

Shafarevich’s Conjecture for CY Manifolds I

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Abstract

Let C be a Riemann surface with fixed finite number of points on it. Given a Calabi-Yau (CY) manifold with fixed topological type, we consider the set of all families of CY manifolds of fixed topological type over C with degenerate fibres over the fixed points up to isomorphism. This set is called the Shafarevich set. The analogue of Shafarevich conjecture for CY manifolds is for which topological types of CY the Shafarevich set is finite. We relate the analogue of Shafarevich problem for CY manifolds to the non-vanishing of the Yukawa coupling. We also study in this paper some moduli problems related to the Shafarevich conjecture. We also give a counter example of the Shafarevich problem for a class of CY manifolds.

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1 Introduction

Let C be a fixed, non-singular algebraic curve, and let E be a fixed effective divisor on C such that all the points in E have multiplicity 1. Define $\text{Sh}(C, E, Z)$ to be the set of all isomorphism classes of projective varieties $\pi : Z \rightarrow C$ with a fibre of "type" Z such that the only singular fibres are over the set E . The general Shafarevich type problem is: "*For which "type" of varieties Z and data (C, E) , is $\text{Sh}(C, E, Z)$ finite?*"

In his article published in the Proceedings of the International Congress of Mathematicians, Stockholm meeting held in 1962, Shafarevich wrote: "*One of the main theorems on algebraic numbers connected with the concept of discriminant is Hermit's theorem, which states that the number of extensions k'/k of a given degree and given discriminant is finite. This theorem may be formulated as follows: the number of extensions k'/k of a given degree whose critical prime divisors belong to a given finite set S is finite.*" Inspired by this result of Hermit, Shafarevich conjectured in [37]: "*There exists only a finite number of fields of algebraic functions K/k of a given genus $g \geq 1$, the critical prime divisors of which belong to a given finite set S .*" In an unpublished work, Shafarevich proved his conjecture in the setting of hyperelliptic curves.

In the case when Z is a curve of genus $g > 1$ and E is empty, Parshin proved in [29] that $\text{Sh}(C, E, Z)$ is finite, and Arakelov proved in [2] the

finiteness in the case E is not empty. Faltings constructed examples showing that $\text{Sh}(C, E, Z)$ is infinite for abelian varieties of dimension ≥ 8 . See [13]. Saito and Zucker extended the construction of Faltings to the setting when Z is an algebraic polarized K3 surface. They were able to classify all cases when the set $\text{Sh}(C, E, Z)$ is infinite. See [33]. Faltings proved the Shafarevich's conjecture over number fields and thus he proved the Mordell conjecture. Yves Andre proved the analogue of Shafarevich's conjectures over the number fields for K3 surfaces. See [1]. Using techniques from harmonic maps Jost and Yau analyzed $\text{Sh}(C, E, Z)$ for a large class of varieties. See [20]. Ch. Peters studied finiteness theorems by considering variations of Hodge structures and utilizing differential geometric aspects of the period map and the associated metrics on the period domain. See [30].

A. Parshin and E. Bedulev have proved the boundedness for families of algebraic surfaces over a fixed algebraic curve assuming that all the fibres are non-singular. E. Bedulev and E. Viehweg have proved the boundedness for families of algebraic surfaces over a fixed algebraic curve, which possibly have singular fibres. See [7]. Migliorini, Kováč and Zhang proved that any family of minimal algebraic surfaces of general type over a curve of genus g and m singular points such that $2g-2+m \leq 0$ is isotrivial. See [21], [27], [48] and [7]. Recently very important results of Viehweg and the last named author appeared in [42], [43] and [44]. Brody hyperbolicity was proved for the moduli space of canonically polarized complex manifolds. In [44] 6.2 a) They proved the boundedness for $\text{Sh}(C, E, Z)$ for arbitrary Z , with ω_Z semi-ample. They also established that the automorphism group of moduli stacks of polarized manifolds is finite. The rigidity property for the generic family of polarized manifolds was also proved. The basis idea in the proof is to consider the non-vanishing property of the maximal iterated Kodaira-Spencer map. One notices that in the case of Calabi-Yau manifolds this maximal iterated Kodaira-Spencer map is related to the Yukawa coupling. In this paper the relation between the Yukawa coupling and rigidity problems for CY manifolds is first formally formulated.

We are going to study the analogue of the Shafarevich's conjecture about finiteness of the families of polarized CY manifolds over a fixed Riemann surface with prescribed points of degenerate fibres. Any family of polarized CY manifolds induces a map from the base without discriminant locus to the coarse moduli space. Thus Shafarevich conjecture is reduced to prove that moduli space of maps of Riemann surface into the moduli space of polarized CY manifolds with some additional properties is a finite set. It is not difficult to show that the existence of Kähler metric with negative bounded holomorphic curvature and the rigidity of any family of CY

manifolds over a Riemann surface with fixed points of degenerations, Yau's form of Schwarz Lemma and the Bishop compactness imply the Shafarevich conjecture. We used the Yukawa coupling to check rigidity:

Condition 1 *Suppose that*

$$\pi : \mathcal{X} \rightarrow C \tag{1}$$

is a family of polarized CY manifolds over a Riemann surface C . Let $t_0 \in C$ such that for $M = \pi^{-1}(t_0)$ the following condition is satisfied: for any non zero $\phi \in H^1(M, T^{1,0}M)$, $\wedge^n \phi \neq 0$ in $H^n(M, (\Omega_M^n)^)$. Then the family (1) is rigid.*

The condition 1 brings up an interesting relation between rigidity and mirror symmetry. We would also like to compare Condition 1 with some of the results obtained in [43]. In [45] some important applications of the Condition 1 are obtained. In [45] the finiteness of those families, whose iterated Kodaira-Spencer maps have the same length as the length of the iterated Kodaira-Spencer map of the moduli space, are obtained. E. Viehweg and the last named author constructed in [46] rigid families of CY hypersurfaces for which the condition 1 does not hold, i.e. the Yukawa coupling is zero. They also have constructed non-rigid families of hypersurfaces in $\mathbb{C}P^n$ for any $n > 2$ and of any degree $d > 3$. See [46]. Recently Y. Zhang obtained some important results. More precisely he proved that any Lefschetz family of CY manifold is rigid. See [49] and [50].

In this paper our approach is close to that of Jost and Yau. Instead of harmonic maps in this paper we are using holomorphic ones. See [20]. Our paper is based on the study of the moduli space of polarized CY manifolds and its Teichmüller space. We will show that the Teichmüller space $\tilde{T}(M)$ of a Calabi-Yau manifold M exists and it has finite number of irreducible components. Each component $\mathcal{T}(M)$ of $\tilde{T}(M)$ is a non-singular complex manifold. Moreover over $\tilde{T}(M)$ there exists a universal family of marked polarized CY manifolds up to the action of a finite group of complex analytic automorphisms which acts trivially on the middle cohomology.

The existence of the coarse moduli space $\mathfrak{M}_L(M)$ of polarized CY manifolds as a quasi-projective variety was proved by Viehweg. See [41]. Yau conjectured that Mumford-Chow stability of the canonically embedded projective manifolds is equivalent to the existence of Kähler-Einstein metric. Recently Donaldson proved that the existence of constant scalar curvature implies Chow stability. By combining the result of Donaldson with the solution of the Calabi conjecture given by Yau, one can reprove the result of Viehweg.

In this paper we will prove the existence of a finite cover $\mathcal{M}_L(M)$ of the coarse moduli space $\mathfrak{M}_L(M)$ of polarized CY manifolds such that $\mathcal{M}_L(M)$ is a non-singular quasi-projective variety and over $\mathcal{M}_L(M)$ there exists a family of polarized CY manifolds. The construction of $\mathcal{M}_L(M)$ and the family over it is based on the existence of family of CY manifolds over the Hilbert scheme. We will apply the results obtained in this paper to the analogue of the Shafarevich's conjecture by studying the moduli space of holomorphic maps from a fixed Riemann surface with prescribed points of degenerate fibres to the moduli space of polarized CY manifolds with the Hodge metric.

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2 Moduli of Polarized CY Manifolds

2.1 Construction of the Teichmüller Space of Marked Polarized CY Manifolds

Theorem 3 *Let $\pi : \mathcal{X} \rightarrow \mathcal{K}$ be the Kuranishi family of a polarized CY manifold $M = \pi^{-1}(\tau_0)$, $\tau_0 \in \mathcal{K}$. Suppose that G is a group of holomorphic automorphisms of M such that G acts trivially on $H_n(M, \mathbb{Z})$ and preserves the polarization. Then G is a finite group of holomorphic automorphisms of all the fibres of the Kuranishi family.*

Proof: Since G acts trivially on $H^n(M, \mathbb{Z})$ and fixes the polarization class L then the uniqueness of the Calabi-Yau metric that corresponds to L implies that the group G is a group of isometries of the CY metric. Since on CY manifolds there does not exist global holomorphic vector fields, we can conclude that G is a discrete subgroup of the orthogonal group. The compactness of the orthogonal group implies that G is finite. The local Torelli Theorem implies that $\mathcal{K} \subset \mathbb{P}(H^n(M, \mathbb{Z}) \otimes \mathbb{C})$. So G acts on \mathcal{K} and fixes

the point τ_0 . Since G acts trivially on $H^n(M, \mathbb{Z})$, then it will act trivially on \mathcal{K} . Next we are going to prove that if $g \in G$ is any element of G then it acts as complex analytic automorphism on each $M_\tau = \pi^{-1}(\tau)$ for any $\tau \in \mathcal{K}$. This means that $g^*(I_\tau) = I_\tau$, where $I_\tau \in C^\infty(M, \text{Hom}(T_M^*, T_M^*))$, $I_\tau^2 = -id$ is the integrable complex structure operator that defines M_τ for $\tau \in \mathcal{K}$. T_N^* is the cotangent C^∞ real bundle. It is a well known fact that $I_\tau = (A_\tau)^{-1} \circ I_0 \circ A_\tau$, where

$$A_\tau = \begin{pmatrix} id & \phi(\tau) \\ \phi(\tau) & id \end{pmatrix}.$$

See [40]. Let us recall that if Θ_M is the holomorphic tangent bundle then

$$\phi(\tau) \in C^\infty \left(M, \text{Hom} \left(\Omega_M^{1,0}, \overline{\Omega_M^{1,0}} \right) \right) \cong C^\infty \left(M, \Theta_M \otimes \Omega_M^{0,1} \right)$$

and $\phi(\tau)$ satisfies the equation that guarantees the integrability

$$\bar{\partial}\phi(\tau) = \frac{1}{2}[\phi(\tau), \phi(\tau)] \text{ and } \bar{\partial}^*\phi(\tau) = 0. \quad (2)$$

of the complex structure operator I_τ defined above. For all the details see [40]. Here $\bar{\partial}^*$ means the conjugate of the operator $\bar{\partial}$ with respect to the Calabi-Yau metric corresponding to the polarization class L . If we prove that for each $g \in G$

$$g^*(\phi(\tau)) = \phi(\tau) \quad (3)$$

then (3) implies $g^*(I_\tau) = I_\tau$ and so $\phi(\tau)$ is a complex analytic automorphism of N_τ .

Proof of (3): According to [40] if we fix a basis ϕ_1, \dots, ϕ_N of harmonic forms of $\mathbb{H}^1(M, \Theta_M)$ then the solution of the equations (2) are given by the power series

$$\phi(\tau) = \sum_{i=1}^N \phi_i \tau^i + \sum_{i_1 + \dots + i_N = m > 1} \phi_{i_1, \dots, i_N} (\tau^1)^{i_1} \times \dots \times (\tau^N)^{i_N} = \sum_{m=1}^N \phi_{[m]}(\tau),$$

where

$$\phi_{[m]}(\tau) = \sum_{i_1 + \dots + i_N = m} \phi_{i_1, \dots, i_N} (\tau^1)^{i_1} \times \dots \times (\tau^N)^{i_N} \quad (4)$$

and $\phi(\tau)$ satisfies the recurrent relation:

$$\phi(\tau) = \sum_{i=1}^N \phi_i \tau^i + \frac{1}{2} (\bar{\partial})^* \circ \mathbb{G}[\phi(\tau), \phi(\tau)]. \quad (5)$$

See [40]. Here \mathbb{G} is the Green operator associated to the Laplacian with respect to the CY metric associated to the polarization class L . Notice that since G is the group of isometries of CY metric, for any $g \in G$ the Green operator will be invariant, i.e. $g^*\mathbb{G} = \mathbb{G}$. In [40] it is proved that if $\phi \in \mathbb{H}^1(N, \Theta_N)$ is a harmonic form with respect to the CY metric then $\phi \lrcorner \omega_M$ will be a harmonic form of type $(n-1, 1)$. This fact together with the fact that G acts trivially on $H^n(M, \mathbb{C})$ imply that the group acts trivially on $\mathbb{H}^1(M, \Theta_M)$. This implies that the linear term of (4) satisfies

$$g^*(\phi_{[1]}(\tau)) = \phi_{[1]}(\tau). \quad (6)$$

The proof of the fact that $g^*\phi(\tau) = \phi(\tau)$ is done by induction on the homogeneity of the terms of the power series (4). Formula (6) shows that $\phi_{[1]}(\tau)$ is invariant under the action of g . Since for the higher order terms of (4) the relation (5) implies

$$\phi_{[k]} = \sum_{p+q=k} [\phi_{[p]}(\tau), \phi_{[q]}(\tau)]$$

and $g^*(\phi_{[p]}) = \phi_{[p]}$ for $p < k$ then we can conclude that $g^*\phi_{[k]}(\tau) = \phi_{[k]}(\tau)$ for any $k > 0$. So the relation(3) is proved. Theorem 3 is proved. ■

Definition 4 *The Teichmüller space $\mathcal{T}(M)$ of a compact complex manifold M is defined as follows: $\mathcal{T}(M) := \mathcal{I}(M)/\text{Diff}_0(M)$, where*

$$\mathcal{I}(M) := \{\text{all integrable complex structures on } M\}$$

and $\text{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity. The group $\text{Diff}(M_0)$ acts on $\mathcal{I}(M)$ as follows: let $\phi \in \text{Diff}_0(M)$ then ϕ acts on integrable complex structures on M by pull-back, i.e. if $I \in C^\infty(M, \text{Hom}(T(M), T(M)))$, then we define $\phi(I_\tau) = \phi^*(I_\tau)$.

Definition 5 *We will call a pair $(M; \gamma_1, \dots, \gamma_{b_n})$ a marked CY manifold if M is a CY manifold where $\{\gamma_1, \dots, \gamma_{b_n}\}$ is a basis of $H_n(M, \mathbb{Z})/\text{Tor}$.*

Remark 6 *Let \mathcal{K} be the Kuranishi space. It is easy to see that if we choose a basis of $H_n(M, \mathbb{Z})/\text{Tor}$ in one of the fibres of the Kuranishi family $\pi : \mathcal{M} \rightarrow \mathcal{K}$ then all the fibres will be marked, since as a C^∞ manifold $\mathcal{X}_{\mathcal{K}} \cong M \times \mathcal{K}$.*

Next we are going to construct a universal family of polarized marked CY manifolds $\pi : \mathcal{U}_L \rightarrow \mathcal{T}_L(M)$ up to the action of the group of complex analytic automorphisms G which acts trivially on $H^2(M, \mathbb{Z})$ on the fibres.

Theorem 7 *There exists a family of marked polarized CY manifolds*

$$\pi : \mathcal{U}_L \rightarrow \mathcal{T}_L(M), \quad (7)$$

with the following properties: **A.** $\mathcal{T}_L(M)$ is a smooth manifold of complex dimension $h^{n-1,1}$. **B.** The holomorphic tangent space $\Theta_{\tau, \mathcal{T}_L(M)}$ at each point $\tau \in \mathcal{T}_L(M)$ is naturally identified with $H^1(M_\tau, \Theta_{M_\tau})$ and **C.** Let $\pi_{\mathcal{C}} : \mathcal{Y} \rightarrow \mathcal{C}$ be any complex analytic family of marked polarized CY manifolds such that there exists a point $x_0 \in \mathcal{C}$ and the fibre $(\pi_{\mathcal{C}})^{-1}(x_0) = M_{x_0}$ as a marked polarized CY manifold is isomorphic to some fibre of the family (7). Then there exists a unique holomorphic map of families

$$\kappa : (\mathcal{Y} \rightarrow \mathcal{C}) \rightarrow (\mathcal{U}_L \rightarrow \mathcal{T}_L(M)) \quad (8)$$

defined up to a biholomorphic map ϕ of the fibres which induces the identity map on $H_n(M, \mathbb{Z})$. The restriction of the map κ on the base \mathcal{C} is unique.

Proof: In this paragraph we will use the following result of Y.-T. Siu. See [9] and [35].

Theorem 8 *Let X be an algebraic variety and let \mathcal{L} be an ample line bundle then $(\mathcal{K}_X)^{\otimes 2} \otimes (\mathcal{L})^{\otimes m}$ is very ample for any*

$$m \geq 2 + \binom{3n+1}{n}, \quad (9)$$

where n is the dimension of X .

Based on Theorem 8 we will prove the following Theorem:

Theorem 9 *Let M be a fixed projective manifold with $c_1(M)=0$. Let us fix its cohomology ring $H^*(M, \mathbb{Z})$ over \mathbb{Z} , its Chern classes and the polarization class $L \in H^2(M, \mathbb{Z})$ corresponding to an ample line bundle. Then there are only finite number of components of the Hilbert scheme that parametrizes all polarized projective manifolds with fixed cohomology ring $H^*(M, \mathbb{Z})$, and Chern classes with a polarization class L .*

Proof: Let us fix the following data; a projective manifold M with canonical class zero, its cohomology ring $H^*(M, \mathbb{Z})$ over \mathbb{Z} , its Chern classed and the polarization class $L \in H^2(M, \mathbb{Z})$ which is the Chern class of an ample line bundle. According to a theorem of Sullivan there are only finite number of C^∞ structures on M if the real dimension of M is greater or equal to 5 with the data mentioned above. See [38]. Suppose that m satisfy the inequality

(9). Since $\mathcal{L}^{\otimes m}$ is a very ample line, $H^k(X, \mathcal{L}^{\otimes m})=0$ for $k > 0$ and Theorem 8, imply that if we consider a CY manifold with fixed Chern classes $c_1 = 0, c_2, \dots, c_n$, fixed cohomology ring $H^*(M, \mathbb{Z})$ and polarization class L , then all these Kähler manifolds can be embedded in a fixed projective space $\mathbb{C}\mathbb{P}^k$. The Hirzebruch-Riemann-Roch theorem implies that for all Kähler manifolds with canonical class zero with fixed cohomology ring $H^*(M, \mathbb{Z})$, fixed Chern classes and the line bundle \mathcal{L} with a fixed ample Chern class L its the Euler characteristics:

$$\chi(M, \mathcal{L}^m) = \dim_{\mathbb{C}} H^0(M, \mathcal{L}^m) = \int_M Td(M) Ch(\mathcal{L}^m)$$

have the same Hilbert polynomial $\chi(M, \mathcal{L}^m)$ for all such manifolds. By using the theory of Hilbert schemes of Grothendieck developed in [12], we can conclude that there are a finite number of components of the Hilbert scheme that parametrizes all polarized Kähler manifolds with fixed cohomology ring $H^*(M, \mathbb{Z})$, Chern classes and the polarization class L . ■

In [17] we proved the following Theorem:

Theorem 10 *Let \mathcal{H}_L be the Hilbert scheme of non-singular CY manifolds embedded by the linear system $|L^m|$ defined by the polarization class L , then \mathcal{H}_L is a non-singular quasi-projective variety.*

We know from the results in [12] that there exists a family of polarized CY manifolds:

$$\mathcal{Y}_L \rightarrow \mathcal{H}_L, \tag{10}$$

where $\mathcal{Y}_L \subset \mathbb{C}\mathbb{P}^N \times \mathcal{H}_L$. Let $\tilde{\mathcal{H}}_L$ be the universal covering of \mathcal{H}_L and let

$$\tilde{\mathcal{Y}}_L \rightarrow \tilde{\mathcal{H}}_L \tag{11}$$

be the pullback family of (10). It is easy to see that the group $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ acts on \mathcal{H}_L . This implies that $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ acts also on $\tilde{\mathcal{H}}_L$. We will need the following lemma and its corollary:

Lemma 11 *Let G be a subgroup of $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ that fixes a point $\tau_0 \in \tilde{\mathcal{H}}_L$, then G is a finite group of complex analytic automorphisms of the CY manifold M_{τ_0} and it is a normal subgroup of $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ that acts trivially on $H_n(M, \mathbb{Z})$.*

Proof: According to Theorem 10 \mathcal{H}_L is a smooth quasi-projective variety. So its universal cover $\tilde{\mathcal{H}}_L$ is a simply connected complex manifold and we may suppose that the family (11) is marked and polarized. The

Definition 5 of the marked family of CY manifolds implies that if G fixes the point $\tau_0 \in \tilde{\mathcal{H}}_L$, then G is a subgroup in $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ that stabilizes M_{τ_0} in $\mathbb{C}\mathbb{P}^N$. This shows that G must be a group of holomorphic automorphisms of N_{τ_0} and it must act trivially on $H_n(M, \mathbb{Z})$. Theorem 3 implies that G acts trivially on $\tilde{\mathcal{H}}_L$. This will imply that it is a normal subgroup in $\mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$. To show this fact we need to prove that for any element $g \in \mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})$ we have $g^{-1}Gg = G$. Let $\tau = g^{-1}(\tau_0)$. Then direct computations show that for any element $h \in G$ we have $g^{-1}hg(\tau) = g^{-1} \circ g(\tau) = \tau$. The last equality shows that G is a normal subgroup. Lemma 11 is proved. ■

Corollary 12 *The group $G_1 := \mathbb{S}\mathbb{L}_{N+1}(\mathbb{C})/G$ acts freely on $\tilde{\mathcal{H}}_L$.*

We will prove that the quotients $\tilde{\mathcal{Y}}_L/G_1$ and $\tilde{\mathcal{H}}_L/G_1$ exist as complex manifolds and that $\mathcal{U}_L := \tilde{\mathcal{Y}}_L/G_1 \rightarrow \mathcal{T}_L(N) := \tilde{\mathcal{H}}_L/G_1$ will be the family (7) with the properties stated in the Theorem.

Palais proved in [28] the following theorem:

Theorem 13 *Suppose that an arbitrary Lie group \mathfrak{G} acts on a complex manifold Y , then the quotient Y/\mathfrak{G} exists in the category of complex spaces, provided that the map*

$$\psi : \mathfrak{G} \times Y \rightarrow Y \times Y \quad (12)$$

defined as $\psi(g, m) = (g(m), m)$ is proper.

In order to prove that the morphism defined by the action of the Lie group $\mathfrak{G}=G_1 := \mathbb{S}\mathbb{L}_{k+1}(\mathbb{C})/G$ acts properly on $\tilde{\mathcal{H}}_L$ and $\tilde{\mathcal{Y}}_L$, we need to use the following criterion for the properness of the map that can be found in [11], Chapter II, 7:

Criterion 14 *Let $\psi : X \rightarrow Z$ be a morphism of algebraic varieties over an algebraically closed field k , \mathcal{O} a discrete valuation ring with a residue class field k , $\phi^* : \text{Spec } \mathcal{O} \rightarrow X$ is a rational morphism and $h : \text{Spec } \mathcal{O} \rightarrow Z$ is a morphism, where $\psi \circ \phi^* = h$. If for such ϕ^* and h there exists a morphism $\phi : \text{Spec } \mathcal{O} \rightarrow X$, coinciding with ϕ^* on the generic point of $\text{Spec } \mathcal{O}$, the morphism ψ is proper.*

To apply the Criterion 14 in our situation we notice that the properness of the map (12) for the manifolds $\tilde{\mathcal{Y}}_L$ and $\tilde{\mathcal{H}}_L$ and the group G_1 is an obvious formal consequence of the analogous fact for the varieties \mathcal{Y}_L and \mathcal{H}_L . In this case we have to deal with algebraic varieties and algebraic action of the group G_1 . Therefore ψ is a morphism in the category of algebraic varieties. The existence of the Grothendieck families $\mathcal{Y}_L \rightarrow \mathcal{H}_L$ defines the

following families $\pi_1 : \mathcal{N}_L \rightarrow G_1 \times \mathcal{H}_L$ and $\pi_2 : \mathcal{R}_L \rightarrow \mathcal{H}_L \times \mathcal{H}_L$, where $\pi_1^{-1}(g, \tau) := N_{g(\tau)} \times N_\tau$ and $\pi_2^{-1}(\tau_1, \tau_2) = N_{\tau_1} \times N_{\tau_2}$.

Next we are going to apply Criterion 14 to the map ψ defined by (12). We will need to use Theorem 2 of the paper [26] of Matsusaka and Mumford. It asserts that if V and W are smooth polarized varieties over $\text{Spec } \mathcal{O}$, neither of which is ruled, and their specialization V_0 and W_0 over the closed point of $\text{Spec } \mathcal{O}$ are also smooth polarized varieties, then the specialization of any isomorphism $\psi : V \rightarrow W$ is an isomorphism $\psi_0 : V_0 \rightarrow W_0$.

It is easy to see that giving a morphism of algebraic varieties $h : \text{Spec } \mathcal{O} \rightarrow \mathcal{H}_L \times \mathcal{H}_L$ and a rational morphism $\phi^* : \text{Spec } \mathcal{O} \rightarrow G_1 \times \mathcal{H}_L$ satisfying the criterion stated in the above lemma, defines varieties V and W over $\text{Spec } \mathcal{O}$ and the isomorphism ψ^* of them. On the other hand ϕ^* gives a rational morphism of $\text{Spec } \mathcal{O}$ into G_1 and therefore a rational morphism ϕ^* of $\text{Spec } \mathcal{O}$ into the projective closure of $\overline{G_1}$ of the projective group G_1 . Since the dimension of the scheme $\text{Spec } \mathcal{O}$ is one, then the rational morphism ϕ^* of $\text{Spec } \mathcal{O}$ into a projective variety $\overline{G_1}$ can be prolonged to a morphism ϕ of $\text{Spec } \mathcal{O}$ to $\overline{G_1}$. Thus we get a family $\overline{W} \rightarrow \text{Spec } \mathcal{O}$ which is defined over the closed point of $\text{Spec } \mathcal{O}$ and contains the family $W \rightarrow \text{Spec } \mathcal{O}$. The fact that the rational morphism ϕ^* can be prolonged to an algebraic morphism ϕ implies that the morphism ψ^* can be prolonged to a morphism ψ between the varieties V and \overline{W} . By the Theorem of Matsusaka and Mumford, the specialization of ψ is an isomorphism, i.e. ψ takes closed point of $\text{Spec } \mathcal{O}$ into a point of G_1 . This means that ψ is an algebraic morphism of $\text{Spec } \mathcal{O}$ into G_1 . Thus the criterion of properness for the morphism ψ holds. As we pointed out this fact implies that the action of G_1 on $\tilde{\mathcal{H}}_L$ is proper too. The quotient $\tilde{\mathcal{H}}_L/G_1$ exists and it is a smooth algebraic variety.

The situation is analogous to the morphism $\pi : \tilde{\mathcal{Y}}_L \rightarrow \mathcal{U}_L = \tilde{\mathcal{Y}}_L/G_1$. Thus we have proved that the quotients $\tilde{\mathcal{Y}}_L/G_1$ and $\tilde{\mathcal{H}}_L/G_1$ exist as complex spaces. It is obvious that they define a family $\mathcal{U}_L \rightarrow \mathcal{T}_L(\mathbb{M})$ of marked polarized CY manifolds. According to Cor. 12, G_1 acts on \mathcal{Y}_L and \mathcal{H}_L without fixed points. Therefore,

$$\tilde{\mathcal{Y}}_L/G_1 = \mathcal{U}_L \text{ and } \tilde{\mathcal{H}}_L/G_1 = \mathcal{T}_L(\mathbb{M}) \quad (13)$$

are manifolds. Thus we have constructed the family (7) as required in Theorem 10. We also proved that the base of the family (7) $\mathcal{T}_L(\mathbb{M})$ is smooth. Next we will prove that the complex dimension of $\mathcal{T}_L(\mathbb{M})$ is $h^{n-1,1}$ and that the tangent space at each point $\tau \in \mathcal{T}_L(\mathbb{M})$ is isomorphic to $H^1(\mathbb{M}_\tau, \Theta_{\mathbb{M}_\tau})$.

We shall prove that the family (7) is effectively parametrized. According to Kodaira this means that for any point $s \in \mathcal{T}(\mathbb{M})$ the mapping $T_{s, \mathcal{T}_L(\mathbb{M})} \rightarrow H^1(\mathbb{M}_s, \Theta_{\mathbb{M}_s})$ is an isomorphism. From the construction of $\mathcal{T}_L(\mathbb{M})$ as defined

in (13) we can conclude that $T_{s, \mathcal{T}_L(M)} \cong H^0(M, \mathcal{N}_{\mathbb{C}\mathbb{P}^N/M})/sl(N+1)$ where $sl(N+1)$ is the Lie algebra of the group $\mathbb{S}\mathbb{L}(N+1)$. In [17] we proved that

$$H^0(M, \mathcal{N}_{\mathbb{C}\mathbb{P}^N/M})/sl(N+1) \cong H^1(M, \Theta_M).$$

Theorem 10 is proved. ■

Corollary 15 *Let $\mathcal{Y} \rightarrow X$ be any family of marked CY manifolds, then there exists a unique holomorphic map $\phi : X \rightarrow \mathcal{T}_L(M)$ up to a biholomorphic map ψ of M which induces the identity map on $H_n(M, \mathbb{Z})$.*

From now on we will denote by $\mathcal{T}(M)$ the irreducible component of the Teichmüller space that contains our fixed CY manifold M .

2.2 Construction of the Moduli Space of Polarized CY Manifolds

Definition 16 *We will define the mapping class group $\Gamma(M)$ of any compact C^∞ manifold M as follows: $\Gamma = \text{Diff}_+(M)/\text{Diff}_0(M)$, where $\text{Diff}_+(M)$ is the group of diffeomorphisms of M preserving the orientation of M and $\text{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity.*

Definition 17 *Let $L \in H^2(M, \mathbb{Z})$ be the imaginary part of a Kähler metric. Let $\Gamma_L := \{\phi \in \Gamma(M) \mid \phi(L) = L\}$.*

It is a well-known fact that the moduli space of polarized algebraic manifolds $\mathfrak{M}_L(M)$ is $\mathcal{T}_L(M)/\Gamma_L$.

Theorem 18 *There exists a subgroup of finite index Γ of Γ_L such that Γ acts freely on $\mathcal{T}(M)$ and $\Gamma \backslash \mathcal{T}_L(M) = \mathcal{M}_L(M)$ is a non-singular quasi-projective variety.*

Proof: There are homomorphisms induced by the action of the diffeomorphism group on the middle homology groups with coefficients in $\mathbb{Z} : \phi : \Gamma_L \rightarrow \text{Sp}(2b_n, \mathbb{Z})$ and $\phi : \Gamma_L \rightarrow \text{SO}(2p, q; \mathbb{Z})$ for odd and even dimensional complex Kähler manifolds correspondingly. An important theorem due to Sullivan proved in [38] states that:

Theorem 19 *Suppose that the real dimension of a C^∞ manifold M is bigger or equal to 5, then the image $\phi(\Gamma_L)$ of the mapping class group is an arithmetic group.*

This result of Sullivan implies that Γ_L is an arithmetic group. This means that the image of Γ_L has a finite index in the groups $\mathbb{S}p(2b_n, \mathbb{Z})$, $\mathbb{S}\mathbb{O}(2p, q; \mathbb{Z})$ and $\ker(\phi)$ is a finite group. A theorem of Borel implies that we can always find a subgroup of finite index Γ in Γ_L such that Γ acts freely on $\mathbb{S}p(2b_n, \mathbb{R})/\mathbb{U}(b_n)$ or on $\mathbb{S}\mathbb{O}_0(2p, q; \mathbb{R})/\mathbb{S}\mathbb{O}(2p) \times \mathbb{S}\mathbb{O}(q)$.

Lemma 20 Γ acts without fixed point on $\mathcal{T}_L(M)$.

Proof: Let \mathcal{K} be the Kuranishi space of the deformations of M . Suppose that there exists an element $g \in \Gamma$, such that $g(\tau) = \tau$ for some $\tau \in \mathcal{K} \subset \mathcal{T}_L(M)$. From the local Torelli theorem we deduce that we may assume that the Kuranishi space \mathcal{K} is embedded in \mathcal{G} , the classifying space of the Hodge structures of weight n on $H^n(M, \mathbb{Z}) \otimes \mathbb{C}$. Griffiths proved in [15] that $\mathcal{G} \approx G/K$ where G in the odd dimensional case is $\mathbb{S}p(2b_n, \mathbb{R})$ and in the even dimensional is $\mathbb{S}\mathbb{O}_0(2p, q; \mathbb{R})$ and K is a compact subgroup of G . Let K_0 be the maximal compact subgroup of G . So we have a natural C^∞ fibration

$$K_0/K \subset G/K \rightarrow G/K_0. \quad (14)$$

Griffith's transversality theorem implies that \mathcal{K} is transversal to the fibres K_0/K of the fibration $G/K \rightarrow G/K_0$. If $g \in \Gamma$ fixes a point $\tau \in G/K_0$, then $g \in K_0 \cap \Gamma$. On the other hand side it is easy to see that the local Torelli theorem implies that the action of Γ on \mathcal{K} is induced from the action Γ on G/K by left multiplications. So we can conclude that the action of Γ preserves the fibration (14). Lemma 20 follows directly from here and the fact that Γ acts without fix point on G/K_0 . ■

The second part of the theorem, namely that the space $\Gamma \backslash \mathcal{T}_L(M)$ is a quasi projective follows directly from the fact that $\Gamma \backslash \mathcal{T}_L(M) \rightarrow \Gamma_L \backslash \mathcal{T}_L(M)$ is a finite map and that $\Gamma_L \backslash \mathcal{T}_L(M)$ is a quasi projective variety according to [41]. Theorem 18 is proved. ■

Theorem 21 *There exists a finite cover $\mathcal{M}_L(M)$ of $\mathfrak{M}_L(M) = \mathcal{T}_L(M)/\Gamma_L$ with the following properties: **A.** $\mathcal{M}_L(M)$ is a smooth algebraic variety, **B.** Over $\mathcal{M}_L(M)$ there exists a family*

$$\mathcal{N}_L \rightarrow \mathcal{M}_L(M) \quad (15)$$

of polarized CY manifolds with the following property; Let

$$\pi_{\mathcal{C}} : \mathcal{Y} \rightarrow \mathcal{C} \quad (16)$$

We suppose that K or K_0 acts on the right on G and Γ acts on the left on G .

be any complex analytic family of polarized CY manifolds with a class of polarization L such that at least one of its fibres of the family (16) is isomorphic as a polarized variety to a fibre of the family (15). Then there exists a unique complex analytic map $\phi : \mathcal{C} \rightarrow \mathcal{M}_L(M)$, which induces a holomorphic map between the families (16) and (15). The map between the families is defined uniquely up to a biholomorphic map ϕ of M which induces the identity map on $H_n(M, \mathbb{Z})$.

Proof: Let $\psi : \Gamma_L \rightarrow H^n(M, \mathbb{Z})$ be the natural representation of the group Γ_L . The results of Sullivan imply that the image $\psi(\Gamma_L)$ is an arithmetic group. According to Theorem 18 there exists a subgroup Γ_L'' of finite index in $\psi(\Gamma_L)$ such that $\psi(\Gamma_L)$ acts freely on the Teichmüller space $\mathcal{T}_L(M)$. From here we deduce that the space $\Gamma_L'' \backslash \mathcal{T}_L(M)$ is smooth. Let us define $\Gamma_L' := \psi^{-1}(\Gamma_L'')$. Clearly Γ_L' is a subgroup of finite index in Γ_L . The local Torelli theorem implies that $\mathcal{T}_L(M)/\Gamma_L'$ will be a non-singular variety. From the definition of ψ , it follows that $\ker \psi = G$ acts trivially on $H^n(M, \mathbb{Z})$. The local Torelli Theorem implies that G acts trivially on the Teichmüller space $\mathcal{T}_L(M)$. The existence and the properties of the family $\mathcal{N}_L \rightarrow \mathcal{M}_L(M)$ follows from the existence and the properties of the family $\mathcal{Z}_L \rightarrow \mathcal{T}_L(M)$ proved in Theorem 15. ■

According to [41] the coarse moduli space $\mathfrak{M}_L(M)$ is a quasi-projective variety. Since $\mathcal{M}_L(M)$ is a finite cover of $\mathfrak{M}_L(M)$, then $\mathcal{M}_L(M)$ will be quasi-projective too. Hironaka's Theorem about resolution of singularities states that we can always compactify both $\mathcal{M}_L(M)$ and $\mathfrak{M}_L(M)$ such that

$$\overline{\mathcal{M}_L(M)} \ominus \mathcal{M}_L(M) = \mathcal{D}_\infty \text{ and } \overline{\mathfrak{M}_L(M)} \ominus \mathfrak{M}_L(M) = \mathfrak{D}_\infty \quad (17)$$

the divisors \mathcal{D}_∞ and \mathfrak{D}_∞ are divisors with normal crossings.

3 Moduli of Maps

3.1 Smoothness of the Moduli Space of Maps

Let C and X be two projective varieties. Let $f : C \rightarrow X$ be a morphism between them. Let $\Gamma_f \subset C \times X$ be the graph of the map $f : C \rightarrow X$. According to the results of Grothendieck, the Hilbert scheme of $\Gamma_f \subset C \times X$ is a projective scheme. See [11].

Definition 22 We will denote the Hilbert scheme of $\Gamma_f \subset C \times X$ by $\mathfrak{M}_f(C, X)$ and we will call it the moduli space of the map f .

The above mentioned results of Grothendieck implies that $\mathfrak{M}_f(C, X)$ is a projective scheme.

Definition 23 **a.** Let $f: C \rightarrow X$ be a morphism of projective varieties with $\dim_{\mathbb{C}} C \leq \dim_{\mathbb{C}} X$. Suppose that the morphism $f: C \rightarrow f(C)$ is finite. We say that f admits a non-trivial one parameter deformations if there is a non-singular projective curve T and a family of algebraic maps $F: T \times C \rightarrow X$ such that for some $t_0 \in T$ we have $F_{t_0} = f$ and the morphism $F: T \times C \rightarrow F(T \times C)$ is finite too. **b.** We will say that the deformation of f is trivial if $F_t = f$ for all $t \in T$. **c.** Two families of maps $F_1: T \times C_1 \rightarrow X$ and $F_2: T \times C_2 \rightarrow X$ are said to be isomorphic if there is a common finite cover C of C_1 and C_2 such that the lifts of F_1 to $T \times C$ and F_2 to $T \times C$ are isomorphic, meaning there exists a biholomorphic map $id \times g$ from $T \times C$ to itself such that $F_1 = F_2 \circ (id \times g)$.

From now on we will consider only one parameter deformations of maps.

Definition 24 Let C be a fixed Riemann surface. Let X be a quasi-projective space and \bar{X} some projective compactification of X such that $\mathcal{D}_{\infty} = \bar{X} \ominus X$ is a divisor with normal crossings. Assume that on X there exists a Kähler metric with non-positive holomorphic bisectional curvature, and negative holomorphic sectional curvature bounded away from zero and logarithmic growth near \mathcal{D}_{∞} . We will consider from now on the set \mathcal{M}_C of all holomorphic maps $f: C \rightarrow \bar{X}$ such that: **a.** the map $f: C \rightarrow f(C)$ is finite map and **b.** $f(C)$ is not contained in \mathcal{D}_{∞} and the pullback $f^*(\mathcal{D}_{\infty}) = \mathcal{D}_S$, where \mathcal{D}_S is a fixed divisor on C .

Theorem 25 Suppose that X is a projective manifold such that on a Zariski open set $X' := \bar{X} \ominus \mathcal{D}_{\infty}$ there exists a Kähler metric with a non-positive holomorphic bi-sectional curvature. Let C be a fixed algebraic curve. Let

$$\mathcal{M}_C := \{f: C \rightarrow X \mid f \text{ is a finite hol map from } C \text{ to } f(C) \text{ and } f(C) \not\subset \mathcal{D}_{\infty}\}. \quad (18)$$

Then the moduli space \mathcal{M}_C is a smooth quasi-projective variety.

Proof: The proof is based on several lemmas.

Lemma 26 Let $f: C \rightarrow X$ be a finite map from an algebraic curve C to $f(C)$ such that $f(C) \not\subset \mathcal{D}_{\infty}$. Let $\sigma \in H^0(C, f^*(T_X))$ be a non-zero section with a bounded norm. Then σ is parallel with respect to the connection induced by the restriction of the Kähler metric on $X \ominus \mathcal{D}_{\infty}$.

Proof: Denote by $C^0 = C \ominus f^*(\mathcal{D}_{\infty})$. Let $\sigma \in H^0(C \ominus f^*(\mathcal{D}_{\infty}), f^*(T_X))$ be a non-zero section. Suppose that $\nabla \sigma \neq 0$, where ∇ is the covariant derivative on the vector bundle $f^*(T_X)|_{C^0}$ induced by the pullback of the Kähler metric on X . Direct computations show that the following formula holds:

$$\frac{d^2}{d\tau d\bar{\tau}} \|\sigma\|^2 = \|\nabla\sigma\|^2 - \left\langle R\left(\frac{d}{d\tau}, \frac{\bar{d}}{d\bar{\tau}}\right)\sigma, \sigma \right\rangle \geq 0, \quad (19)$$

where $\left\langle R\left(\frac{d}{d\tau}, \frac{\bar{d}}{d\bar{\tau}}\right)\sigma, \sigma \right\rangle$ is the corresponding holomorphic bi-sectional curvature on the vector bundle $f^*(T_X)|_{C^0}$ with a metric induced by the pullback of the Kähler metric on X . Formula (19) and the assumption that we can prolong the finite map f from $C^0 = C \ominus f^*(\mathcal{D}_\infty)$ to a projective map show that the function $\|\sigma\|^2$ is a bounded plurisubharmonic function on C^0 . Therefore by maximal principle it is the constant function because formula (19) implies

$$\frac{d^2}{d\tau d\bar{\tau}} \|\sigma\|^2 = 0. \quad (20)$$

So we get a contradiction. This implies that $\sigma = 0$ if σ is not parallel. So if σ is a holomorphic non-zero section with finite L^2 norm, then it must be parallel. Lemma 26 is proved. ■

Corollary 27 *The tangent space at the point $f \in \mathcal{M}_C$ is isomorphic to all parallel sections $\sigma \in H^0(C, f^*(T_X))$.*

Lemma 28 *Let $\sigma \in H^0(C \ominus f^*(f(C) \cap \mathcal{D}_\infty), f^*(T_X))$. be a parallel section. Then there exists a family of maps $F_t : C \rightarrow X$ where $t \in \mathcal{D}$ and \mathcal{D} is the unit disk such that $F_0 = f$ and $\frac{d}{dt} F_t|_{t=0} = \sigma$.*

Proof: Since $\sigma \in H^0(C \ominus f^*(f(C) \cap \mathcal{D}_\infty), f^*(T_X))$ is a parallel section, there exists a vector $\vec{\tau} \in T_{x,X}$ at each point $x \in f(C) \ominus \mathcal{D}_\infty$ such that $f^*(\vec{\tau}) = \sigma$ and $x \in f(C)$. Since the metric g on X is Kähler we can define as in the Appendix I 7, Prop. 56 the exponential complex map: $\exp_x : T_{x,X} \rightarrow X$ at each point $x \in f(C)$ such that $\exp_x(t\vec{\tau}) = D_x$ is a totally geodesic disk when $|t| < 1$ and the tangent vector of the disk at the point x is $\vec{\tau}$. The existence of a metric with holomorphic non positive bi-sectional curvature on the moduli space of polarized CY implies that the complex exponential map is injective for $|t| < 1$. Since we assumed that σ is a parallel section of the vector bundle f^*T_X on $C \ominus f^*(f(C) \cap \mathcal{D}_\infty)$ we deduce immediately that by defining $f_t(s) = \exp_{f(s)}(t\vec{\tau})$ and taking the exponential map at each point $x \in f(C)$, we have constructed a one parameter deformation $f_t : C \times \mathcal{D} \rightarrow X$ of $f : C \rightarrow X$ as required in lemma 28. ■

Lemma 28 and Corollary 27 imply that the moduli space of all the maps $f : C \rightarrow X$ such that $f(C) \subsetneq \mathcal{D}_\infty$ is smooth. Indeed we know that the tangent space T_{f, \mathcal{M}_C} of the moduli space \mathcal{M}_C at $f \in \mathcal{M}_C$ is isomorphic to $H^0(C, f^*(T_X))$. Cor. 27 tells us that T_{f, \mathcal{M}_C} consists of all parallel sections.

From Lemma 28 it follows that each parallel section can be integrated to a family of maps parametrized by the unit disk. Grothendieck's theory of Hilbert schemes tells us that the Hilbert scheme of all maps $f:C \rightarrow X$ such that $f(C) \not\subseteq \mathcal{D}_\infty$ is a quasi-projective space. Theorem 25 is proved. ■

3.2 Some Applications

Let $(C; x_1, \dots, x_n)$ be a Riemann surface with n distinct fixed points on it. Let $\mathcal{X}_C \rightarrow C$ be a family of polarized CY manifolds with degenerate fibres over the points x_1, \dots, x_n . It is easy to see that the family $\mathcal{X} \rightarrow C$ defines a map

$$f : C \ominus (x_1 \cup \dots \cup x_n) \rightarrow \mathcal{T}_L(M)/\Gamma_L. \quad (21)$$

Lemma 29 *The map f can be prolonged to a map $\bar{f} : C \rightarrow \overline{\Gamma_L \backslash \mathcal{T}_L(M)} = \overline{\mathfrak{M}_L(M)}$, where the compactification $\overline{\mathfrak{M}_L(M)}$ of $\mathfrak{M}_L(M)$ is such that $\overline{\mathfrak{M}_L(M)}$ is a projective variety and $\overline{\mathfrak{M}_L(M)} \ominus \mathfrak{M}_L(M)$ is a divisor with normal crossings.*

Proof: The proof of Lemma 29 is exactly Proposition 9.10 proved in [15]. ■

Let Γ_L and Γ be the arithmetic groups defined in Theorem 18. The map

$$\pi : \mathcal{T}_L(M)/\Gamma = \mathfrak{M}_L(M) \rightarrow \mathcal{T}_L(M)/\Gamma_L = \mathcal{M}_L(M)$$

is a finite morphism since Γ_L/Γ is a finite set and its cardinality is N . We will prove the following lemma:

Lemma 30 *There exists a finite affine cover $\psi : C_1 \rightarrow C \ominus (x_1 \cup \dots \cup x_n)$ of degree $N = \#\Gamma_L/\Gamma$ such that the map f defined by (21) can be lifted to a map $f_1 : C_1 \rightarrow \mathcal{T}_L(M)/\Gamma$.*

Proof: The construction of the affine Riemann surface C_1 is done in a standard way, namely

$$C_1 = (C \ominus (x_1 \cup \dots \cup x_n)) \times_{f(C \ominus (x_1 \cup \dots \cup x_n))} \pi^{-1}(f(C \ominus (x_1 \cup \dots \cup x_n))).$$

We know from the local Torelli theorem and the fact $\Gamma_L \backslash \mathcal{T}_L(M)$ is the moduli space of polarized CY manifolds, that the map f defined by (21) exists. We know that, since $\Gamma \subset \Gamma_L$ is a subgroup of finite index, the map $\pi : \Gamma_L \backslash \mathcal{T}_L(M) \rightarrow \Gamma \backslash \mathcal{T}_L(M)$ is a finite map. Therefore the map f can be lifted to a map $f' : C_1 \rightarrow \mathcal{T}_L(M)/\Gamma_L(M)$. This follows directly from the construction of the open Riemann surface C' . Lemma 30 is proved. ■

Lemma 31 *The map f_1 constructed in Lemma 30 can be prolonged to a map $f : \overline{C_1} \rightarrow \overline{\Gamma \backslash \mathcal{T}_L(M)} = \overline{\mathcal{M}_L(M)}$ where $\overline{C_1}$ is the closure of C_1 in the compactification $\overline{\mathcal{M}_L(M)}$ of $\mathcal{M}_L(M)$ as defined in (17).*

Proof: The proof of Lemma 31 is the same as Lemma 29 and is due to Griffiths. See [15]. ■

Definition 32 *We will say that a family of polarized CY manifolds $\pi : \mathcal{X}_C \rightarrow C$ over a Riemann surface C is rigid if the moduli space of the map $f : C \rightarrow \overline{\mathcal{M}_L(M)}$ induced by the natural holomorphic map of the base of the family into the moduli space of polarized CY manifolds is a discrete set.*

The proof of the following remark is obvious and we will omit it.

Let $\mathcal{X} \rightarrow C$ be a family of polarized CY manifolds over a fixed Riemann surface C with a fixed points of degenerations. Let $\pi_1 : C_1 \rightarrow C$ be the finite cover constructed in Lemmas 30 and 31. Let $\mathcal{X}_1 \rightarrow C_1$ is the pullback of the family $\mathcal{X} \rightarrow C$. Then the family $\mathcal{X} \rightarrow C$ is rigid if and only if the pullback family $\mathcal{X}_1 \rightarrow C_1$ is rigid.

4 Yukawa Coupling and Rigidity

Definition 33 *Suppose that ω_τ is a family of holomorphic forms on the local universal family of CY manifolds over the Kuranishi space of polarized CY manifold. The expression*

$$\left\langle \underbrace{\left(\nabla_{\frac{\partial}{\partial \tau^{i_1}}} \circ \left(\nabla_{\frac{\partial}{\partial \tau^{i_2}}} \circ \dots \circ \left(\nabla_{\frac{\partial}{\partial \tau^{i_n}}} (\omega_\tau) \right) \right) \right)}_n, \omega_\tau \right\rangle$$

is a holomorphic section of $\left(\left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right)^ \right)^{\otimes 2} \otimes S^{\otimes n}(\mathcal{T}_{\mathcal{M}(M)})$. This section is called the Yukawa coupling.*

Condition 34 *Suppose that for every $\phi \in H^1(M, \Theta_M)$ different from zero the following condition holds:*

$$\wedge^n \phi \neq 0 \in H^n(M, \wedge^n \Theta_M) \quad (22)$$

Then we will prove that any family $\pi : \mathcal{X}_C \rightarrow C$ over a Riemann surface C is rigid if one of its fibre is M .

The condition 34 is difficult to check. Now we will formulate an equivalent condition which is much easier to verify. Let $\pi : \mathcal{X} \rightarrow \mathcal{D}$ be a non-trivial family of non singular n complex dimensional CY manifolds over the unit disk \mathcal{D} . We will denote by t the local coordinate in \mathcal{D} . Let us denote by $\nabla_{\frac{\partial}{\partial t}}$ the covariant differentiation induced by the Gauss-Manin connection. According to Kodaira-Spencer-Kuranishi theory the tangent vectors at $0 \in \mathcal{D}$ can be identified with non-zero elements $\phi \in H^1(M_0, \Theta_{M_0})$. Let $\Omega_{\mathcal{X}/\mathcal{D}}^n$ be the relative dualizing sheaf. Let $\omega_t \in H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{D}}^n)$ be a family of holomorphic n forms ω_t . We may suppose that $\omega_0 \neq 0$.

Theorem 35 *Let $\phi \in H^1(M, \Theta_M)$, then the following condition*

$$\left\langle \underbrace{\left(\nabla_{\frac{\partial}{\partial t}} \circ \left(\nabla_{\frac{\partial}{\partial t}} \circ \dots \circ \left(\nabla_{\frac{\partial}{\partial t}} (\omega_t) \right) \right) \right)}_n \right\rangle_{t=0, \omega_0} \neq 0, \quad (23)$$

is equivalent to (22), i.e. $\wedge^n \phi \neq 0$ in $H^n(M, \wedge^n \Theta_M)$.

Proof: It is a standard fact that for any family of forms ω_s the covariant differentiation given by the Gauss-Manin connection is given by the formula:

$$\nabla_{\frac{\partial}{\partial t}} (\omega_t) |_{t=0} = \omega_0 \lrcorner \phi \quad (24)$$

where \lrcorner means contraction of tensors. From (24) we derive directly:

$$\underbrace{\left(\nabla_{\frac{\partial}{\partial t}} \circ \left(\nabla_{\frac{\partial}{\partial t}} \circ \dots \circ \left(\nabla_{\frac{\partial}{\partial t}} (\omega_t) \right) \right) \right)}_n \Big|_{t=0} = (\wedge^n \phi) \lrcorner \omega_0. \quad (25)$$

(25) means that the form $(\wedge^n \phi) \lrcorner \omega_0$ is of type $(0, n)$. This implies directly that (23) and (22) are equivalent. Theorem 35 is proved. ■

Theorem 36 *Let $\pi : \mathcal{X}_C \rightarrow C$ be a non-isotrivial family of CY manifolds over a compact Riemann surface C . Suppose that there exists a point $\tau_0 \in C$ such that the fibre $\pi^{-1}(\tau_0) = M_{\tau_0}$ satisfies Condition 34. Suppose that \mathcal{X}_C is a projective manifold. Then the family $\pi : \mathcal{X}_C \rightarrow C$ is rigid.*

Proof: Theorem 36 follows from the fact that the moduli space of the deformations of the map f defined in Definition 32 is smooth by Theorem 25 and Lemma 37. The rigidity therefore implies that the tangent space of the moduli space at the point $f: C \rightarrow \mathcal{M}_L(M)$ is zero dimensional.

Lemma 37 *Suppose that the family $\pi : \mathcal{X}_C \rightarrow C$ satisfies the conditions of Theorem 36, then the set of the parallel sections of the bundle $f^*(T_{\mathcal{M}_L(M)})$ consists only of the zero section.*

Proof: The proof of Lemma 37 is done by contradiction. According to [30] the non-rigidity of the family $\pi : \mathcal{X}_C \rightarrow C$ means the existence of $\phi \in H^0(C \oplus S, f^*(T_{\mathcal{M}(M)}))$, $\phi \neq 0$ such that ϕ is parallel section with respect to the pullback of the Hodge metric on $\mathcal{M}(M)$. Since the tangent space at a point $\tau \in \mathcal{M}(M)$ can be identified with $H^1(M_\tau, T_M^{1,0})$, the restriction of the section ϕ at each point $\tau \in C \oplus S$ can be viewed as a Kodaira-Spencer class $\phi_\tau \in H^1(M_\tau, T_M^{1,0})$. Locally on M_τ , ϕ_τ is given by

$$\phi_\tau := \sum_{i,j} (\phi_\tau)_j^i \overline{dz^j} \otimes \frac{d}{dz^i}.$$

On the other hand $\wedge^n \phi_\tau$ is represented on M_τ by

$$\wedge^n \phi_\tau = \det \left((\phi_\tau)_j^i \right) \overline{dz^1} \wedge \dots \wedge \overline{dz^n} \otimes \frac{d}{dz^1} \wedge \dots \wedge \frac{d}{dz^n}. \quad (26)$$

Since $\phi_\tau \in C^\infty \left(M, \text{Hom} \left(\Omega_{M_\tau}^{1,0}, \Omega_{M_\tau}^{0,1} \right) \right)$ then

$$\begin{aligned} \wedge^n \phi_\tau &\in C^\infty \left(M, \text{Hom} \left(\wedge^n \left(\Omega_{M_\tau}^{1,0} \right), \wedge^n \left(\Omega_{M_\tau}^{0,1} \right) \right) \right) \cong \\ &C^\infty \left(M, \text{Hom} \left(\Omega_{M_\tau}^{n,0}, \Omega_{M_\tau}^{0,n} \right) \right) \cong C^\infty \left(M_\tau, \left(\left(\Omega_{M_\tau}^{n,0} \right)^* \right)^{\otimes 2} \right). \end{aligned} \quad (27)$$

Combining (26) and (27) we get that

$$\wedge^n \phi_\tau \in H^0 \left(C, \left(\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^* \right)^{\otimes 2} \right) \quad (28)$$

is a non zero class. Moreover according to Lemma 26 ϕ_τ is a parallel section with respect to the pullback of the Hodge metric. This implies that $\wedge^n \phi_\tau$ is a global parallel non zero section on C of the line bundle $\left(\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^* \right)^{\otimes 2}$ which means that it is a trivial line bundle on C . This is impossible since we know from [25] and the Grothendieck Riemann relative Riemann Roch Theorem that the Chern class $c_1 \left(\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^* \right)$ of the line bundle $\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^*$ is proportional to the imaginary part of the Weil-Petersson metric. We also know that the Chern form of the L^2 metric on $\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^*$ is the imaginary part of the Weil-Petersson metric. So

$$\int_C c_1 \left(\left(R^0 \pi_* \left(\Omega_{\mathcal{X}/\mathcal{M}(M)}^n \right) \right)^* \right) > 0. \quad (29)$$

The inequality (29) implies that $R^0\pi_*\left(\Omega_{\mathcal{X}/\mathcal{M}(\mathcal{M})}^n\right)$ is not a trivial line bundle on \mathcal{C} . So we get a contradiction. Lemma 37 is proved. ■

Lemma 37 implies Theorem 36. ■

Remark. Theorem 36 presumably also follows from the methods used in [43], and it seems to extend to non-isotrivial morphisms $\pi : \mathcal{X}_{\mathcal{C}} \rightarrow \mathcal{C}$ whose generic fibre has a semi-ample canonical sheaf.

Theorem 38 *Suppose that $\pi : \mathcal{X} \rightarrow \mathcal{C}$ is a family of polarized CY manifold over a Riemann surfaces \mathcal{C} . Suppose that the family contains a point $\tau_0 \in \mathcal{C}$ such that around it the monodromy operator \mathbb{T} has a maximal index of unipotency, i.e.*

$$(\mathcal{T}^N - id)^{n+1} = 0 \text{ and } (\mathcal{T}^N - id)^n \neq 0. \quad (30)$$

Then the family $\pi : \mathcal{X} \rightarrow \mathcal{C}$ is rigid.

Proof: We will prove that the Yukawa coupling is non-zero in an open neighborhood of the point $\tau_0 \in \mathcal{C}$ around which the condition (30) holds. Theorem 38 follows from the existence of limit mixed Hodge structure related to the monodromy operator \mathcal{T} established in [36]. Indeed the condition (30) combined with Schmid's Theorem implies the existence of the filtration

$$\mathbb{W}_0 \subset \mathbb{W}_1 \subset \dots \subset \mathbb{W}_{2n-1} \subset \mathbb{W}_{2n} = \mathbb{V} \otimes \mathbb{C} = H^n(\mathbb{M}_{\tau}, \mathbb{C}) \quad (31)$$

such that the nilpotent operator $\mathcal{N} = \log(\mathcal{T}^N - id)$ acts on the filtration (31) as follows: $\mathcal{N}(\mathbb{W}_k) \subseteq \mathbb{W}_{k-2}$. In case of CY manifolds Schmid Theorem about the existence of the limit mixed Hodge structure and condition (30) imply that

$$\mathcal{N}^n(\mathbb{W}_{2n}/\mathbb{W}_{2n-1}) \cong \mathbb{W}_0 \quad (32)$$

(See [36].) The nilpotent orbit theorem proved in [36] combined with (32) implies that for each point in a "small disk" containing the point $\tau_0 \in \mathcal{C}$ we have that

$$\underbrace{\nabla_{\frac{\partial}{\partial \tau_{j_1}}} \circ \dots \circ \nabla_{\frac{\partial}{\partial \tau_{j_n}}}}_n \omega_{\tau} \neq 0 \quad (33)$$

and the $(0, n)$ component of (33) is a non-zero class of cohomology. Here ω_{τ} is a family of holomorphic n forms around the point $\tau \in \mathcal{K}$ in the Kuranishi space that corresponds to the fibre of the point on \mathcal{C} close to τ_0 . Theorem 38 follows directly from the definition 33 and Theorem 36. ■

5 Yau's Form of Schwartz Lemma and Boundedness

S.-T. Yau proved in [47] the following theorem, which is a generalization of the Schwarz Lemma:

Theorem 39 *Let N be a complex manifold and let h be a Hermitian metric on N such that the holomorphic sectional curvature K of h has the following property; there exists a positive constant $c > 0$ such that $K \leq -c$. Let C be either a compact or an affine Riemann surface of hyperbolic type. Let $\phi : C \rightarrow N$ be a holomorphic map different from the constant one, then we have $\phi^*(h) \leq c^{-1}g_P$, where g_P is the Poincare metric on C , i.e. the metric with a constant curvature.*

This lemma was used in [23] to derive various height inequalities. Here we will use the same method to prove the following Theorem:

Theorem 40 *Let $\phi : C \rightarrow \overline{\mathcal{M}_L(M)}$ be a holomorphic map such that $\phi(C) \not\subseteq \mathcal{D}_\infty$ and let x_1, \dots, x_n be the only n different points on C such that $\phi(x_i) \in \mathcal{D}_\infty$. Let $C_1 := C \ominus (x_1 \cup \dots \cup x_n)$. Then we have the following estimate $\text{vol}(\phi(C_1)) \leq c^{-1}|\chi(C_1)|$ for the volume of $\phi(C_1)$ in $\mathcal{M}(M)$ with respect to the Hodge metric h on $\mathcal{M}(M)$.*

Proof: From Yau's form of the Schwarz Lemma we obtain:

$$\text{vol}(\phi(C_1)) = \int_{C_1} h \leq c^{-1} \int_{C_1} g_P.$$

Since the Poincare metric has a constant curvature -1 and from the Gauss-Bonnet theorem we obtain

$$\int_{C_1} g_P = |\chi(C_1)|.$$

From this Theorem 40 follows directly. ■

Definition 41 *Let $(C; x_1, \dots, x_n)$ be a given Riemann surface with fixed points x_1, \dots, x_n . Let S be the divisor $x_1 + \dots + x_n$ on C . Let us fix a CY manifold M . We will define $Sh(C; S, M)$ to be the set of all possible families $\mathcal{X} \rightarrow C$ of CY manifolds defined up to an isomorphism over C with fixed degenerate fibres over the points x_1, \dots, x_n , where \mathcal{X} is a projective manifold and the generic fibre is a CY manifold C^∞ equivalent to M .*

Theorem 42 *Suppose that Condition 34 or its equivalent (23) holds for some fibre M_τ of a family of CY manifolds over a fix Riemann surface C with fixed points of degenerations Then the set $Sh(C;S)$ is finite.*

Proof: The proof of Theorem 42 will be done in two steps. The first step will be to prove that the set of maps $Sh(C;S)$ is discrete. The second step is to show that for each $\phi \in Sh(C;S)$ the volume of $\phi(C)$ is bounded by a universal constant. Then Theorem 42 will follow directly from the theorem of Bishop proved in [8], which implies that the set $Sh(C,S)$ is compact.

Lemma 43 *The set of maps $Sh(C,S)$ is discrete.*

Proof: Since we assumed that the CY manifold M satisfies Condition (22), Lemma 43 follows directly from Theorem 36. ■

We will need the following Lemma for establishing the second step described above:

Lemma 44 *Let $Sh(C,S)$ be the set defined in Definition 24, where S is the divisor $x_1 + \dots + x_n$ on C . Then $Sh(C,S)$ is compact.*

Proof: Each point $z \in Sh(C,S)$ is represented by a map $\phi_z : C \rightarrow \mathcal{T}(M)/\Gamma$ which satisfies the conditions of Definition 24. Let g be the Kähler (1,1) form of the Hodge metric on $\mathcal{T}(M)/\Gamma$, then we can define a height function h on $Sh(C,S)$ as follows:

$$h(\phi_z) = \int_{\phi(C_z)} \text{Im } g = \text{vol}(\phi_z(C)). \quad (34)$$

Since the metric g is a Kähler metric with logarithmic growth on X , then

$$h(\phi_z) = \int_{\overline{\mathcal{T}(M)/\Gamma}} \text{Im}(g) \wedge \mathcal{P}(\phi(C)) = \text{vol}(\phi_z(C)). \quad (35)$$

where $\mathcal{P}(\phi_z(C))$ is the Poincare dual of the homology class of $\overline{\phi_z(C)}$ in $\overline{\mathcal{T}(M)/\Gamma}$. Since $\phi_z \in \overline{Sh(C,S)}$ then it is a deformation of the map f . So for all $\phi_z \in \overline{Sh(C,S)}$, $\overline{\phi_z(C)}$ realizes a fixed class of homology in $\overline{\mathcal{T}(M)/\Gamma}$. The integrals that appeared in (34) and (34) are finite. This follows from Theorem 5.1. proved in [25]. Theorem 40 implies that $h(\phi_z)$ is a bounded function on $Sh(C,S)$.

We will prove now that $Sh(C,S)$ is a compact set. Since $Sh(C,S)$ is a discrete set, the compactness of $Sh(C,S)$ will imply that it is a finite set.

The compactness of $Sh(C,S)$ will follow if we prove that from any sequence $\{\phi_n\}$ in $Sh(C,S)$ there exists a subsequence which converges weakly to an algebraic map, i.e. the corresponding subsequence of images of S converges to an algebraic subvariety of X . Bishop's theorem implies that. See page 292 of [8] and also the Appendix II. Lemma 44 is proved. ■

Theorem 42 follows directly from Lemmas 43 and 44. ■

6 Higgs Bundles, VHS and Rigidity

We are going to study the relationship between Higgs bundles, Variations of Hodge Structures (VHS) and the existence of non rigid families of CY manifolds. Let S^0 and T^0 be smooth quasi-projective manifolds. Let

$$\pi : \mathcal{X} \rightarrow S^0 \times T^0 = \mathcal{Y}_0 \quad (36)$$

be a non rigid family of Calabi-Yau 3-folds, such that the induced map into the moduli space is generically finite. Let $\mathbb{V} = R^3\pi_*\mathbb{Z}$ denote the flat bundle associated with the third cohomology group. The Variation of Hodge Structures (VHS) of weight 3 of π associated with the family (36) will be $\mathbb{V} \otimes \mathbb{C}$. The corresponding Hodge bundles are then

$$\mathbb{V} \otimes \mathbb{C} = \pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^3 \oplus R^1\pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^2 \oplus R^2\pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^1 \oplus R^3\pi_*\mathcal{O}_{\mathcal{X}}.$$

The cup product of the Kodaira-Spencer class of π $\theta_0^{p,q} : E_0^{p,q} \rightarrow E_0^{p,q} \otimes \Omega_{\mathcal{Y}_0}^1$, where

$$E_0^{3,0} = \pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^3, \quad E_0^{2,1} = R^1\pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^2, \quad E_0^{1,2} = R^2\pi_*\Omega_{\mathcal{X}/\mathcal{Y}_0}^2 \quad \text{and} \quad E_0^{0,3} = R^3\pi_*\mathcal{O}_{\mathcal{X}}.$$

will provide $\left(\bigoplus_{p+q=3} E_0^{p,q}, \bigoplus_{p+q=3} \theta_0^{p,q} \right)$ with the Higgs bundle structure on \mathbb{V} corresponding to the VHS.

Theorem 45 *Suppose that S^0 and T^0 are Zariski open sets in projective varieties S and T . Suppose that*

$$\pi : \mathcal{X} \rightarrow \mathcal{Y}_0 = S^0 \times T^0 \quad (37)$$

is a non-rigid family of Calabi-Yau manifolds, such that the induced map to the moduli space is generically finite. Let $\mathbb{V} \rightarrow \mathcal{Y}_0 = S^0 \times T^0$ be the VHS induced by the family (37). Suppose that (s_i, t_j) is a point of $\mathcal{D}_\infty = (S \ominus S^0) \times (T \ominus T^0)$. Let γ_{s_i} and γ_{t_j} be two short loops around the divisors T_{s_i} and S_{t_j} . Let S_{t_0} be the curve in S -direction passing through the point (s_0, t_0) . Suppose that the monodromy operators $\rho(\gamma_{s_i}) = \mathcal{T}_i$ and $\rho(\gamma_{t_j}) = \mathcal{T}_j$ have infinite order, then the endomorphisms $\mathcal{T}_i : \mathbb{V}|_{(s_0, t_0)} \rightarrow \mathbb{V}|_{(s_0, t_0)}$ and $\mathcal{T}_j : \mathbb{V}|_{(s_0, t_0)} \rightarrow \mathbb{V}|_{(s_0, t_0)}$ are distinct.

Proof: The idea of the proof is to construct on the restrictions of VHS $\mathbb{V}|_{S^0}$ and $\mathbb{V}|_{T^0}$ two endomorphisms by using the fact that the family of CY manifold $\mathcal{X} \rightarrow S^0 \times T^0$ is not rigid. According to Lemma 26 these two endomorphisms are parallel sections of the Hodge bundle $R^1\pi_*\Omega_{\mathcal{X}/S^0 \times T^0}^{n-1}$. Theorem 45 will follow from the way the weight filtrations of nilpotent endomorphisms introduced by W. Schmid are constructed, the fact that the two filtrations will be different and the theorem about the complete reducibility of local systems of VHS over quasi-projective manifolds due to Deligne.

There are two ways to obtain endomorphisms on $\mathbb{V}|_{S_t}$ and $\mathbb{V}|_{T_j}$, where $S_t := S \times t_0$ and $T_j = s_0 \times T$.

1. Consider the two projections $p_S : S \times T \rightarrow S$ and $p_T : S \times T \rightarrow T$. One has

$$\Omega_{S \times T}^1(\log \mathcal{D}_{S \times T}) = p_S^*(\Omega_S^1(\log \mathcal{D}_S)) \oplus p_T^*(\Omega_T^1(\log \mathcal{D}_T))$$

where $\mathcal{D}_{S \times T}$ is the discriminant locus of the family (37). Then the restriction of the Higgs map to S_t defines a natural map

$$\theta|_{S_t} : (p_S^*(\Theta_S(-\log \mathcal{D}_S)) \oplus p_T^*(\Theta_T(-\log \mathcal{D}_T)))_{S_t} \rightarrow \text{End}(\mathbb{V})|_{S_t}, \quad (38)$$

by identifying the sections of $(p_S^*(\Theta_S(-\log \mathcal{D}_S)) \oplus p_T^*(\Theta_T(-\log \mathcal{D}_T)))_{S_t}$ with Kodaira-Spencer classes. Note that

$$p_T^*\Theta_T(-\log \mathcal{D}_T)|_{S_t} \simeq \mathcal{O}_{S_t} \text{ and } p_S^*\Theta_S(-\log \mathcal{D}_S)|_{T_j} \simeq \mathcal{O}_{T_j}.$$

Let $1_T \in p_T^*\Theta_T(-\log \mathcal{D}_T)|_{S_t}$ and $1_S \in p_S^*\Theta_S(-\log \mathcal{D}_S)|_{T_j}$ be the constant section. Then by using a Theorem of Jost and Yau proved in [20] one obtains endomorphisms

$$\theta|_{S_t}(1_{S_t}) : \mathbb{V}_{S_t^0} \rightarrow \mathbb{V}_{S_t^0} \text{ and } \theta|_{T_j}(1_{T_j}) : \mathbb{V}_{T_j^0} \rightarrow \mathbb{V}_{T_j^0}. \quad (39)$$

See also [51] for another proof. This proof is based on the observation that the image of the map (38) is contained in the kernel of the induced Higgs map on $\text{End}(\mathbb{V})$. Therefore, the image is a Higgs subsheaf with the trivial Higgs field. The Higgs poly-stability of $\text{End}(\mathbb{V})|_{S_t}$ implies that any section in this subsheaf is flat. One sees also that this flat section is of Hodge type $(-1,1)$.

2. Let \mathbb{V} be an arbitrary Variation of Hodge Structure over some quasi-projective space by taking the local monodromy by the following consideration. Let $S^0 = S \setminus \{s_0, \dots, s_m\}$ and $T^0 = T \setminus \{t_0, \dots, t_n\}$ be two smooth projective curves. The local system \mathbb{V} corresponds to a representation

$$\rho : \pi_1(S^0, *) \times \pi_1(T^0, *) \rightarrow GL(V),$$

and the restriction $\rho|_{S_t}$ corresponds to the restricted representation $\rho|_{\pi_1(S^0, *)} : \pi_1(S^0, *) \rightarrow GL(V)$. Let γ_{t_j} be a short loop around some point t_j , then it commutes with $\pi_1(S^0, *)$. Hence, $\rho(\gamma_t)$ descends to an endomorphism, say $\rho(\gamma_{t_j}) : \mathbb{V}_{S_t^0} \rightarrow \mathbb{V}_{S_t^0}$. Similarly, one gets also an endomorphism $\rho(\gamma_{s_i})$ on $\mathbb{V}_{T_s} \rightarrow \mathbb{V}_{T_s}$. It is well known that those two endomorphisms $\rho(\gamma_{t_j})$ and $\rho(\gamma_{s_i})$ are unipotent. Let $\mathcal{N}_{s_i} = \log(\mathcal{T}_i - id)$ and $\mathcal{N}_{t_j} = \log(\mathcal{T}_j - id)$ be the nilpotent part of \mathcal{T}_i and \mathcal{T}_j respectively which are non-trivial by assumption. Let $\theta|_{S_t}(1_{S_t}) : \mathbb{V}_{S_t^0} \rightarrow \mathbb{V}_{S_t^0}$ be the endomorphism of the variations of the Hodge structure induced by $\theta|_{S_t}(1_{S_t})$ in (39). We noticed that $\theta|_{S_t}(1_{S_t})$ and $\theta|_{T_j}(1_{T_s})$ are nilpotent. Similar to what Schmid did for nilpotent endomorphisms on vector space we see that $\theta|_{S_t}(1_{S_t})$ induces a weight filtration of the local system $0 \subset \mathbb{W}_0 \subset \mathbb{W}_1 \subset \dots \subset \mathbb{W}_{2k-1} \subset \mathbb{W}_{2k} = \mathbb{V}_{S_{t_0}^0}$, such that $\theta|_{S_t}(1_{S_t})(\mathbb{W}_l) \subset \mathbb{W}_{l-2}$. Since the local system $\mathbb{V}_{S_{t_0}^0}$ is completely reducible by Deligne, one gets an isomorphism $\mathbb{V}|_{s_j} = \bigoplus_j \mathbb{W}_j / \mathbb{W}_{j-1}$. The isomorphism $\theta|_{S_t}(1_{S_t})$ shifts $\mathbb{W}_l / \mathbb{W}_{l-1}$ to $\bigoplus_{j \leq l-2} \mathbb{W}_j / \mathbb{W}_{j-1}$. On the other hand, regarding \mathcal{T}_i as an element in the image $\rho : \pi_1(S_{t_0}, s_0) \rightarrow GL(V)$ the above decomposition on the fibre \mathbb{V}_{s_0, t_0} decomposes $\mathcal{N}_{s_i} = \log(\mathcal{T}_i - id)$ as a direct sum of endomorphisms on $(\mathbb{W}_j)_{(s_0, t_0)} / (\mathbb{W}_{j-1})_{(s_0, t_0)}$. In particular, \mathcal{N}_{s_i} can not be equal to \mathcal{N}_{t_j} as endomorphism on the fibre $\mathbb{V}_{(s_0, t_0)}$ since the filtrations of those two parallel endomorphisms of VHS will be different since the families obtained from the restrictions of the family (37) on S and T are not isomorphic. This follows from local Torelli Theorem. The two filtrations will be invariant under the actions of $\log(\mathcal{T}_i - id)$ and $\log(\mathcal{T}_j - id)$ respectively. Theorem 45 is proved. ■

Corollary 46 *Suppose that S^0 and T^0 are Zariski open sets in projective varieties S and T . Suppose that $\pi : \mathcal{X} \rightarrow \mathcal{Y} = S \times T$ is the family of Calabi-Yau manifolds (37), such that the induced map to the moduli space $\overline{\mathcal{M}}_L(M)$ is generically finite. Suppose that the discriminant locus $\mathcal{D}_\infty = \overline{\mathcal{M}}_L(M) \ominus \mathcal{M}_L(M)$ is a irreducible divisor and at some point (s_0, t_0) of $\mathcal{D}_\infty = (S \ominus S^0) \times (T \ominus T^0)$ the local monodromy operators of the restrictions of the family (37) on S and T around the points $s_0 \in S$ and $t_0 \in T$ are infinite. Then the family (37) is rigid.*

Proof: Since we assumed that the discriminant locus $\mathfrak{D}_\infty = \overline{\mathcal{M}_L(\mathbb{M})} \ominus \mathcal{M}_L(\mathbb{M})$ is an irreducible divisor in $\overline{\mathcal{M}_L(\mathbb{M})}$ then the monodromy operators acting on the middle cohomology of a fixed CY manifold induced by any loops γ in $\mathcal{M}_L(\mathbb{M})$ around the image of the point (s_0, t_0) in \mathfrak{D}_∞ will be the same. Since we assumed that the monodromy operators are infinite we get a contradiction with Theorem 45. ■

Remark 47 *It is a well known fact that the discriminant locus \mathfrak{D}_∞ in the Hilbert scheme of hypersurfaces in $\mathbb{C}\mathbb{P}^n$ for $n \geq 2$ is an irreducible divisor.*

The motivation of the next Theorem is based on a question raised by S.-T. Yau and this question is closely related to the example constructed in **Appendix I**. We will use the following definitions in the formulation of Theorem 48: We will say that \mathbb{V} is a K3 like, abelian or elliptic local system if it is a VHS of weight two induced by an Euclidean lattice with signature $(2, k)$ where $0 < k \leq 19$, or a VHS of an abelian variety or elliptic curve respectively.

Theorem 48 *Suppose that*

$$\pi : \mathcal{X} \rightarrow S^0 \times T^0 = \mathcal{Y}_0 \quad (40)$$

is a family of Calabi-Yau manifolds, such that the induced map into the moduli space is generically finite where S^0 and T^0 are Zariski open sets in projective varieties S and T . Suppose that $\mathbb{V} \rightarrow S^0 \times T^0 = \mathcal{Y}_0$ is the VHS induced by the family (37). Fixing $t \in T^0$, then the local system $\mathbb{V} \otimes \mathbb{C}|_{S_t}$ has a splitting over \mathbb{C} and the splittings of the local systems are either

$$\mathbb{V} \otimes \mathbb{C}|_{S_t} = \mathbb{G}_0 \oplus \mathbb{G}_1 \oplus \mathbb{G}_2 \quad (41)$$

where \mathbb{G}_0 and \mathbb{G}_2 are K3 liked local systems which are isomorphic to each other and \mathbb{G}_1 is unitary local system or

$$\mathbb{V} \otimes \mathbb{C}|_{S_t} = \mathbb{G}_0 \oplus \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \mathbb{G}_3 \oplus \mathbb{G}_4 \quad (42)$$

where \mathbb{G}_0 and \mathbb{G}_4 are isomorphic elliptic-like local systems, \mathbb{G}_2 is an abelian-like local system and \mathbb{G}_1 and \mathbb{G}_3 are unitary local systems that are isomorphic to each other.

Proof: Consider the endomorphism $\sigma : \mathbb{V} \otimes \mathbb{C}|_{S_t^0} \rightarrow \mathbb{V} \otimes \mathbb{C}|_{S_t^0}$, which is the image of

$$1_{S_t} : \mathcal{O}_{S_t} \rightarrow p_T^* \Theta_T(-\log \mathcal{D}_T)|_{S_t} \rightarrow \text{End}(\mathbb{V}|_{S_t}).$$

Since we assumed that the family (40) is non-rigid, we have that $\sigma^3 = 0$. Otherwise if $\sigma^3 \neq 0$ then by Theorem 38 the family (40) will be rigid. Clearly σ is of $(-1,1)$ type. We know from Lemma 26 that any non rigid deformation corresponds to a parallel section of the pullback of the tangent bundle of the moduli space. Thus this parallel section can be identified with some Kodaira-Spencer class θ . Applying Cor. 6.5 in [43] to non-rigid families of CY 3-folds, the three times iterated Kodaira-Spencer map

$$(i_{T_s}^* \theta)^3 : \mathbb{V}|_{T_s} \rightarrow \mathbb{V}|_{T_s} \otimes S^3 \Omega_{T_s}^1(\log D_{T_s}), \quad \forall s \in S^0$$

must be zero. Fixing $t \in T$, and varying s , the restriction $(i_{T_s}^* \theta)|_{t,s}$ is precisely σ . Next we will consider two different cases:

Case 1. Suppose that $\sigma^2 = 0$. We consider the weight filtration studied in [36] defined by $\sigma : \mathbb{W}_0 \subset \mathbb{W}_1 \subset \mathbb{W}_2 = \mathbb{V}|_{S_t}$. We set $\mathbb{W}_0 = \sigma(\mathbb{W}_2)$, $\mathbb{W}_1 = \ker(\sigma : \mathbb{W}_2 \rightarrow \mathbb{W}_0)$. One has $\sigma : \mathbb{W}_2/\mathbb{W}_1 \simeq \mathbb{W}_0$. Note that the Hodge bundle corresponding to \mathbb{W}_0 has the form

$$\mathbb{W}_0 = \sigma(E^{3,0}) \oplus \sigma(E^{2,1}) \oplus \sigma(E^{1,2}),$$

setting $\mathbb{G}_0 = \mathbb{W}_0$, $\mathbb{G}_2 = \mathbb{W}_2/\mathbb{W}_1$ we obtain $\sigma : \mathbb{G}_2 \simeq \mathbb{W}_0 = \mathbb{G}_0$. Since from the definition of the mixed Hodge structure it follows that the pure Hodge structure on $\mathbb{W}_2/\mathbb{W}_1$ is of weight two then the VHS on \mathbb{G}_0 and \mathbb{G}_2 are K3 liked local system since the fibre of the family (40) is a CY manifold and $\text{sodim}_{\mathbb{C}} \sigma(E^{3,0}) = 1$. Note that the Hodge bundle corresponding to \mathbb{W}_1 has the form

$$\ker(\sigma : E^{2,1} \rightarrow E^{1,2}) \oplus \ker(\sigma : E^{1,2} \rightarrow E^{0,3}) \oplus E^{0,3}.$$

The Hodge bundle corresponding to $\mathbb{G}_1 = \mathbb{W}_1/\mathbb{W}_0$ has the form

$$\ker(\sigma : E^{2,1} \rightarrow E^{1,2})/\sigma(E^{3,0}) \oplus \ker(\sigma : E^{1,2} \rightarrow E^{0,3})/\sigma(E^{2,1}).$$

Since $\mathbb{V}|_{S_t}$ is complete reducible by Deligne, we obtain $\mathbb{V}|_{S_t} \simeq \bigoplus_{i=0}^2 \mathbb{G}_i$.

Case 2) Suppose now $\sigma^2 \neq 0$, $\sigma^3 = 0$. Consider again the weight filtration

$$0 \subset \mathbb{W}_0 \subset \mathbb{W}_1 \subset \mathbb{W}_2 \subset \mathbb{W}_3 \subset \mathbb{W}_4 = \mathbb{V}|_{S_t}.$$

Set $\mathbb{W}_0 := \sigma^2(\mathbb{W}_4)$, $\mathbb{W}_3 := \ker(\sigma^2 : \mathbb{W}_4 \rightarrow \mathbb{W}_0)$ Note that σ is of $(-1,1)$ -type, so the only possible non-zero parts of σ^2 are

$$\sigma^2(E^{2,1}) \subseteq E^{0,3}, \quad \sigma^2(E^{3,0}) \subset E^{1,2}$$

they are dual to each other. Thus, if $\sigma^2 \neq 0$, then

$$\sigma^2(E^{2,1}) = E^{0,3} \text{ and } \sigma^2(E^{3,0}) \subset E^{1,2}.$$

Therefore, $\sigma^2 : \mathbb{W}_4/\mathbb{W}_3 \simeq \mathbb{W}_0$. And the Hodge bundle corresponding to \mathbb{W}_0 has the form $\theta : F_0^{1,0} \rightarrow F_0^{0,1}$, where $F_0^{1,0} = \sigma^2(E^{3,0})$, $F_0^{0,1} = E^{0,3}$ and θ is the original Kodaira-Spencer class restricted to $\bigoplus_{p+q=1} F_0^{p,q} \subset \bigoplus_{p+q=3} E^{p,q}$.

Thus the VHS on \mathbb{W}_0 is like the VHS on an elliptic curve. We set now $\mathbb{W}_3 = \ker(\sigma^2 : \mathbb{W}_4 \rightarrow \mathbb{W}_0)$ whose Hodge bundle has the form

$$\mathbb{W}_3 = \ker(\sigma^2 : E^{2,1} \rightarrow E^{0,3}) \oplus E^{1,2} \oplus E^{0,3}.$$

Let $\mathbb{W}_1 = \sigma(\mathbb{W}_3)$. Then \mathbb{W}_1 as a Hodge bundle has the form

$$\mathbb{W}_1 = \sigma(\ker(\sigma^2 : E^{2,1} \rightarrow E^{0,3})) \oplus \sigma(E^{1,2}) = \sigma(\ker(\sigma^2 : E^{2,1} \rightarrow E^{0,3})) \oplus E^{0,3}.$$

Let $\mathbb{W}_2 = \{v \in \mathbb{W}_1 \mid \sigma(v) \in \mathbb{W}_0\}$. Clearly we have $\sigma : \mathbb{W}_3/\mathbb{W}_2 \simeq \mathbb{W}_1/\mathbb{W}_0$. Remember that the Hodge bundle corresponding to \mathbb{W}_0 has weight one and thus the form $\sigma^2(E^{3,0}) \oplus E^{0,3}$. The Hodge bundle corresponding to $\mathbb{W}_1/\mathbb{W}_0$ has the form

$$F^{0,0} = \sigma(\ker(\sigma^2 : E^{2,1} \rightarrow E^{0,3})/\sigma^2(E^{3,0})).$$

So, we see that $\sigma : \mathbb{W}_3/\mathbb{W}_2 \simeq \mathbb{W}_1/\mathbb{W}_0$ are unitary local systems. Finally we want to determine the Hodge bundle corresponding to \mathbb{W}_2 . It is just the preimage of the Hodge bundle corresponding to \mathbb{W}_0 , namely

$$\sigma^{-1}(\sigma^2(E^{3,0})) \cap E^{2,1} \oplus E^{1,2} \oplus E^{0,3}.$$

So, the Hodge bundle corresponding to $\mathbb{W}_2/\mathbb{W}_1$ has the form

$$\sigma^{-1}(\sigma^2(E^{3,0})) \cap E^{2,1} \oplus E^{1,2} / \sigma(\ker(\sigma^2 : E^{2,1} \rightarrow E^{0,3})).$$

It is easy to see that it is again abelian variety liked local system. Theorem 48 is proved. ■

7 Appendix I. Complex Exponential Maps in Kähler Geometry

It is stated in [16] that the following conditions are equivalent: **1.** $g_{i,\bar{j}}$ is a Kähler metric on a complex manifold M . **2.** The $(1,1)$ form $\omega := \text{Im}(g)$ is

closed. **3.** The complex structure operator J_M on M is a parallel tensor with respect to the Levi-Cevita connection, i.e. $\nabla J_M = 0$. **4.** Around any point $m_0 \in M$ there exist holomorphic coordinates $\{z^1, \dots, z^n\}$ in an open set $\mathcal{U} \subset M$ such that locally the metric g is given by:

$$g_{i,\bar{j}} = \delta_{ij} + \frac{1}{4} \sum_{k,l} R_{i\bar{j},k\bar{l}} z^k \bar{z}^l + \dots$$

where $R_{i\bar{j},k\bar{l}}$ is the curvature tensor.

The coordinates $\{z^1, \dots, z^n\}$ will be called flat coordinates with respect to the Kähler metric g . Let us fix a point m_0 in the complex Kähler manifold M . Let T_{m_0} be the tangent space at the point $m_0 \in M$. We will consider for the moment the tangent space T_{m_0} as a real $2n$ dimensional vector space. Let $e_1 \in T_{m_0}$ be a vector of length 1. We will define $e_{n+1} := J_M e_1$. Let e_2 be a vector perpendicular to the vectors e_1 and e_{n+1} and $\|e_2\| = 1$. We will define $e_{n+2} := J_M e_2$. By continuing this process we obtain a basis in T_{m_0} consisting of vectors $\{e_1, \dots, e_n, J_M e_1 = e_{n+1}, \dots, J_M e_n = e_{2n}\}$ such that they satisfy $\langle e_i, e_j \rangle = \delta_{ij}$. Let $\gamma_i(t)$ be geodesics for $|t| < \varepsilon$ with respect to the metric g on M such that $\gamma_i(0) = m_0 \in M$ and $\frac{d\gamma_i(t)}{dt} |_{t=0} = e_i$ for $i = 1, \dots, n$.

Definition 49 We will define two dimensional distributions \mathcal{D}_i in the tangent bundle $T(M)$ in a small neighborhood \mathcal{U} of the point $m_0 \in M$. \mathcal{D}_i by parallel transport along the geodesics containing the point m_0 of the two dimensional subspaces $\mathcal{E}_i(0) \subset T_{m_0, M}$ span by e_i and $J_M e_i$ for $i = 1, \dots, n$.

The following theorem is probably well known. We will outline its proof only.

Theorem 50 Locally around the point $m_0 \in M$ there exist one dimensional complex manifolds Z_i for each $1 \leq i \leq n$ such that **1.** For each point $x \in \gamma_i(t) \subset Z_i$ the tangent space T_{x, Z_i} of Z_i is equal to $\mathcal{D}_i(t)$. **2.** Z_i is a totally geodesic two dimensional real submanifold.

Proof: We will construct Z_i for $i = 1, \dots, n$ based on the following Proposition:

Proposition 51 The distributions \mathcal{D}_i are integrable distributions.

Proof: The proof of Proposition 51 follows almost directly from the definition of the integrability. ■

Proposition 52 There exists a unique totally geodesic two dimensional real submanifold Z_i such that at each point $m \in Z_i$ we have: $T_{m, Z_i} = \mathcal{D}_i |_m$.

Proof: The proof is obvious. ■

Proposition 52 implies Theorem 50. ■

Remark 53 *Theorem 50 implies that to the pairs $(e_i, J_M e_i)$ of vectors in T_{m_0, M_0} we can associate a system of local holomorphic coordinates (z^1, \dots, z^n) in a small neighborhood of $m_0 \in M_0$. The complex curves Z_i are defined by the system of equations: $z^1 = \dots = z^{i-1} = z^{i+1} = \dots = z^n = 0$.*

Theorem 54 *Let g be a Kähler metric on M . Let (z^1, \dots, z^n) be the coordinate system as defined in Remark 53 in a small neighborhood of $m_0 \in M$. Then we have the following expression for $g_{i,\bar{j}}$:*

$$g_{i,\bar{j}}(z, \bar{z}) = \delta_{i,\bar{j}} + \frac{1}{6} R_{i,\bar{j},k,\bar{l}} z^k \bar{z}^l + \dots$$

where $R_{i,\bar{j},k,\bar{l}}$ is the curvature tensor.

Proof: Let us consider the geodesic coordinates $(x^1, y^1; \dots, x^n, y^n)$ associated to the orthonormal frame $(e_1, J_M e_1; \dots, e_n, J_M e_n)$. In the geodesic coordinate system associated to $(e_1, J_M e_1; \dots, e_n, J_M e_n)$, we have:

$$\frac{\partial}{\partial x^m} (g_{i,j})|_{m_0} = \frac{\partial}{\partial y^m} (g_{i,j})|_{m_0} = 0$$

for $m = 1, \dots, n$. This follows from the following Proposition:

Proposition 55 *At the origin of a geodesic coordinate system, the metric has the following Taylor expansion:*

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{ip,qj} x^p x^q + O(|x|^3),$$

where $R_{i,p,q,j}$ is the curvature tensor of g . (See Proposition 1.14 on page 8 of the book [32].)

On the other hand, Theorem 50 implies that we have the following relations between the coordinate systems $(x^1, y^2; \dots, x^n, y^n)$ and (z^1, \dots, z^n) $z^i = x^i + \sqrt{-1}y^i$. So we deduce that $\frac{\partial}{\partial z^k} (g_{i,\bar{j}})|_{m_0} = \frac{\bar{\partial}}{\partial z^i} (g_{i,\bar{j}})|_{m_0} = 0$. From here and Proposition 55 we can conclude that

$$g_{i,\bar{j}}(z, \bar{z}) = \delta_{i,\bar{j}} + \frac{1}{6} R_{i,\bar{j},k,\bar{l}} z^k \bar{z}^l + \dots$$

where $R_{i,\bar{j},k,\bar{l}}$ is the curvature tensor. This proves Theorem 54. ■

Corollary 56 *Let M be a Kähler Manifold with a Kähler metric $g_{i,\bar{j}}(z, \bar{z})$ such that $g_{i,\bar{j}}(z, \bar{z})$ are real analytic functions with respect to a local holomorphic coordinates, then we can identify the totally geodesic one complex dimensional manifolds $\gamma(\tau)$ in M at a point $m \in M$ with respect to the Kähler metric $g_{i,\bar{j}}(z, \bar{z})$ with the complex lines $l \subset T_{m,M} = \mathbb{C}^N$ through the origin $0 \in \mathbb{C}^N$ with directions defined by $\frac{d}{d\tau}\gamma(\tau)|_0$.*

Definition 57 *The map defined in Cor. 56 we will be called the complex exponential map in Kähler Geometry.*

8 Appendix II. Applications of Bishop's Convergence Theorem

We will prove the following Theorem following the arguments in [34]:

Theorem 58 *Let $\pi : \mathcal{N} \rightarrow \mathcal{K}$ and $\pi' : \mathcal{N}' \rightarrow \mathcal{K}$ be two holomorphic families of Kähler manifolds over a complex manifold \mathcal{K} as a parameter space. Suppose that both families can be identified with the C^∞ trivial family $\mathcal{K} \times N_0$ by some diffeomorphism ϕ isotopic to the identity. Let $0 \in \mathcal{K}$ be a fixed point. Let $\tau_n \in \mathcal{K}$ be a sequence of points in \mathcal{K} such that*

$$\lim_{k \rightarrow \infty} \tau_k = \tau_0.$$

Suppose that for all k there exist biholomorphic maps $\phi_k : N_{\tau_k} \rightarrow N'_{\tau_k}$ such that ϕ_k induces the identity map on $H^2(N_0, \mathbb{Z})$. Then N_{τ_0} and N'_{τ_0} are bimeromorphic

Proof: For each τ_k we have a biholomorphic map $\phi_k : N'_{\tau_k} \rightarrow N_{\tau_k}$ then the graphs of ϕ_k are submanifolds $\Gamma_k \subset N'_{\tau_k} \times N_{\tau_k}$. The idea of the proof is to show that

$$\lim_{k \rightarrow \infty} \Gamma_k = \Gamma_0 \subset N'_{\tau_0} \times N_{\tau_0} \tag{43}$$

exists and from this fact to deduce Theorem 58.

Lemma 59 *There exists a subsequence $\{k_n\}$ of the sequence $\{k\}$ such that the limit of the currents $[\Gamma_{k_n}]$ $\lim_{n \rightarrow \infty} [\Gamma_{k_n}] = [\Gamma_0]$ exists, where Γ_0 is a complex analytic subspace in $N_{\tau_0} \times N_{\tau_0}$.*

Proof: The proof of Lemma 59 is based on the following Theorem of Bishop. See [8]:

Theorem 60 *Let M be a complex analytic manifold and let $[N_n]$ be a sequence of complex analytic subspaces considered as currents in M . Suppose that h is a Hermitian metric on M such that $\text{vol}(N_n) \leq C$ then there exists a subsequence of current $[N_{n_k}]$ such that the limits of the currents $\lim_{k \rightarrow \infty} [N_{n_k}] = [N_0]$ exists and as a current $[N_0]$ is defined by a complex analytic subspace N_0 in M .*

In order to apply Theorem 60 we need to construct a Hermitian metric h on the complex manifold $\mathcal{M} = \mathcal{N}' \times_{\mathcal{K}} \mathcal{N}$ and prove that the volume of the graphs Γ_k of the isomorphisms ϕ_k are uniformly bounded

Construction of an Hermitian metric on \mathcal{M} . We assumed that $\mathcal{N}' \rightarrow \mathcal{K}'$ and $\mathcal{N} \rightarrow \mathcal{K}$ are families of Kähler manifolds. So we may we have two families of Kähler metrics $\omega^{1,1}(\tau)$ and $\omega_1^{1,1}(\tau)$ respectively defined on the fibres of \mathcal{N}' and \mathcal{N} and the metrics depend on $\tau \in \mathcal{K}$ in a C^∞ manner. Let

$$\eta = \sqrt{-1} \sum_{i=1}^N d\tau^i \wedge \overline{d\tau^i}$$

be a positive definite form on \mathcal{U} . The collection of forms $\omega^{1,1}(\tau)$, $\omega_1^{1,1}(\tau)$ and η define a Hermitian metric H on $\mathcal{M} = \mathcal{N}' \times_{\mathcal{K}} \mathcal{N}$.

Proposition 61 *The submanifolds Γ_k defined by (43) have a constant volume with respect to the family of metrics $\omega^{1,1}(\tau) + \omega_1^{1,1}(\tau)$ on $N_\tau \times N_{1,\tau}$, i.e. $\text{vol}(\Gamma_k) = c$ for all k .*

Proof: It is easy to see that

$$\text{vol}(\Gamma_k) = \int_{N'_k} (\phi_k^*(\omega_1^{1,1}(\tau_k)) + \omega^{1,1}(\tau_k))^{2n}.$$

Remember that ϕ_k^* is the identity map on $H^2(N, \mathbb{Z})$. Here we are using the fact that as C^∞ manifolds $\mathcal{N} = \mathcal{K} \times N_0$ and $\mathcal{N}' = \mathcal{K}' \times N_0$ are diffeomorphic and the diffeomorphism which acts fibrewise is isotopic to the identity on N_0 . Using this fact we see that the classes of cohomology of $\phi_k^*(\omega_1^{1,1}(\tau_k))$ and $\omega^{1,1}(\tau_k)$ are fixed. Let us denote them by $[\omega_1]$ and $[\omega]$ respectively. So we get that

$$\int_{N'_k} (\phi_k^*(\omega_1^{1,1}(\tau_k)) + \omega^{1,1}(\tau_k))^{2n} = \int_{N'_k} ([\omega_1] + [\omega])^{2n} = c_0.$$

Proposition 61 is proved. ■

The end of the proof of Lemma 59: For a subvariety Z of pure codimension m in a complex manifold N_{τ_0} , we denote by $[Z]$ the current on N_{τ_0} defined by Z . Now we can apply Bishop's Theorem 60 and conclude that the sequence of currents converges weakly to a current $[\Gamma_0]$ in $N_{\tau_0} \times N'_{\tau_0}$ of the form

$$[\Gamma_0] = \sum_i m_i [\Gamma_i],$$

where m_i are positive integers and Γ_i are irreducible complex analytic subspaces in $N_{\tau_0} \times N'_{\tau_0}$. Lemma 59 is proved ■.

The end of the proof of Theorem 58: Any closed current $[Z]$ on $N_{\tau_0} \times N'_{\tau_0}$ defines a linear map:

$$[Z]_* : H^*(N'_{\tau_0}, \mathbb{C}) \rightarrow H^*(N_{\tau_0}, \mathbb{C})$$

between the cohomology rings of N'_{τ_0} and N_{τ_0} as follows: Let $\alpha \in H^*(N'_{\tau_0}, \mathbb{C})$, then

$$[Z]_*(\alpha) := (pr_2)_* ([Z] \wedge (pr_1)^* \alpha)$$

where pr_i are respectively the projections of $N_{\tau_0} \times N'_0$ onto the first and the second factor and $(pr_1)_*$ and $(pr_2)^*$ are the pushforward and the pullback maps. Similarly we define a linear map

$$[Z]^* : H^*(N_{\tau_0}, \mathbb{C}) \rightarrow H^*(N'_{\tau_0}, \mathbb{C}).$$

The map $[\Gamma_0]_*$ defined by the current $[\Gamma_0]$ clearly agrees with the map

$$\lim_{k \rightarrow \infty} [\Gamma_k] = \lim_{k \rightarrow \infty} id = id = [\Gamma_0]_*$$

on $H^{2n}(N, \mathbb{C})$. From here we deduce that $[\Gamma_0]_*$ is the identity map on $H^{2n}(N, \mathbb{C})$. This implies

$$[\Gamma_0]_*(\wedge^n [\omega_1^{1,1}]) = (pr_2)_* \left(\sum_i m_i [\Gamma_i] \wedge (pr_1)^*(\wedge^n [\omega_1^{1,1}]) \right) = \wedge^n [\omega_1^{1,1}].$$

Hence there must be some Γ_j among the components of which is projected both onto N'_0 and N_{τ_0} . There can be only one Γ_j in Γ_0 with multiplicity $m_j = 1$, because both

$$\left(\sum_i m_i [\Gamma_i] \right)_* \text{ and } \left(\sum_i m_i [\Gamma_i] \right)^*$$

must leave fixed the class in $H^0(N, \mathbb{C})$ defined by the function 1. This implies that the projection maps from Γ_j to N_{τ_0} and to N'_{τ_0} have degree one. So N_{τ_0} and N'_{τ_0} are bimeromorphically equivalent. **Theorem 58** is proved. ■

Theorem 62 *Suppose that the map $f : N_{\tau_0} \rightarrow N'_{\tau_0}$ is bimeromorphic. Let \mathcal{L}_1 be an ample line bundle on N'_{τ_0} such that $f^*(\mathcal{L}_1)$ is also an ample line bundle on N_{τ_0} . Then f is a biholomorphic map.*

Proof: Since $f : N_{\tau_0} \rightarrow N'_{\tau_0}$ is a bimeromorphic map, there exist complex analytic subspaces $Z_1 \subset N_{\tau_0}$ and $Z_2 \subset N'_{\tau_0}$ of $\text{codim} \geq 2$ such that $f : N_{\tau_0} \ominus Z_1 \rightarrow N'_{\tau_0} \ominus Z_2$ is a biholomorphic map between those two Zariski open sets. From here it follows that $f^*(\mathcal{L}_1)$ is a well defined line bundle on $U := N_{\tau_0} \ominus Z_1$. Since \mathcal{L}_1 is an ample bundle on N'_{τ_0} , $(\mathcal{L}_1)^{\otimes m}$ will be a very ample bundle on N'_{τ_0} for some positive integer m . Let $\sigma_0, \dots, \sigma_k \in H^0(N'_{\tau_0}, (\mathcal{L}_1)^{\otimes m})$ be a basis in $H^0(N'_{\tau_0}, (\mathcal{L}_1)^{\otimes m})$. Since $(\mathcal{L}_1)^{\otimes m}$ is a very ample line bundle then $\sigma_0, \dots, \sigma_k$ define an embedding of N'_{τ_0} into \mathbb{P}^k . Clearly $f^*(\sigma_0), \dots, f^*(\sigma_k)$ will define an embedding of the Zariski open set $U_0 := N_{\tau_0} \ominus Z_1$ in N_{τ_0} into \mathbb{P}^k . Since Z_1 has a complex codimension ≥ 2 Hartogs Theorem implies that the sections $f^*(\sigma_0), \dots, f^*(\sigma_k)$ of the line bundle $f^*(\mathcal{L}_1)^{\otimes m}$ are well defined holomorphic sections on N_{τ_0} . We assumed that $f^*(\mathcal{L}_1)$ is an ample line bundle on N_{τ_0} so we can choose m to be large enough to conclude that $f^*(\mathcal{L}_1)^{\otimes m}$ is a very ample line bundle on N_{τ_0} . From here we conclude that

$$x \in N_{\tau_0} \rightarrow (f^*(\sigma_0)(x), \dots, f^*(\sigma_k)(x))$$

defines a holomorphic embedding $\varphi : N_{\tau_0} \subset \mathbb{P}^k$. So we deduce that $f : N_{\tau_0} \rightarrow N'_{\tau_0}$ is an isomorphism. Theorem 62 is proved. ■

Remark 63 *The arguments that we used to prove Theorem 58 are similar to the arguments used by Siu in [34]. These arguments were suggested by Deligne to D. Burns and Rapoport.*

9 Appendix III. Counter Examples to the Analogue of Shafarevich Conjecture of CY Manifolds

Let us consider K3 surfaces on which there exists an automorphism σ of order two such that σ has no fixed points. Such algebraic K3 surfaces exist and the quotient $X/\sigma = Y$ is an Enriques surface. It is a well known fact that σ acts on the holomorphic two form ω_X on the K3 surface X as follows: $\sigma^*(\omega_X) = -\omega_X$. Let us consider an elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau := \{m + n\tau | m, n \in \mathbb{Z}, \tau \in \mathbb{C} \text{ and } \text{Im } \tau > 0\}$. We know that E_τ can be embedded in $\mathbb{C}\mathbb{P}^2$ and in one of the standard affine open sets E_τ

is given by the equation: $y^2 = 4x^3 + g_2x + g_3$, where $g_2(\tau) = 60E_4(\tau)$ and $g_3(\tau) = 140E_6(\tau)$. $E_4(\tau)$ and $E_6(\tau)$ are the Eisenstein series. Let σ_1 be the involution defined on the equation of the elliptic curve as follows: $\sigma_1(x, y) = (-x, y)$. Clearly σ_1 acts on E_τ and $E_\tau/\sigma_1 = \mathbb{CP}^1$. Thus we obtain that σ_1 acts on the holomorphic one form $dz_\tau = \frac{dx}{y}$ on E_τ as follows: $\sigma_1^*(dz_\tau) = -dz_\tau$. From here it is very easy to see that the quotient of the product $X \times E_\tau/\sigma \times \sigma_1 = M$ will be a CY manifold. These CY manifolds are called Borcea-Voisin manifolds. From the theory of moduli of Enriques surfaces and of elliptic curves it follows that the moduli space $\mathcal{M}(M)$ of M is isomorphic to $\Gamma \backslash \mathfrak{h}_{2,10} \times \mathbb{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$. For more detail about moduli of Enriques surfaces see [6]. Let $\mathcal{X} \rightarrow C$ be a non isotrivial family of K3 surfaces with an involution σ without fixed points acting on the non singular fibres of the above family. In [18] it was proved that the set \mathcal{D}_C of points in C over which the fibres are singular is not empty. We will call the set \mathcal{D}_C the discriminant locus.

Let E_λ be the family of elliptic curves: $y^2 = x(x-1)(x-\lambda)$. Clearly we have defined a family of elliptic curves over the projective line. Let us denote it by $\tilde{E} \rightarrow \mathbb{CP}^1$. Let us take the product

$$\mathcal{X} \times \tilde{E} \rightarrow C \times \mathbb{CP}^1. \quad (44)$$

On the family (44) we can define the action of σ and σ_1 . By taking the quotient we will get a family

$$\mathcal{Y} \rightarrow C \times \mathbb{CP}^1. \quad (45)$$

of three dimensional CY manifolds over the product space $C \times \mathbb{CP}^1$. Clearly that for fixed $\lambda \in \mathbb{CP}^1$ and $\lambda \neq 0, 1 \ \& \ \infty$ we can identify all curves $C_\lambda = C \times \lambda$. Moreover under this identification the discriminant locuses \mathcal{D}_{C_λ} are identified too. This means that the discriminant locus of the family (45) satisfies: $\mathcal{D}_{\mathbb{CP}^1 \times C} = p_1^* \mathcal{D}_{\mathbb{CP}^1} + p_2^* \mathcal{D}_C$. So the family (45) is not rigid and gives a counter example to Shafarevich's conjecture.

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