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Isometric Embedding of Negatively Curved Disks in the Minkowski Space

Bo Guan

Dedicated to Professor Leon Simon on his 60th birthday

1. Introduction

The hyperbolic plane $\mathbb{H}^2$ has a canonical isometric embedding in the Minkowski space $\mathbb{R}^{2,1}$ given by the hyperboloid

$$x_3 = \sqrt{1 + |x'|^2}, \quad x' \in \mathbb{R}^2. \tag{1.1}$$

It seems an interesting question whether a two-dimensional simply connected complete Riemannian manifold (M, g) of negative curvature always admits an isometric embedding in $\mathbb{R}^{2,1}$. This is equivalent (see Section 2) to solving the Monge-Ampère type equation on (M, g):

$$\det \nabla^2 u = -K_g (1 + |\nabla u|^2) \det g \tag{1.2}$$

where K_g is the (intrinsic) curvature of g. Note that this equation is elliptic when $K_g < 0$. One can also ask other questions concerning local or global isometric embedding in $\mathbb{R}^{2,1}$ for two-dimensional Riemannian manifolds. In this note we shall consider the problem for compact disks with negative curvature.

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Theorem 1.1. *Let g be a smooth metric of negative curvature on a compact 2-disk $\mathcal{D}$ with smooth boundary $\partial\mathcal{D}$. Suppose $\partial\mathcal{D}$ has positive geodesic curvature. Then there exists a smooth isometric embedding $\mathbf{x} : (\mathcal{D}, g) \to \mathbb{R}^{2,1}$ with $\mathbf{x}(\partial\mathcal{D}) \subset \{x_3 = 0\}$.*

This can be viewed as a counterpart to a theorem of Pogorelov [23] and Hong [14], which states that a positively curved 2-disk with positive geodesic curvature along its boundary admits an isometric embedding in $\mathbb{R}^3$ with planar boundary.

Without the assumption of positive geodesic curvature along $\partial\mathcal{D}$ we shall prove a weaker existence result which seems to have no counterpart in the positive curvature case.

Theorem 1.2. *Let $(\mathcal{D}, g)$ be a smooth compact disk of negative curvature with smooth boundary $\partial\mathcal{D}$. Suppose $(\mathcal{D}, g)$ is geodesically star-shaped with respect to an interior point in $\mathcal{D}$. Then $(\mathcal{D}, g)$ admits a smooth isometric embedding into $\mathbb{R}^{2,1}$.*

We say $(\mathcal{D}, g)$ is *geodesically star-shaped* with respect to $x_0 \in \mathcal{D}$ if the exponential map $\exp_{x_0}$ is a diffeomorphism from a star-shaped domain (with respect to the origin) in the tangent plane $T_{x_0}\mathcal{D}$ onto $\mathcal{D}$.

Problems concerning isometric embedding of surfaces in the Euclidean space $\mathbb{R}^3$ have received much attention. In 1906 Weyl considered the problem whether a smooth metric on $\mathbb{S}^2$ with positive curvature always admits an isometric embedding in $\mathbb{R}^3$. This problem, known as the Weyl problem, was studied subsequently by Lewy, Alexanderov, and finally solved by Nirenberg [19] and Pogorelov [22] independently; their results were extended by Guan-Li [8] and Hong-Zuily [18] to the nonnegative curvature case. In [25] Yau posed the question of finding isometric embedding with prescribed boundary or with boundary contained in a given surface in $\mathbb{R}^3$ for compact disks of positive curvature. Important contributions to the area were made by Pogorelov [23] and Hong [12]-[17] who established remarkable existence results for compact disks of positive curvature, and for complete noncompact surfaces of nonnegative curvature. Hong [14] also considered the problem for complete noncompact surfaces of negative curvature. We should also mention the breakthroughs of Lin [20], [21], and the more recent work of Han-Hong-Lin [11] and Han [9], [10] for local isometric embedding in $\mathbb{R}^3$. It would be interesting to study corresponding questions for isometric embedding in $\mathbb{R}^{2,1}$ of negatively or nonpositively curved surfaces.

We shall derive equation (1.2) in Section 2 and show the equivalence of its solvability to finding isometric embedding in $\mathbb{R}^{2,1}$ of a surface, therefore reducing the proof of Theorems 1.1 and 1.2 to the Dirichlet problems for (1.2). In Section 3 we construct subsolutions for (1.2) when $K_g < 0$, which implies the existence of solutions according to the general theory of Monge-Ampère equations. We shall consider more general equations and boundary data in higher dimensions.

2. Basic Formulas

In this section we derive equation (1.2) for isometric embedding of a surface into $\mathbb{R}^{2,1}$. We begin with some basic notation and formulas.

Let $\mathbb{R}^{n,1}$ $(n \geq 2)$ be the $(n+1)$-dimensional Minkowski space which is $\mathbb{R}^{n+1}$ equipped with the Lorentz metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

We use $\langle \cdot, \cdot \rangle$ to denote the Lorentz pairing, i.e.

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i - v_{n+1} w_{n+1}.$$

A hypersurface Σ in $\mathbb{R}^{n,1}$ is *spacelike* if ds^2 induces a Riemannian metric on Σ, that is, the restriction of the Lorentz pairing to the tangent plane of Σ at any point is positive definite.

Let Σ^n be a spacelike hypersurface in $\mathbb{R}^{n,1}$. We shall use ∇ and Δ to denote the Levi-Civita connection and the Laplace-Beltrami operator on Σ, respectively, while D the standard connection of $\mathbb{R}^{n,1}$. Let ν be the timelike unit normal of Σ, i.e.

$$\langle \nu, \nu \rangle = -1.$$

Let $e_1, \ldots, e_n$ be a local orthonormal frame on Σ. The second fundamental form of Σ is defined as

$$h_{ij} = \langle D_{e_i} \nu, e_j \rangle, \;\; 1 \leq i, j \leq n.$$

We have $h_{ij} = h_{ji}$,

$$D_{e_i}\nu = h_{ij}e_j, \quad D_{e_i}e_j = h_{ij}\nu + \Gamma_{ij}^k e_k, \tag{2.1}$$

where $\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle$ are the Christoffel symbols, and the Codazzi equation

$$\nabla_k h_{ij} = \nabla_i h_{jk}. \tag{2.2}$$

The Riemannian curvature tensor is given by the Gauss equation (see e.g. [4])

$$R_{ijkl} = -h_{ik}h_{jl} + h_{il}h_{jk}. \tag{2.3}$$

The mean and Gauss curvatures of Σ are

$$H = \frac{1}{n}\sum h_{ii}, \quad K = \det h_{ij} \tag{2.4}$$

respectively, while the norm of the second fundamental form is given by

$$|A|^2 = \sum h_{ij}^2.$$

Thus for a spacelike surface in $\mathbb{R}^{2,1}$ its intrinsic curvature is $R_{1212} = -K$.

Let $\mathbf{x}$ be the position vector of Σ in $\mathbb{R}^{n,1}$ and define

$$u = -\langle \mathbf{x}, \mathbf{e} \rangle, \quad \eta = \langle \mathbf{x}, \nu \rangle, \quad z = \langle \mathbf{x}, \mathbf{x} \rangle$$

which are called the *height, support*, and *extrinsic distance* functions of Σ, respectively. Here $\mathbf{e} = (0, \ldots, 0, 1) \in \mathbb{R}^{n,1}$ is the unit vector in the x_{n+1} (time) direction:

$$\langle \mathbf{e}, \mathbf{e} \rangle = -1.$$

We have $D_{e_i}\mathbf{x} = e_i$,

$$\nabla_{ij}\mathbf{x} := D_{e_i}D_{e_j}\mathbf{x} - \Gamma_{ij}^k D_{e_k}\mathbf{x} = h_{ij}\nu. \tag{2.5}$$

Thus

$$\nabla_i u = -\langle e_i, \mathbf{e} \rangle,$$

$$\nabla_{ij} u = -\langle \nu, \mathbf{e} \rangle h_{ij}, \tag{2.6}$$

and

$$|\nabla u|^2 = \sum_{i=1}^n \langle e_i, \mathbf{e} \rangle^2 = \langle \mathbf{e}, \mathbf{e} \rangle + \langle \nu, \mathbf{e} \rangle^2 = -1 + \langle \nu, \mathbf{e} \rangle^2. \tag{2.7}$$

Consequently, u satisfies the Monge-Ampère type equation

$$\det \nabla_{ij} u = K(1 + |\nabla u|^2)^{\frac{n}{2}}. \tag{2.8}$$

Let $(\mathcal{D}, g)$ be a two-dimensional Riemannian manifold and $\mathbf{x} : (\mathcal{D}, g) \to \Sigma \subset \mathbb{R}^{2,1}$ an isometric embedding. Thus Σ is naturally spacelike in $\mathbb{R}^{2,1}$ and we see that the function $u := \langle \mathbf{x}, \mathbf{e} \rangle$ satisfies equation (1.2) in $\mathcal{D}$. Conversely, a solution of (1.2) in $\mathcal{D}$ yields an isometric embedding of $(\mathcal{D}, g)$ into $\mathbb{R}^{2,1}$. This is a consequence of the following fact.

Lemma 2.1. *For $u \in C^2(\mathcal{D}, g)$ the Gauss curvature of the metric $g_1 = g + du^2$ is*

$$K_{g_1} = \frac{1}{1 + |\nabla u|^2}\Big(K_g + \frac{\det \nabla^2 u}{(1 + |\nabla u|^2) \det g}\Big).$$

In particular, $K_{g_1} = 0$ if u is a solution of (1.2).

This lemma guarantees that for a smooth solution u of (1.2) we can always find a smooth isometry from $(\mathcal{D}, g + du^2)$ into $\mathbb{R}^2$ when $\mathcal{D}$ is simply connected since $g + du^2$ is flat. If we use $(x, y) : \mathcal{D} \to \mathbb{R}^2$ to denote this isometry then the desired isometric embedding $\mathbf{x} : (\mathcal{D}, g) \to \Sigma \subset \mathbb{R}^{2,1}$ is given by $\mathbf{x} = (x, y, u)$.

For the proof of Lemma 2.1 one can follow, for example, that of Lemma 1 in [16] with slight modifications. So we omit it here.

In the rest of this section we derive equations for geometric quantities of $(\mathcal{D}, g)$. Let $\mathcal{L}$ be the linear operator defined as

$$\mathcal{L}v = h^{ij} \nabla_{ij} v, \quad v \in C^2(\mathcal{D}).$$

First, differentiating $K = \det h_{ij}$ we obtain

$$h^{ij} h_{ijk} = (\log K)_k \tag{2.9}$$

and

$$h^{ij} h_{ijkk} - h^{ib} h^{aj} h_{ijk} h_{abk} = (\ln K)_{kk} \tag{2.10}$$

where $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk} h_{ij}$, etc. Next, we calculate

$$\begin{aligned} \nabla_{ij} \nu &:= D_{e_i} D_{e_j} \nu - \Gamma_{ij}^k D_{e_k} \nu \\ &= D_{e_i}(h_{jk} e_k) - \Gamma_{ij}^k h_{kl} e_l \\ &= h_{ik} h_{jk} \nu + h_{jki} e_k \\ &= h_{ik} h_{jk} \nu + h_{ijk} e_k \end{aligned} \tag{2.11}$$

by the Codazzi equation, and

$$\nabla_i z = 2\langle \mathbf{x}, e_i \rangle$$

$$\nabla_{ij} z = 2\delta_{ij} + 2\eta h_{ij}. \tag{2.12}$$

Thus

$$|\nabla z|^2 = 4 \sum_{i=1}^{n} \langle \mathbf{x}, e_i \rangle^2 = 4z + 4\eta^2, \tag{2.13}$$

$$\mathcal{L} z = 2 \sum_i h^{ii} + 2n\eta, \tag{2.14}$$

$$\mathcal{L} \mathbf{x} = n\nu \tag{2.15}$$

by (2.5), and

$$\mathcal{L} \nu = nH\nu + \nabla \ln K. \tag{2.16}$$

Therefore,

$$\begin{aligned} \mathcal{L}\eta &= \langle \mathbf{x}, \mathcal{L}\nu \rangle + 2h^{ij} \langle \nabla_i \mathbf{x}, \nabla_j \nu \rangle + \langle \nu, \mathcal{L}\mathbf{x} \rangle \\ &= nH\eta + n + \langle \mathbf{x}, \nabla \ln K \rangle. \end{aligned} \tag{2.17}$$

Note that

$$h^{ij} \eta_i \eta_j = h_{ij} \langle \mathbf{x}, e_i \rangle \langle \mathbf{x}, e_j \rangle = \frac{1}{4} h_{ij} z_i z_j$$

So

$$\frac{1}{4} \mathcal{L}(|\nabla z|^2) = \mathcal{L}(z + \eta^2) = 2 \sum h^{ii} + 4n\eta + 2nH\eta^2 + 2\eta \langle \mathbf{x}, \nabla \ln K \rangle + \frac{1}{2} h_{ij} z_i z_j.$$

Finally, from

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl}.$$

and the Codazzi and Gauss equations, we derive

$$\begin{aligned} \mathcal{L} H &= \frac{1}{n} \sum_k h^{ij} h_{kkij} = \frac{1}{n} \sum_k h^{ij} h_{ikkj} \\ &= \frac{1}{n} \sum_k h^{ij} h_{ikjk} + \frac{1}{n} \sum_{k,m} h^{ij} h_{im} R_{mkkj} + \frac{1}{n} \sum_{k,m} h^{ij} h_{mk} R_{mikj} \\ &= \frac{1}{n} \sum_k h^{ij} h_{ijkk} + nH^2 - |A|^2 \\ &= \frac{1}{n} \Delta \ln K + \frac{1}{n} \sum_k h^{il} h^{mj} h_{ijk} h_{lmk} + nH^2 - |A|^2, \end{aligned} \tag{2.18}$$

and

(2.19)

$$\begin{aligned}\frac{1}{2}\mathcal{L}|A|^2 &= \sum_{k,l} h^{ij}h_{kl}h_{klij} + \sum_{k,l} h^{ij}h_{kli}h_{klj}\\ &= \sum_{k,l} h^{ij}h_{kl}h_{ijkl} + \sum_{k,l} h^{ij}h_{kli}h_{klj} + \sum_{k,l,m} h^{ij}h_{kl}(h_{im}R_{mkjl} + h_{mk}R_{mijl})\\ &= nH|A|^2 + \sum_{k,l} h^{ij}h_{kli}h_{klj} + \sum_{k} h^{it}h^{sj}h_{kl}h_{ijk}h_{stl}\\ &\quad + \sum_{k,l} h_{kl}(\ln K)_{kl} - \sum_{k,l,m} h_{kl}h_{km}h_{ml}.\end{aligned}$$

The last inequality should be compared with the Calabi identity [3] (see also [4])

$$\frac{1}{2}\Delta|A|^2 = |A|^4 + \sum_{i,j,k} h_{ijk}^2 + n\sum_{i,j} h_{ij}\nabla_{ij}H - nH\sum_{i,j,k} h_{ij}h_{jk}h_{ki}. \tag{2.20}$$

3. Construction of subsolutions

In this section we construct subsolutions for equation (1.2) from which we conclude Theorems 1.1 and 1.2. We shall do this for a general class of Hessian equations that include (1.2). Throughout this section we assume $(\mathcal{D}^n, g)$ to be a compact simply connected Riemannian manifold of dimension n ($n \geq 2$) with nonpositive sectional curvature and smooth boundary $\partial\mathcal{D}$.

Let us first consider a radially symmetric function $u(x) = u(|x|)$ in $\mathbb{R}^n$. A straightforward calculation shows that

$$\det D^2u = \big(\frac{u'}{r}\big)^{n-1}u'', \quad r = |x|, \tag{3.1}$$

Thus the Monge-Ampère equation in $\mathbb{R}^n$

$$\det D^2u = \psi(x, u, Du)$$

takes the form

$$(u')^{n-1}u'' = r^{n-1}\psi(x, u, u') \tag{3.2}$$

for radially symmetric functions.

Lemma 3.1. *Let $f \in C^l(\mathbb{R}_+)$ be a nonnegative function defined on $\mathbb{R}_+ := \{r \geq 0\}$. Then there exists a unique convex function $\phi \in C^{l+2}(\mathbb{R}_+)$ with $\phi(0) = 0$, $\phi'(0) = 0$ and*

$$|\phi'|^{n-1}\phi'' = r^{n-1}f(r)(1 + |\phi'|^n), \ \ \forall\, r > 0. \tag{3.3}$$

Moreover, ϕ is strictly convex where $f > 0$, and

$$\lim_{r\to+\infty} \phi(r) = +\infty \tag{3.4}$$

unless $f \equiv 0$ on $\mathbb{R}$.

Proof. Integrating equation (3.3), we have

$$\log\big(1 + (\phi')^n\big) = n\int_0^r r^{n-1}f(r)dr, \ \ r \geq 0.$$

Therefore,

$$\phi'(r) = \big(e^{h(r)} - 1\big)^{\frac{1}{n}}, \ \ r \geq 0 \tag{3.5}$$

where

$$h(r) := n\int_0^r r^{n-1}f(r)dr.$$

Integrating again,

$$\phi(r) = \int_0^r \big(e^{h(r)} - 1\big)^{\frac{1}{n}}dr, \ \ \forall\, r \geq 0.$$

Finally, the convexity of ϕ and (3.4) follow from the fact that $\phi'(0) = 0$ and $\phi"(r) > 0$ whenever $f(r) > 0$. □

Remark 3.2. When f is constant Lemma 3.1 is a special case of Lemma 3.7 in [7].

We now suppose that $(\mathcal{D}, g)$ is geodesically star-shaped with respect to $x_0 \in \mathcal{D}$. Given any positive function $\psi \in C^\infty(\overline{\mathcal{D}})$, define

$$w(x) := \phi(r(x)), \ \ x \in \mathcal{D} \tag{3.6}$$

where ϕ is obtained from Lemma 3.1 with $f := A\max_{\mathcal{D}}\psi$, $A > 0$, and r is the distance function from x_0

$$r(x) := \mathrm{dist}_g(x, x_0), \ \ x \in \mathcal{D}.$$

We calculate

$$\nabla^2 w = \phi'\nabla^2 r + \phi'' dr \otimes dr.$$

Since g has nonpositive curvature, by the Hessian comparison principle (see, e.g., [24]) we see that $\nabla^2 w$ is positive definite and

$$\det \nabla^2 w \geq \big(\frac{\phi'}{r}\big)^{n-1}\phi'' = f(r)(1+|\phi'|^n) \geq A\psi(1+|\nabla w|^n) \;\text{ in } \mathcal{D}. \tag{3.7}$$

Therefore w is a subsolution of the following Monge-Ampère equation

$$\det \nabla^2 u = \psi(1+|\nabla u|^2)^{\frac{n}{2}} \det g \;\text{ in } \overline{\mathcal{D}} \tag{3.8}$$

when A is chosen sufficiently large. By Theorem 5.1 of [5] we obtain a locally strictly convex solution $u \in C^\infty(\overline{\mathcal{D}})$ of (3.8) satisfying the Dirichlet boundary condition $u = w$ on $\partial\mathcal{D}$. In particular, for $\psi = -K_g$ this implies Theorem 1.2.

Using the function w constructed above it is possible to solve the Dirichlet problem for equation (3.8) with arbitrary smooth boundary data when $\mathcal{D}$ is strictly convex, i.e., the second fundamental form of $\partial\mathcal{D}$ is positive definite. (When $\mathcal{D}$ is a strictly convex bounded domain in $\mathbb{R}^n$ this was observed by P. L. Lions; see [1].) More generally, we have the following existence result for the Hessian equation

$$\sigma_k(\nabla^2 u) = \psi(x,u)(1+|\nabla u|^2)^{\frac{k}{2}} \;\text{ in } \overline{\mathcal{D}}, \tag{3.9}$$

where $\sigma_k(\nabla^2 u) = \sigma_k(\lambda(\nabla^2 u))$ is the k-th elementary symmetric function of the eigenvalues of $\nabla^2 u$ with respect to metric g.

Theorem 3.3. *Let $\psi \in C^\infty(\overline{\mathcal{D}} \times \mathbb{R})$, $\psi \geq 0$, $\psi_u \geq 0$, and $\varphi \in C^\infty(\partial\mathcal{D})$. Suppose that $\partial\mathcal{D}$ satisfies the condition*

$$(\kappa_1, \ldots, \kappa_{n-1}) \in \Gamma_{k-1} \;\text{ on } \partial\mathcal{D}, \tag{3.10}$$

where $(\kappa_1, \ldots, \kappa_{n-1})$ are the principal curvatures of $\partial\mathcal{D}$. Then equation (3.9) has a unique admissible solution $u \in C^\infty(\overline{\mathcal{D}})$ which satisfies the boundary condition

$$u = \varphi \;\text{ on } \partial\mathcal{D}. \tag{3.11}$$

Here Γ_k denotes the open convex cone in $\mathbb{R}^n$ defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

A function $u \in C^2(\mathcal{D})$ is *admissible* if $\lambda(\nabla^2 u) \in \Gamma_k$. Equation (3.9) is elliptic at an admissible solution; see e.g. [2].

By Theorem 1.3 in [6], in order to prove Theorem 3.3 we only need to construct an admissible subsolution attaining the same boundary data.

Lemma 3.4. *Suppose $\partial\mathcal{D}$ satisfies (3.10). Then for any $\varphi \in C^\infty(\overline{\mathcal{D}})$ and $A > 0$ there exists an admissible function $\underline{u} \in C^\infty(\overline{\mathcal{D}})$ with*

$$\sigma_k^{1/k}(\nabla^2 \underline{u}) \geq A(1 + |\nabla \underline{u}|^2)^{\frac{1}{2}} \ \ \text{in } \overline{\mathcal{D}}, \quad \underline{u} = \varphi \ \ \text{on } \partial\mathcal{D}. \tag{3.12}$$

Proof. We shall modify the constructions in [2] and [6]. For convenience we assume σ_k is normalized so that

$$\sigma_k(1, \ldots, 1) = 1. \tag{3.13}$$

Let d denote the distance to $\partial\mathcal{D}$,

$$d(x) = \text{dist}_{\mathcal{D}}(x, \partial\mathcal{D}) \ \ \text{for } x \in \mathcal{D}.$$

We may choose $\delta_0 > 0$ sufficiently small such that d is a smooth function in

$$N_\delta \equiv \{x \in \bar{\mathcal{D}} : 0 \leq d \leq \delta\} \quad \forall\, 0 < \delta \leq \delta_0,$$

and for each point $x \in N_{\delta_0}$ there is a unique point $y = y(x) \in \partial\mathcal{D}$ with

$$d(x) = \text{dist}_{\mathcal{D}}(x, y).$$

The eigenvalues of the Hessian of d in N_{δ_0} are given by

$$\lambda(\nabla^2 d(x)) = (-\kappa_1(y(x)) + O(d), \ldots, -\kappa_{n-1}(y(x)) + O(d), 0)$$

where $\kappa_1(y), \ldots, \kappa_{n-1}(y)$ are the principal curvatures of $\partial\mathcal{D}$ at $y \in \partial\mathcal{D}$.

For $t > 0$ consider the function in N_{δ_0}

$$\eta = \frac{1}{t}(e^{-td} - 1);$$

We have

$$\nabla^2 \eta = e^{-td}(-\nabla^2 d + t\nabla d \otimes \nabla d)$$

and

$$\sigma_j(\nabla^2 \eta) = (e^{-td})^j \sigma_j(-\nabla^2 d) + t(e^{-td})^{j-1}\sigma_{j-1}(-\nabla^2 d), \ \ \forall\, 1 \leq j \leq k.$$

By assumption (3.10) there exists $t_0 > 0$ sufficiently large such that

$$\nabla^2 \eta \in \Gamma_k \ \ \text{and} \ \ \sigma_k(\nabla^2 \eta) \geq \frac{t}{2} e^{-ktd} \ \ \text{in } N_\delta, \ \ \forall\, t \geq t_0 \tag{3.14}$$

for $\delta > 0$ sufficiently small.

On the other hand,

$$|\nabla \eta| = e^{-td} \leq 1 \ \ \text{in } N_\delta. \tag{3.15}$$

Fixing $t \geq 2(16eA)^k$ and $\delta = t^{-1}$, we see that

$$\sigma_k^{1/k}(\nabla^2 \eta) \geq 8A(1 + |\nabla \eta|) \text{ in } N_\delta. \tag{3.16}$$

Let $h(s)$ be a smooth convex function on $s \leq 0$ satisfying

$$h(s) = \begin{cases} s & \text{for } -\varepsilon_2 \leq s \leq 0 \\ \frac{1}{2}(\varepsilon_1 + \varepsilon_2) & \text{for } s \leq -\varepsilon_1 \end{cases}$$

and

$$h'(s) > 0 \text{ for } -\varepsilon_1 < s < -\varepsilon_2$$

where $\varepsilon_1 = \frac{1}{t}(1 - e^{-1})$ and $\varepsilon_2 = \frac{1}{t}(1 - e^{-1/2})$. The function $\zeta := h(\eta)$ is smooth in $\mathcal{D}$ and

$$\nabla^2 \zeta = h' \nabla^2 \eta + h'' \nabla \eta \otimes \nabla \eta. \tag{3.17}$$

Since $h' \geq 0$ and $h'' \geq 0$ we have

$$\lambda(\nabla^2 \zeta) \in \overline{\Gamma_k} \text{ in } \overline{\mathcal{D}}.$$

Let $\rho \geq 0$ be a smooth cutoff function with compact support in $\mathcal{D}$ such that $\rho \equiv 1$ in the complement of $N_{\delta/2}$. We consider the function

$$v := \zeta + c\rho w$$

where w is a smooth convex function satisfying

$$(\det \nabla^2 w)^{\frac{1}{n}} \geq 2A\psi(1 + |\nabla w|) \text{ in } \mathcal{D}, \tag{3.18}$$

which can be constructed as in (3.6) with a suitable choice of f. Note that $v = 0$ on $\partial \mathcal{D}$. Moreover,

$$\nabla v = \nabla \zeta + c(\rho \nabla w + w \nabla \rho)$$

and

$$\nabla^2 v = \nabla^2 \zeta + c\rho \nabla^2 w + c(\nabla \rho \otimes \nabla w + \nabla w \otimes \nabla \rho + w \nabla^2 \rho). \tag{3.19}$$

Since $\zeta = \eta$ in $N_{\delta/2}$, by (3.14) and (3.16) we may fix $c > 0$ sufficiently small (which may depend on A, however) such that $\lambda(\nabla^2 v) \in \Gamma_k$ in $N_{\delta/2}$ and

$$\sigma_k^{1/k}(\nabla^2 v) \geq \frac{1}{2}\sigma_k^{1/k}(\nabla^2 \zeta) \geq 4A(1 + |\nabla \eta|) \geq 2A(1 + |\nabla v|) \text{ in } N_{\delta/2} \tag{3.20}$$

Since $\rho \equiv 1$ in the complement of $N_{\delta/2}$ and $0 \leq h' \leq 1$, we have

$$|\nabla v| \leq |\nabla \zeta| + c|\nabla w| = h'|\nabla \eta| + c|\nabla w| \text{ in } \overline{\mathcal{D}} \setminus N_{\delta/2}$$

and

$$\nabla^2 v = \nabla^2 \zeta + c\nabla^2 w \in \Gamma_k, \quad \text{in } \overline{\mathcal{D}} \setminus N_{\delta/2}.$$

By (3.17), (3.16), (3.18) and the concavity of $\sigma_k^{1/k}$ in Γ^k,

$$\begin{aligned} \sigma_k^{1/k}(\nabla^2 v) &\geq \sigma_k^{1/k}(h'\nabla^2\eta + c\nabla^2 w) \\ &\geq h'\sigma_k^{1/k}(\nabla^2\eta) + c\sigma_k^{1/k}(\nabla^2 w) \\ &\geq 8Ah'(1+|\nabla\eta|) + 2Ac(1+|\nabla w|) \\ &\geq 2A(c+|\nabla v|) \qquad\qquad \text{in } \overline{\mathcal{D}} \setminus N_{\delta/2}. \end{aligned} \tag{3.21}$$

Here we have used the Newton-MacLaurin inequality

$$\sigma_k^{1/k}(\nabla^2 w) \geq \sigma_n^{1/n}(\nabla^2 w) = (\det(\nabla^2 w))^{\frac{1}{n}}.$$

(Recall that σ_k is normalized; see (3.13).) Finally, choosing B sufficiently large we see from (3.20) and (3.21) that the function $\underline{u} := Bv + \varphi$ is admissible and satisfies (3.12). □

Note that any admissible function in $\mathcal{D}$ is subharmonic and therefore satisfies the maximum principle. Since $\psi_u \geq 0$, taking

$$A = \max_{x\in\overline{\mathcal{D}}} \psi(x, \bar{\varphi}), \quad \bar{\varphi} = \max_{\partial\mathcal{D}} \varphi$$

in Lemma 3.4 we obtain an admissible subsolution for the Dirichlet problem (3.9), (3.11). Theorem 3.3 now follows from Theorem 1.3 in [6]. The proof of Theorem 3.3 is complete and therefore so is that of Theorem 1.2.

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Bo Guan
Department of Mathematics, Ohio State University
Columbus, OH 43210, USA
E-mail: guan@math.osu.edu