{n-1} + a_{n-2} + \cdots + a_1 + b_1 = 2 + \sum_{1 \leq i \leq n-1} \frac{1}{2^{i-1}} v_{i-1}^+(\{c, 1\}).$$

Question A.6. For $n \geq 2$ and $c \in (0, 1]$, is it true that $v_n^+(\{c, 1\}) \leq \frac{1}{2^{n-1}} v_{n-1}^+(\{c, 1\})$?

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