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The Space of Complete Quotients

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Abstract: We introduce *complete quotients* over the projective line and prove that they form smooth projective varieties. The resulting parameter spaces coincide with the varieties constructed in [HLS11] and [Shao11]. Hence they provide modular smooth compactifications with normal crossing boundaries of the spaces of algebraic maps from the projective line to Grassmannian varieties, resolving the singularities of the boundaries of the Quot scheme compactifications.
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1. Introduction

We work throughout over an algebraically closed field $\Bbbk$ of characteristic 0.

Fix the Grassmannian $\mathrm{Gr}(k, V)$ of k-dimensional subspace in a vector space $V \cong \Bbbk^n$. The set $\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(k, V))$ of degree d algebraic maps from $\mathbb{P}^1$ to $\mathrm{Gr}(k, V)$ has a natural structure of a smooth quasi-projective variety. It comes with a natural compactification, the Grothendieck Quot scheme $Q_d := \mathrm{Quot}^{d,n-k}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\Bbbk}$, parametrizing all equivalence classes $[V_{\mathbb{P}^1} \twoheadrightarrow F]$ of quotients of degree d and rank $n-k$, where $V_{\mathbb{P}^1} := V \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^1}$. The subset

$$\mathring{Q}_d := \{[V_{\mathbb{P}^1} \twoheadrightarrow F] \in Q_d \,|\, F \text{ is locally free}\}$$

is open in Q_d and can be identified with the variety $\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(k, V))$.

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The compactification Q_d is smooth but the boundary $Q_d \setminus \mathring{Q}_d$ has rather intricate singularities. It comes equipped with a natural filtration by closed subsets

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = Q_d \setminus \mathring{Q}_d$$

where $Z_{d,r} = \{[V_{\mathbb{P}^1} \twoheadrightarrow F] \in Q_d \,|\, \text{the torsion of } F \text{ has degree} \geq d-r\}$, $r = 0, \ldots, d-1$. The subsets $Z_{d,r}$ admit natural subscheme structures (cf. Section 3, [Shao11]).

Theorem 1.1 ([HLS11, Shao11])**.** *The singularities of the subscheme $Z_{d,r}$ can be resolved by repeatedly blowing up $Z_{d,0}$, $Z_{d,1}, \cdots, Z_{d,r-1}$. Consequently, by iteratively blowing-up the Quot scheme Q_d along $Z_{d,0}$, $Z_{d,1}, \cdots, Z_{d,d-1}$, we obtain a smooth compactification $\widetilde{Q}_d$ such that the boundary $\widetilde{Q}_d \setminus \mathring{Q}_d$ is a simple normal crossing divisor.*

The authors of [HLS11] proved the case when $k = 1$. In his Ph.D thesis [Shao11], the second named author proved it for all Grassmannians $\mathrm{Gr}(k, V)$, $k \geq 1$.

The main purpose of this paper is to provide $\widetilde{Q}_d$ the following modular interpretation.

For any coherent sheaf X over $\mathbb{P}^1$, we let $X^{\mathbf{t}}$ denote the torsion subsheaf of X and $X^{\mathbf{f}} = X/X^{\mathbf{t}}$ denote the free part of X. For any coherent sheaves A and B on $\mathbb{P}^1$, an extension of A by B is a short exact sequence $0 \to B \to X \to A \to 0$, which will also be shorthanded by $B \rightarrowtail X \twoheadrightarrow A$. Two extensions $B \rightarrowtail X_1 \twoheadrightarrow A$ and $B \rightarrowtail X_2 \twoheadrightarrow A$ differ by a scalar multiple λ if there is an isomorphism $X_1 \simeq X_2$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \lambda\cdot\downarrow & & \downarrow\simeq & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & X_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commutes, where $\lambda\cdot$ stands for the multiplication by the scalar λ. We use $[B \rightarrowtail X \twoheadrightarrow A]$ to denote the equivalence class of the extension $B \rightarrowtail X \twoheadrightarrow A$ modulo scalar multiplication.

Definition 1.2. *A complete quotient of $V_{\mathbb{P}^1}$ of degree d and rank $n-k$ on $\mathbb{P}^1$ is either*

- *a quotient $[V_{\mathbb{P}^1} \twoheadrightarrow X_1] \in Q_d$ such that X_1 is locally free; or,*

- *a sequence* $([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X_1^{\mathbf{f}} \rightarrowtail X_2 \twoheadrightarrow X_1^{\mathbf{t}}], \ldots, [X_m^{\mathbf{f}} \rightarrowtail X_{m+1} \twoheadrightarrow X_m^{\mathbf{t}}])$ *with* $m \geq 1$ *such that* $[V_{\mathbb{P}^1} \twoheadrightarrow X_1] \in Q_d$*, for every* $1 \leq i \leq m$*,* X_{i+1} *is a non-split extension of* $X_i^{\mathbf{t}}$ *by* $X_i^{\mathbf{f}}$*, and further, the last sheaf* X_{m+1} *is the unique one that is locally free.*

It follows from the definition that all the sheaves $X_1, \ldots, X_{m+1}$ are coherent and have degree d and rank $n-k$.

The main theorem of this paper is

Theorem 1.3. *The projective variety* $\widetilde{Q}_d$ *parameterizes all complete quotients of* $V_{\mathbb{P}^1}$ *of degree* d *and rank* $n-k$*.*

To prove this theorem, we calculate the normal bundle of the blowup center in each blowup of $\widetilde{\mathcal{Q}}_d \longrightarrow Q_d$ as presented in [HLS11, Shao11], and modularly interpret the points in the normal bundle as the desired extensions.

This work is the genus zero case of a larger project. The higher genus case will appear in forthcoming papers.

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The notations in this paper closely follow those in [Shao11] with occasional modifications.

2. The space of relative extensions

2.1. **The space of non-split extensions.** Let F and T be two coherent sheaves over $\mathbb{P}^1$ with F being locally free and T torsion. The vector space $E = \mathrm{Ext}^1_{\mathbb{P}^1}(T, F)$ can be identified with the set of all (isomorphism classes of) extensions $F \rightarrowtail X \twoheadrightarrow T$ of T by F. The zero element corresponds to the split extension $F \rightarrowtail F \oplus T \twoheadrightarrow T$ while the remainders correspond to non-split ones. Therefore the projective space $\mathbb{P}(E) := \mathrm{Proj}(\mathrm{Sym}^*(E^\vee))$, where $E^\vee$ is the linear dual of the vector space E, parametrizes the set of all non-split extensions of T by F up to scalar multiplication. Here again, as in the introduction, two extensions $F \rightarrowtail$

$X_1 \twoheadrightarrow T$ and $F \rightarrowtail X_2 \twoheadrightarrow T$ differ by a nonzero scalar multiple λ if and only if there is an isomorphism $X_1 \simeq X_2$ that makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & X_1 & \longrightarrow & T & \longrightarrow & 0 \\ & & \lambda\cdot\downarrow & & \downarrow\simeq & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & X_2 & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

where $\lambda\cdot$ is the scalar multiplication by λ. We denote by $[F \rightarrowtail X \twoheadrightarrow T]$ the point of $\mathbb{P}(E)$ corresponding to a non-split extension $F \rightarrowtail X \twoheadrightarrow T$. One checks that an extension $F \rightarrowtail X \twoheadrightarrow T$ splits if and only if $\deg X^{\mathbf{t}} = \deg T$. For a discussion on universal extensions, see Example 2.1.12, [HL97] .

We need to introduce a relative version of the space of non-split extensions. For this, we begin with a lemma.

Lemma 2.1. *Let S be a noetherian scheme, $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{T} \to 0$ a short exact sequence of coherent sheaves on S with $\mathcal{T}$ torsion and $\mathcal{E}_0$ and $\mathcal{E}_1$ locally free, and $\mathcal{F}$ a coherent torsion-free sheaf on S. Then*

(1) $H^0(S, \mathcal{E}xt^1(\mathcal{T}, \mathcal{F})) = \mathrm{Ext}^1(\mathcal{T}, \mathcal{F})$;
(2) *for any morphism $f : R \to S$, $f^* \mathcal{E}xt^1_S(\mathcal{T}, \mathcal{F}) = \mathcal{E}xt^1_R(f^*\mathcal{T}, f^*\mathcal{F})$.*

Proof. (1) Since $\mathcal{T}$ is torsion and $\mathcal{F}$ is torsion-free, we have $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and $\mathcal{H}om(\mathcal{T}, \mathcal{F}) = 0$. Applying $\mathrm{Hom}(-, \mathcal{F})$ and $\mathcal{H}om(-, \mathcal{F})$ to the locally free presentation of $\mathcal{T}$, we obtain a long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}_0, \mathcal{F}) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{T}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{F})$$

and a short exact sequence

$$0 \to \mathcal{H}om(\mathcal{E}_0, \mathcal{F}) \to \mathcal{H}om(\mathcal{E}_1, \mathcal{F}) \to \mathcal{E}xt^1(\mathcal{T}, \mathcal{F}) \to 0$$

Taking global sections of the above short exact sequence, we obtain a long exact sequence:

$$\begin{aligned} 0 \to H^0(\mathcal{H}om(\mathcal{E}_0, \mathcal{F})) \to H^0(\mathcal{H}om(\mathcal{E}_1, \mathcal{F})) \to H^0(\mathcal{E}xt^1(\mathcal{T}, \mathcal{F})) \\ \to H^1(\mathcal{H}om(\mathcal{E}_0, \mathcal{F})) \to H^1(\mathcal{H}om(\mathcal{E}_1, \mathcal{F})) \end{aligned}$$

We have identifications

$$\mathrm{Hom}(\mathcal{E}_i, \mathcal{F}) = H^0(\mathcal{H}om(\mathcal{E}_i, \mathcal{F})), \quad i = 0, 1$$

and a natural homomorphism

$$\mathrm{Ext}^1(\mathcal{T},\mathcal{F}) \to H^0(\mathcal{E}xt^1(\mathcal{T},\mathcal{F}))$$

Using the Grothendieck spectral sequence for the composition of the two functors H^0 and $\mathcal{H}om^1(\mathcal{E}_i,-)$, we obtain exact sequences of low degrees:

$$0 \to H^1(\mathcal{H}om(\mathcal{E}_i,\mathcal{F})) \to \mathrm{Ext}^1(\mathcal{E}_i,\mathcal{F}) \to H^0(\mathcal{E}xt^1(\mathcal{E}_i,\mathcal{F})) \to \cdots, \quad i=0,1.$$

Since $\mathcal{E}_i$ are locally free, we have $H^0(\mathcal{E}xt^1(\mathcal{E}_i,\mathcal{F})) = 0$. Therefore we obtain identifications:

$$H^1(\mathcal{H}om(\mathcal{E}_i,\mathcal{F})) = \mathrm{Ext}^1(\mathcal{E}_i,\mathcal{F}), \quad i=0,1$$

Thus we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathcal{E}_0,\mathcal{F}) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1,\mathcal{F}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{T},\mathcal{F}) & \longrightarrow \\
\| & & \| & & \downarrow & \\
H^0(\mathcal{H}om(\mathcal{E}_0,\mathcal{F})) & \longrightarrow & H^0(\mathcal{H}om(\mathcal{E}_1,\mathcal{F})) & \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{T},\mathcal{F})) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Ext}^1(\mathcal{E}_0,\mathcal{F}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{E}_1,\mathcal{F}) \\
\| & & \| \\
H^1(\mathcal{H}om(\mathcal{E}_0,\mathcal{F})) & \longrightarrow & H^1(\mathcal{H}om(\mathcal{E}_1,\mathcal{F}))
\end{array}$$

By the Five Lemma, we obtain the identification

$$H^0(\mathcal{E}xt^1(\mathcal{T},\mathcal{F})) = \mathrm{Ext}^1(\mathcal{T},\mathcal{F})$$

The proof of (2) is parallel to that of [Shao11], Proposition 2.3. So, we omit the details. □

2.2. **The space of relative non-split extensions.** To proceed, we fix a set of notations.

Notation. *For any* $\Bbbk$*-schemes* R *and* S*,*

(1) *we denote the projection* $\mathbb{P}^1 \times R \to R$ *by* π_R *or simply* π*;*
(2) *for any coherent sheaf* $\mathcal{H}$ *over* R*, denote by* $\mathcal{H}^\vee$ *the dual sheaf* $\mathcal{H}om_R(\mathcal{H},\mathcal{O}_R)$*;*
(3) *for any coherent sheaf* $\mathcal{F}$ *on* $\mathbb{P}^1 \times R$*, we set* $\mathcal{F}_x := \mathcal{F}|_{\mathbb{P}^1\times\{x\}}$ *for any point* $x \in R$*;*
(4) *for any morphism* $f : R \to S$*, we set* $\bar{f} := 1 \times f : \mathbb{P}^1 \times R \to \mathbb{P}^1 \times S$*.*

We now introduce a relative version of the space of non-split extensions. Let S be a noetherian scheme over $\Bbbk$ and let $\mathcal{F}$ and $\mathcal{T}$ be two coherent sheaves on $\mathbb{P}^1 \times S$, both flat over S, with $\mathcal{F}$ locally free and $\mathcal{T}$ torsion (i.e., having rank 0). Let $\pi_S : \mathbb{P}^1 \times S \to S$ be the projection, and set

$$\mathcal{E} := \pi_{S*}\, \mathcal{E}xt^1_{\mathbb{P}^1 \times S}(\mathcal{T}, \mathcal{F}).$$

As in [Shao11], Proposition 2.4 (1), one checks that $\mathcal{E}xt^1(\mathcal{T}, \mathcal{F})$ is a torsion sheaf and is flat over S. By Cohomology and Base Change, we can show $\mathcal{E}$ is locally free and for any point $s \in S$,

$$\mathcal{E}|_s = H^0(\mathcal{E}xt^1(\mathcal{T}, \mathcal{F})_s) = H^0(\mathcal{E}xt^1(\mathcal{T}_s, \mathcal{F}_s)) = \mathrm{Ext}^1(\mathcal{T}_s, \mathcal{F}_s).$$

where the second equality holds by Lemma 2.1 (2) and the third holds by Lemma 2.1 (1). So the projective bundle

$$\mathbb{P}(\mathcal{E}) := \mathbf{Proj}(\mathrm{Sym}^*(\mathcal{E}^\vee))$$

over S is a family of spaces of non-split extensions: for each point $s \in S$, the fiber of $\mathbb{P}(\mathcal{E})$ over s is $\mathbb{P}(\mathcal{E}|_s) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{T}_s, \mathcal{F}_s))$.

2.3. The universal extension.

Notation. *For any coherent sheaf H on $\mathbb{P}^1 \times \mathbb{P}(\mathcal{E})$, we denote by $H(m,n)$ the sheaf*

$$H \otimes p^*\mathcal{O}_{\mathbb{P}^1}(m) \otimes \pi^*_{\mathbb{P}(\mathcal{E})}\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$$

where $p : \mathbb{P}^1 \times \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ and $\pi_{\mathbb{P}(\mathcal{E})} : \mathbb{P}^1 \times \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ are the two projections.

Let $a : \mathbb{P}(\mathcal{E}) \to S$ be the structure morphism. The space $\mathbb{P}(\mathcal{E})$ comes equipped with a universal quotient $a^*\mathcal{E}^\vee \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Dualizing the universal quotient and tensoring the result with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, we obtain a line subbundle

$$\begin{aligned}\mathcal{O}_{\mathbb{P}(\mathcal{E})} \hookrightarrow a^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \pi_{\mathbb{P}(\mathcal{E})*}\bar{a}^*\, \mathcal{E}xt^1(\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \\ \pi_{\mathbb{P}(\mathcal{E})*}(\mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F})(0,1)) = \pi_{\mathbb{P}(\mathcal{E})*}\, \mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1))\end{aligned}$$

Here the first equality holds because the morphism a is flat, and the second holds by Lemma 2.1 (2). This line subbundle corresponds to a nonzero element of

$$\begin{aligned}\Gamma(\mathbb{P}(\mathcal{E}), \pi_{\mathbb{P}(\mathcal{E})*}\, \mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1))) &= \Gamma(\mathbb{P}^1 \times \mathbb{P}(\mathcal{E}), \mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1))) \\ &= \mathrm{Ext}^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1))\end{aligned}$$

where the second equality holds by Lemma 2.1 (1). We can write this element as an extension

$$0 \to \bar{a}^*\mathcal{F}(0,1) \to \mathcal{X} \to \bar{a}^*\mathcal{T} \to 0. \tag{2.1}$$

which we call the *universal extension*. Note that the universal extension is *nowhere-split*, which means that, for each $\Bbbk$-point $s \in S$ and each $\Bbbk$-point $x \in a^{-1}(\{s\}) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{T}_s, \mathcal{F}_s))$, the extension

$$0 \to \mathcal{F}_s \to \mathcal{X}_x \to \mathcal{T}_s \to 0 \tag{2.2}$$

obtained by pulling back the universal extension to $\mathbb{P}^1 \times \{x\} \simeq \mathbb{P}^1$ is non-split.

Conversely, the universal quotient can be recovered from the universal extension by reversing the above process. In fact, the universal extension itself is a nonzero element of

$$\begin{aligned}\mathrm{Ext}^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1)) &= \Gamma(\mathbb{P}^1 \times \mathbb{P}(\mathcal{E}), \mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1)) \\ &= \Gamma(\mathbb{P}(\mathcal{E}), \pi_{\mathbb{P}(\mathcal{E})*}\mathcal{E}xt^1(\bar{a}^*\mathcal{T}, \bar{a}^*\mathcal{F}(0,1))) = \Gamma(\mathbb{P}(\mathcal{E}), a^*\mathcal{E}(1))\end{aligned}$$

hence determines a line subbundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})} \hookrightarrow a^*\mathcal{E}(1)$. Tensoring the line subbundle with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ and taking the dual, we obtain the universal quotient.

Thus, based on the universal property of the universal quotient, we have

Theorem 2.2. *Let R be an S-scheme with structure morphism $\rho : R \to S$, L a line bundle on R. If $0 \to \bar{\rho}^*\mathcal{F} \otimes \pi_R^* L \to \mathcal{Y} \to \bar{\rho}^*\mathcal{T} \to 0$ is a nowhere-split extension on $\mathbb{P}^1 \times R$, then there is a unique S-morphism $f : R \to \mathbb{P}(\mathcal{E})$ such that there are isomorphisms $L \overset{g}{\simeq} f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, $\mathcal{Y} \overset{h}{\simeq} \bar{f}^*\mathcal{X}$ that make the following diagram commute:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{\rho}^*\mathcal{F} \otimes \pi_R^* L & \longrightarrow & \mathcal{Y} & \longrightarrow & \bar{r}^*\mathcal{T} & \longrightarrow & 0 \\ & & {\scriptstyle 1\otimes\pi_R^* g}\downarrow\simeq & & {\scriptstyle h}\downarrow\simeq & & \| & & \\ 0 & \longrightarrow & \bar{f}^*\bar{a}^*\mathcal{F} \otimes \pi_R^* f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) & \longrightarrow & \bar{f}^*\mathcal{X} & \longrightarrow & \bar{f}^*\bar{a}^*\mathcal{T} & \longrightarrow & 0 \end{array}$$

where the second row is the pullback of the universal extension via $\bar{f}$.

3. A flattening stratification

In this section, we continue to follow the previous notations, and we impose an additional assumption on the sheaf $\mathcal{F}$ of Theorem 2.2 (also Lemma 2.1):

$$R^1\pi_{S*}(\mathcal{F} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0, \tag{3.1}$$

where $p : \mathbb{P}^1 \times S \to \mathbb{P}^1$ is the projection.

We also need the following notation (cf. Introduction).

Notation. *For any coherent sheaf H on $\mathbb{P}^1$, we denote by $H^{\mathbf{t}}$ the torsion subsheaf of H. The quotient $H/H^{\mathbf{t}}$ is locally free and we will denote it by $H^{\mathbf{f}}$. Since $H \simeq H^{\mathbf{t}} \oplus H^{\mathbf{f}}$ on $\mathbb{P}^1$, we call $H^{\mathbf{t}}$ and $H^{\mathbf{f}}$ the torsion part and the locally free part of H, respectively.*

Let d_1 and d_2 be the relative degrees of $\mathcal{F}$ and $\mathcal{T}$ over S, respectively. Then the universal extention $\mathcal{X}$ (see (2.1)) is of relative degree $d := d_1 + d_2$ over S. The set

$$\mathring{\mathbb{P}}(\mathcal{E}) := \{x \in \mathbb{P}(\mathcal{E}) \,|\, \mathcal{X}_x \text{ is locally free}\}$$

is an open subset of $\mathbb{P}(\mathcal{E})$. Its complement $\mathbb{P}(\mathcal{E}) \setminus \mathring{\mathbb{P}}(\mathcal{E})$ has a sequence of nested closed subsets:

$$\varnothing = Y_{d_1} \subset Y_{d_1+1} \subset \cdots \subset Y_{d-1} = \mathbb{P}(\mathcal{E}) \setminus \mathring{\mathbb{P}}(\mathcal{E})$$

where d is the relative degree of $\mathcal{T}$ over S, and

$$Y_r = \{x \in \mathbb{P}(\mathcal{E}) \,|\, \deg((\mathcal{X}_x)^{\mathbf{t}}) \geq d - r\}, \quad r = d_1, \ldots, d-1.$$

Note that an extension $0 \to \mathcal{F}_s \to X \to \mathcal{T}_s \to 0$ is split if and only if the torsion part of X is of degree d_2. Hence $Y_{d_1} = \varnothing$.

For $m \geq 0$, applying $\mathrm{Hom}(-, \mathcal{O}(m))$ to the extension (2.2), we obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{X}_x, \mathcal{O}(m)) \to \mathrm{Hom}(\mathcal{F}_s, \mathcal{O}(m)) \stackrel{\delta_{x,m}}{\to} \mathrm{Ext}^1(\mathcal{T}_s, \mathcal{O}(m)) \to \mathrm{Ext}^1(\mathcal{X}_x, \mathcal{O}(m)) \to 0$$

where $\delta_{x,m}$ denotes the connecting homomorphism. Then we have

Proposition 3.1. $\mathrm{rank}\,\delta_{x,m} = d_2 - \deg(\mathcal{X}_x)^{\mathbf{t}}$ *for any $m \geq d$.*

Proof. Suppose $\mathcal{F}_s$ has rank k. Then $\mathcal{X}_x$ has rank k as well. We have non-canonical isomorphisms

$$\mathcal{F}_s \simeq \bigoplus_{i=1}^{r} \mathcal{O}(a_i), \quad \mathcal{X}_x \simeq (\mathcal{X}_x)^{\mathbf{t}} \oplus \bigoplus_{i=1}^{r} \mathcal{O}(b_i)$$

where $\sum_{i=1}^r a_i = d_1$ and $\sum_{i=1}^r b_i = d - \deg(\mathcal{X}_x)^{\mathbf{t}}$. By the assumption (3.1), we have $H^1(\mathbb{P}^1, \mathcal{F}_s(-1)) = 0$ and also $H^1(\mathbb{P}^1, \mathcal{X}_x(-1)) = 0$. It follows that $a_i \geq 0$

and $b_i \geq 0$ for all i. Hence we have $a_i \leq d_1 < d$ and $b_i \leq d$ for all i. So

$$\mathrm{Hom}(\mathcal{F}_s, \mathcal{O}(m)) \simeq \bigoplus_{i=1}^{k} \mathrm{Hom}(\mathcal{O}(a_i), \mathcal{O}(m)) = \bigoplus_{i=1}^{k} H^0(\mathcal{O}(m-a_i)), \text{ and}$$

$$\mathrm{Hom}(\mathcal{X}_x, \mathcal{O}(m)) \simeq \bigoplus_{i=1}^{k} \mathrm{Hom}(\mathcal{O}(b_i), \mathcal{O}(m)) = \bigoplus_{i=1}^{k} H^0(\mathcal{O}(m-b_i)).$$

For $m \geq d$, we have $a_i \leq m$ and $b_i \leq m$, therefore

$$\begin{aligned}\operatorname{rank} \delta_{x,m} &= \dim \mathrm{Hom}(\mathcal{F}_s, \mathcal{O}(m)) - \dim \mathrm{Hom}(\mathcal{X}_x, \mathcal{O}(m)) \\ &= \sum_{i=1}^{k} (m - a_i + 1) - (m - b_i + 1) \\ &= \sum_{i=1}^{k} b_i - \sum_{i=1}^{k} a_i = d - \deg(\mathcal{X}_x)^{\mathbf{t}} - d_1 = d_2 - \deg(\mathcal{X}_x)^{\mathbf{t}}\end{aligned}$$

□

Corollary 3.2. *For any integers m, r with $m \geq d$ and $d_1 \leq r \leq d$, we have $\deg(\mathcal{X}_x)^{\mathbf{t}} \geq d - r$ if and only if $\operatorname{rank} \delta_{x,m} \leq r - d_1$.*

Notation. *For any scheme R and any coherent sheaf $\mathcal{H}$ on R, we will use the following abbreviations:*

$$\mathcal{H}^{\varepsilon} := \mathcal{E}xt^1_R(\mathcal{H}, \mathcal{O}_R)$$

Corollary 3.2 suggests us that we can define the scheme structure of Y_r in the following way. Let $m \gg 0$. Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}(\mathcal{E})}(m,1))$ to the universal extension (2.1) to obtain an exact sequence

$$0 \to \mathcal{X}^{\vee}(m,1) \to \bar{a}^*\mathcal{F}^{\vee}(m,0) \stackrel{\delta_m}{\to} \bar{a}^*\mathcal{T}^{\varepsilon}(m,1) \to \mathcal{X}^{\varepsilon}(m,1) \to 0$$

where δ_m is the connecting homomorphism. Next, applying $\pi_{\mathbb{P}(\mathcal{E})*}$ to δ_m and using identifications

$$\begin{aligned}\pi_*(\bar{a}^*\mathcal{F}^{\vee}(m,0)) &= \pi_*\bar{a}^*(\mathcal{F}^{\vee}(m)) = a^*(\pi_*\mathcal{F}^{\vee}(m)), \\ \pi_*(\bar{a}^*\mathcal{T}^{\varepsilon}(m,1)) &= \pi_*\bar{a}^*(\mathcal{T}^{\varepsilon}(m))(1) = a^*(\pi_*\mathcal{T}^{\varepsilon}(m))(1)\end{aligned}$$

we obtain a nowhere-vanishing homomorphism

$$\pi_*\delta_m : a^*(\pi_*\mathcal{F}^{\vee}(m)) \to a^*(\pi_*\mathcal{T}^{\varepsilon}(m))(1) \tag{3.2}$$

Note that both $\pi_*\mathcal{F}^\vee(m)$ and $\pi_*\mathcal{T}^\varepsilon(m)$ are locally free sheaves on S for $m \gg 0$. Applying the exterior power $\bigwedge^{l+1}$ ($l \geq 0$) to $\pi_*\delta_m$, we obtain

$$\bigwedge^{l+1} \pi_*\delta_m : \bigwedge^{l+1} a^*(\pi_*\mathcal{F}^\vee(m)) \to \bigwedge^{l+1} a^*(\pi_*\mathcal{T}^\varepsilon(m))(l+1)$$

By Corollary 3.2, Y_{d_1+l} is exactly the (set-theoretic) zero locus of $\bigwedge^{l+1} \pi_*\delta_m$, for each l with $0 \leq l < d_2$. The section $\bigwedge^{l+1} \pi_*\delta_m$ induces a homomorphism

$$\mathcal{H}om\left(\bigwedge^{l+1} a^*(\pi_*\mathcal{F}^\vee(m)), \bigwedge^{l+1} a^*(\pi_*\mathcal{T}^\varepsilon(m))(l+1)\right)^\vee \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$$

The image, which we denote by $I_{l,m}$, is an ideal sheaf. In a similar way as in [Shao11], Proposition 3.4, we can prove that

Proposition 3.3. *There exists an integer $N > 0$ such that $I_{l,m} = I_{l,N}$ as subsheaves of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$ for all $m > N$ and for all l with $0 \leq l < d_2$.*

Obviously, the subscheme defined by the ideal $I_{l,N}$ is supported on the closed subset Y_{d_1+l}. For simplicity, we denote this subscheme still by Y_{d_1+l}, that is,

(3.3)

$$Y_r = \text{ the closed subscheme defined by the ideal } I_{r-d_1+1,N}, \quad r = d_1+1, \ldots, d-1$$

In addition, we set $\mathring{Y}_r := Y_r \setminus Y_{r-1}$ for $d_1 < r < d$. Note that $\mathring{Y}_{d_1+1} = Y_{d_1+1}$ since $Y_{d_1} = \varnothing$.

Again, in a similar way as in [Shao11], Theorem 3.5, we can prove that

Proposition 3.4. *The locally closed subschemes $Y_{d_1+1}, \mathring{Y}_{d_1+2}, \cdots, \mathring{Y}_{d-1}$ of $\mathbb{P}(\mathcal{E})$ form the flattening stratification of $\mathbb{P}(\mathcal{E})$ by the sheaf $\mathcal{X}^\varepsilon$, which means that, for any noetherian $\Bbbk$-scheme R and any morphism $f : R \to \mathbb{P}(\mathcal{E})$, the sheaf $\bar{f}^*\mathcal{X}^\varepsilon$ on $\mathbb{P}^1 \times R$ is flat over R with relative degree $d - r$ if and only if f factors through the inclusion $\mathring{Y}_r \hookrightarrow \mathbb{P}(\mathcal{E})$. In particular, the restriction of $\mathcal{X}^\varepsilon$ to $\mathbb{P}^1 \times \mathring{Y}_r$ is flat over $\mathring{Y}_r$ with relative degree $d - r$.*

4. The normal bundles: a first case

Let V be a vector space of dimension n over $\Bbbk$, and $\mathrm{Gr}(k, V)$ be the Grassmannian parametrizing all the k-dimensional subspaces of V. For any $d \geq 0$, the space $\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(k, V))$ of degree d maps from $\mathbb{P}^1$ to $\mathrm{Gr}(k, V)$ is a nonsingular quasi-projective variety. A smooth compactification of $\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(k, V))$ is given by the Quot scheme $Q_d := \mathrm{Quot}^{n-k,d}_{V_{\mathbb{P}^1}/\mathbb{P}^1/\Bbbk}$, parametrizing all rank-$(n-k)$, degree-d quotients $V_{\mathbb{P}^1} \twoheadrightarrow F$ of the trivial vector bundle $V_{\mathbb{P}^1}$ of rank n on $\mathbb{P}^1$. It comes with a universal exact sequence of sheaves on $\mathbb{P}^1 \times Q_d$:

$$0 \to \mathcal{E}_d \to V_{\mathbb{P}^1 \times Q_d} \to \mathcal{F}_d \to 0$$

Here $\mathcal{F}_d$ is flat over Q_d with rank $n-k$ and relative degree d. It follows that $\mathcal{E}_d$ is locally free with rank k and relative degree $-d$ over Q_d. The open subvariety $\mathring{Q}_d := \{x \in Q_d \,|\, (\mathcal{F}_d)_x \text{ is locally free}\}$ coincides with $\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(k, V))$.

Recall from the introduction that for any $d > r \geq 0$, we have the closed subscheme

$$Z_{d,r} = \{[V_{\mathbb{P}^1} \twoheadrightarrow F] \in Q_d \mid \deg(F^{\mathbf{t}}) \geq d-r\}.$$

We refer the reader to Section 3 in [Shao11] for the details on the subscheme structure of $Z_{d,r}$. These are the subschemes that are blown up to yield the variety $\widetilde{Q}_d$. Below we analyze the normal bundle of the locally closed subsets $\mathring{Z}_{d,r} := Z_{d,r} \setminus Z_{d,r-1}$.

The subscheme $Z_{d,r}$ is closely related to the following relative Quot scheme over Q_r:

$$Q_{d,r} := \mathrm{Quot}^{0,d-r}_{\mathcal{E}_r/\mathbb{P}^1 \times Q_r/Q_r}$$

If a point of Q_r is represented by the exact sequence $E \rightarrowtail V_{\mathbb{P}^1} \twoheadrightarrow F$, then the fiber of $Q_{d,r}$ over the point consists of points represented by the quotient $E \twoheadrightarrow T$ with T torsion of degree $d-r$. Let $\theta_{d,r} : Q_{d,r} \to Q_r$ be the structure morphism. (Note that the notation for this morphism is simply θ in [Shao11]. We add sub-index in this paper because we will deal with multiple $Q_{d,r}$'s together with their structure morphisms simultaneously. The same reason applies to the other similar situations below.) In [Shao11], we showed that $Q_{d,r}$ is relatively smooth over Q_r, hence is a nonsingular variety. It comes equipped with a universal exact sequence on $\mathbb{P}^1 \times Q_{d,r}$:

$$0 \to \mathcal{E}_{d,r} \to \theta^*_{d,r}\mathcal{E}_r \to \mathcal{T}_{d,r} \to 0 \tag{4.1}$$

Here $\mathcal{T}_{d,r}$ is flat over $Q_{d,r}$ with rank 0 and relative degree $d-r$. Since $\theta^*_{d,r}\mathcal{E}_r$ is locally free of rank k and of relative degree $-r$ over $Q_{d,r}$, it follows that $\mathcal{E}_{d,r}$ is locally free of rank k and of relative degree $-d$ over $Q_{d,r}$. We set $\mathring{Q}_{d,r} := \theta^{-1}_{d,r}(\mathring{Q}_r)$ and write $\mathring{\theta}_{d,r} : \mathring{Q}_{d,r} \to Q_r$ for the restriction of $\theta_{d,r}$ on $\mathring{Q}_{d,r}$.

We form a commutative diagram on $\mathbb{P}^1 \times Q_{d,r}$:

(4.2)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_{d,r} & \longrightarrow & \bar{\theta}^*_{d,r}\mathcal{E}_r & \longrightarrow & \mathcal{T}_{d,r} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \vdots & & \\
0 & \longrightarrow & \mathcal{E}_{d,r} & \longrightarrow & V_{\mathbb{P}^1\times Q_{d,r}} & \longrightarrow & \mathcal{F}_{d,r} & \longrightarrow & 0 \\
 & & & & \downarrow & & \vdots & & \\
 & & & & \bar{\theta}^*_{d,r}\mathcal{F}_r & = & \bar{\theta}^*_{d,r}\mathcal{F}_r & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the middle column is the pullback of the universal exact sequence of Q_r via $\bar{\theta}_{d,r}$, and $\mathcal{F}_{d,r}$ is defined to be the cokernel of the composite map $\mathcal{E}_{d,r} \to \bar{\theta}^*_{d,r}\mathcal{E}_r \to V_{\mathbb{P}^1\times Q_{d,r}}$. One checks that $\mathcal{F}_{d,r}$ is flat over $Q_{d,r}$ of rank $n-k$ and relative degree d. By the universal property of Q_d, the middle row determines a morphism $\phi_{d,r} : Q_{d,r} \to Q_d$ (it is denoted by ϕ in [Shao11]) such that the following diagram commutes:

(4.3)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_{d,r} & \longrightarrow & V_{\mathbb{P}^1\times Q_{d,r}} & \longrightarrow & \mathcal{F}_{d,r} & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow\simeq & & \\
0 & \longrightarrow & \bar{\phi}^*_{d,r}\mathcal{E}_d & \longrightarrow & \bar{\phi}^*_{d,r}V_{\mathbb{P}^1\times Q_d} & \longrightarrow & \bar{\phi}^*_{d,r}\mathcal{F}_d & \longrightarrow & 0
\end{array}$$

The morphism $\phi_{d,r}$ maps $Q_{d,r}$ onto $Z_{d,r}$ (cf. Proposition 4.6 of [Shao11]). We denote by $\mathring{\phi}_{d,r} : \mathring{Q}_{d,r} \to Q_d$ the restriction of $\phi_{d,r}$ on $\mathring{Q}_{d,r}$. Note that $\mathring{\phi}_{d,0} = \phi_{d,0}$ since $\mathring{Q}_{d,0} = Q_{d,0}$. We showed that $\mathring{\phi}_{d,r}$ maps $\mathring{Q}_{d,r}$ into $\mathring{Z}_{d,r} \subset Q_d$ in [Shao11], where $\mathring{Z}_{d,r} = Z_{d,r} \backslash Z_{d,r-1}$, and we set $Z_{d,-1} := \varnothing$. We denote by $\mathring{\varphi}_{d,r} : \mathring{Q}_{d,r} \to \mathring{Z}_{d,r}$ the map obtained by restricting the codomain of $\mathring{\phi}_{d,r}$ to $\mathring{Z}_{d,r}$. In [Shao11], we showed that $\mathring{\varphi}_{d,r} : \mathring{Q}_{d,r} \to \mathring{Z}_{d,r}$ is in fact an isomorphism of schemes, hence $\mathring{\phi}_{d,r} : \mathring{Q}_{d,r} \to Q_d$ is an embedding. (cf. Proposition 4.8, [Shao11])

Proposition 4.1 ([Str87], Theorem 7.1)**.** *The tangent bundle $\mathcal{T}_{Q_d}$ of Q_d is naturally isomorphic to $\pi_* \mathcal{H}om(\mathcal{E}_d, \mathcal{F}_d)$. The relative tangent bundle $\mathcal{T}_{Q_{d,r}/Q_r}$ of $Q_{d,r}$ over Q_r is naturally isomorphic to $\pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \mathcal{T}_{d,r})$.*

Proof. The first assertion is proved in [Str87], Theorem 7.1. The second, which is a relative version of the first, can be proved by slightly modifying the proof of [Str87], Theorem 7.1. We omit the details. $\square$

The tangent bundle $\mathcal{T}_{Q_{d,r}}$ of $Q_{d,r}$ fits into the following exact sequence

$$0 \to \mathcal{T}_{Q_{d,r}/Q_r} \to \mathcal{T}_{Q_{d,r}} \to \theta_{d,r}^* \mathcal{T}_{Q_r} \to 0 \tag{4.4}$$

The morphism $\mathring{\phi}_{d,r} : \mathring{Q}_{d,r} \to Q_d$ is an embedding and is factored as

$$\mathring{Q}_{d,0} \overset{\mathring{\varphi}_{d,r}}{\tilde{\to}} \mathring{Z}_{d,r} \hookrightarrow Q_d$$

(cf. Proposition 4.8 of [Shao11]). Let $\mathcal{N}_{\mathring{Q}_{d,r}/Q_d}$ denote the normal sheaf of $\mathring{Q}_{d,0}$ in Q_d. Since $Q_{d,r}$ is nonsingular, $\mathcal{N}_{\mathring{Q}_{d,r}/Q_d}$ is locally free.

Proposition 4.2. *We have a natural identification*

$$\mathcal{N}_{\mathring{Q}_{d,r}/Q_d} = \pi_* \mathcal{E}xt^1(\mathcal{T}_{d,r}, \bar{\theta}_{d,r}^* \mathcal{F}_r)|_{\mathring{Q}_{d,r}}.$$

In particular, letting $r = 0$*, we have* $\mathcal{N}_{Q_{d,0}/Q_d} = \pi_* \mathcal{E}xt^1(\mathcal{T}_{d,0}, \bar{\theta}_{d,0}^* \mathcal{F}_0)$.

Proof. Since $\mathcal{N}_{\mathring{Q}_{d,r}/Q_d}$ is locally free, we have the following exact sequence of sheaves on $\mathring{Q}_{d,r}$:

$$0 \to \mathcal{T}_{\mathring{Q}_{d,r}} \to \mathring{\phi}_{d,r}^* \mathcal{T}_{Q_d} \to \mathcal{N}_{\mathring{Q}_{d,r}/Q_d} \to 0$$

Note that the restriction of the exact sequence (4.4) to $\mathring{Q}_{d,r}$ gives an exact sequence

$$0 \to \mathcal{T}_{\mathring{Q}_{d,r}/\mathring{Q}_r} \to \mathcal{T}_{\mathring{Q}_{d,r}} \to \mathring{\theta}_{d,r}^* \mathcal{T}_{Q_r} \to 0.$$

Combining the above two sequences, we can form a commutative diagram of sheaves on $\mathring{Q}_{d,r}$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \mathcal{T}_{\mathring{Q}_{d,r}/\mathring{Q}_r} & \longrightarrow & \mathcal{T}_{\mathring{Q}_{d,r}} & \longrightarrow & \mathring{\theta}_{d,r}^* \mathcal{T}_{Q_r} & \longrightarrow & 0 \\
\| & & \downarrow & & \vdots & & \\
0 \to \mathcal{T}_{\mathring{Q}_{d,r}/\mathring{Q}_r} & \longrightarrow & \mathring{\phi}_{d,r}^* \mathcal{T}_{Q_d} & \longrightarrow & \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \bar{\theta}_{d,r}^* \mathcal{F}_r)|_{\mathring{Q}_{d,r}} & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \\
 & & \mathcal{N}_{\mathring{Q}_{d,r}/Q_d} & = & \mathcal{N}_{\mathring{Q}_{d,r}/Q_d} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the middle row comes from the natural identifications

$$\mathcal{T}_{\mathring{Q}_{d,r}/\mathring{Q}_r} = \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \mathcal{T}_{d,r})|_{\mathring{Q}_{d,r}},$$
$$\mathring{\phi}_{d,r}^* \mathcal{T}_{Q_d} = \mathring{\phi}_{d,r}^* \pi_* \mathcal{H}om(\mathcal{E}_d, \mathcal{F}_d) = \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \mathcal{F}_{d,r})|_{\mathring{Q}_{d,r}}$$

by Proposition 4.1 and the exact sequence

$$0 \to \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \mathcal{T}_{d,r}) \to \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \mathcal{F}_{d,r}) \to \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \bar{\theta}^*_{d,r}\mathcal{F}_r) \to 0$$

obtained by applying $\pi_* \mathcal{H}om(\mathcal{E}_{d,r}, -)$ to the third column of the diagram (4.2). The dotted arrows in the third column are induced maps on the quotients. Since all rows and the middle column are exact, the third column is forced to be exact as well.

Using the identification

$$\mathring{\theta}^*_{d,r}\mathcal{T}_{Q_r} = \mathring{\theta}^*_{d,r}\pi_* \mathcal{H}om(\mathcal{E}_r, \mathcal{F}_r) = \pi_* \mathcal{H}om(\bar{\theta}^*_{d,r}\mathcal{E}_r, \bar{\theta}^*_{d,r}\mathcal{F}_r)|_{\mathring{Q}_{d,r}}$$

and comparing the third column with the short exact sequence

$$0 \to \pi_* \mathcal{H}om(\bar{\theta}^*_{d,r}\mathcal{E}_r, \bar{\theta}^*_{d,r}\mathcal{F}_r) \to \pi_* \mathcal{H}om(\mathcal{E}_{d,r}, \bar{\theta}^*_{d,r}\mathcal{F}_r) \to \pi_* \mathcal{E}xt^1(\mathcal{T}_{d,r}, \bar{\theta}^*_{d,r}\mathcal{F}_r) \to 0$$

obtained by applying $\pi_* \mathcal{H}om(-, \bar{\theta}^*_{d,0}\mathcal{F}_0)$ to the exact sequence (4.1), we obtain a natural identification

$$\mathcal{N}_{\mathring{Q}_{d,r}/Q_d} = \pi_* \mathcal{E}xt^1(\mathcal{T}_{d,r}, \bar{\theta}^*_{d,r}\mathcal{F}_r)|_{\mathring{Q}_{d,r}}.$$

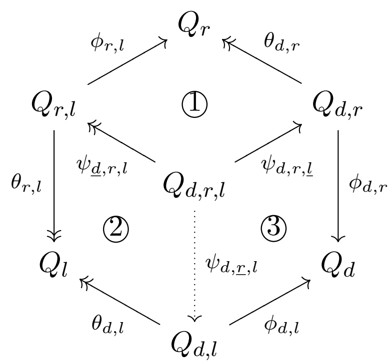
□

5. The normal bundles: the general case

In this technical section, we introduce and analyze the properties of a set of auxiliary schemes. These will be used in the final section to identify with and to derive the desired modular properties of the exceptional divisors created in the sequence of blowups $\widetilde{Q}_d \to Q_d$.

5.1. **The schemes $Q_{d,r,l}$ and morphisms $\psi_{d,\underline{r},l}$.** Let $d > r > l \geq 0$, and we consider $Q_{d,r,l} := Q_{d,r} \times_{Q_r} Q_{r,l}$. Let $\psi_{d,r,\underline{l}} : Q_{d,r,l} \to Q_{d,r}$ and $\psi_{\underline{d},r,l} : Q_{d,r,l} \to Q_{r,l}$ be the two projections. (Here the underscored subscript $\underline{l}$ in the map $\psi_{d,r,\underline{l}}$ indicates that the subscript l shows up in the source ($Q_{d,r,l}$) but not in the target ($Q_{d,r}$). Ditto for $\underline{d}$, and for $\underline{r}$ below.) First of all, we see that $Q_{d,r,l}$ is smooth over $Q_{r,l}$ because $Q_{d,r}$ is smooth over Q_r. It follows that $Q_{d,r,l}$ is a nonsingular variety. Next, we will define a finite morphism $\psi_{d,\underline{r},l} : Q_{d,r,l} \to Q_{d,l}$ based on the morphisms in the following diagram

(5.1)



The parallelogram ① is commutative by the definition of $Q_{d,r,l}$. On $\mathbb{P}^1 \times Q_{d,r,l}$ we have two short exact sequences:

$$0 \to \bar{\psi}^*_{\underline{d},r,l}\mathcal{E}_{r,l} \to \bar{\psi}^*_{\underline{d},r,l}\bar{\theta}^*_{r,l}\mathcal{E}_l \to \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l} \to 0$$
$$0 \to \bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} \to \bar{\psi}^*_{d,r,\underline{l}}\bar{\theta}^*_{d,r}\mathcal{E}_r \to \bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r} \to 0$$

which are pullbacks of the universal exact sequences of $Q_{r,l}$ and $Q_{d,r}$ via $\bar{\psi}_{\underline{d},r,l}$ and $\bar{\psi}_{d,r,\underline{l}}$ respectively. Note that $\bar{\psi}^*_{d,r,\underline{l}}\bar{\theta}^*_{d,r}\mathcal{E}_r = \bar{\psi}^*_{\underline{d},r,l}\bar{\phi}^*_{r,l}\mathcal{E}_r = \bar{\psi}_{\underline{d},r,l}\mathcal{E}_{r,l}$. Putting the two sequences together, we can form a commutative diagram as follows:

(5.2)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} & \longrightarrow & \bar{\psi}^*_{\underline{d},r,l}\mathcal{E}_{r,l} & \longrightarrow & \bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \vdots & & \\
0 & \longrightarrow & \bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} & \longrightarrow & \bar{\psi}^*_{\underline{d},r,l}\bar{\theta}^*_{r,l}\mathcal{E}_l & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\
 & & & & \downarrow & & \vdots & & \\
 & & & & \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l} & = & \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathcal{T} = \mathrm{Coker}(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} \to \bar{\psi}^*_{\underline{d},r,l}\mathcal{E}_{r,l} \to \bar{\psi}^*_{\underline{d},r,l}\bar{\theta}^*_{r,l}\mathcal{E}_l)$, and the dotted arrows are the induced maps on quotients. Since the upper two rows and the middle column are exact, the last column are forced to be exact as well. Note that $\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}$ and $\bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}$ are both flat over $Q_{d,r,l}$ with rank 0 but with relative degree $d-r$ and $r-l$ respectively. Hence $\mathcal{T}$ is flat over $Q_{d,r,l}$ as well and is of rank 0 and of relative degree $d-l$. Thus the quotient $\bar{\psi}^*_{\underline{d},r,l}\bar{\theta}^*_{r,l}\mathcal{E}_l \twoheadrightarrow \mathcal{T}$ from the middle row determines a Q_l-morphism

$$\psi_{d,\underline{r},l} : Q_{d,r,l} \to Q_{d,l}$$

such that the pullback of the universal exact sequence of $Q_{d,l}$ is the middle row of (5.2), i.e., we have the following identifications:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} & \longrightarrow & \bar{\psi}^*_{\underline{d},r,l}\bar{\theta}^*_{r,l}\mathcal{E}_l & \longrightarrow & \mathcal{T} & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & \bar{\psi}^*_{d,\underline{r},l}\mathcal{E}_{d,l} & \longrightarrow & \bar{\psi}^*_{d,\underline{r},l}\bar{\theta}^*_{d,l}\mathcal{E}_l & \longrightarrow & \bar{\psi}^*_{d,\underline{r},l}\mathcal{T}_{d,l} & \longrightarrow 0 \end{array} \tag{5.3}$$

In particular, the identification $\mathcal{T} = \bar{\psi}^*_{d,\underline{r},l}\mathcal{T}_{d,l}$ allows us to rewrite the third column of the diagram (5.2) as

$$0 \to \bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r} \to \bar{\psi}^*_{d,\underline{r},l}\mathcal{T}_{d,l} \to \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l} \to 0 \tag{5.4}$$

By the definition of $\psi_{d,\underline{r},l}$, the parallelogram ② in the diagram (5.1) automatically commutes. We also have

Proposition 5.1. *The parallelogram ③ in diagram (5.1) commutes.*

Proof. We need to show $\phi_{d,r}\psi_{d,r,\underline{l}} = \phi_{d,l}\psi_{d,\underline{r},l}$. Because both $\phi_{d,r}\psi_{d,r,\underline{l}}$ and $\phi_{d,l}\psi_{d,\underline{r},l}$ map into Q_d, by the universal property of the Quot scheme Q_d (see [Shao11], Theorem 2.1), it suffices to show that there is an isomorphism $(\overline{\phi_{d,r}\psi_{d,r,\underline{l}}})^*\mathcal{F}_d \simeq (\overline{\phi_{d,l}\psi_{d,\underline{r},l}})^*\mathcal{F}_d$ that makes the following diagram commute:

$$\begin{array}{ccccc} V_{\mathbb{P}^1\times Q_{d,r,l}} & = & (\overline{\phi_{d,r}\psi_{d,r,\underline{l}}})^*V_{\mathbb{P}^1\times Q_d} & \longrightarrow & (\overline{\phi_{d,r}\psi_{d,r,\underline{l}}})^*\mathcal{F}_d \\ \| & & & & \downarrow\simeq \\ V_{\mathbb{P}^1\times Q_{d,r,l}} & = & (\overline{\phi_{d,l}\psi_{d,\underline{r},l}})^*V_{\mathbb{P}^1\times Q_d} & \longrightarrow & (\overline{\phi_{d,l}\psi_{d,\underline{r},l}})^*\mathcal{F}_d \end{array}$$

By the diagram (4.3), we have an exact sequence

$$0 \to \mathcal{E}_{d,r} \to V_{\mathbb{P}^1\times Q_{d,r}} \to \bar{\phi}^*_{d,r}\mathcal{F}_d \to 0$$

Replacing r with l, we obtain another one

$$0 \to \mathcal{E}_{d,l} \to V_{\mathbb{P}^1\times Q_{d,l}} \to \bar{\phi}^*_{d,l}\mathcal{F}_d \to 0$$

Applying $\bar{\psi}^*_{d,r,\underline{l}}$ and $\bar{\psi}^*_{d,\underline{r},l}$ to the above two exact sequences respectively, we obtain two exact sequences

$$0 \to \bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r} \to V_{\mathbb{P}^1\times Q_{d,r,l}} \to \bar{\psi}^*_{d,r,\underline{l}}\bar{\phi}^*_{d,r}\mathcal{F}_d \to 0$$
$$0 \to \bar{\psi}^*_{d,\underline{r},l}\mathcal{E}_{d,l} \to V_{\mathbb{P}^1\times Q_{d,r,l}} \to \bar{\psi}^*_{d,\underline{r},l}\bar{\phi}^*_{d,l}\mathcal{F}_d \to 0$$

Using the identification $\bar\psi^*_{d,r,\underline{l}}\mathcal{E}_{d,r} = \bar\psi^*_{d,\underline{r},l}\mathcal{E}_{d,l}$ from the diagram (5.3), we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar\psi^*_{d,r,\underline{l}}\mathcal{E}_{d,r} & \longrightarrow & V_{\mathbb{P}^1\times Q_{d,r,l}} & \longrightarrow & \bar\psi^*_{d,r,\underline{l}}\bar\phi^*_{d,r}\mathcal{F}_d & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow \simeq & & \\ 0 & \longrightarrow & \bar\psi^*_{d,\underline{r},l}\mathcal{E}_{d,l} & \longrightarrow & V_{\mathbb{P}^1\times Q_{d,r,l}} & \longrightarrow & \bar\psi^*_{d,\underline{r},l}\bar\phi^*_{d,l}\mathcal{F}_d & \longrightarrow & 0 \end{array}$$

which induces an isomorphism $\bar\psi^*_{d,r,\underline{l}}\bar\phi^*_{d,r}\mathcal{F}_d \simeq \bar\psi^*_{d,\underline{r},l}\bar\phi^*_{d,l}\mathcal{F}_d$ in the third column. This isomorphism is the desired one. □

The map $\psi_{d,\underline{r},l}$ is both proper and quasi-finite, hence it is a finite morphism.

Notation. *For any integer $m \geq 1$, let Σ_m be the set of all strictly decreasing sequences of nonnegative integers of length m:*

$$\Sigma_m = \{(r_1, \cdots, r_m) \,|\, r_1 > \cdots > r_m\}$$

and let Σ be the set of all strictly decreasing sequences of nonnegative integers of any finite length:

$$\Sigma = \bigcup_{m=1}^{\infty} \Sigma_m$$

For any sequence $\sigma = (r_1, \cdots, r_m) \in \Sigma$, the first term r_1 is called the leading term and is denoted as $\mathrm{lt}(\sigma)$. For any integer $r > \mathrm{lt}(\sigma)$, by (r, σ) we mean the new sequence $(r, r_1, \cdots, r_m)$. The length of a sequence σ is denoted by $|\sigma|$.

When we use a sequence of Σ as sub-index, we would omit the parentheses. For example, if $\sigma = (8, 5, 3, 1, 0)$, $\tau = (5, 3, 1, 0)$ and $\lambda = (3, 1, 0)$, then the notations Q_σ, $Q_{8,\tau}$ and $Q_{8,5,\lambda}$ all mean the same thing: $Q_{8,5,3,1,0}$.

5.2. **The schemes P_σ and their properties.** We now introduce a set of spaces P_σ together with a set of coherent sheaves $\mathcal{X}_\sigma$ on $\mathbb{P}^1 \times P_\sigma$, indexed by $\sigma \in \Sigma$. We will need a set of auxiliary spaces R_σ indexed by $\sigma \in \Sigma$ with $|\sigma| \geq 2$. First, we define P_σ for $\sigma \in \Sigma_1$. Suppose $\sigma = (d)$. In this case, we set

$$P_\sigma = P_d := Q_d, \quad \mathring{P}_\sigma = \mathring{P}_d := \mathring{Q}_d, \quad \mathcal{X}_\sigma = \mathcal{X}_d := \mathcal{F}_d$$

We will denote by $\mathring{\mathcal{X}}_d$ the restriction of $\mathcal{X}_d$ on $\mathbb{P}^1 \times \mathring{P}_d$. We know that $\mathring{\mathcal{X}}_d$ is locally free.

Next we define P_σ for $\sigma \in \Sigma_2$. Suppose $\sigma = (d,r)$, $d > r$. We set

$$R_{d,r} := Q_{d,r} \times_{Q_r} P_r, \quad \mathring{R}_{d,r} := Q_{d,r} \times_{Q_r} \mathring{P}_r \subset R_{d,r}$$

Since $P_r = Q_r$ and $\mathring{P}_r = \mathring{Q}_r$, we actually have $R_{d,r} = Q_{d,r}$ and $\mathring{R}_{d,r} = \mathring{Q}_{d,r}$. Let $q_{d,r}$ and $p_{d,r}$ be the projections from $R_{d,r}$ to $Q_{d,r}$ and to P_r, respectively, and let $\mathring{q}_{d,r}$ and $\mathring{p}_{d,r}$ be the projections from $\mathring{R}_{d,r}$ to $Q_{d,r}$ and to $\mathring{P}_r$, respectively. We define $P_{d,r}$ to be a (relative) space of non-split extensions:

$$P_{d,r} = \mathbb{P}(\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,r}\mathcal{T}_{d,r}, \bar{\mathring{p}}^*_{d,r}\mathring{\mathcal{X}}_r))$$

and let $a_{d,r} : P_{d,r} \to \mathring{R}_{d,r}$ be the structure morphism. We denote the universal extension on $\mathbb{P}^1 \times P_{d,r}$ by

$$0 \to (\bar{a}^*_{d,r}\bar{\mathring{p}}^*_{d,r}\mathring{\mathcal{X}}_r)(0,1) \to \mathcal{X}_{d,r} \to \bar{a}^*_{d,r}\bar{\mathring{q}}^*_{d,r}\mathcal{T}_{d,r} \to 0$$

Let $\mathring{P}_{d,r} \subset P_{d,r}$ be the open subset defined as

$$\mathring{P}_{d,r} := \{x \in P_{d,r} \,|\, (\mathcal{X}_{d,r})_x \text{ is locally free}\}$$

We denote by $\mathring{\mathcal{X}}_{d,r}$ the restriction of $\mathcal{X}_{d,r}$ to $\mathbb{P}^1 \times \mathring{P}_{d,r}$. Then $\mathring{\mathcal{X}}_{d,r}$ is locally free, and $P_{d,r}$ can be considered as a Q_d-scheme through the composition $P_{d,r} \to \mathring{R}_{d,r} \to Q_{d,r} \to Q_d$. So we have defined P_σ, $\mathcal{X}_\sigma$, R_σ, etc., for any $\sigma \in \Sigma$ with $|\sigma| = 2$. In the following, we will define P_σ, $\mathcal{X}_\sigma$, etc., for any $\sigma \in \Sigma$ with $|\sigma| \geq 3$ inductively.

Assume that, for each $\sigma \in \Sigma_m$ for some $m \geq 2$, the space P_σ of non-split extensions is defined and the sheaf $\mathcal{X}_\sigma$ is the middle term from the universal extension on $\mathbb{P}^1 \times P_\sigma$. Assume also that a morphism $P_\sigma \to Q_l$ ($l = \mathrm{lt}(\sigma)$) has been specified so that P_σ can be considered as a Q_l-scheme.

Let $\sigma \in \Sigma_{m+1}$, $d = \mathrm{lt}(\sigma)$, τ be the sequence formed from σ by removing the leading term d, and $r = \mathrm{lt}(\tau)$. So $\sigma = (d,\tau) = (d,r,\cdots)$. By induction hypothesis, the space P_τ of non-split extensions is defined and is a Q_r-scheme. We set

$$R_\sigma := Q_{d,r} \times_{Q_r} P_\tau, \quad \mathring{R}_\sigma := Q_{d,r} \times_{Q_r} \mathring{P}_\tau \subset R_\sigma$$

Let q_σ and p_σ be the two projections from R_σ to $Q_{d,r}$ and to P_τ, respectively, and let $\mathring{q}_\sigma$ and $\mathring{p}_\sigma$ be the two projections from $\mathring{R}_\sigma$ to $Q_{d,r}$ and to $\mathring{P}_\tau$, respectively. We define $P_\sigma = P_{d,\tau}$ to be a space of non-split extensions over $\mathring{R}_\sigma$ by

$$P_\sigma := \mathbb{P}(\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_\sigma\mathcal{T}_{d,r}, \bar{\mathring{p}}^*_\sigma\mathring{\mathcal{X}}_\tau))$$

and let $a_\sigma : P_\sigma \to \mathring{R}_\sigma$ be the structure morphism. We denote the universal extension on $\mathbb{P}^1 \times P_\sigma$ by

$$0 \to (\bar{a}_\sigma^* \bar{\mathring{p}}_\sigma^* \mathring{\mathcal{X}}_\tau)(0,1) \to \mathcal{X}_\sigma \to \bar{a}_\sigma^* \bar{\mathring{q}}_\sigma^* \mathcal{T}_{d,r} \to 0$$

P_σ can be considered as a Q_d-scheme through $P_\sigma \to \mathring{R}_\sigma \to Q_{d,r} \to Q_d$. We define the open subset $\mathring{P}_\sigma \subset P_\sigma$ as

$$\mathring{P}_\sigma := \{x \in P_\sigma \mid (\mathcal{X}_\sigma)_x \text{ is locally free}\}$$

Then, $\mathring{\mathcal{X}}_\sigma$, the restriction of $\mathcal{X}_\sigma$ on $\mathbb{P}^1 \times \mathring{P}_\sigma$ is locally free. By induction, we have defined P_σ, $\mathcal{X}_\sigma$, R_σ, etc., for all $\sigma \in \Sigma$.

Lemma 5.2. *For each σ,*

(1) *$\mathcal{X}_\sigma$ is flat over $\mathbb{P}_\sigma$ with relative degree* $\mathrm{lt}(\sigma)$*.*
(2) *$R^1\pi_*(\mathcal{X}_\sigma(-1)) = 0$.*

The closed subset $P_\sigma \setminus \mathring{P}_\sigma$ of P_σ has a sequence of nested closed subschemes

$$\varnothing = Y_{d,r,\tau} \subset Y_{d,r+1,\tau} \subset \cdots \subset Y_{d,d-1,\tau} = P_\sigma \setminus \mathring{P}_\sigma$$

where $Y_{d,e,\tau} = \{x \in P_\sigma \mid \deg((\mathcal{X}_\sigma)_x^{\mathbf{t}}) \geq d-e\}$ and the subscheme structure on $Y_{d,e,\tau}$ is defined by (3.3). We set $\mathring{Y}_{d,e,\tau} := Y_{d,e,\tau} \setminus Y_{d,e-1,\tau}$ for $e = r+1, \cdots, d-1$. That is, $\mathring{Y}_{d,e,\tau} = \{x \in P_\sigma \mid \deg((\mathcal{X}_\sigma)_x^{\mathbf{t}}) = d-e\}$.

Lemma 5.3. *The space P_σ is nonsingular for any $\sigma \in \Sigma$.*

Proof. We prove it by induction on $|\sigma|$. When $|\sigma| = 1$, say $\sigma = (d)$ for some d, we have $P_\sigma = P_d = Q_d$. So this case is obvious since Q_d is nonsingular. Assume that the statement holds true for all $\sigma \in \Sigma_m$ for some m. Let $\sigma \in \Sigma_{m+1}$ and suppose $\sigma = (d, \tau)$ where $\tau \in \Sigma_m$. By definition, $P_\sigma = P_{d,\tau}$ is a projective bundle over $Q_{d,r} \times_{Q_r} \mathring{P}_\tau$ where $r = \mathrm{lt}(\tau)$. Hence P_σ is smooth over $Q_{d,r} \times_{Q_r} \mathring{P}_\tau$. We know that $Q_{d,r}$ is smooth over Q_r, hence $Q_{d,r} \times_{Q_r} \mathring{P}_\tau$ is smooth over $\mathring{P}_\tau$. Since smoothness is transitive, P_σ is smooth over $\mathring{P}_\tau$. By induction hypothesis, P_τ is nonsingular, and so is $\mathring{P}_\tau$ and hence P_σ is nonsingular. This completes the proof. $\square$

Let $\sigma \in \Sigma$, $l = \mathrm{lt}(\sigma)$ and let $d > r > l$. We define a morphism $\phi_{d,r,\sigma} : R_{d,r,\sigma} \to P_{d,\sigma}$ using the following commutative diagram

(5.5)
$$\begin{array}{ccccc} P_{r,\sigma} & \xleftarrow{\;p_{d,r,\sigma}\;} & Q_{d,r} \times_{Q_r} P_{r,\sigma} = R_{d,r,\sigma} & \xrightarrow{\;\phi_{d,r,\sigma}\;} & P_{d,\sigma} \\ \downarrow{\scriptstyle a_{r,\sigma}} & & \downarrow{\scriptstyle 1\times a_{r,\sigma}} & & \downarrow{\scriptstyle a_{d,\sigma}} \\ Q_{r,l} \times_{Q_l} \mathring{P}_\sigma & \xleftarrow{\;\psi_{\underline{d},r,l}\times 1\;} & Q_{d,r} \times_{Q_r} Q_{r,l} \times_{Q_l} \mathring{P}_\sigma & \xrightarrow{\;\psi_{d,\underline{r},l}\times 1\;} & Q_{d,l} \times_{Q_l} \mathring{P}_\sigma \\ \downarrow{\scriptstyle \mathring{q}_{r,\sigma}} & & \downarrow{\scriptstyle 1\times \mathring{q}_{r,\sigma}} & & \downarrow{\scriptstyle \mathring{q}_{d,\sigma}} \\ Q_{r,l} & \xleftarrow{\;\psi_{\underline{d},r,l}\;} & Q_{d,r} \times_{Q_r} Q_{r,l} & \xrightarrow{\;\psi_{d,\underline{r},l}\;} & Q_{d,l} \end{array}$$

By the base change property, $R_{d,r,\sigma}$ is a projective bundle over $Q_{d,r} \times_{Q_r} \mathring{R}_{r,\sigma} = Q_{d,r} \times_{Q_r} Q_{r,l} \times_{Q_l} \mathring{P}_\sigma$

$$R_{d,r,\sigma} = (Q_{d,r} \times_{Q_r} \mathring{R}_{r,\sigma}) \times_{\mathring{R}_{r,\sigma}} P_{r,\sigma} = \mathbb{P}((\psi_{\underline{d},r,l} \times 1)^* \pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{r,\sigma} \mathcal{T}_{r,l}, \bar{\mathring{p}}^*_{r,\sigma} \mathring{\mathcal{X}}_\sigma))$$

with $\mathcal{O}_{R_{d,r,\sigma}}(1) = p^*_{d,r,\sigma} \mathcal{O}_{P_{r,\sigma}}(1)$ and structure morphism $1 \times a_{r,\sigma}$. On $\mathbb{P}^1 \times R_{d,r,\sigma}$, we have a commutative diagram of sheaves

(5.6)
$$\begin{array}{ccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bar{q}^*_{d,r,\sigma}\mathcal{T}_{d,r} & = & \overline{{}^{\backprime}a_{r,\sigma}}^{\,*}\,\overline{{}^{\backprime}\mathring{q}_{r,\sigma}}^{\,*}\,\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\overline{{}^{\backprime}a_{r,\sigma}}^{\,*}\,\overline{\psi'_{d,\underline{r},l}}^{\,*}\,\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1) & \longrightarrow & \mathcal{X} & \longrightarrow & \overline{{}^{\backprime}a_{r,\sigma}}^{\,*}\,\overline{\psi'_{d,\underline{r},l}}^{\,*}\,\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{p}^*_{d,r,\sigma}((\bar{a}^*_{r,\sigma}\bar{\mathring{p}}^*_{r,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) & \longrightarrow & \bar{p}^*_{d,r,\sigma}\mathcal{X}_{r,\sigma} & \longrightarrow & \bar{p}^*_{d,r,\sigma}\bar{a}^*_{r,\sigma}\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{r,l} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where

- for short notations, we have set ${}^{\backprime}a_{r,\sigma} := 1 \times a_{r,\sigma}$, ${}^{\backprime}\mathring{q}_{r,\sigma} := 1 \times q_{r,\sigma}$, $\psi'_{d,\underline{r},l} := \psi_{d,\underline{r},l} \times 1$,
- the last row is the pullback of the universal extension of $P_{r,\sigma}$ via $\bar{p}_{d,r,\sigma} : \mathbb{P}^1 \times R_{d,r,\sigma} \to \mathbb{P}^1 \times P_{r,\sigma}$,
- the last column is the pullback of the exact sequence of torsion sheaves via $\overline{{}^{\backprime}a_{r,\sigma}}\,\overline{{}^{\backprime}\mathring{q}_{r,\sigma}} : \mathbb{P}^1 \times R_{d,r,\sigma} \to \mathbb{P}^1 \times (Q_{d,r} \times_{Q_r} Q_{r,l})$, and
- $\mathcal{X}$ is the fiber product $(\bar{p}^*_{d,r,\sigma}\mathcal{X}_{r,\sigma}) \times_{(\bar{p}^*_{d,r,\sigma}\bar{a}^*_{r,\sigma}\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{r,l})} (\overline{{}^{\backprime}a_{r,\sigma}}^{\,*}\,\overline{\psi'_{d,\underline{r},l}}^{\,*}\,\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{d,l})$ in the category of coherent sheaves.

Recall that the universal extension of $P_{d,\sigma}$ is the exact sequence

$$0 \to (\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1) \to \mathcal{X}_{d,\sigma} \to \bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} \to 0 \tag{5.7}$$

on $\mathbb{P}^1 \times P_{d,\sigma}$. By Theorem 2.2, the middle row determines a $(Q_{d,l} \times_{Q_l} \mathring{P}_\sigma)$-morphism

$$\phi_{d,r,\sigma} : R_{d,r,\sigma} \to P_{d,\sigma}$$

such that there are isomorphisms $\mathcal{O}_{R_{d,r,\sigma}}(1) \simeq \phi^*_{d,r,\sigma}\mathcal{O}_{P_{d,\sigma}}(1)$ and $\mathcal{X} \simeq \bar{\phi}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma}$ that make the following diagram commute:

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\overline{`a_{r,\sigma}}^*\overline{\psi'_{d,\underline{r},l}}^*\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1) & \longrightarrow & \mathcal{X} & \longrightarrow & \overline{`a_{r,\sigma}}^*\overline{\psi'_{d,\underline{r},l}}^*\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} & \longrightarrow 0 \\
& \downarrow\simeq & & \downarrow\simeq & & \| & \\
0 \longrightarrow & \bar{\phi}^*_{d,r,\sigma}((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) & \longrightarrow & \bar{\phi}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma} & \longrightarrow & \bar{\phi}^*_{d,r,\sigma}\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} & \longrightarrow 0
\end{array}$$

where the first row is the middle row of diagram (5.6) and the second row is the pullback of the universal extension (5.7) of $P_{d,\sigma}$ via $\bar{\phi}_{d,r,\sigma} : \mathbb{P}^1 \times R_{d,r,\sigma} \to \mathbb{P}^1 \times P_{d,\sigma}$. For simplicity, we make identifications

$$\mathcal{O}_{R_{d,r,\sigma}}(1) = \phi^*_{d,r,\sigma}\mathcal{O}_{P_{d,\sigma}}(1), \qquad \mathcal{X} = \bar{\phi}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma}. \tag{5.8}$$

Let $\mathring{\phi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \to P_{d,\sigma}$ be the restriction of $\phi_{d,r,\sigma}$ to $\mathring{R}_{d,r,\sigma}$. The restriction of the middle column of diagram (5.6) to $\mathbb{P}^1 \times \mathring{R}_{d,r,\sigma}$ is the exact sequence

$$0 \to \bar{\mathring{q}}^*_{d,r,\sigma}\mathcal{T}_{d,r} \to \bar{\mathring{\phi}}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma} \to \bar{\mathring{p}}^*_{d,r,\sigma}\mathring{\mathcal{X}}_{r,\sigma} \to 0 \tag{5.9}$$

Proposition 5.4. *The map $\mathring{\phi}_{d,r,\sigma}$ factors through the inclusion $\mathring{Y}_{d,r,\sigma} \hookrightarrow P_{d,\sigma}$.*

Proof. Taking the dual of the sequence (5.9), we obtain $(\bar{\mathring{\phi}}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma})^\varepsilon \simeq (\bar{\mathring{q}}^*_{d,r,\sigma}\mathcal{T}_{d,r})^\varepsilon$. Since $(\bar{\mathring{\phi}}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma})^\varepsilon = \bar{\mathring{\phi}}^*_{d,r,\sigma}(\mathcal{X}^\varepsilon_{d,\sigma})$, we have that $\bar{\mathring{\phi}}^*_{d,r,\sigma}(\mathcal{X}^\varepsilon_{d,\sigma})$ is flat over $\mathring{R}_{d,r,\sigma}$ with relative degree $d - r$. By Proposition 3.4, the map $\mathring{\phi}_{d,r,\sigma}$ factors through the inclusion $\mathring{Y}_{d,r,\sigma} \hookrightarrow P_{d,\sigma}$. □

We denote by $\mathring{\varphi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \to \mathring{Y}_{d,r,\sigma}$ the map factored out from $\mathring{\phi}_{d,r,\sigma}$.

Proposition 5.5. *The morphism $\mathring{\varphi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \to \mathring{Y}_{d,r,\sigma}$ is an isomorphism.*

Proof. We prove by constructing an inverse of $\mathring{\varphi}_{d,r,\sigma}$. Let $i : \mathring{Y}_{d,r,\sigma} \hookrightarrow P_{d,\sigma}$ be the inclusion map. The pullback of the universal extension (5.7) via i is the exact sequence

$$0 \to \bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) \to \bar{i}^*\mathcal{X}_{d,\sigma} \to \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} \to 0$$

on $\mathbb{P}^1 \times \mathring{Y}_{d,r,\sigma}$. Taking dual, we obtain a long exact sequence

$$0 \to (\bar{i}^*\mathcal{X}_{d,\sigma})^\vee \to (\bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)))^\vee \to (\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l})^\varepsilon \to (\bar{i}^*\mathcal{X}_{d,\sigma})^\varepsilon \to 0$$

We break it into two short exact sequences

$$0 \to (\bar{i}^*\mathcal{X}_{d,\sigma})^\vee \to (\bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)))^\vee \to \mathcal{T} \to 0$$

$$0 \to \mathcal{T} \to (\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l})^\varepsilon \to (\bar{i}^*\mathcal{X}_{d,\sigma})^\varepsilon \to 0$$

We have $(\bar{i}^*\mathcal{X}_{d,\sigma})^\varepsilon = \bar{i}^*(\mathcal{X}^\varepsilon_{d,\sigma})$ and $(\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l})^\varepsilon = \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}(\mathcal{T}^\varepsilon_{d,l})$. By Proposition 3.4, $\bar{i}^*(\mathcal{X}^\varepsilon_{d,\sigma})$ is flat over $\mathring{Y}_{d,r,\sigma}$ with relative degree $d-r$. Since $\mathcal{T}^\varepsilon_{d,l}$ is also flat over $Q_{d,l}$ with relative degree $d-l$, we have $\mathcal{T}$ is flat over $\mathring{Y}_{d,r,\sigma}$ with relative degree $r-l$. We also know that both $(\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l})^\varepsilon$ and $(\bar{i}^*\mathcal{X}_{d,\sigma})^\varepsilon$ are torsion, hence $\mathcal{T}$ is also torsion. Since the middle term of the first sequence is locally free and the last term is flat over $\mathring{Y}_{d,r,\sigma}$, the first term, $(\bar{i}^*\mathcal{X}_{d,\sigma})^\vee$, is locally free as well.

Now dualizing both of the above sequences, we obtain another two exact sequences

$$0 \to \bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) \to (\bar{i}^*\mathcal{X}_{d,\sigma})^{\vee\vee} \to \mathcal{T}^\varepsilon \to 0 \tag{5.10}$$

$$0 \to (\bar{i}^*\mathcal{X}_{d,\sigma})^{\varepsilon\varepsilon} \to \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} \to \mathcal{T}^\varepsilon \to 0 \tag{5.11}$$

The Quot scheme $Q_{d,l}$ has a universal quotient $\bar{\theta}^*_{d,l}\mathcal{E}_l \twoheadrightarrow \mathcal{T}_{d,l}$. Applying $\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}$ to this quotient, we obtain a quotient $\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\bar{\theta}^*_{d,l}\mathcal{E}_l \twoheadrightarrow \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}$. Its composition with the quotient $\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} \twoheadrightarrow \mathcal{T}^\varepsilon$ from the sequence (5.11) yields a quotient

We form a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{E}_{d,l} & \dashrightarrow & & \mathcal{E} & \dashrightarrow & (\bar{i}^*\mathcal{X}_{d,\sigma})^{\varepsilon\varepsilon} & \longrightarrow 0 & \\
 & \| & & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{E}_{d,l} & \longrightarrow & & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\bar{\theta}^*_{d,l}\mathcal{E}_l & \longrightarrow & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} & \longrightarrow 0 & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{T}^\varepsilon & = & \mathcal{T}^\varepsilon & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the last column is the sequence (5.11), the middle row is the pullback of the universal exact sequence of $Q_{d,l}$ via $\bar{\mathring{q}}_{d,\sigma}\bar{a}_{d,\sigma}\bar{i} : \mathbb{P}^1 \times \mathring{Y}_{d,r,\sigma} \to \mathbb{P}^1 \times Q_{d,l}$, and $\mathcal{E}$ is defined to be the kernel of the composition $\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\bar{\theta}^*_{d,l}\mathcal{E}_l \twoheadrightarrow \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l} \twoheadrightarrow \mathcal{T}^\varepsilon$. The dotted arrows in the first row are the induced maps. Since $\mathcal{T}^\varepsilon$ is torsion and flat over $\mathring{Y}_{d,r,\sigma}$ with relative degree $r - l$, the quotient $\bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\bar{\theta}^*_{d,l}\mathcal{E}_l \twoheadrightarrow \mathcal{T}^\varepsilon$ from the middle column induces a Q_l-morphism $\lambda : \mathring{Y}_{d,r,\sigma} \to Q_{r,l}$ such that the pullback of the universal exact sequence of $Q_{r,l}$ via $\bar{\lambda}$ is the same as the middle column:

$$
\begin{array}{ccccccccc}
0 & \to & \bar{\lambda}^*\mathcal{E}_{r,l} & \longrightarrow & \bar{\lambda}^*\bar{\theta}^*_{r,l}\mathcal{E}_l & \longrightarrow & \bar{\lambda}^*\mathcal{T}_{r,l} & \to & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\bar{\theta}^*_{d,l}\mathcal{E}_l & \longrightarrow & \mathcal{T}^\varepsilon & \longrightarrow & 0
\end{array}
\tag{5.12}
$$

We have a morphism $\phi_{r,l} : Q_{r,l} \to Q_r$ and an identification $\mathcal{E}_{r,l} = \bar{\phi}^*_{r,l}\mathcal{E}_r$ as in diagram (4.3). Thus we have an identification

$$\mathcal{E} = \bar{\lambda}^*\mathcal{E}_{r,l} = \bar{\lambda}^*\bar{\phi}^*_{r,l}\mathcal{E}_r$$

and we can rewrite the first row as

$$0 \to \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{E}_{d,l} \to \bar{\lambda}^*\bar{\phi}^*_{r,l}\mathcal{E}_r \to (\bar{i}^*\mathcal{X}_{d,\sigma})^{\varepsilon\varepsilon} \to 0$$

Since $(\bar{i}^*\mathcal{X}_{d,\sigma})^{\varepsilon\varepsilon}$ is torsion and flat over $\mathring{Y}_{d,r,\sigma}$ with relative degree $d-r$, the above sequence induces a morphism $\mu : \mathring{Y}_{d,r,\sigma} \to Q_{d,r}$ such that the pullback of the universal exact sequence of $Q_{d,r}$ via $\bar{\mu}$ is the same as the above sequence:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{\mu}^*\mathcal{E}_{d,r} & \longrightarrow & \bar{\mu}^*\bar{\theta}^*_{d,r}\mathcal{E}_r & \longrightarrow & \bar{\mu}^*\mathcal{T}_{d,r} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \to & \bar{i}^*\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{E}_{d,l} & \longrightarrow & \bar{\lambda}^*\bar{\phi}^*_{r,l}\mathcal{E}_r & \longrightarrow & (\bar{i}^*\mathcal{X}_{d,\sigma})^{\varepsilon\varepsilon} & \longrightarrow & 0
\end{array}
$$

Using the identification $\mathcal{T}^\varepsilon = \bar{\lambda}^*\mathcal{T}_{r,l}$ from sequence (5.12), we can rewrite the sequence (5.10) as

$$0 \to \bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) \to (\bar{i}^*\mathcal{X}_{d,\sigma})^{\vee\vee} \to \bar{\lambda}^*\mathcal{T}_{r,l} \to 0$$

By Theorem 2.2, the above exact sequence determines a morphism $\nu : \mathring{Y}_{d,r,\sigma} \to P_{r,\sigma}$ such that the pullback of the universal extension of $P_{r,\sigma}$ is the same as the above sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \bar{\nu}^*((\bar{a}^*_{r,\sigma}\bar{\mathring{p}}^*_{r,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) & \longrightarrow & \bar{\nu}^*\mathcal{X}_{r,\sigma} & \longrightarrow & \bar{\nu}^*\bar{a}^*_{r,\sigma}\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{r,l} & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & \bar{i}^*((\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)(0,1)) & \longrightarrow & (\bar{i}^*\mathcal{X}_{d,\sigma})^{\vee\vee} & \longrightarrow & \bar{\lambda}^*\mathcal{T}_{r,l} & \longrightarrow 0 \end{array}$$

But because the middle term $(\bar{i}^*\mathcal{X}_{d,\sigma})^{\vee\vee}$ is locally free, the map ν actually maps $\mathring{Y}_{d,r,\sigma}$ into $\mathring{P}_{r,\sigma}$. So we obtain a morphism

$$\mu \times \nu : \mathring{Y}_{d,r,\sigma} \to Q_{d,r} \times_{Q_r} \mathring{P}_{r,\sigma} := \mathring{R}_{d,r,\sigma}$$

It is now routine to check that $\mu \times \nu$ is the inverse of $\mathring{\varphi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \to \mathring{Y}_{d,r,\sigma}$, and this complete the proof. □

The above proposition shows that $\mathring{\phi}_{d,r,\sigma}$ is an embedding. So we can identify $\mathring{R}_{d,r,\sigma}$ with the subscheme $\mathring{Y}_{d,r,\sigma}$ of $P_{d,\sigma}$ and identify $\mathring{\phi}_{d,r,\sigma}$ with the inclusion map. We also have $\bar{\mathring{\phi}}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma} = \mathcal{X}_{\mathring{R}_{d,r,\sigma}}$. So the exact sequence (5.9) can be rewritten as:

$$0 \to \bar{\mathring{q}}^*_{d,r,\sigma}\mathcal{T}_{d,r} \to \bar{\mathring{\phi}}^*_{d,r,\sigma}\mathcal{X}_{d,\sigma} \to \bar{\mathring{p}}^*_{d,r,\sigma}\mathring{\mathcal{X}}_{r,\sigma} \to 0$$

Since $R_{d,r,\sigma}$ and $P_{d,\sigma}$ are both nonsingular, we can talk about their tangent bundles as well as the normal bundle of the embedding $\mathring{\phi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \hookrightarrow P_{d,\sigma}$.

Proposition 5.6. *The normal bundle $\mathcal{N}_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}}$ of the embedding $\mathring{\phi}_{d,r,\sigma} : \mathring{R}_{d,r,\sigma} \hookrightarrow P_{d,\sigma}$ is isomorphic to $\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,r,\sigma}\mathcal{T}_{d,r}, \bar{\mathring{p}}^*_{d,r,\sigma}\mathring{\mathcal{X}}_{r,\sigma})$.*

Proof. Since $P_{d,\sigma}$ is a projective bundle over $\mathring{R}_{d,\sigma}$, the relative cotangent bundle $\Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}}$ fits into the following exact sequence

$$0 \to \Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}} \to (a^*_{d,\sigma}\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma))^\vee(-1) \to \mathcal{O}_{P_{d,\sigma}} \to 0$$

Pulling the sequence back to $R_{d,r,\sigma}$ via $\phi_{d,r,\sigma}$, we obtain an exact sequence (5.13)

$$0 \to \phi^*_{d,r,\sigma}\Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}} \to (\phi^*_{d,r,\sigma}a^*_{d,\sigma}\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma))^\vee(-1) \to \phi^*_{d,r,\sigma}\mathcal{O}_{P_{d,\sigma}} \to 0$$

We can rewrite $\phi^*_{d,r,\sigma}a^*_{d,\sigma}\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma)$ as

$$\begin{aligned} &\phi^*_{d,r,\sigma}a^*_{d,\sigma}\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma) \\ =\ &\pi_* \mathcal{E}xt^1(\bar{\phi}^*_{d,r,\sigma}\bar{a}^*_{d,\sigma}\bar{\mathring{q}}^*_{d,\sigma}\mathcal{T}_{d,l}, \bar{\phi}^*_{d,r,\sigma}\bar{a}^*_{d,\sigma}\bar{\mathring{p}}^*_{d,\sigma}\mathring{\mathcal{X}}_\sigma) \\ =\ &\pi_* \mathcal{E}xt^1(\bar{f}^*\bar{\psi}^*_{d,\underline{r},l}\mathcal{T}_{d,l}, \bar{g}^*\mathring{\mathcal{X}}_\sigma) \end{aligned}$$

where

$$f := (1 \times \mathring{q}_{r,\sigma})(1 \times a_{r,\sigma}), \quad g := \mathring{p}_{d,\sigma} a_{d,\sigma} \phi_{d,r,\sigma}.$$

Set $QQP := Q_{d,r} \times_{Q_r} Q_{r,l} \times_{Q_l} \mathring{P}_\sigma$. Since $R_{d,r,\sigma}$ is a projective bundle over QQP, the relative cotangent bundle $\Omega_{R_{d,r,\sigma}/QQP}$ fits into the following exact sequence

(5.14)

$$0 \to \Omega_{R_{d,r,\sigma}/QQP} \to ((1 \times a_{r,\sigma})^*(\psi_{\underline{d},r,l} \times 1)^* \pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{r,l}, \bar{\mathring{p}}^*_{r,\sigma}\mathring{\mathcal{X}}_\sigma))^\vee(-1) \to \mathcal{O}_{R_{d,r,\sigma}} \to 0$$

We have

$$(1 \times a_{r,\sigma})^*(\psi_{\underline{d},r,l} \times 1)^* \pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{r,\sigma}\mathcal{T}_{r,l}, \bar{\mathring{p}}^*_{r,\sigma}\mathring{\mathcal{X}}_\sigma) = \pi_* \mathcal{E}xt^1(\bar{f}^* \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}, \bar{g}^*\mathring{\mathcal{X}}_\sigma)$$

based on the equalities $\mathring{q}_{r,\sigma}(\psi_{\underline{d},r,l} \times 1)(1 \times a_{r,\sigma}) = \psi_{\underline{d},r,l} f$ and $\mathring{p}_{r,\sigma}(\psi_{\underline{d},r,l} \times 1)(1 \times a_{r,\sigma}) = g$.

We make a diagram

(5.15)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{K} : & = & (\pi_* \mathcal{E}xt^1(\bar{f}^* \bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{g}^*\mathring{\mathcal{X}}_\sigma))^\vee(-1) & & & & \\
 & & \vdots & & \downarrow & & & & \\
0 & \longrightarrow & \phi^*_{d,r,\sigma}\Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}} & \longrightarrow & (\pi_* \mathcal{E}xt^1(\bar{f}^* \bar{\psi}^*_{d,\underline{r},l}\mathcal{T}_{d,l}, \bar{g}^*\mathring{\mathcal{X}}_\sigma))^\vee(-1) & \longrightarrow & \phi^*_{d,r,\sigma}\mathcal{O}_{P_{d,\sigma}} & \longrightarrow & 0 \\
 & & \vdots & & \downarrow & \textcircled{1} & \| & & \\
0 & \longrightarrow & \Omega_{R_{d,r,\sigma}/QQP} & \longrightarrow & (\pi_* \mathcal{E}xt^1(\bar{f}^* \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}, \bar{g}^*\mathring{\mathcal{X}}_\sigma))^\vee(-1) & \longrightarrow & \mathcal{O}_{R_{d,r,\sigma}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where

- the last row is the exact sequence (5.13),
- the middle row is the exact sequence (5.14), and
- the middle column is obtained by applying $\pi_* \mathcal{E}xt^1(\bar{f}^*(-), \bar{g}^*\mathring{\mathcal{X}}_\sigma)$ to the exact sequence (5.4), then taking dual, and lastly twisting by $\mathcal{O}_{R_{d,r,\sigma}}(-1)$.

The commutativity of the rectangle ① in the diagram (5.15) follows from diagram (5.6) and the identifications (5.8). Thus, we have induced maps (the dotted arrows) in the first column and the first column is exact. Restricting the first column to $\mathring{R}_{d,r,\sigma}$, we obtain an exact sequence

$$0 \to \mathring{\mathcal{K}} \to \mathring{\phi}^*_{d,r,\sigma}\Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}} \to \Omega_{\mathring{R}_{d,r,\sigma}/QQP} \to 0$$

where $\mathring{\mathcal{K}} := \mathcal{K}|_{\mathring{R}_{d,r,\sigma}}$. We have

$$\mathring{\mathcal{K}} = (\pi_* \mathcal{E}xt^1(\bar{\mathring{f}}^* \bar{\psi}^*_{d,r,\underline{l}} \mathcal{T}_{d,r}, \bar{\mathring{g}}^* \mathring{\mathcal{X}}_\sigma))^\vee(-1)$$

where $\mathring{f}$ and $\mathring{g}$ are the restrictions of f and g to $\mathring{R}_{d,r,\sigma}$.

Recall that $\psi_{d,r,l} : Q_{d,r,l} \to Q_{d,l}$ is a finite morphism. Since $\Bbbk$ is assumed to be of characteristic 0, the function field extension $K(Q_{d,r,l})/K(Q_{d,l})$ is separable. Therefore, we have an exact sequence of relative cotangent sheaves

$$0 \to \psi^*_{d,\underline{r},l}\Omega_{Q_{d,l}} \to \Omega_{Q_{d,r,l}} \to \Omega_{Q_{d,r,l}/Q_{d,l}} \to 0$$

where both $\Omega_{Q_{d,l}}$ and $\Omega_{Q_{d,r,l}}$ are locally free of the same rank while $\Omega_{Q_{d,r,l}/Q_{d,l}}$ is torsion.

We form the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\psi^*_{\underline{d},r,l}\theta^*_{r,l}\Omega_{Q_l} & & 0 & & 0 & \longrightarrow & (\pi_* \mathcal{H}om(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\vee & & \\
 & \diagdown\!\!\diagdown & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \psi^*_{d,\underline{r},l}\theta^*_{d,l}\Omega_{Q_l} & \longrightarrow & \psi^*_{d,\underline{r},l}\Omega_{Q_{d,l}} & \longrightarrow & \psi^*_{d,\underline{r},l}\Omega_{Q_{d,l}/Q_l} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \psi^*_{\underline{d},r,l}\Omega_{Q_{r,l}} & \longrightarrow & \Omega_{Q_{d,r,l}} & \longrightarrow & \Omega_{Q_{d,r,l}/Q_{r,l}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \psi^*_{\underline{d},r,l}\Omega_{Q_{r,l}/Q_l} & \dashrightarrow & \Omega_{Q_{d,r,l}/Q_{d,l}} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Applying the Snake Lemma to the third row and the fourth row of the above diagram, we obtain an exact sequence by connecting the second row with the last row:

$$0 \to (\pi_* \mathcal{H}om(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\vee \to \psi^*_{\underline{d},r,l}\Omega_{Q_{r,l}/Q_l} \to \Omega_{Q_{d,r,l}/Q_{d,l}} \to 0$$

This sequence fits into the following commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
(\pi_* \mathcal{H}om(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\vee & = & (\pi_* \mathcal{H}om(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\vee \\
\downarrow & & \downarrow \\
\psi^*_{\underline{d},r,l}\Omega_{Q_{r,l}/Q_l} & = & (\pi_* \mathcal{H}om(\bar{\psi}^*_{\underline{d},r,l}\mathcal{E}_{r,l}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\vee \\
\downarrow & & \downarrow \\
\Omega_{Q_{d,r,l}/Q_{d,l}} & \cdots\cdots & (\pi_* \mathcal{E}xt^1(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\varepsilon \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the identification in the second row follows from the canonical identification $\mathcal{T}_{Q_{r,l}/Q_l} = \pi_* \mathcal{H}om(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{E}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l})$. So we obtain an identification of the quotients

$$\Omega_{Q_{d,r,l}/Q_{d,l}} = (\pi_* \mathcal{E}xt^1(\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\varepsilon$$

Since $Q_{d,r,l}$ is smooth over Q_l, $QQP = Q_{d,r,l} \times_{Q_l} \mathring{P}_\sigma$ is smooth over $\mathring{P}_\sigma$. The morphism $\psi_{d,\underline{r},l} \times 1 : QQP \to \mathring{R}_{d,\sigma}$ is obtained from $\psi_{d,\underline{r},l}$ by the base change $\mathring{q}_{d,\sigma} : \mathring{R}_{d,\sigma} \to Q_{d,l}$, hence we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\psi_{d,\underline{r},l} \times 1)^*\Omega_{\mathring{P}_{d,\sigma}} & \longrightarrow & \Omega_{QQP} & \longrightarrow & \Omega_{QQP/\mathring{P}_{d,\sigma}} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & (1 \times \mathring{q}_{d,\sigma})^*\psi^*_{d,\underline{r},l}\Omega_{Q_{d,l}} & \longrightarrow & (1 \times \mathring{q}_{d,\sigma})^*\Omega_{Q_{d,r,l}} & \longrightarrow & (1 \times \mathring{q}_{d,\sigma})^*\Omega_{Q_{d,r,l}/Q_{d,l}} & \longrightarrow & 0
\end{array}$$

The conormal bundle $\mathcal{N}^\vee_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}}$ fits into the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{N}^\vee_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}} & \longrightarrow & \mathring{\mathcal{K}} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathring{\phi}^*_{d,r,\sigma} a^*_{d,\sigma}\Omega_{\mathring{R}_{d,\sigma}} & \longrightarrow & \mathring{\phi}^*_{d,r,\sigma}\Omega_{P_{d,\sigma}} & \longrightarrow & \mathring{\phi}^*_{d,r,\sigma}\Omega_{P_{d,\sigma}/\mathring{R}_{d,\sigma}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (1\times \mathring{a}_{r,\sigma})^*\Omega_{QQP} & \longrightarrow & \Omega_{\mathring{R}_{d,r,\sigma}} & \longrightarrow & \Omega_{\mathring{R}_{d,r,\sigma}/QQP} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathring{f}^*\Omega_{Q_{d,r,l}/Q_{d,l}} & \longrightarrow & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Applying the Snake Lemma to the third and fourth rows, we obtain an exact sequence by connecting the second row with the fifth row:

$$0 \to \mathcal{N}^\vee_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}} \to \mathring{\mathcal{K}} \to \mathring{f}^*\Omega_{Q_{d,r,l}/Q_{d,l}} \to 0$$

The above sequence fits into the following commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{N}^\vee_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}} & \cdots\cdots & (\pi_*\,\mathcal{E}xt^1(\bar{\mathring{f}}^*\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{\mathring{p}}^*_{d,r,\sigma}\mathring{\mathcal{X}}_{r,\sigma}))^\vee \\
\downarrow & & \downarrow \\
\mathring{\mathcal{K}} & = & (\pi_*\,\mathcal{E}xt^1(\bar{\mathring{f}}^*\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{g}^*\mathring{\mathcal{X}}_{\sigma}))^\vee(-1) \\
\downarrow & & \downarrow \\
\mathring{f}^*\Omega_{Q_{d,r,l}/Q_{d,l}} & = & (\pi_*\,\mathcal{E}xt^1(\bar{\mathring{f}}^*\bar{\psi}^*_{d,r,\underline{l}}\mathcal{T}_{d,r}, \bar{\mathring{f}}^*\bar{\psi}^*_{\underline{d},r,l}\mathcal{T}_{r,l}))^\varepsilon \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where the second column is obtained as follows: first we can rewrite the last row of diagram (5.6) as

$$0 \to (\bar{g}^*\mathring{\mathcal{X}}_\sigma)(0,1) \to \bar{p}^*_{d,r,\sigma}\mathcal{X}_{r,\sigma} \to \bar{f}^*\bar{\psi}_{\underline{d},r,l}\mathcal{T}_{r,l} \to 0;$$

next the restriction to $\mathbb{P}^1 \times \mathring{R}_{d,r,\sigma}$ is

$$0 \to (\bar{\mathring{g}}^*\mathring{\mathcal{X}}_\sigma)(0,1) \to \bar{\mathring{p}}^*_{d,r,\sigma}\mathring{\mathcal{X}}_{r,\sigma} \to \bar{\mathring{f}}^*\bar{\psi}_{\underline{d},r,l}\mathcal{T}_{r,l} \to 0;$$

and next applying $\pi_* \mathcal{H}om(\bar{\mathring{f}}^* \bar{\psi}^*_{d,r,\underline{l}} \mathcal{T}_{d,r}, -)$ to the above sequence, and lastly taking dual, we obtain the second column in the diagram. Thus there is an induced identification as in the first row, which gives the natural identification

$$\mathcal{N}_{\mathring{R}_{d,r,\sigma}/P_{d,\sigma}} = \pi_* \mathcal{E}xt^1(\bar{\mathring{f}}^* \bar{\psi}^*_{d,r,\underline{l}} \mathcal{T}_{d,r}, \bar{\mathring{p}}^*_{d,r,\sigma} \mathring{\mathcal{X}}_{r,\sigma}) = \pi_* \mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,r,\sigma} \mathcal{T}_{d,r}, \bar{\mathring{p}}^*_{d,r,\sigma} \mathring{\mathcal{X}}_{r,\sigma})$$

□

6. The modular interpretation

The compactification $\widetilde{Q}_d$ is obtained by successively blowing up the Quot scheme Q_d along $Z_{d,0}, \cdots, Z_{d,d-1}$. We illustrate the process in the following diagram

$$\begin{array}{cccccccl}
Z^{d-1}_{d,0} & Z^{d-1}_{d,1} & Z^{d-1}_{d,2} & \cdots & Z^{d-1}_{d,d-1} \subset & Q^{d-1}_d = \widetilde{Q}_d & & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \leftarrow & \text{along } Z^{d-2}_{d,d-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \leftarrow & \text{along } Z^{1}_{d,2} \\
Z^1_{d,0} & Z^1_{d,1} & Z^1_{d,2} \subset & \cdots \subset & Z^1_{d,d-1} \subset & Q^1_d & & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \leftarrow & \text{along } Z^{0}_{d,1} \\
Z^0_{d,0} & Z^0_{d,1} \subset & Z^0_{d,2} \subset & \cdots \subset & Z^0_{d,d-1} \subset & Q^0_d & & \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \leftarrow & \text{along } Z_{d,0} \\
Z_{d,0} \subset & Z_{d,1} \subset & Z_{d,2} \subset & \cdots \subset & Z_{d,d-1} \subset & Q_d & &
\end{array}$$

Here, inductively, $Z^j_{d,0}, \cdots, Z^j_{d,j}$ are the exceptional divisors created by the sequence of blowups $Q^j_d \to Q_d$ $(0 \le j \le d-1)$; the nested subschemes $Z^j_{d,j+1} \subset \cdots \subset Z^j_{d,d-1}$ are the proper transforms of the subschemes $Z^{j-1}_{d,j+1} \subset \cdots \subset Z^{j-1}_{d,d-1}$ (respectively); these are the subschemes lining up to be blown-up in the next steps. Below, we will provide modular meanings to the points of $Z^j_{d,0}, \cdots, Z^j_{d,j}$ for all j. Thus every intermediate space Q^j_d, $j = 0, \ldots, d-1$, also compactification of $\mathring{Q}_d$, admits parameter space interpretation (Proposition. 6.4). The case Q^{d-1}_d is our final space $\widetilde{Q}_d$ (Corollary 6.5).

To proceed, we introduce the following

$$\hat{Z}^l_{d,r} := \begin{cases} \varnothing, & \text{if } r = l \\ Z^l_{d,r}, & \text{if } l < r < d \end{cases}, \quad \mathring{Z}^l_{d,r} := Z^l_{d,r} \setminus \hat{Z}^l_{d,r-1}, \quad r = l+1, \ldots, d-1$$

Let $d > r \geq 0$. For each subsequence $\sigma \subset [r] := (r, r-1, \cdots, 1, 0) \in \Sigma$, we set

$$E^r_{d,\sigma} := \bigcap_{i \in \sigma} Z^r_{d,i} \setminus \bigcup_{i \in [r]\setminus\sigma} Z^r_{d,i}.$$

Then there is a stratification of Q^r_d:

$$Q^r_d = (Q^r_d \setminus \bigcup_{i \in [r]} Z^r_{d,i}) \sqcup \bigsqcup_{\sigma \subset [r]} E^r_{d,\sigma} = (Q_d \setminus Z_{d,r}) \sqcup \bigsqcup_{\sigma \subset [r]} E^r_{d,\sigma}.$$

Lemma 6.1. *Let $\tau \in \Sigma$, $t = \mathrm{lt}(\tau)$. We have*

(1) $E^j_{d,\tau} = E^m_{d,\tau} \setminus Z^m_{d,j}$, *for any m and j with $t \leq m < j < d$.*
(2) *if $l > t$ and $\sigma = (l, \tau)$, then $E^l_{d,\sigma}$ is the exceptional divisor of the blowup of $E^t_{d,\tau} \setminus \hat{Z}^t_{d,l-1}$ along $E^t_{d,\tau} \cap \mathring{Z}^t_{d,l}$.*

Proof. (1) Consider the composite blowup $b : Q^j_d \to Q^m_d$, $m < j$. We have $b^{-1}(Z^m_{d,i}) = Z^j_{d,i}$ for $i = 0, \ldots, m$, and $b^{-1}(Z^m_{d,j}) = \bigcup_{i=m+1}^{j} Z^j_{d,i}$. Moreover, b is an isomorphism away from $Z^m_{d,j}$. These facts give us identifications

$$Q^j_d \setminus b^{-1}(Z^m_{d,j}) = Q^m_d \setminus Z^m_{d,j}, \text{ and } Z^j_{d,i} \setminus b^{-1}(Z^m_{d,j}) = Z^m_{d,i} \setminus Z^m_{d,j}, \text{ for } i = 0, \ldots, m$$

Therefore

$$\begin{aligned} E^j_{d,\tau} &= \bigcap_{i \in \tau} Z^j_{d,i} \setminus \bigcup_{i \in [j]\setminus\tau} Z^j_{d,i} = \left(\bigcap_{i \in \tau} Z^j_{d,i} \setminus \bigcup_{i \in [m]\setminus\tau} Z^j_{d,i} \right) \setminus \bigcup_{i=m+1}^{j} Z^j_{d,i} \\ &= \left(\bigcap_{i \in \tau} Z^j_{d,i} \setminus \bigcup_{i \in [m]\setminus\tau} Z^j_{d,i} \right) \setminus b^{-1}(Z^m_{d,j}) \\ &= \left(\bigcap_{i \in \tau} Z^m_{d,i} \setminus \bigcup_{i \in [m]\setminus\tau} Z^m_{d,i} \right) \setminus Z^m_{d,j} = E^m_{d,\tau} \setminus Z^m_{d,j} \end{aligned}$$

(2) By definition,

$$E^l_{d,\sigma} = \bigcap_{i \in \sigma} Z^l_{d,i} \setminus \bigcup_{i \in [l]\setminus\sigma} Z^l_{d,i} = Z^l_{d,l} \cap \bigcap_{i \in \tau} Z^l_{d,i} \setminus \bigcup_{i \in [l]\setminus\sigma} Z^l_{d,i}$$

We know $Z^l_{d,l}$ is the exceptional divisor of the blowup $Q^l_d \to Q^{l-1}_d$, which is along $Z^{l-1}_{d,l}$. Since $(\bigcap_{i \in \tau} Z^l_{d,i} \setminus \bigcup_{i \in [l]\setminus\sigma} Z^l_{d,i}) \subset Q^l_d$ is exactly the preimage of $\bigcap_{i \in \tau} Z^{l-1}_{d,i} \setminus \bigcup_{i \in [l]\setminus\sigma} Z^{l-1}_{d,i} \subset Q^{l-1}_d$ under the blowup, we have that $E^l_{d,\sigma}$ is the exceptional divisor of the (induced) blowup of $\bigcap_{i\in\tau} Z^{l-1}_{d,i} \setminus \bigcup_{i\in[l]\setminus\sigma} Z^{l-1}_{d,i}$ along $Z^{l-1}_{d,l} \cap \bigcap_{i\in\tau} Z^{l-1}_{d,i} \setminus \bigcup_{i\in[l]\setminus\sigma} Z^{l-1}_{d,i}$. Note that

$$\bigcap_{i\in\tau} Z^{l-1}_{d,i} \setminus \bigcup_{i\in[l]\setminus\sigma} Z^{l-1}_{d,i} = \bigcap_{i\in\tau} Z^{l-1}_{d,i} \setminus \left(\bigcup_{i\in[t]\setminus\tau} Z^{l-1}_{d,i} \cup \bigcup_{i=t+1}^{l-1} Z^{l-1}_{d,i} \right) = E^{l-1}_{d,\tau} \setminus \bigcup_{i=t+1}^{l-1} Z^{l-1}_{d,i}$$

If $l-1=t$, then $E^l_{d,\sigma}$ is the exceptional divisor of the blowup of $E^t_{d,\tau}$ along $Z^t_{d,t+1}$, and we are done. Now suppose $l-1>t$. Then under the identification $Q^{l-1}_d \setminus \bigcup_{i=t+1}^{l-1} Z^{l-1}_{d,i} = Q^t_d \setminus Z^t_{d,l-1}$, we have identifications $E^{l-1}_{d,\tau} \setminus \bigcup_{i=t+1}^{l-1} Z^{l-1}_{d,i} = E^t_{d,\tau} \setminus Z^t_{d,l-1}$ and

$$Z^{l-1}_{d,l} \setminus \bigcup_{i=t+1}^{l-1} Z^{l-1}_{d,i} = Z^t_{d,l} \setminus Z^t_{d,l-1} = \mathring{Z}^t_{d,l}.$$

□

Proposition 6.2. *There is a collection of isomorphisms*

$$i_{d,\sigma} : P_{d,\sigma} \xrightarrow{\sim} E^l_{d,\sigma},$$

one for each $\sigma \in \Sigma$ with $l := \mathrm{lt}(\sigma) < d$, such that the following properties hold:

(1) *$i_{d,\sigma}$ maps $Y_{d,r,\sigma}$ onto $E^l_{d,\sigma} \cap Z^l_{d,r}$ for all r, $l < r < d$;;*

(2) *$i_{d,\sigma}$ maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E^l_{d,\sigma} \cap \mathring{Z}^l_{d,r}$ for all r, $l < r < d$;*

(3) *The following diagram commutes:*

$$\begin{array}{ccccc} P_{d,\sigma} & & \xrightarrow[\sim]{i_{d,\sigma}} & & E^l_{d,\sigma} \\ \downarrow & & & & \downarrow \\ \mathring{R}_{d,l,\tau} & \xrightarrow{\sim} & \mathring{Y}_{d,l,\tau} & \xrightarrow[i_{d,\tau}]{\sim} & E^t_{d,\tau} \cap \mathring{Z}^t_{d,l} \end{array}$$

where $\sigma = (l,\tau)$ and $t = \mathrm{lt}(\tau)$;

(4) *Let $e_{d,\sigma} : P_{d,\sigma} \hookrightarrow Q^l_d$ be the embedding obtained through the composition $P_{d,\sigma} \xrightarrow{\sim} E^l_{d,\sigma} \hookrightarrow Q^l_d$. Then the following diagram commutes*

$$\begin{array}{ccc} Q_{d,r} \times_{Q_r} P_{r,\sigma} & \xhookrightarrow{1\times e_{r,\sigma}} & Q_{d,r} \times_{Q_r} Q^l_r \\ \phi_{d,r,\sigma} \downarrow & & \downarrow \phi^l_{d,r} \\ P_{d,\sigma} & \xhookrightarrow{e_{d,\sigma}} & Q^l_{d,\sigma} \end{array}$$

for all r, $l < r < d$.

Proof. We prove by constructing the isomorphisms $i_{d,\sigma}$ explicitly, and this is done by induction on the length of σ. We first deal with the base case: $\sigma \in \Sigma_1$ or $\sigma = (l)$. In this case, we construct isomorphisms

$$P_{d,l} \xrightarrow{\sim} E^l_{d,l} = Z^l_{d,l} \setminus \bigcup_{0 \le i \le l-1} Z^l_{d,i}$$

which map $Y_{d,r,l}$ onto $E^l_{d,l} \cap Z^l_{d,r}$ and map $\mathring{Y}_{d,r,l}$ isomorphically onto $E^l_{d,l} \cap \mathring{Z}^l_{d,r}$ for all d, r and l such that $l < r < d$.

Recall that $Z^l_{d,l}$ is the exceptional divisor in the blowup $b : Q^l_d \to Q^{l-1}_d$ along $Z^{l-1}_{d,l}$. We have $b^{-1}(Z^{l-1}_{d,i}) = Z^l_{d,i}$ for $i = 0, \dots, l-1$, hence

$$b : Q^l_d \setminus \bigcup_{i=0}^{l-1} Z^l_{d,i} \to Q^{l-1}_d \setminus \bigcup_{i=0}^{l-1} Z^{l-1}_{d,i}$$

is the blowup along $Z^{l-1}_{d,l} \setminus \bigcup_{i=0}^{l-1} Z^{l-1}_{d,i}$ with exceptional divisor $Z^l_{d,l} \setminus \bigcup_{i=0}^{l-1} Z^l_{d,i} = E^l_{d,l}$. Note that $Q^{l-1}_d \setminus \bigcup_{i=0}^{l-1} Z^{l-1}_{d,i} = Q_d \setminus Z_{d,l-1}$ and $Z^{l-1}_{d,l} \setminus \bigcup_{i=0}^{l-1} Z^{l-1}_{d,i} = Z_{d,l} \setminus Z_{d,l-1} = \mathring{Z}_{d,l}$. Therefore $E^l_{d,l}$ is the projective bundle $\mathbb{P}(\mathcal{N}_{\mathring{Z}_{d,l}/Q_d})$ over $\mathring{Z}_{d,l}$. Since $\mathring{Z}_{d,l} \simeq \mathring{Q}_{d,l}$ and by Proposition 4.2

$$\mathcal{N}_{\mathring{Q}_{d,l}/Q_d} = \pi_* \,\mathcal{E}xt^1(\mathcal{T}_{d,l}, \bar{\theta}^*_{d,l}\mathcal{F}_l)|_{\mathring{Q}_{d,l}} = \pi_* \,\mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,l}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,l}\mathring{\mathcal{X}}_l),$$

we know $E^l_{d,l}$ is isomorphic to the projective bundle

$$P_{d,l} = \mathbb{P}(\pi_* \,\mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,l}\mathcal{T}_{d,l}, \bar{\mathring{p}}^*_{d,l}\mathring{\mathcal{X}}_l))$$

over $\mathring{Q}_{d,l}$. So we obtain an embedding $e_{d,l} : P_{d,l} \hookrightarrow Q^l_d$ through the composition

$$P_{d,l} \xrightarrow{\sim} E^l_{d,l} \hookrightarrow Q^l_d.$$

For each r with $l < r < d$, we have a commtative diagram

$$\begin{array}{ccc}
Q_{d,r} \times_{Q_r} P_{r,l} & \xrightarrow{1 \times e_{r,l}} & Q_{d,r} \times_{Q_r} Q^l_r \\
\downarrow{\scriptstyle \phi_{d,r,l}} & & \downarrow{\scriptstyle \phi^l_{d,r}} \\
P_{d,l} & \xrightarrow{e_{d,l}} & Q^l_d
\end{array}$$

We have that $\mathrm{Im}(\phi_{d,r,l}) = Y_{d,r,l}$, that $\mathrm{Im}(1 \times e_{r,l}) = Q_{d,r} \times_{Q_r} E^l_{r,l}$, and that $\phi^l_{d,r}$ maps $Q_{d,r} \times_{Q_r} E^l_{r,l}$ onto $E^l_{d,l} \cap Z^l_{d,r}$. It follows that $e_{d,l}$ maps $Y_{d,r,l}$ onto $E^l_{d,l} \cap Z^l_{d,r}$.

Next, we show that $e_{d,l} : P_{d,l} \to Q_d^l$ maps $\mathring{Y}_{d,r,l}$ isomorphically onto $E_{d,l}^l \cap \mathring{Z}_{d,r}^l$ for $l < r < d$. We have a commutative diagram

$$\begin{array}{ccccc}
Q_{d,r} \times_{Q_r} \mathring{P}_{r,l} & \overset{1 \times e_{r,l}}{\hookrightarrow} & Q_{d,r} \times_{Q_r} (Q_r^l \setminus Z_{r,r-1}^l) & = & Q_{d,r} \times_{Q_r} (Q_r^{r-1} \setminus \bigcup_{i=l+1}^{r-1} Z_{r,i}^{r-1}) \\
\downarrow \phi_{d,r,l} & & \downarrow \phi_{d,r}^l & & \downarrow \phi_{d,r}^{r-1} \\
P_{d,l} \setminus Y_{d,r-1,l} & \overset{e_{d,l}}{\hookrightarrow} & Q_d^l \setminus Z_{d,r-1}^l & = & Q_d^{r-1} \setminus \bigcup_{i=l+1}^{r-1} Z_{d,i}^{r-1}
\end{array}$$

We see that $e_{r,l}$ maps $\mathring{P}_{r,l}$ isomorphically onto $E_{r,l}^l \setminus Z_{r,r-1}^l$ in $Q_r^l \setminus Z_{r,r-1}^l$, $E_{r,l}^l \setminus Z_{r,r-1}^l$, which can be identified with $E_{r,l}^{r-1} \subset Q_r^{r-1} \setminus \bigcup_{i=l+1}^{r-1} Z_{r,i}^{r-1}$ (by Lemma 6.1), and $\phi_{d,r}^{r-1}$ maps $Q_{d,r} \times_{Q_r} E_{r,l}^{r-1}$ isomorphically onto $E_{d,l}^{r-1} \cap Z_{d,r}^{r-1}$, which is identified with $E_{d,l}^l \cap \mathring{Z}_{d,r}^l$. On the other hand, $\phi_{d,r,l}$ maps $Q_{d,r} \times_{Q_r} \mathring{P}_{r,l}$ isomorphically onto $\mathring{Y}_{d,r,l} \subset P_{d,l} \setminus Y_{d,r-1,l}$. It follows that $e_{d,l}$ maps $\mathring{Y}_{d,r,l}$ isomorphically onto $E_{d,l}^l \cap \mathring{Z}_{d,r}^l$. Thus the case that $\sigma \in \Sigma_1$ is constructed.

Suppose we have constructed the isomorphisms for all $\sigma \in \Sigma_m$ for some m. We now construct the isomorphisms for $\sigma \in \Sigma_{m+1}$. Now write $\sigma = (l, \tau)$ and let $t = \mathrm{lt}(\tau)$. By induction hypothesis, for any $d > t$, we have an isomorphism $P_{d,\tau} \xrightarrow{\sim} E_{d,\tau}^t$ which maps $Y_{d,r,\tau}$ onto $E_{d,\tau}^t \cap Z_{d,r}^t$ and maps $\mathring{Y}_{d,r,\tau}$ isomorphically onto $E_{d,\tau}^t \cap \mathring{Z}_{d,r}^t$. Let $d > l$. By Lemma 6.1, $E_{d,\sigma}^l$ is the exceptional divisor of the blowup of $E_{d,\tau}^t \setminus \tilde{Z}_{d,l-1}^t$ along $E_{d,\tau}^t \cap \mathring{Z}_{d,l}^t$. Since the isomorphism $P_{d,\tau} \xrightarrow{\sim} E_{d,\tau}^t$ maps $\mathring{Y}_{d,l,\tau}$ isomorphically onto $E_{d,\tau}^t \cap \mathring{Z}_{d,l}^t$, we have an isomorphism

$$\mathrm{Bl}_{\mathring{Y}_{d,l,\tau}}(P_{d,\tau} \setminus Y_{d,l-1,\tau}) \xrightarrow{\sim} \mathrm{Bl}_{E_{d,\tau}^t \cap \mathring{Z}_{d,l}^t}(E_{d,\tau}^t \setminus \tilde{Z}_{d,l-1}^t)$$

which maps the exceptional divisor $E_{d,\sigma}^l$ of the blowup $\mathrm{Bl}_{\mathring{Y}_{d,l,\tau}}(P_{d,\tau} \setminus Y_{d,l-1,\tau})$ isomorphically onto the exceptional divisor $E_{d,\sigma}^l$ of $\mathrm{Bl}_{E_{d,\tau}^t \cap \mathring{Z}_{d,l}^t}(E_{d,\tau}^t \setminus Z_{d,l-1}^t)$. On the other hand, the exceptional divisor of $\mathrm{Bl}_{\mathring{Y}_{d,l,\tau}}(P_{d,\tau} \setminus Y_{d,l-1,\tau})$ is isomorphic to the projective normal bundle of $\mathring{Y}_{d,l,\tau}$ in $P_{d,\tau} \setminus Y_{d,l-1,\tau}$ or just in $P_{d,\tau}$. We know that $\phi_{d,l,\tau} : Q_{d,l} \times_{Q_l} P_{l,\tau} \to P_{d,\tau}$ maps $Q_{d,l} \times_{Q_l} \mathring{P}_{l,\tau}$ isomorphically onto $\mathring{Y}_{d,l,\tau}$, and by Proposition 5.6 the normal bundle of $Q_{d,l} \times_{Q_l} \mathring{P}_{l,\tau}$ in $P_{d,\tau}$ is $\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}_{d,l,\tau}^* \mathcal{T}_{d,l}, \bar{\mathring{p}}_{d,l,\tau}^* \mathring{\mathcal{X}}_{l,\tau})$. Hence the projective normal bundle of $\mathring{Y}_{d,l,\tau}$ in $P_{d,\tau}$ is isomorphic to $\mathbb{P}(\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}_{d,l,\tau}^* \mathcal{T}_{d,l}, \bar{\mathring{p}}_{d,l,\tau}^* \mathring{\mathcal{X}}_{l,\tau})) = P_{d,l,\tau} = P_{d,\sigma}$. Thus we obtain an isomorphism $P_{d,\sigma} \xrightarrow{\sim} E_{d,\sigma}^l$. Next we show that this isomorphism maps $Y_{d,r,\sigma}$ onto $E_{d,\sigma}^l \cap Z_{d,r}^l$ and maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E_{d,\sigma} \cap \mathring{Z}_{d,r}^l$.

Let $e_{d,\sigma} : P_{d,\sigma} \hookrightarrow Q^l_d$ denote the embedding obtained from the composition $P_{d,\sigma} \xrightarrow{\sim} E^l_{d,\sigma} \hookrightarrow Q^l_{d,\sigma}$ for each $d > l$. Let $d > r > l$. Then we have a commutative diagram

$$\begin{array}{ccc} Q_{d,r} \times_{Q_r} P_{r,\sigma} & \xhookrightarrow{1\times e_{r,\sigma}} & Q_{d,r} \times_{Q_r} Q^l_r \\ \downarrow{\scriptstyle \phi_{d,r,\sigma}} & & \downarrow{\scriptstyle \phi^l_{d,r}} \\ P_{d,\sigma} & \xhookrightarrow{e_{d,\sigma}} & Q^l_{d,\sigma} \end{array}$$

We have that $\mathrm{Im}(\phi_{d,r,\sigma}) = Y_{d,r,\sigma}$, that $\mathrm{Im}(1 \times e_{r,\sigma}) = Q_{d,r} \times_{Q_r} E^l_{r,\sigma}$, and that $\phi^l_{d,r}$ maps $Q_{d,r} \times_{Q_r} E^l_{r,\sigma}$ onto $E^l_{d,\sigma} \cap Z^l_{d,r}$. It follows that $e_{d,\sigma}$ maps $Y_{d,r,\sigma}$ onto $E^l_{d,\sigma} \cap Z^l_{d,r}$.

Next, we show that $e_{d,\sigma} : P_{d,\sigma} \hookrightarrow Q^l_d$ maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E^l_{d,\sigma} \cap \mathring{Z}^l_{d,r}$ for any r, $l < r < d$. We have a commutative diagram

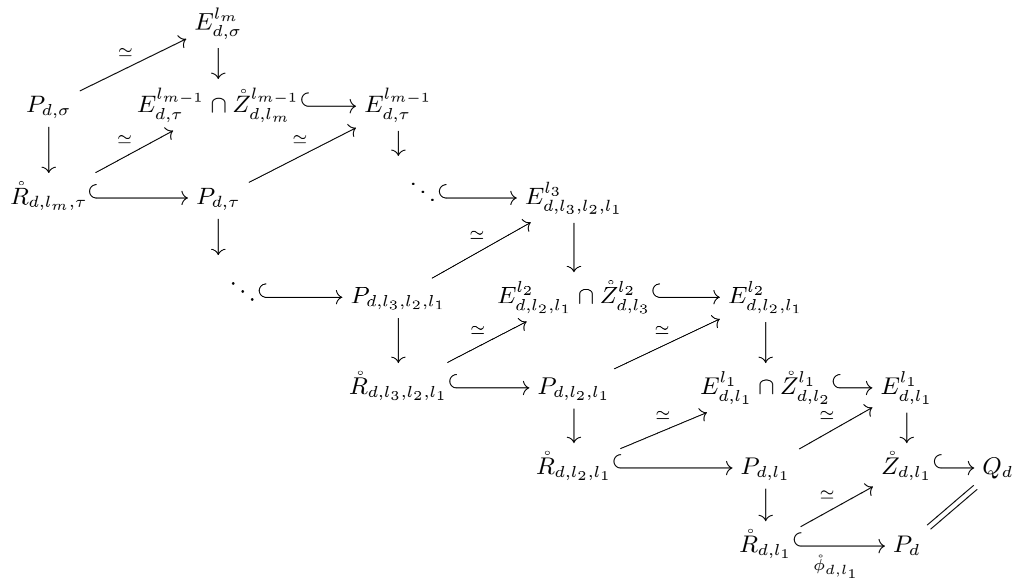
$$\begin{array}{ccccc} Q_{d,r} \times_{Q_r} \mathring{P}_{r,\sigma} & \xhookrightarrow{1\times e_{r,\sigma}} & Q_{d,r} \times_{Q_r} (Q^l_r \setminus Z^l_{r,r-1}) & = & Q_{d,r} \times_{Q_r} (Q^{r-1}_r \setminus \bigcup_{i=l+1}^{r-1} Z^{r-1}_{r,i}) \\ \downarrow{\scriptstyle \phi_{d,r,\sigma}} & & \downarrow{\scriptstyle \phi^l_{d,r}} & & \downarrow{\scriptstyle \phi^{r-1}_{d,r}} \\ P_{d,\sigma} \setminus Y_{d,r-1,\sigma} & \xhookrightarrow{e_{d,\sigma}} & Q^l_d \setminus Z^l_{d,r-1} & = & Q^{r-1}_d \setminus \bigcup_{i=l+1}^{r-1} Z^{r-1}_{d,i} \end{array}$$

We see that $e_{r,\sigma}$ maps $\mathring{P}_{r,\sigma}$ isomorphically onto $E^l_{r,\sigma} \setminus Z^l_{r,r-1}$ in $Q^l_r \setminus Z^l_{r,r-1}$, that $E^l_{r,\sigma} \setminus Z^l_{r,r-1}$ can be identified with $E^{r-1}_{r,\sigma} \subset Q^{r-1}_r \setminus \bigcup_{i=l+1}^{r-1} Z^{r-1}_{r,i}$ (by Lemma 6.1), and that $\phi^{r-1}_{d,r}$ maps $Q_{d,r} \times_{Q_r} E^{r-1}_{r,l}$ isomorphically onto $E^{r-1}_{d,l} \cap Z^{r-1}_{d,r}$, which is identified with $E^l_{d,l} \cap \mathring{Z}^l_{d,r}$. On the other hand, $\phi_{d,r,\sigma}$ maps $Q_{d,r} \times_{Q_r} \mathring{P}_{r,\sigma}$ in isomorphically onto $\mathring{Y}_{d,r,\sigma} \subset P_{d,\sigma} \setminus Y_{d,r-1,\sigma}$. It follows that $e_{d,\sigma}$ maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E^l_{d,\sigma} \cap \mathring{Z}^l_{d,r}$. Thus the case that $\sigma \in \Sigma_{m+1}$ is constructed. □

Proposition 6.3. *Let $\sigma \in \Sigma$. Then for any d and l, $d > l \geq \mathrm{lt}(\sigma)$, there is an isomorphism $P_{d,\sigma} \setminus Y_{d,l,\sigma} \xrightarrow{\sim} E^l_{d,\sigma}$ which maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E^l_{d,\sigma} \cap \mathring{Z}^l_{d,r}$ for any r, $l < r < d$;*

Proof. The case that $l = \mathrm{lt}(\sigma)$ is proved in the above proposition. We now prove the case that $l > \mathrm{lt}(\sigma)$. Let $t = \mathrm{lt}(\sigma)$. Then by the above proposition, we have an isomorphism $P_{d,\sigma} \xrightarrow{\sim} E^t_{d,\sigma}$ which maps $Y_{d,l,\sigma}$ onto $E^t_{d,\sigma} \cap Z^t_{d,l}$ for each r. This isomorphism restricts to an isomorphism $P_{d,\sigma} \setminus Y_{d,l,\sigma} \xrightarrow{\sim} E^t_{d,\sigma} \setminus Z^t_{d,l}$. Under the identification $Q_d^t \setminus Z_{d,l}^t = Q_d^l \setminus \bigcup_{i=t+1}^l Z_{d,i}^l$, $E_{d,\sigma}^t \setminus Z_{d,l}^t$ is identified with $E_{d,\sigma}^l$, and $E_{d,\sigma}^t \cap \mathring{Z}_{d,r}^t$ is identified with $E_{d,\sigma}^l \cap \mathring{Z}_{d,r}^l$ for any r, $l < r < d$. Hence we obtain an isomorphism $P_{d,\sigma} \setminus Y_{d,l,\sigma} \xrightarrow{\sim} E_{d,\sigma}^l$ which maps $\mathring{Y}_{d,r,\sigma}$ isomorphically onto $E_{d,\sigma}^l \cap \mathring{Z}_{d,r}^l$ for any r, $l < r < d$. $\square$

Let $\sigma = (l_m, \cdots, l_1) \in \Sigma$ and $d > l_m$. We have the following commutative diagram



where $\tau = (l_{m-1}, \cdots, l_1, l_0)$. Thus we obtain a sequence of canonical identifications:

$$\mathring{R}_{d,l_0} = \mathring{Z}_{d,l_0}, \quad P_{d,l_0} = E_{d,l_0}^{l_0}, \quad \mathring{R}_{d,l_1,l_0} = E_{d,l_0}^{l_0} \cap \mathring{Z}_{d,l_1}^{l_0}, \quad \cdots, \quad P_{d,\sigma} = E_{d,\sigma}^{l_m}.$$

Recall that Q_d parametrizes quotient of the form $V_{\mathbb{P}^1} \twoheadrightarrow X_1$ with $\deg X_1 = d$, and the subset $\mathring{Z}_{d,l_1}$ of Q_d parametrizes such quotients with $\deg X_1^{\mathfrak{t}} = d - l_1$.

Let $x_1 = [V_{\mathbb{P}^1} \twoheadrightarrow X_1] \in \mathring{Z}_{d,l_1} = \mathring{R}_{d,l_1}$. Using the identification,

$$E_{d,l_1}^{l_1} = P_{d,l_1} = \mathbb{P}(\pi_* \mathcal{E}xt^1(\bar{\mathring{q}}_{d,l_1}^* \mathcal{T}_{d,l_1}, \bar{\mathring{p}}_{d,l_1}^* \mathring{\mathcal{X}}_{l_1})),$$

we see that the fiber of $E_{d,l_1}^{l_1} \to \mathring{Z}_{d,l_1}$ over x_1 is $\mathbb{P}(\mathrm{Ext}^1((\bar{\mathring{q}}_{d,l}^* \mathcal{T}_{d,l})_{x_1}, (\bar{\mathring{p}}_{d,l}^* \mathring{\mathcal{X}}_l)_{x_1}))$. On the other hand, we have an exact sequence

$$0 \to \bar{\mathring{q}}_{d,l_1}^* \mathcal{T}_{d,l_1} \to \bar{\bar{\mathring{\phi}}}_{d,l_1}^* \mathcal{X}_d \to \bar{\mathring{p}}_{d,l_1}^* \mathring{\mathcal{X}}_{l_1} \to 0$$

whose restriction to $\mathbb{P}^1 \times \{x_1\}$,

$$0 \to (\bar{\mathring{q}}^*_{d,l_1} \mathcal{T}_{d,l_1})_{x_1} \to (\bar{\mathring{\phi}}^*_{d,l_1} \mathcal{X}_d)_{x_1} \to (\bar{\mathring{p}}^*_{d,l_1} \mathring{\mathcal{X}}_{l_1})_{x_1} \to 0$$

is also an exact sequence. Since $\mathring{\phi}_{d,l_1}$ is an inclusion map, we have that $(\bar{\mathring{\phi}}^*_{d,l_1} \mathcal{X}_d)_{x_1} = X_1$. Hence $(\bar{\mathring{q}}^*_{d,l_1} \mathcal{T}_{d,l_1})_{x_1} = X_1^{\mathbf{t}}$ and $(\bar{\mathring{p}}^*_{d,l_1} \mathring{\mathcal{X}}_{l_1})_{x_1} = X_1^{\mathbf{f}}$, and the fiber over x_1 is $\mathbb{P}(\mathrm{Ext}^1(X_1^{\mathbf{t}}, X_1^{\mathbf{f}}))$, which parametrizes non-split extensions of the form $[X_1^{\mathbf{f}} \rightarrowtail X_2 \twoheadrightarrow X_1^{\mathbf{t}}]$. Thus $E^{l_1}_{d,l_1}$ parametrizes sequences of the form

$$([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X_1^{\mathbf{f}} \rightarrowtail X_2 \twoheadrightarrow X_1^{\mathbf{t}}]).$$

with $\deg X_1^{\mathbf{t}} = d - l_1$ and $\deg X_2^{\mathbf{t}} < d - l_1$. Using the identification

$$\mathring{R}_{d,l_2,l_1} = \mathring{Y}_{d,l_2,l_1} = E^{l_1}_{d,l_1} \cap \mathring{Z}^{l_1}_{d,l_2},$$

we see that $E^{l_1}_{d,l_1} \cap \mathring{Z}^{l_1}_{d,l_2}$ parametrizes such sequences with $\deg X_1^{\mathbf{t}} = d - l_1$ and $\deg X_2^{\mathbf{t}} = d - l_2$, by the definition of $\mathring{Y}_{d,l_2,l_1}$.

Let $x_2 = ([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X_1^{\mathbf{f}} \rightarrowtail X_2 \twoheadrightarrow X_1^{\mathbf{t}}]) \in E^{l_1}_{d,l_1} \cap \mathring{Z}^{l_1}_{d,l_2} = \mathring{R}_{d,l_2,l_1}$. Using the identification $E^{l_2}_{d,l_2,l_1} = P_{d,l_2,l_1} = \mathbb{P}(\mathcal{E}xt^1(\bar{\mathring{q}}^*_{d,l_2,l_1} \mathcal{T}_{d,l_2}, \bar{\mathring{p}}^*_{d,l_2,l_1} \mathring{\mathcal{X}}_{l_2,l_1}))$, we have that the fiber of $E^{l_2}_{d,l_2,l_1} \to E^{l_1}_{d,l_1} \cap \mathring{Z}^{l_1}_{d,l_2}$ over the point x_2 is

$$\mathbb{P}(\mathcal{E}xt^1((\bar{\mathring{q}}^*_{d,l_2,l_1} \mathcal{T}_{d,l_2})_{x_2}, (\bar{\mathring{p}}^*_{d,l_2,l_1} \mathring{\mathcal{X}}_{l_2,l_1})_{x_2})).$$

On the other hand, we have an exact sequence

$$0 \to \bar{\mathring{q}}^*_{d,l_2,l_1} \mathcal{T}_{d,l_2} \to \bar{\mathring{\phi}}^*_{d,l_2,l_1} \mathcal{X}_{d,l_1} \to \bar{\mathring{p}}^*_{d,l_2,l_1} \mathring{\mathcal{X}}_{l_2,l_1} \to 0$$

whose restriction to $\mathbb{P}^1 \times \{x_2\}$,

$$0 \to (\bar{\mathring{q}}^*_{d,l_2,l_1} \mathcal{T}_{d,l_2})_{x_2} \to (\bar{\mathring{\phi}}^*_{d,l_2,l_1} \mathcal{X}_{d,l_1})_{x_2} \to (\bar{\mathring{p}}^*_{d,l_2,l_1} \mathring{\mathcal{X}}_{l_2,l_1})_{x_2} \to 0$$

is also an exact sequence. Since $\bar{\mathring{\phi}}^*_{d,l_2,l_1}$ is an inclusion map, we have

$$(\bar{\mathring{\phi}}^*_{d,l_2,l_1} \mathcal{X}_{d,l_1})_{x_2} = X_2.$$

Hence $(\bar{\mathring{q}}^*_{d,l_2,l_1} \mathcal{T}_{d,l_2})_{x_2} = X_2^{\mathbf{t}}$, $(\bar{\mathring{p}}^*_{d,l_2,l_1} \mathring{\mathcal{X}}_{l_2,l_1})_{x_2} = X_2^{\mathbf{f}}$, and the fiber over x_2 is $\mathbb{P}(\mathrm{Ext}^1(X_2^{\mathbf{t}}, X_2^{\mathbf{f}}))$, which parametrizes non-split extensions of the form $[X_2^{\mathbf{f}} \rightarrowtail X_3 \twoheadrightarrow X_2^{\mathbf{t}}]$. Thus $E^{l_2}_{d,l_2,l_1}$ parametrizes sequences of the form:

$$([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X_1^{\mathbf{f}} \rightarrowtail X_2 \twoheadrightarrow X_1^{\mathbf{t}}], [X_2^{\mathbf{f}} \rightarrowtail X_3 \twoheadrightarrow X_2^{\mathbf{t}}]).$$

with $\deg X_1^{\mathbf{t}} = d - l_1$, $\deg X_2^{\mathbf{t}} = d - l_2$ and $\deg X_3^{\mathbf{t}} < d - l_2$. Since

$$\mathring{R}_{d,l_3,l_2,l_1} = \mathring{Y}_{d,l_3,l_2,l_1} = E^{l_2}_{d,l_2,l_1} \cap \mathring{Z}^{l_2}_{d,l_3},$$

by the definition of $\mathring{Y}_{d,l_3,l_2,l_1}$, $E^{l_2}_{d,l_2,l_1} \cap \mathring{Z}^{l_2}_{d,l_3}$ parametrizes such sequences with $\deg X^{\mathbf{t}}_1 = d - l_1$, $\deg X^{\mathbf{t}}_2 = d - l_2$ and $\deg X^{\mathbf{t}}_3 = d - l_3$.

Continuing this argument, we eventually obtain the parameter-space interpretation for $E^{l_m}_{d,\sigma}$: $E^{l_m}_{d,\sigma}$ parametrizes sequences of the form

$$([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X^{\mathbf{f}}_1 \rightarrowtail X_2 \twoheadrightarrow X^{\mathbf{t}}_1], \cdots, [X^{\mathbf{f}}_m \rightarrowtail X_{m+1} \twoheadrightarrow X^{\mathbf{t}}_m])$$

with $\deg X^{\mathbf{t}}_i = d - l_i$, $(i = 1, \cdots, m)$ and $\deg X^{\mathbf{t}}_{m+1} < d - l_m$. For any r with $d > r > l_m$, $E^{l_m}_{d,\sigma} \cap \mathring{Z}^{l_m}_{d,r}$ parametrizes such sequences with $\deg X^{\mathbf{t}}_i = d - l_i$, $(i = 1, \cdots, m)$ and $\deg X^{\mathbf{t}}_{m+1} = d - r$.

Next, we deal with the modular interpretation of $E^r_{d,\sigma}$ for any r, $d > r > l_m$. By the lemma before, we have $E^r_{d,\sigma} = E^{l_m}_{d,\sigma} \setminus Z^{l_m}_{d,r}$. Since $Z^{l_m}_{d,r} = \bigcup_{i=l_m+1}^{r} \mathring{Z}^{l_m}_{d,i}$, we know that $E^{l_m}_{d,\sigma} \cap Z^{l_m}_{d,r}$ parametrizes sequences as above with $\deg X^{\mathbf{t}}_i = d - l_i$, $(i = 1, \cdots, m)$ and $\deg X^{\mathbf{t}}_{m+1} \geq d - r$. Therefore, $E^r_{d,\sigma}$, which equals $E^{l_m}_{d,\sigma} \setminus Z^{l_m}_{d,r}$, parametrizes such sequences with $\deg X^{\mathbf{t}}_i = d - l_i$, $(i = 1, \cdots, m)$ and $\deg X^{\mathbf{t}}_{m+1} < d - r$.

In summary, we have

Proposition 6.4. *Let $d > r$. For any $\sigma = (l_m, \cdots, l_2, l_1) \in \Sigma$ with $l_m < r$, the stratum $E^r_{d,\sigma}$ of Q^r_d parametrizes the sequences of the form*

$$([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X^{\mathbf{f}}_1 \rightarrowtail X_2 \twoheadrightarrow X^{\mathbf{t}}_1], \cdots, [X^{\mathbf{f}}_m \rightarrowtail X_{m+1} \twoheadrightarrow X^{\mathbf{t}}_m])$$

with $\deg X^{\mathbf{t}}_i = d - l_i$, $(i = 1, \cdots, m)$ *and* $\deg X^{\mathbf{t}}_{m+1} < d - r$.

Corollary 6.5. *The boundary $\widetilde{Q}_d \backslash \mathring{Q}_d$ comes equipped with a natural stratification with strata $E_{d,\sigma}$ indexed by $\sigma = (l_m, \cdots, l_2, l_1)$ with $d > l_m > \cdots > l_1 \geq 0$,. The stratum $E_{d,\sigma}$ parametrizes the sequences of the form*

$$([V_{\mathbb{P}^1} \twoheadrightarrow X_1], [X^{\mathbf{f}}_1 \rightarrowtail X_2 \twoheadrightarrow X^{\mathbf{t}}_1], \cdots, [X^{\mathbf{f}}_m \rightarrowtail X_{m+1} \twoheadrightarrow X^{\mathbf{t}}_m])$$

such that $\deg X^{\mathbf{t}}_i = d - l_i$, *$i = 1, \cdots, m$, and the last sheaf X_{m+1} is the unique one that is locally free.*

This proves Theorem 1.3.

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