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The Rallis-Schiffmann Lifting and Arthur Packets of G_2

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§1. Introduction

The purpose of this paper is to give a natural (unconditional) construction of a family of non-tempered Arthur packets of G_2, and to construct the submodule in the space of square-integrable automorphic forms associated to these Arthur packets. A surprising aspect of our definition is that a representation in one of these local Arthur packets can actually be reducible. To the best of our knowledge, this is the first instance of such a phenomenon for split p-adic groups.

Our construction is based on an earlier paper of Rallis and Schiffmann [RS] which we recall briefly. In the paper [RS], Rallis and Schiffmann constructed a lifting of cuspidal automorphic forms from the metaplectic group $\widetilde{SL}_2$ to the split exceptional group of type G_2 over a number field F. This was achieved by exploiting the fact that $SL_2 \times G_2$ is a subgroup of $SL_2 \times O_7$, which is the classical dual pair in Sp_{14}. The lifting is then defined using the theta kernel furnished by the Weil representation $\omega_\psi^{(7)}$ of $\widetilde{Sp}_{14}$ (which depends on the choice of an additive character ψ).

The surprising discovery of Rallis-Schiffmann is that, despite restricting from O_7 to the smaller group G_2, one still obtains a correspondence of representations. More precisely, if σ is an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$, let $V(\sigma)$ be the theta lift of σ; it is a non-zero subspace of the space of automorphic forms on G_2. Then the main results of Rallis-Schiffmann are:

- $V(\sigma)$ is contained in the space of cusp forms if and only if the theta lift (associated to ψ) of σ to SO_3 (studied by Waldspurger) is zero.
- The cuspidal representations obtained as lifts from $\widetilde{SL}_2$ are characterized as those having a non-zero period with respect to some quasi-split SU_3 (which is a subgroup of G_2).

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- The local correspondence of *unramified* representations is precisely determined. In particular, when $V(\sigma)$ is cuspidal, the local components of each irreducible constituent of $V(\sigma)$ are determined for *almost all* places v, in terms of the local components of σ.

- As a consequence of the unramified correspondence, the irreducible cuspidal representations contained in $V(\sigma)$ are non-generic and CAP with respect to the Heisenberg parabolic or the Borel subgroup of G_2. This gives the first construction of CAP representations of G_2.

In this paper, we complete the study initiated in [RS] by giving a precise determination of the representation $V(\sigma)$. The first step in this is the complete determination of the *local* theta correspondence. Since the archimedean correspondence has to a large extent been determined by Li-Schwermer [LS], we shall only discuss the non-archimedean case here. More precisely, if v is a p-adic place of F and σ_v an irreducible representation of $\widetilde{SL_2}(F_v)$, the maximal σ_v-isotypic quotient of $\omega_{\psi_v}^{(7)}$ can be expressed as $\sigma_v \otimes \theta(\sigma_v)$, where $\theta(\sigma_v)$ is a smooth representation of $G_2(F_v)$. Let $\Theta(\sigma_v)$ be the maximal semisimple quotient of $\theta(\sigma_v)$. Our main local result is:

(1.1) Theorem $\Theta(\sigma_v)$ *can be completely determined for any* σ_v *(to the extent that classification of representations of* $G_2(F_v)$ *is known). It turns out that* $\Theta(\sigma_v)$ *is irreducible except when* $\sigma_v = \omega_{\psi_v}^{\pm}$ *(the even and odd Weil representations of* $\tilde{S}L_2(F_v)$ *associated to* ψ_v*). In these two exceptional cases,* $\Theta(\sigma_v)$ *is the sum of 2 unipotent representations.*

A precise statement of the results is given in Theorem 2.16.

We now turn to the global situation. For any cuspidal σ on $\widetilde{SL}_2$, one can show that $V(\sigma)$ is contained in the space of square-integrable automorphic forms. Thus $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) := \otimes_v \Theta(\sigma_v)$. Our precise local result immediately shows that $V(\sigma) \cong \Theta(\sigma)$, whenever σ_v is not the even or odd Weil representations associated to ψ_v for any place v, since $\Theta(\sigma)$ is irreducible then. However, when $\Theta(\sigma)$ is reducible, there are more than one possibilities for $V(\sigma)$. The determination of $V(\sigma)$ in this case is easily the trickiest part of the paper. In any case, our main global result (Theorem 3.7) says:

(1.2) Theorem *If* $\sigma \subset \mathcal{A}_{00}$*, i.e.* σ *is not an irreducible summand of any Weil representation, then* $V(\sigma) \cong \Theta(\sigma)$*.*

In fact, one can define the theta lift of any square-integrable automorphic representation of $\widetilde{SL}_2$ by a regularization of the theta integral. Thus, one can speak of the regularized theta lift of the orthogonal complement of $\mathcal{A}_{00}$, which consists of the Weil representations of $\widetilde{SL}_2$. One can show that the space of automorphic forms of G_2 thus obtained is precisely equal to that constructed in [GGJ], by restriction of the minimal representation of the various quasi-split $Spin_8$'s.

Let us highlight a corollary of the global theorem above. It pertains to the question of whether there are cuspidal representations of G_2 with non-zero SL_3-period. Such cuspidal representations should be very scarce, but can be obtained by restriction from the minimal representation of split $Spin_8$ [GGJ]. It wasn't known previously if other SL_3-distinguished cuspidal representations exist. Now, as a consequence of our global theorem, we know that they do. Indeed, when $\sigma \subset \mathcal{A}_{00}$ is such that its theta lift to SO_3 is non-zero, we know from [RS] that $V(\sigma)$ is not totally contained in $\mathcal{A}_{cusp}(G_2)$. However, this does not exclude the possibility that $V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$ is non-zero. In [RS, Pg. 823], Rallis-Schiffmann remarked that they do not know if such a possibility can actually happen. Our global theorem implies that it does, and the cuspidal representations thus obtained are SL_3-distinguished.

This paper serves as an announcement of some of the results of the longer paper [GG]. In particular, though we provide a precise statement of the local theorem, we do not give its proof here. We do, however, give the proof of the global theorem, since it provides a justification for our definition of the local Arthur packets, especially in the case when the local packet contains a reducible representation. The details can be found in Section 4. In [GG], we provide further justification of the correctness of our local packets by showing that the spaces we constructed are full near equivalence classes; this last aspect will not be discussed here.

§2. Local Results

In this section, we shall state our precise local results. For this, we need to introduce a number of notations and recall a number of background facts. Throughout this section, F will denote a non-archimedean local field of characteristic zero, and we fix a non-trivial additive character ψ of F. For any $a \in F^\times$, we let ψ_a be the character defined by $\psi_a(x) = \psi(ax)$.

(2.1) The group $\tilde{SL}(2)$. The group $\widetilde{SL}_2(F)$ is a topological central extension of $SL_2(F)$ by $\{\pm 1\}$. As usual, we shall let T denote the diagonal torus of SL_2 and N the group of unipotent upper triangular matrices. Hence $B = TN$ is the usual Borel subgroup. For a subgroup H of $SL_2(F)$, let $\tilde{H}$ be its inverse image in $\widetilde{SL}_2(F)$.

One can define a character χ_ψ of $\tilde{T}$ by:

$$\chi_\psi(t(a), \epsilon) = \epsilon \cdot \gamma_\psi / \gamma_{\psi_a}$$

where $t(a) = diag(a, a^{-1})$, $\epsilon = \pm 1$ and γ_ψ is the 8th root of unity associated to ψ by Weil. Let us recall the classification of irreducible genuine representations of $\widetilde{SL}_2(F)$.

(2.2) The Weil representations of $\widetilde{SL}_2(F)$. Let χ be a quadratic character of $F^\times$ (possibly trivial). Then χ corresponds to an element $a_\chi \in F^\times / F^{\times 2}$. Associated to χ is a Weil representation ω_χ of $\widetilde{SL}_2(F)$. As a representation of $\widetilde{SL}_2(F)$, ω_χ is reducible. In fact, it is the direct sum of two irreducible representations:

$$\omega_\chi = \omega_\chi^+ \oplus \omega_\chi^-,$$

where ω_χ^- is supercuspidal and ω_χ^+ is not.

(2.3) The principal series. The principal series representations of $\widetilde{SL}_2(F)$ can be parametrized by the characters μ of $F^\times$ (cf. [W2, Pg. 225]). The representation associated to μ is the induced representation

$$\tilde{\pi}(\mu) = Ind_{\tilde{B}}^{\tilde{SL}_2} \chi_\psi \cdot \delta_{\tilde{B}}^{1/2} \cdot \mu$$

Note that this parametrization depends on the additive character ψ. Now we have:

(2.4) Proposition *(i) $\tilde{\pi}(\mu)$ is irreducible if and only if $\mu^2 \neq |-|^{\pm 1}$, in which case $\tilde{\pi}(\mu) \cong \tilde{\pi}(\mu^{-1})$.*

(ii) If $\mu = \chi \cdot |-|^{1/2}$ where χ is a quadratic character, then we have a short exact sequence:

$$0 \longrightarrow sp_\chi \longrightarrow \tilde{\pi}(\mu) \longrightarrow \omega_\chi^+ \longrightarrow 0.$$

We call sp_χ the special representation associated to χ.

(iii) If $\mu = \chi \cdot |-|^{-1/2}$, then we have a short exact sequence,

$$0 \longrightarrow \omega_\chi^+ \longrightarrow \tilde{\pi}(\mu) \longrightarrow sp_\chi \longrightarrow 0.$$

The proposition gives all the non-supercuspidal genuine representations of $\tilde{SL}_2(F)$. The other irreducible representations of $\tilde{SL}_2(F)$ are all supercuspidal, including the ω_χ^-'s introduced above.

(2.5) Whittaker functionals. For any $a \in F^\times$, let ψ_a be the character of N defined by:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \psi(ax).$$

It is known that

$$\dim(\sigma_{N,\psi_a}) \leq 1.$$

We say that σ has a ψ_a-Whittaker functional if $\sigma_{N,\psi_a} \neq 0$. It is also known that any genuine σ has a ψ_a-Whittaker functional for some a. We let

$$\widehat{F}(\sigma) = \{a \in F^\times : \ \sigma \text{ has a } \psi_a\text{-Whittaker functional}\}.$$

Clearly, $\widehat{F}(\sigma)$ is a non-empty union of square classes.

(2.6) The Weil representation of $\widetilde{SL}_2(F) \times SO(V,q)$. If (V,q) is a quadratic space, then as is well-known, one can define a Weil representation $\omega_{\psi,q}$ of the group $\widetilde{SL}_2(F) \times SO(V,q)$. For example, if χ is a quadratic character of $F^\times$ and (V,q) is the rank 1 quadratic space $\langle a_\chi \rangle$, then $\omega_{\psi,q}$ is simply the representation of $\widetilde{SL}_2(F)$ denoted by ω_χ in 2.2.

Let (V_m, q_m) be the $(2m+1)$-dimensional quadratic space $\langle 1 \rangle \oplus \mathbb{H}^m$, where $\mathbb{H}$ is the rank 2 hyperbolic space. We shall write $\omega_\psi^{(m)}$ for the Weil representation of $\widetilde{SL}_2(F) \times SO(V_m, q_m)$.

(2.7) Waldspurger's lift and packets for $\widetilde{SL}_2(F)$. By a detailed study of the representation $\omega_{\psi,q}$ as (V,q) ranges over all 3-dimensional quadratic spaces [W1,2], Waldspurger defined a map Wd_ψ from the set of irreducible representations of $\widetilde{SL}_2(F)$ which are not equal to ω_χ^+ to the set of infinite dimensional representations of $PGL_2(F)$. This leads to a partition of the set of such representations of $\widetilde{SL}_2(F)$ indexed by the infinite dimensional representations of $PGL_2(F)$. Namely, if τ is such a representation of $PGL_2(F)$, we set

$$\tilde{A}_\tau = \text{ inverse image of } \tau \text{ under } Wd_\psi.$$

It turns out that

$$\#\tilde{A}_\tau = \begin{cases} 2 \text{ if } \tau \text{ is discrete series;} \\ 1 \text{ if } \tau \text{ is not.} \end{cases}$$

In the first case, the set $\tilde{A}_\tau$ has a distinguished element σ_τ^+, which is characterized by the fact that $\tau \otimes \sigma_\tau^+$ is a quotient of $\omega_\psi^{(3)}$. The other element of $\tilde{A}_\tau$ will be denoted by σ_τ^-. In the second case, we shall let $\sigma(\tau)^+$ be the unique element in $\tilde{A}_\tau$ and set $\sigma(\tau)^- = 0$.

(2.8) The group G_2. Now we come to the split group G_2. It is the automorphism group of the octonion algebra $\mathbb{O}$. Like the quaternion algebra, the octonion algebra carries a quadratic norm form N and a linear trace form Tr and these are preserved by G_2. Let V be the space of trace zero elements in $\mathbb{O}$, equipped with the quadratic form $q = -N$. Then (V, q) is isomorphic to (V_7, q_7) and G_2 acts as automorphisms of (V, q). This gives us an embedding $G_2 \hookrightarrow SO(V_7, q_7)$.

The group G_2 has two conjugacy classes of maximal parabolic subgroups. One of them is the Heisenberg parabolic $P_2 = L_2 \cdot U_2$, with U_2 a 5-dimensional Heisenberg group. Denote the other maximal parabolic by $P_1 = L_1 \cdot U_1$. Its unipotent radical U_1 is a 3-step nilpotent group. In both cases, the Levi subgroups are isomorphic to GL_2 and we fix these isomorphisms.

Now let us recall some facts about representations of $G_2(F)$.

(2.9) Langlands quotients. Let τ be a tempered representation of $GL_2(F)$ and $s > 0$. In the following, we shall use standard notions for the representations of PGL_2. For example, St denotes the Steinberg representation, St_χ the twist of St by the quadratic character χ and $\pi(\mu_1, \mu_2)$ for a principal series representation. Now the induced representations

$$I_{P_i}(\tau, s) = Ind_{P_i}^{G_2} \delta_{P_i}^{1/2} \cdot \tau \cdot |det|^s$$

has a unique irreducible quotient $J_{P_i}(\tau, s)$. The reducibility points of these induced representations are known by [M, Thm. 3.1 and Thm. 5.3].

(2.10) Degenerate principal series. Consider now the induced representation

$$I_{P_1}(\mu) = Ind_{P_1}^{G_2} \delta_{P_1}^{1/2} (\mu \circ det),$$

where μ is a character of $F^\times$. The following was shown in [M, Thm. 3.1 and Props. 4.1, 4.3, 4.4]:

(2.11) Lemma *Assume that $|\mu| = |-|^s$ with $Re(s) \geq 0$. Then $I_{P_1}(\mu)$ is irreducible unless $\mu^2 = |-|$ or $\mu = |-|^{5/2}$. For these exceptional cases, we have the following non-split exact sequences:*

(i) If $\mu = \chi|-|^{1/2}$, with $\chi \neq 1$ a quadratic character, then we have:

$$0 \longrightarrow J_{P_2}(St_\chi, 1/2) \longrightarrow I_{P_1}(\mu) \longrightarrow J_{P_1}(\pi(1,\chi), 1) \longrightarrow 0$$

(ii) If $\mu = |-|^{1/2}$, then we have:

$$0 \longrightarrow J_{P_1}(St, 1/2) \longrightarrow I_{P_1}(\mu) \longrightarrow J_{P_1}(\pi(1,1),1) \oplus J_{P_2}(St, 1/2) \longrightarrow 0$$

(iii) If $\mu = |-|^{5/2}$,

$$0 \longrightarrow J_{P_2}(St, 3/2) \longrightarrow I_{P_1}(\mu) \longrightarrow 1 \longrightarrow 0$$

(2.12) **U_2-spectrum.** The $L_2(F)$-orbits of characters of $U_2(F)$ can be naturally parametrized by cubic algebras over F (cf. [GGJ]). For a cubic algebra E, let us write ψ_E for a character in the corresponding orbit. The U_2-spectrum of a smooth representation π of G_2 is the set of those cubic algebras E such that the corresponding twisted Jacquet module π_{U_2,ψ_E} is non-zero. In this paper, we shall only look at the cubic algebras of the form $F \times K$ where K is an étale quadratic algebra. We set

$$\widehat{F}(\pi) = \{a \in F^\times : \ K_a = F(\sqrt{a}) \text{ is in the } U_2\text{-spectrum of } \pi\}.$$

Clearly, $\widehat{F}(\pi)$ is a union of square classes.

(2.13) Some unipotent representations. We recall the results of [HMS] concerning the restriction of the (unique) unitarizable minimal representation Π_K of the quasi-split $Spin^K(8)$ to the subgroup G_2, where K is an étale quadratic algebra. The representation Π_K is trivial on the center of $Spin_8^K$ and can be extended to a representation of SO_8^K. Any such extension will be called a minimal representation of SO_8^K and each has the same restriction to G_2. Now we have:

(2.14) Proposition *(i) When $K = F \times F$,*

$$\Pi_K = J_{P_1}(\pi(1,1),1) \oplus 2 \cdot J_{P_2}(St, 1/2) \oplus \pi_\epsilon$$

where π_ϵ is supercuspidal.

(ii) When K is a quadratic field, with associated quadratic character χ,

$$\Pi_K = J_{P_1}(\pi(1,\chi),1) \oplus \pi(\chi)$$

where $\pi(\chi)$ is supercuspidal.

For a given K, the irreducible constituents of Π_K obtained in the above proposition make up a unipotent Arthur packet, as explained in [GGJ].

(2.15) The Weil representation for $\widetilde{SL}_2(F)\times G_2(F)$. Finally, we are in a position to describe our main local theorem. Since G_2 is a subgroup of $SO(V_7,q_7)$, we may restrict the representation $\omega_\psi^{(7)}$ to $\widetilde{SL}_2(F)\times G_2(F)$. Given a representation σ of $\widetilde{SL}_2(F)$, we have defined the smooth representation $\theta(\sigma)$ and $\Theta(\sigma)$ in the introduction. The main local theorem is:

(2.16) Theorem *The representation $\theta(\sigma)$ is non-zero and admissible. Moreover, we have:*

(a) (Principal series) If $\sigma=\tilde{\pi}(\mu)$ is an irreducible principal series (so that $\mu^2\neq|-|^{\pm1}$) with $\mu\neq|-|^{5/2}$, then

$$\theta(\sigma)\cong I_{P_1}(\mu^{-1}).$$

In particular, $\theta(\sigma)=\Theta(\sigma)$ is irreducible, unless $\sigma=\tilde{\pi}(|-|^{\pm5/2})$, in which case $\Theta(\sigma)$ is the trivial representation of G_2.

(b) (Special representations) If $\sigma=sp_\chi$, then

$$\Theta(\sigma)\cong\begin{cases}J_{P_2}(St_\chi,1/2) & \text{if } \chi\neq1;\\ J_{P_1}(St,1/2) & \text{if } \chi=1.\end{cases}$$

(c) (Weil representations) If $\sigma=\omega_\chi^+$ where χ is a quadratic character of $F^\times$, then

$$\theta(\sigma)=\begin{cases}J_{P_1}(\pi(1,\chi),1) & \text{if } \chi\neq1;\\ J_{P_1}(\pi(1,1),1)\oplus J_{P_2}(St,1/2) & \text{if } \chi=1.\end{cases}$$

If $\sigma=\omega_\chi^-$, then

$$\theta(\sigma)=\begin{cases}\pi(\chi) & \text{if } \chi\neq1;\\ J_{P_2}(St,1/2)\oplus\pi_\epsilon & \text{if } \chi=1.\end{cases}$$

Here $\pi(\chi)$ and π_ϵ were defined in Prop. 2.14.

(d) (Supercuspidals) Suppose that σ is supercuspidal and $\sigma\neq\omega_\chi^-$ for any χ. Let $\tau=Wd_\psi(\sigma)$. If $\sigma=\sigma_\tau^+$, then

$$\theta(\sigma)=J_{P_2}(\tau,1/2).$$

If $\sigma=\sigma_\tau^-$, then $\theta(\sigma)$ is an irreducible non-generic supercuspidal representation such that

$$\widehat{F}(\theta(\sigma))=\widehat{F}(\sigma).$$

Moreover, if $\theta(\sigma) \cong \theta(\sigma')$, then $\sigma \cong \sigma'$.

The main ingredients in the proof of the theorem are the computation of the Jacquet and Fourier-Jacobi functors of $\omega_\psi^{(7)}$ with respect to various unipotent subgroups of $\widetilde{SL}_2$ and G_2, as well as the the study of the U_2-spectrum of $\theta(\sigma)$ in terms of the N-spectrum of σ.

(2.17) Remark Even though we have restricted ourselves to the non-archimedean case in this section, the archimedean correspondence can also be completely determined. To a large extent, this was already done by Li-Schwermer [LS].

§3. Global Results

In this section, we let F be a number field with adele ring $\mathbb{A}$ and fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$. We shall describe our main global results below.

(3.1) Cusp forms of $\widetilde{SL}_2(\mathbb{A})$. Let $\mathcal{A}^2$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\mathcal{A}^2 = \mathcal{A}_{00} \bigoplus \left(\bigoplus_\chi \mathcal{A}_\chi \right).$$

Here, χ runs over all quadratic characters of $F^\times\backslash\mathbb{A}^\times$.

Let us describe the space $\mathcal{A}_\chi$ more concretely. If $\omega_\chi = \otimes_v \omega_{\chi_v}$ is the global Weil representation attached to χ, then the formation of theta series gives a map $\theta_\chi : \omega_\chi \to \mathcal{A}^2$, whose image is the space $\mathcal{A}_\chi$. To describe the decomposition of $\mathcal{A}_\chi$, for a finite set S of places of F, let us set

$$\omega_{\chi,S} = (\otimes_{v\in S}\omega_{\chi_v}^-) \otimes (\otimes_{v\notin S}\omega_{\chi_v}^+)$$

so that

$$\omega_\chi = \bigoplus_S \omega_{\chi,S}.$$

Then we have

$$\mathcal{A}_\chi \cong \bigoplus_{\#S \text{ even}} \omega_{\chi,S}.$$

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if S is non-empty.

(3.2) Near equivalence classes. In a profound piece of work [W2], Waldspurger has described the near equivalence clases of representations in $\mathcal{A}_{00}$. Earlier, in [W1], he has shown that $\mathcal{A}_{00}$ satisfies multiplicity one. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of PGL_2, we define a set of irreducible unitary representations of $\tilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place v, we have a local packet

$$\tilde{A}_{\tau_v} = \{\sigma^+_{\tau_v}, \sigma^-_{\tau_v}\}$$

where $\sigma^-_{\tau_v} = 0$ if τ_v is not discrete series. Now set

$$\tilde{A}_\tau = \{\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is called the global packet associated to τ.

For $\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} \in \tilde{A}_\tau$, let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have:

$$\mathcal{A}_{00} = \bigoplus_{\text{cuspidal } \tau} \mathcal{A}(\tau)$$

where each $\mathcal{A}(\tau)$ is a near equivalence class of cuspidal representations and

$$\mathcal{A}(\tau) = \bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \sigma.$$

(3.3) Fourier coefficients. For a character χ of $N(F)\backslash N(\mathbb{A})$, the χ-Fourier coefficient of an automorphic form f of $\widetilde{SL}(\mathbb{A})$ is the function defined by

$$f_\chi(h) = \int_{N(F)\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot f(nh)\, dn.$$

Say that σ has missing χ-coefficient if $f_\chi = 0$ for all $f \in \sigma$.

(3.4) Global theta lift. Let $\omega^{(7)}_\psi = \otimes_v \omega^{(7)}_{\psi_v}$ be the global Weil representation of $\widetilde{Sp}_{14}(\mathbb{A})$ associated to ψ. By the formation of theta series, we have a map

$$\theta : \omega^{(7)}_\psi \longrightarrow \mathcal{A}(\tilde{S}p_{14}).$$

Now if $\sigma \subset \mathcal{A}^2$ is a cuspidal representation of $\tilde{SL}_2(\mathbb{A})$, then we let $V(\sigma)$ denote the linear span of all functions on $G_2(\mathbb{A})$ of the form

$$\theta(\varphi, f)(g) = \int_{\widetilde{SL}_2(F)\backslash \widetilde{SL}_2(\mathbb{A})} \theta(\varphi)(gh) \cdot \overline{f(h)} dh, \qquad \text{for } \varphi \in \omega_\psi^{(7)} \text{ and } f \in \sigma.$$

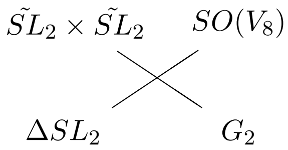
The complex conjugate over $f(h)$ is introduced to ensure the compatibility of global and local theta lifts.

From the results of [RS], one deduces:

(3.5) Proposition *The space $V(\sigma)$ is non-zero and is contained in the space of square-integrable automorphic forms on G_2, so that $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) = \otimes_v \Theta(\sigma_v)$. It is contained in the space of cusp forms if and only if σ has missing ψ-coefficient.*

(3.6) Regularized theta lift. It is desirable to extend the definition of the theta lift to all summands of $\mathcal{A}^2$, i.e. for the non-cuspidal representations $\omega_{\chi,S}$ (S empty). Let us explain how this is done.

For simplicity, let us take the case when $\chi = 1$ is trivial, so that $\omega_1 = \omega_\psi^{(1)}$. With $V_8 := V_7 \oplus (-V_1) \cong \mathbb{H}^4$, we have the following seesaw diagram:



As a representation of $G_2(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$, $\omega_\psi^{(7)} \otimes \overline{\omega_\psi^{(1)}}$ is (a dense subspace of) the restriction to $SO(V_8)(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$ of the Weil representation W_ψ of $\widetilde{Sp}_{16}(\mathbb{A})$. Let $\Theta : W_\psi \to \mathcal{A}(\widetilde{Sp}_{16})$ be the usual theta map. In particular, for $\varphi \in \omega_\psi^{(7)}$ and $f \in \omega_\psi^{(1)}$, the function

$$(g, h) \mapsto \theta(\varphi)(gh) \cdot \overline{f(h)}$$

is the restriction to $G_2(\mathbb{A}) \times \Delta SL_2(\mathbb{A})$ of the element $\Theta(\varphi \otimes f)$. It follows that the absolute convergence of $\theta(\varphi, f)$ is equivalent to the absolute convergence of

$$\int_{\Delta SL_2(F)\backslash \Delta SL_2(\mathbb{A})} \Theta(\varphi \otimes f)(gh) dh$$

for all $g \in G_2(\mathbb{A})$.

Now the convergence of this latter integral is a well-studied problem in the topic of regularized Siegel-Weil formula. In our case, if we realize W_ψ in the Schrodinger model $\mathcal{S}(V_8(\mathbb{A}))$, it is easy to see that the integral of $\Theta(\Phi)$ (for $\Phi \in W_\psi$) over ΔSL_2 converges absolutely iff $W_\psi(h)\Phi(0) = 0$ for all $h \in \Delta SL_2(\mathbb{A})$. So if $\varphi \otimes f$ has this property, then the integral $\theta(\varphi, f)$ converges. This is the case, for example, if f lies in $\omega_{\chi,S}$ with S non-empty.

In general, the map sending Φ to the function $h \mapsto W_\psi(h)\Phi(0)$ gives a $(SO(V_8) \times \Delta SL_2)$-equivariant map

$$T : W_\psi \longrightarrow Ind_B^{SL_2} \delta_B^2 \qquad \text{(unnormalized induction).}$$

Now fix an archimedean place v_0. It is clear that one can find an element Z in the center of the universal enveloping algebra of $\Delta SL_2(F_{v_0})$ such that

$$Z = \begin{cases} 1 \text{ on the trivial representation;} \\ 0 \text{ on the above principal series.} \end{cases}$$

Then $W_\psi(Z)\Phi$ lies in $ker(T)$ and this allows us to define the regularized theta lift:

$$\theta^{reg}(\varphi, f)(g) := \int_{\Delta SL_2(F)\backslash \Delta SL_2(\mathbb{A})} \Theta(W_\psi(Z)(\varphi \otimes f))(gh)dh.$$

Note that there is a unique equivariant extension of the theta integral from $Ker(T)$ to W_ψ. Hence the regularized theta lift defined here is canonical.

Let

$$V_\chi = \text{regularized theta lift of } \mathcal{A}_\chi.$$

Then a consideration of the above see-saw diagram and [GRS, Theorem 6.9] (which says that the regularized theta lift of the trivial representation of ΔSL_2 is a minimal representation of SO_8) gives the first part of our main global theorem:

(3.7) Theorem *(i) The space V_χ is equal to the space of automorphic forms obtained by restricting the automorphic minimal representation of the quasi-split $Spin_8^\chi$. The latter space consists of square-integrable automorphic forms and was determined in [GGJ] as an abstract representation.*

(ii) If $\sigma \subset \mathcal{A}_{00}$, then $V(\sigma) \cong \Theta(\sigma)$.

(3.8) Sketch proof. We give a sketch of the proof of Theorem 3.7(ii). Clearly, there is nothing to prove if $\Theta(\sigma)$ is irreducible. In general, the proof is achieved by studying the Fourier coefficients of $\theta(\varphi, f)$, as we now explain.

We begin with some generalities which hold for any cuspidal σ. Recall that, with the character ψ as a base point, the $\tilde{T}$-orbits of Fourier coefficients for $\widetilde{SL}_2$ are parametrized by quadratic F-algebras. If $\sigma \subset \mathcal{A}_{00}$, then σ has at least 2 non-vanishing ($\tilde{T}$-orbits of) Fourier coefficients, whereas the representation $\omega_{\chi,S}$ supports only one Fourier coefficient, namely the one determined by the quadratic character χ. For a quadratic field K, we shall let $\tilde{\psi}_K$ denote a character of $N(F)\backslash N(\mathbb{A})$ in the orbit indexed by K.

As for G_2, we shall consider Fourier expansion along U_2, in which case the L_2-orbits of Fourier coefficients are parametrized by cubic F-algebras. For a quadratic field K, we shall let ψ_K denote a character of $U_2(\mathbb{A})$, trivial on $U_2(F)$, which lies in the orbit indexed by $F \times K$. For $\theta(\varphi, f) \in V(\sigma)$, we set

$$\theta(\varphi, f)_{\psi_K}(g) = \int_{U_2(F)\backslash U_2(\mathbb{A})} \overline{\psi_K(u)} \cdot \theta(\varphi, f)(ug)\, du, \qquad g \in G_2(\mathbb{A}).$$

Let $\mathcal{W}(\sigma, \psi_K)$ denote the span of all the functions $\theta(\varphi, f)_{\psi_K}$ with varying φ and $f \in \sigma$. Then we have a $G_2(\mathbb{A})$-equivariant surjective map from $V(\sigma)$ to $\mathcal{W}(\sigma, \psi_K)$, so that $\mathcal{W}(\sigma, \psi_K)$ is a semisimple representation of $G_2(\mathbb{A})$ and is a summand of $\Theta(\sigma)$.

(3.9) Proposition *The space $\mathcal{W}(\sigma, \psi_K)$ is non-zero iff σ has non-zero $\tilde{\psi}_K$-Fourier coefficient. In this case, if $f = \otimes_v f_v$ and $\varphi = \otimes_v \varphi_v$, then one has an expression:*

$$\theta(\varphi, f)_{\psi_K}(g) = \prod_v \mathcal{W}_{\psi_{K_v}}(\varphi_v, f_v, g_v)$$

where $\mathcal{W}_{\psi_{K_v}}(\varphi_v, f_v, g_v)$ is a local expression depending only on $\varphi_v \in \omega^{(7)}_{\psi_v}$ and $f_v \in \sigma_v$ (and the character ψ_{K_v}).

Let $\mathcal{W}(\sigma_v, \psi_{K_v})$ denote the span of all the functions $\mathcal{W}_{\psi_{K_v}}(\varphi_v, f_v, -)$, with varying φ_v and f_v. Then as a corollary, we have:

(3.10) Corollary *As a representation of $G_2(\mathbb{A})$,*

$$\mathcal{W}(\sigma, \psi_K) \cong \otimes_v \mathcal{W}(\sigma_v, \psi_{K_v})$$

and $\mathcal{W}(\sigma_v, \psi_{K_v})$ is a non-zero summand of $\Theta(\sigma_v)$.

(3.11) The proof. Now suppose that $\sigma \subset \mathcal{A}_{00}$. Choose a quadratic field K so that σ supports a $\tilde{\psi}_K$-Fourier coefficient. Then to prove the theorem for σ, it suffices to show that:

$$\mathcal{W}(\sigma_v, \psi_{K_v}) \cong \Theta(\sigma_v) \qquad \text{for all } v.$$

Again this is clear if $\Theta(\sigma_v)$ is irreducible. Hence, we are reduced to showing this for the representation $\omega^-_{\psi_v}$ with $K_v = F_v \times F_v$. More precisely, we need to show that

$$\mathcal{W}(\omega^-_{\psi_v}, \psi_{K_v}) \cong \Theta(\omega^-_{\psi_v}) = J_{P_2}(St_v, 1/2) \oplus \pi_{\epsilon_v}.$$

It is likely that there is a purely local proof of this statement. However, we shall present a local-global argument which we find rather amusing.

(3.12) A local-global argument. Suppose we want to prove the local statement above for a place v_0. Choose a quadratic field K split at v_0, and let χ_K be the quadratic character corresponding to K. Let v_1 be another place where K is split and set $S_0 = \{v_0, v_1\}$.

Consider the two representations π_1 and π_2 of $G_2(\mathbb{A})$ defined as follows:

$$(\pi_1)_v = \begin{cases} \pi_{\epsilon_v} \text{ if } v \in S_0; \\ J_{P_1}(\pi(1,\chi_v), 1) \text{ if } v \notin S_0 \end{cases} \qquad \text{and} \quad (\pi_2)_v = \begin{cases} \pi_{\epsilon_v} \text{ if } v = v_1; \\ J_{P_2}(St_v, 1/2) \text{ if } v = v_0; \\ J_{P_1}(\pi(1,\chi_v), 1) \text{ if } v \notin S_0. \end{cases}$$

By the results of [GGJ], we know that π_1 and π_2 occur with multiplicity one in the restriction of the minimal representation of $Spin_8^K$. Hence, by Theorem 3.7(i), π_1 and π_2 occur with multiplicity one in V_{χ_K}. So they must occur in $V(\omega_{\chi_K,S})$ for some S of even cardinality. By our local results, one sees that the only possibility for S is S_0. Hence we have:

$$\pi_1 \oplus \pi_2 \subset V(\omega_{\chi_K,S_0}).$$

Now since ω_{χ_K,S_0} has non-zero $\tilde{\psi}_K$-coefficient, we have a surjective map from $V(\omega_{\chi_K,S_0})$ onto the non-zero space $\mathcal{W}(\omega_{\chi_K,S_0}, \psi_K)$. In fact, by results of [GGJ], this map is non-zero when restricted to any irreducible summand of $V(\omega_{\chi_K,S_0})$. Hence, for $i = 1$ or 2, $\pi_i \subset \mathcal{W}(\omega_{\chi_K,S_0}, \psi_K)$ and thus $(\pi_i)_{v_0} \subset \mathcal{W}(\omega^-_{\psi_{v_0}}, \psi_{K_{v_0}})$, which is what we desire to prove.

§4. Arthur Packets

In this section, we shall explain how our main results allow us to give a definition of a family of Arthur packets for G_2.

(4.1) Arthur parameters. Let L_F be the conjectural Langlands group of F. We shall be considering a family of Arthur parameters for G_2:

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow G_2(\mathbb{C}).$$

To write down the relevant family, let us observe that

$$SL_{2,l} \times_{\mu_2} SL_{2,s} \subset G_2$$

where $(SL_{2,l}, SL_{2,s})$ is a pair of commuting SL_2's corresponding to a pair of mutually orthogonal long and short roots. Now suppose that τ is a cuspidal representation of PGL_2. Conjecturally, τ corresponds to an irreducible representation

$$\phi_\tau : L_F \longrightarrow SL_2(\mathbb{C}).$$

We define an Arthur parameter by

$$\psi_\tau : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times id} SL_{2,l}(\mathbb{C}) \times SL_{2,s}(\mathbb{C}) \xrightarrow{\quad i \quad} G_2(\mathbb{C}).$$

If S_{ψ_τ} is the component group of the centralizer of ψ_τ, then $S_{\psi_\tau} \cong \mathbb{Z}/2\mathbb{Z}$.

The global parameter gives rise to local parameters $\psi_{\tau,v}$ for each place v. The local component groups $S_{\psi_{\tau,v}}$ are given by

$$S_{\psi_\tau,v} = \begin{cases} 1, & \text{if } \phi_{\tau,\mathrm{v}} \text{ is reducible} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \phi_{\tau,\mathrm{v}} \text{ is irreducible.} \end{cases}$$

Note the the condition $\phi_{\tau,v}$ is irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F_v)$.

(4.2) Local Arthur packets. Now Arthur's conjecture (cf. [A1,2]) predicts that for each place v, there is a finite set $A_{\tau,v}$ of unitary representations of $G_2(F_v)$ associated to $\psi_{\tau,v}$. The representations should be indexed by the irreducible characters of $S_{\psi_{\tau,v}}$. Hence, in our case, $A_{\tau,v}$ has the form:

$$A_{\tau,v} = \begin{cases} \{\pi^+_{\tau_v}\}, & \text{if } \tau_\mathrm{v} \text{ is not discrete series} \\ \{\pi^+_{\tau_v}, \pi^-_{\tau_v}\}, & \text{if } \tau_\mathrm{v} \text{ is discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v}$ is indexed by the trivial character of $S_{\tau,v}$.

The set $A_{\tau,v}$ is called a local A-packet, and should satisfy

- for almost all v, $\pi^+_{\tau_v}$ is irreducible and unramified with Satake parameter

$$s_{\tau,v} = i\left(t_{\tau,v} \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix}\right) \in G_2(\mathbb{C})$$

  where $t_{\tau,v} \in SL_{2,l}(\mathbb{C})$ is the Satake parameter of τ_v.

- the distribution $\pi^+_{\tau_v} - \pi^-_{\tau_v}$ is stable.

- certain identities involving transfer of character distributions to endoscopic groups of $G(F_v)$ should hold.

(4.3) Definition of local Arthur packets. Now we can use Theorem 2.16 to give a natural candidate for the packet $A_{\tau,v}$. Recall that τ_v determines a set $\tilde{A}_{\tau_v}$ of representations of $\widetilde{SL}_2(F_v)$. This has 2 or 1 elements $\sigma^{\pm}_{\tau_v}$, depending on whether τ_v is discrete series or not. We set

$$\pi^{\pm}_{\tau_v} = \Theta(\sigma^{\pm}_{\tau_v}).$$

This defines the packet $A_{\tau,v}$.

Why is this a reasonable definition? For one thing, when τ_v is unramified, then $\Theta(\sigma^+_{\tau_v})$ is indeed irreducible and unramified with the required Satake parameter $s_{\tau,v}$. As a second justification, we consider the following special case.

(4.4) A special case. We would like to highlight the case when τ_v is the Steinberg representation St. In this case, according to our definition,

$$\begin{cases} \pi^+_{\tau_v} = \Theta(\omega^-_{\psi_v}) = J_{P_2}(St, 1/2) + \pi_\epsilon \\ \pi^-_{\tau_v} = \Theta(sp_1) = J_{P_1}(St, 1/2). \end{cases}$$

For the case of split p-adic groups, this is the first instance we know in which the representation in a packet can be reducible, and this is quite surprising at first sight. The initial guess would be to take $\pi^+_{\tau_v}$ simply as $J_{P_2}(St, 1/2)$. However, we have:

(4.5) Proposition *Assume that the packet of unipotent representations in Prop. 2.14(i) is indeed an Arthur packet, so that $J_{P_1}(\pi(1,1),1) + 2J_{P_2}(St,1/2) + \pi_\epsilon$ is stable. Then $(J_{P_2}(St,1/2) + \pi_\epsilon) - J_{P_1}(St,1/2)$ is stable.*

The proposition justifies our definition of $\pi^+_{\tau_v}$. A more powerful justification is given by our global theorem 3.7, as we explain below.

(4.6) Global A-packets. With the local packets $A_{\tau,v}$ at hand, we may define the global A-packet by:

$$A_\tau = \{\pi = \otimes_v \pi^{\epsilon_v} : \pi^{\epsilon_v} \in A_{\tau,v}, \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

It is a set of nearly equivalent representations of $G_2(\mathbb{A})$. If S is the set of places v where τ_v is discrete series, then $\#A_\tau = 2^{\#S}$.

To each $\pi \in A_\tau$, one can attach a multiplicity $m(\pi)$ as follows. Arthur attached to ψ_τ a quadratic character ϵ_{ψ_τ} of the component group S_{ψ_τ}. In the case at hand, ϵ_{ψ_τ} is the non-trivial character of $S_{\psi_\tau} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau, 1/2) = -1$. Now if $\pi = \otimes_v \pi^{\epsilon_v} \in A_\tau$, set

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(\tau, 1/2); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(\tau, 1/2). \end{cases}$$

If we let

$$V_\tau = \bigoplus_{\pi \in A_\tau : \epsilon_\pi = \epsilon(\pi, 1/2)} \pi,$$

then Arthur conjectures that there is a $G_2(\mathbb{A})$-equivariant embedding

$$\iota_\tau : V_\tau \hookrightarrow L^2_d(G_2(F) \backslash G_2(\mathbb{A})).$$

Now our global theorem 3.7 says that for the given τ, the global theta correspondence constructs a subspace of $L^2_d(G_2(F)\backslash G_2(\mathbb{A}))$ isomorphic to

$$\bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \Theta(\sigma).$$

This is isomorphic to V_τ with our definition of the local packets $A_{\tau,v}$. This provides compelling global justification for our definition, especially for taking $\pi^+_{\tau_v}$ to be reducible when τ_v is Steinberg.

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