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Connected sum of orientable surfaces and Reidemeister torsion

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Abstract: Let $\Sigma_{g,n}$ be an orientable surface with genus $g \geq 2$ bordered by $n \geq 1$ curves homeomorphic to circle. As is well known that one-holed torus $\Sigma_{1,1}$ is the building block of such surfaces. By using the notion of symplectic chain complex, homological algebra techniques and considering the double of the building block, the present paper proves a novel formula for computing Reidemeister torsion of one-holed torus. Moreover, applying this result and considering $\Sigma_{g,n}$ as the connected sum $\Sigma_{1,n}\#(g-1)\Sigma_{1,0}$, the present paper establishes a novel formula to compute Reidemeister torsion of $\Sigma_{g,n}$.

Keywords: Reidemeister torsion, symplectic chain complex, homological algebra, orientable surfaces.

1. Introduction

The topological invariant Reidemeister torsion was introduced by K. Reidemeister in [18], where by using this invariant he was able to classify 3-dimensional lens spaces. This invariant has many interesting applications in several branches of mathematics and theoretical physics, such as topology [7, 11, 12, 18], differential geometry [3, 14, 17], representation spaces [19, 22, 26], knot theory [6], Chern-Simon theory [25], 3-dimensional Seiberg-Witten theory [10], algebraic K-theory [13], dynamical systems [8], theoretical physics and quantum field theory [25, 26]. The reader is referred to [16, 24] for more information about this invariant.

Symplectic chain complex is an algebraic topological tool and was introduced by E. Witten [25], where using Reidemeister torsion and symplectic chain complex he computed the volume of several moduli spaces of representations from a Riemann surface to a compact gauge group.

It is well known that one-holed torus $\Sigma_{1,1}$ is a building block of orientable surfaces $\Sigma_{g,n}, g \geq 2, n \geq 0$. Let us note that closed orientable surface $\Sigma_{2,0}$

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of genus 2 can be obtained by gluing two one-holed torus along the common boundary circle. With the help of homological algebra computations and the notion of symplectic chain complex, we establish a novel formula (Theorem 3.1.1) for computing Reidemeister torsion of one-holed torus. Furthermore, considering orientable surface $\Sigma_{g,n}$ as $\Sigma_{1,n}\#(g-1)\Sigma_{1,0}$, and applying the obtained Reidemeister torsion formula of $\Sigma_{1,1}$, we prove novel formulas (Theorem 3.2.6–Theorem 3.2.10) for computing Reidemeister torsion of $\Sigma_{g,n}$. Here, $\#$ is the connected sum and $(g-1)$ denotes $(g-1)$ copies.

2. Preliminaries

In this section, we give the basic definitions and facts about Reidemeister torsion and symplectic chain complex. For further information and the detailed proof, the reader is referred to [15, 16, 19–25] and the references therein.

Let $C_* : 0 \to C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$ be a chain complex of finite dimensional vector spaces over the field $\mathbb{R}$ of real numbers. For $p = 0, \ldots, n$, let $B_p(C_*) = \mathrm{Im}\{\partial_{p+1} : C_{p+1} \to C_p\}$, $Z_p(C_*) = \mathrm{Ker}\{\partial_p : C_p \to C_{p-1}\}$, and $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ be $p-$th homology group of the chain complex. Using the definition of $Z_p(C_*), B_p(C_*)$, and $H_p(C_*)$, we have the following short-exact sequences

$$0 \longrightarrow Z_p(C_*) \stackrel{\imath}{\longrightarrow} C_p \stackrel{\partial_p}{\longrightarrow} B_{p-1}(C_*) \longrightarrow 0, \tag{2.0.1}$$

$$0 \longrightarrow B_p(C_*) \stackrel{\imath}{\longrightarrow} Z_p(C_*) \stackrel{\varphi_p}{\longrightarrow} H_p(C_*) \longrightarrow 0. \tag{2.0.2}$$

Here, $\imath$ and φ_p are the inclusion and the natural projection, respectively.

Let $s_p : B_{p-1}(C_*) \to C_p$, $\ell_p : H_p(C_*) \to Z_p(C_*)$ be sections of $\partial_p : C_p \to B_{p-1}(C_*)$, $\varphi_p : Z_p(C_*) \to H_p(C_*)$, respectively. The short-exact sequences (2.0.1) and (2.0.2) yield

$$C_p = B_p(C_*) \oplus \ell_p(H_p(C_*)) \oplus s_p(B_{p-1}(C_*)). \tag{2.0.3}$$

If $\mathbf{c_p}$, $\mathbf{b_p}$, and $\mathbf{h_p}$ are bases of C_p, $B_p(C_*)$, and $H_p(C_*)$, respectively, then by equation (2.0.3), we obtain a new basis, more precisely $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p, $p = 0, \ldots, n$.

Definition 2.0.1. Reidemeister torsion (R-torsion) of chain complex C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined as the alternating product

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^n \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}},$$

where $[\mathbf{e}_p, \mathbf{f}_p]$ is the determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of C_p.

If $0 \to A_* \stackrel{\imath}{\to} B_* \stackrel{\pi}{\to} D_* \to 0$ is a short-exact sequence of chain complexes, then by the Snake Lemma we have the long-exact sequence of vector spaces

$$C_* : \cdots \to H_p(A_*) \xrightarrow{\imath_p} H_p(B_*) \xrightarrow{\pi_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \to \cdots$$

Here, $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(B_*)$, and $C_{3p+2} = H_p(A_*)$. Clearly, one can consider bases $\mathbf{h}_p^D$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^A$ for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively.

Theorem 2.0.2. *([13]) Let $0 \to A_* \stackrel{\imath}{\to} B_* \stackrel{\pi}{\to} D_* \to 0$ be a short-exact sequence of chain complexes. Let C_* be the corresponding long-exact sequence of vector spaces obtained by the Snake Lemma. Suppose that $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Suppose also that $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, and $\mathbf{c}_p^D$ are compatible in the sense that $[\mathbf{c}_p^B, \mathbf{c}_p^A \sqcup \widetilde{\mathbf{c}_p^D}] = \pm 1$, where $\pi_p\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$. Then, the following formula holds:*

$$\begin{aligned}\mathbb{T}(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n) &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\, \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n) \\ &\quad \times \mathbb{T}(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}).\end{aligned}$$

From Theorem 2.0.2 it follows that

Lemma 2.0.3. *If A_*, D_* are two chain complexes, and if $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ are bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively, then*

$$\begin{aligned}T(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n) \\ &\quad \times \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).\end{aligned}$$

We refer the reader to [23] for detailed proof and further information.

Definition 2.0.4. Let $(C_*, \partial_*, \{\omega_{*,q-*}\}) : 0 \to C_q \stackrel{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$ be a chain complex of real vector spaces with the following properties:

1) $q \equiv 2 (\text{mod } 4)$,
2) There is a non-degenerate bilinear form $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$ for $p = 0, \ldots, q/2$ such that
   - $\partial-$compatible: $\omega_{p,q-p}(\partial_{p+1} a, b) = (-1)^{p+1} \omega_{p+1,q-(p+1)}(a, \partial_{q-p} b)$,
   - anti-symmetric: $\omega_{p,q-p}(a, b) = (-1)^{p(q-p)} \omega_{q-p,p}(b, a)$.

Then, $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is called a symplectic chain complex of length q.

By the fact $q \equiv 2 (\text{mod } 4)$, we have $\omega_{p,q-p}(a,b) = (-1)^p \omega_{q-p,p}(b,a)$. From the $\partial-$compatibility of $\omega_{p,q-p}$, it follows that they can be extended to homologies [19].

Assume that $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a symplectic chain complex. Assume also that $\mathbf{c}_p$ and $\mathbf{c}_{q-p}$ are bases of C_p and C_{q-p}, respectively. We say that these bases are $\omega-$*compatible*, if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ is equal to the $k \times k$ identity matrix $I_{k\times k}$ when $p \neq q/2$ and is equal to $\begin{pmatrix} 0_{l\times l} & I_{l\times l} \\ -I_{l\times l} & 0_{l\times l} \end{pmatrix}$ otherwise. Here, $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$. Note that every symplectic chain complex has $\omega-$compatible bases.

Using the existence of $\omega-$compatible bases, the following formula was proved for calculating R-torsion of symplectic chain complex.

Theorem 2.0.5. *([19]) If $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a symplectic chain complex and if $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ are $\omega-$compatible bases of C_p, C_{q-p}, and $\mathbf{h}_p$ is a basis of $H_p(C_*)$, $p = 0, \ldots, q$, then*

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q) = \prod_{p=0}^{(q/2)-1} (\det[\omega_{p,q-p}])^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{(-1)^{q/2}}.$$

Here, $\det[\omega_{p,q-p}]$ is the determinant of the matrix of the non-degenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R}$ in the bases $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

Let M be a smooth m-manifold with a cell decomposition K. Let $\mathbf{c}_i$ be the geometric basis for the $i-$cells $C_i(K)$, $i = 0, \ldots, m$. Note that associated to M there is the chain-complex $0 \to C_m(K) \stackrel{\partial_m}{\to} C_{m-1}(K) \to \cdots \to C_1(K) \stackrel{\partial_1}{\to} C_0(K) \to 0$, where ∂_i is the boundary operator. $\mathbb{T}(C_*(K), \{\mathbf{c}_i\}_{i=0}^m, \{\mathbf{h}_i\}_{i=0}^m)$ is called *Reidemeister torsion (R-torsion)* of M, where $\mathbf{h}_i$ is a basis for $H_i(K)$, $i = 0, \ldots, m$.

Following the arguments introduced in [19, Lemma 2.0.5], one can conclude that R-torsion of a manifold M is independent of the cell-decomposition K of M. Hence, instead of $\mathbb{T}(C_*(K), \{\mathbf{c}_i\}_{i=0}^m, \{\mathbf{h}_i\}_{i=0}^m)$, we write $\mathbb{T}(M, \{\mathbf{h}_i\}_{i=0}^m)$.

Theorem 2.0.5 yields the following result, which suggests a formula for computing R-torsion of a manifold.

Theorem 2.0.6 ([23])**.** *Assume that M is an orientable closed connected $2m-$manifold ($m \geq 1$). Assume also that $\mathbf{h}_p$ is a basis of $H_p(M)$ for $p = 0, ..., 2m$. Then, R-torsion of M satisfies the following formula:*

$$|\mathbb{T}(M, \{\mathbf{h}_p\}_{p=0}^{2m})| = \prod_{p=0}^{m-1} |\det \Delta_{p,2m-p}(M)|^{(-1)^p} \sqrt{|\det \Delta_{m,m}(M)|}^{(-1)^m}.$$

Here, $\det \Delta_{p,2m-p}(M)$ *is determinant of matrix of the intersection pairing* $(\cdot,\cdot)_{p,2m-p} : H_p(M) \times H_{2m-p}(M) \to \mathbb{R}$ *in bases* $\mathbf{h}_p$, $\mathbf{h}_{2m-p}$.

Theorem 2.0.7 ([23])**.** *Suppose* M *is an orientable closed connected* $(2m+1)-$*manifold* $(m \geq 0)$ *and* $\mathbf{h}_p$ *is a basis of* $H_p(M)$, $p = 0,\dots,2m+1$. *Then,* $|\mathbb{T}(M, \{\mathbf{h}_p\}_{p=0}^{2m+1})| = 1$.

Remark 2.0.8. *Let* $\mathbb{S}^1$ *be the unit circle and* $\mathbf{h}_0$, $\mathbf{h}_1$ *be bases of* $H_0(\mathbb{S}^1)$, $H_1(\mathbb{S}^1)$ *respectively. Then, by Theorem 2.0.7, we have* $\left|\mathbb{T}(\mathbb{S}^1, \{\mathbf{h}_0, \mathbf{h}_1\})\right| = 1$.

We refer the reader [15, 19–23] for further applications of Theorem 2.0.5.

3. Main result

In this section, by considering orientable surface $\Sigma_{g,n}, g \geq 2, n \geq 0$ as the connected sum $\Sigma_{1,0}\#\cdots\#\Sigma_{1,0}\#\Sigma_{1,n}$ (see, Fig.1), we establish a formula to compute R-torsion of $\Sigma_{g,n}, g \geq 2, n \geq 0$ in terms of R-torsion of $\Sigma_{1,1}$. To obtain this formula, we first prove a formula for computing R-torsion of $\Sigma_{1,1}$ (Theorem 3.1.1), then we establish a formula (Proposition 3.2.1) for R-torsion of $\Sigma_{1,n}, n \geq 2$, and finally using these results we prove the formulas (Theorem 3.2.6–Theorem 3.2.10) to compute R-torsion of $\Sigma_{g,n}, g \geq 2, n \geq 0$ in terms of R-torsion of $\Sigma_{1,1}$.

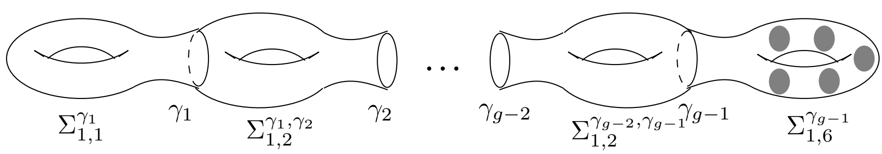

Figure 1: Orientable surface $\Sigma_{g,5}$ with genus $g \geq 2$, bordered by $n = 5$ curves homeomorphic to circle.

3.1. R-torsion of torus with one boundary circle $\Sigma_{1,1}$

Let $\Sigma_{1,1}$ be a torus with boundary circle γ. Note that the double of $\Sigma_{1,1}$ is $\Sigma_{2,0}$. Clearly, there is the following short-exact sequence of the chain complexes

$$0 \to C_*(\gamma) \longrightarrow C_*(\Sigma_{1,1}) \oplus C_*(\Sigma_{1,1}) \longrightarrow C_*(\Sigma_{2,0}) \to 0. \tag{3.1.1}$$

The sequence (3.1.1), the Snake Lemma, and homology groups of $\Sigma_{1,1}$, $\Sigma_{2,0}$, γ yield the long-exact sequence

$$\mathcal{H}_* : 0 \ \to H_2(\Sigma_{2,0}) \stackrel{f}{\to} H_1(\gamma) \stackrel{g}{\to} H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1}) \stackrel{h}{\to} H_1(\Sigma_{2,0})$$

$$\xrightarrow{i} H_0(\gamma) \xrightarrow{j} H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}) \xrightarrow{k} H_0(\Sigma_{2,0}) \xrightarrow{\ell} 0. \tag{3.1.2}$$

From exactness of the sequence (3.1.2) and the First Isomorphism Theorem it follows that $\mathrm{Im}(g) = \mathrm{Im}(i) = 0$, $\mathrm{Im}(k) = H_0(\Sigma_{2,0})$, and the isomorphisms: $\mathrm{Im}(f) \cong H_2(\Sigma_{2,0})$, $\mathrm{Im}(h) \cong H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$, $\mathrm{Im}(j) \cong H_0(\gamma)$.

Theorem 3.1.1. *Suppose $\Sigma_{1,1}$ is a torus with boundary circle γ and $\Sigma_{2,0}$ is the double of $\Sigma_{1,1}$. If $\mathbf{h}_i^{\Sigma_{1,1}}$ is a basis of $H_i(\Sigma_{1,1})$ and $\mathbf{h}_i^{\gamma}$ is an arbitrary basis of $H_i(\gamma)$, $i = 0, 1$, then there exists a basis $\mathbf{h}_i^{\Sigma_{2,0}}$ of $H_i(\Sigma_{2,0})$, $i = 0, 1, 2$ such that R-torsion of $\mathcal{H}_*$ in the corresponding bases equals to 1. Furthermore, the following formula holds*

$$|\mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_i^{\Sigma_{1,1}}\}_0^1)| = \sqrt{\frac{|\det \Delta_{0,2}(\Sigma_{2,0})|}{\sqrt{|\det \Delta_{1,1}(\Sigma_{2,0})|}}},$$

where $\det \Delta_{i,2-i}(\Sigma_{2,0})$ *denotes the determinant of matrix of the intersection pairing $(\cdot,\cdot)_{i,2-i} : H_i(\Sigma_{2,0}) \times H_{2-i}(\Sigma_{2,0}) \to \mathbb{R}$ in the bases $\mathbf{h}_i^{\Sigma_{2,0}}$, $\mathbf{h}_{2-i}^{\Sigma_{2,0}}$.*

Proof. Let us first explain the method we will use to show that there exists a basis $\mathbf{h}_i^{\Sigma_{2,0}}$ of $H_i(\Sigma_{2,0})$, $i = 0, 1, 2$ so that R-torsion of the chain complex (3.1.2) in the corresponding bases becomes 1.

For $p = 0, \ldots, 6$, let us denote by $C_p(\mathcal{H}_*)$ the vector spaces in the long-exact sequence (3.1.2). Consider the following short-exact sequences:

$$0 \to Z_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \xrightarrow{\partial_p} B_{p-1}(\mathcal{H}_*) \to 0, \tag{3.1.3}$$

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow Z_p(\mathcal{H}_*) \overset{\varphi_p}{\twoheadrightarrow} H_p(\mathcal{H}_*) \to 0. \tag{3.1.4}$$

Here, "$\hookrightarrow$" and "$\twoheadrightarrow$" are the inclusion and the natural projection, respectively. Assume $s_p : B_{p-1}(\mathcal{H}_*) \to C_p(\mathcal{H}_*)$ and $\ell_p : H_p(\mathcal{H}_*) \to Z_p(\mathcal{H}_*)$ are sections of $\partial_p : C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$ and $\varphi_p : Z_p(\mathcal{H}_*) \to H_p(\mathcal{H}_*)$, respectively. By the exactness of $\mathcal{H}_*$, we have $Z_p(\mathcal{H}_*) = B_p(\mathcal{H}_*)$ for all p. Hence, the sequence (3.1.3) becomes

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*) \to 0. \tag{3.1.5}$$

Recall that if for $p = 0, \ldots, 6$, $\mathbf{h_p}$, $\mathbf{b_p}$, and $\mathbf{h}_p^*$ are bases of $C_p(\mathcal{H}_*)$, $B_p(\mathcal{H}_*)$, and $H_p(\mathcal{H}_*)$, respectively, then R-torsion of $\mathcal{H}_*$ with respect to bases $\{\mathbf{h}_p\}_{p=0}^6$, $\{\mathbf{h}_p^*\}_{p=0}^6$ is the alternating product

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_0^6, \{\mathbf{h}_p^*\}_0^6\right) = \prod_{p=0}^{6} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p^*) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{h}_p\right]^{(-1)^{(p+1)}}.$$

From the fact that for $p = 0, \dots, 6$, $H_p(\mathcal{H}_*)$ is zero, it follows that $\mathbf{h}_p^* = 0$ and all ℓ_p are the zero map. Thus, R-torsion of $\mathcal{H}_*$ can be rewritten as:

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_0^6, \{0\}_0^6\right) = \prod_{p=0}^{6} [\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{h}_p]^{(-1)^{(p+1)}}. \tag{3.1.6}$$

Note that J. Milnor proved in [13] that R-torsion does not depend on bases $\mathbf{b}_p$ and sections s_p, ℓ_p. Therefore, in the following method we will choose suitable bases $\mathbf{b}_p$ and sections s_p so that (3.1.6) is equal to 1. For each p, we will denote the obtained basis $\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$ by $\mathbf{h}'_p$.

First, let us consider the space $C_0(\mathcal{H}_*) = H_0(\Sigma_{2,0})$ in the sequence (3.1.5), we get

$$0 \to B_0(\mathcal{H}_*) \hookrightarrow C_0(\mathcal{H}_*) \xrightarrow{\ell} B_{-1}(\mathcal{H}_*) \to 0. \tag{3.1.7}$$

Clearly, we can consider the zero map $s_0 : B_{-1}(\mathcal{H}_*) \longrightarrow C_0(\mathcal{H}_*)$ as a section of ℓ, because $B_{-1}(\mathcal{H}_*)$ is zero. From Splitting Lemma it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(k) \oplus s_0(B_{-1}(\mathcal{H}_*)) = \mathrm{Im}(k). \tag{3.1.8}$$

Let us take the basis of $\mathrm{Im}(k)$ as $\mu_{11}k(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + \mu_{12}k(0, \mathbf{h}_0^{\Sigma_{1,1}})$, where $(\mu_{11}, \mu_{12}) \neq (0,0)$. By equation (3.1.8), $\mu_{11}k(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + \mu_{12}k(0, \mathbf{h}_0^{\Sigma_{1,1}})$ becomes the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Letting the beginning basis $\mathbf{h}_0$ (namely, $\mathbf{h}_0^{\Sigma_{2,0}}$) of $H_0(\Sigma_{2,0})$ be $\mathbf{h}'_0$, we obtain

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{3.1.9}$$

Now, the sequence (3.1.5) for $C_1(\mathcal{H}_*) = H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$ becomes

$$0 \to \mathrm{Im}(j) \hookrightarrow C_1(\mathcal{H}_*) \xrightarrow{k} \mathrm{Im}(k) \to 0 \tag{3.1.10}$$

for $B_1(\mathcal{H}_*)$, $B_0(\mathcal{H}_*)$ being $\mathrm{Im}(j)$, $\mathrm{Im}(k)$, respectively.

By the First Isomorphism Theorem, $\mathrm{Im}(k)$ and $(H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}))/\mathrm{Ker}(k)$ are isomorphic. Therefore, we can consider the inverse of this isomorphism, namely $s_1 : \mathrm{Im}(k) \to (H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}))/\mathrm{Ker}(k)$, as a section of k. By Splitting Lemma, we get

$$C_1(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_1(\mathrm{Im}(k)). \tag{3.1.11}$$

Note that the given basis $\mathbf{h}_1$ of $H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$ is $\{(\mathbf{h}_0^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_0^{\Sigma_{1,1}})\}$. Using the fact that $\mathrm{Im}(j)$ is isomorphic to $H_0(\gamma)$, $K_1 \cdot j(\mathbf{h}_0^{\gamma})$ is a basis of

$\mathrm{Im}(j)$, where the non-zero constant K_1 will be chosen. From the fact that $\mathrm{Im}(j)$ and $s_{_1}(\mathrm{Im}(k))$ are $1-$dimensional subspaces of the $2-$dimensional space $H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$ it follows that there exist non-zero vectors $(e_{_{i1}}, e_{_{i2}}), i = 1, 2$ in the plane such that

$$j(\mathbf{h}_0^{\gamma}) = e_{_{11}}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + e_{_{12}}(0, \mathbf{h}_0^{\Sigma_{1,1}}), \tag{3.1.12}$$

$$s_{_1}(\mathbf{h}^{\mathrm{Im}(k)}) = e_{_{21}}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + e_{_{22}}(0, \mathbf{h}_0^{\Sigma_{1,1}}). \tag{3.1.13}$$

Let us choose the basis $\mathbf{h}^{\mathrm{Im}(j)}$ of $\mathrm{Im}(j)$ as $K_1 j(\mathbf{h}_0^{\gamma})$, where $K_1 = 1/\det E$ and E is the 2×2 real matrix $[e_{ij}]$. By equations (3.1.11)–(3.1.13),

$$\left\{K_1[e_{_{11}}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + e_{_{12}}(0, \mathbf{h}_0^{\Sigma_{1,1}})], e_{_{21}}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + e_{_{22}}(0, \mathbf{h}_0^{\Sigma_{1,1}})\right\}$$

is the obtained basis $\mathbf{h}_1'$ for $C_1(\mathcal{H}_*)$. Hence, we have

$$[\mathbf{h}_1', \mathbf{h}_1] = K_1(\det(E)) = 1. \tag{3.1.14}$$

We now consider the short-exact sequence (3.1.5) for $C_2(\mathcal{H}_*) = H_0(\gamma)$. The fact that $B_2(\mathcal{H}_*)$ and $B_1(\mathcal{H}_*)$ are respectively equal to $\mathrm{Im}(i)$ and $\mathrm{Im}(j)$ yields

$$0 \to \mathrm{Im}(i) \hookrightarrow C_2(\mathcal{H}_*) \xrightarrow{j} \mathrm{Im}(j) \to 0. \tag{3.1.15}$$

For $j : H_0(\gamma) \to \mathrm{Im}(j)$ being an isomorphism, we can take the inverse of j as a section $s_{_2} : \mathrm{Im}(j) \to H_0(\gamma)$ of j. From Splitting Lemma it follows

$$C_2(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_{_2}(\mathrm{Im}(j)). \tag{3.1.16}$$

Recall that in the previous step, we chose $j(K_1\mathbf{h}_0^{\gamma})$ as a basis of $\mathrm{Im}(j)$. By equation (3.1.16) and the fact that $\mathrm{Im}(i) = 0$, we have that the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$ is $K_1\mathbf{h}_0^{\gamma}$. Thus, by the fact that the given basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$ is also $\mathbf{h}_0^{\gamma}$, we get

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{3.1.17}$$

Considering the space $C_3(\mathcal{H}_*) = H_1(\Sigma_{2,0})$ in the sequence (3.1.5) and using the fact that $B_3(\mathcal{H}_*)$, $B_2(\mathcal{H}_*)$ equal to $\mathrm{Im}(h)$, $\mathrm{Im}(i)$, respectively, we obtain

$$0 \to \mathrm{Im}(h) \hookrightarrow C_3(\mathcal{H}_*) \xrightarrow{i} \mathrm{Im}(i) \to 0. \tag{3.1.18}$$

Since $\mathrm{Im}(i)$ is zero, we can take zero map $s_3 : \mathrm{Im}(i) \to H_1(\Sigma_{2,0})$ as a section of i. By Splitting Lemma, we have

$$C_3(\mathcal{H}_*) = \mathrm{Im}(h) \oplus s_3(\mathrm{Im}(i)) = \mathrm{Im}(h). \tag{3.1.19}$$

The given basis $\mathbf{h}^{H_1(\Sigma_{1,1})\oplus H_1(\Sigma_{1,1})}$ of $H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$ is

$$\left\{(\mathbf{h}_{1,1}^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_{1,1}^{\Sigma_{1,1}}), (\mathbf{h}_{1,2}^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_{1,2}^{\Sigma_{1,1}})\right\}.$$

From the fact that $\mathrm{Im}(h)$ is isomorphic to $H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$ it follows that we can choose the basis $\mathbf{h}^{\mathrm{Im}(h)}$ of $\mathrm{Im}(h)$ as

$$\left\{h((\mathbf{h}_{1,1}^{\Sigma_{1,1}}, 0)), h((0, \mathbf{h}_{1,1}^{\Sigma_{1,1}})), h((\mathbf{h}_{1,2}^{\Sigma_{1,1}}, 0)), h((0, \mathbf{h}_{1,2}^{\Sigma_{1,1}}))\right\}.$$

By equation (3.1.19), we have that the obtained basis $\mathbf{h}'_3$ of $C_3(\mathcal{H}_*)$ is $\mathbf{h}^{\mathrm{Im}(h)}$. If we let the beginning basis $\mathbf{h}_3$ (namely, $\mathbf{h}_1^{\Sigma_{2,0}}$) of $C_3(\mathcal{H}_*)$ as $\mathbf{h}'_3$, then we get

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1. \tag{3.1.20}$$

Let us consider the sequence (3.1.5) for the space $C_4(\mathcal{H}_*) = H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$. By the fact that $B_4(\mathcal{H}_*)$, $B_3(\mathcal{H}_*)$ are equal to $\mathrm{Im}(g)$, $\mathrm{Im}(h)$, respectively, we obtain

$$0 \to \mathrm{Im}(g) \hookrightarrow C_4(\mathcal{H}_*) \stackrel{h}{\to} \mathrm{Im}(h) \to 0. \tag{3.1.21}$$

From the fact that h is an isomorphism it follows that we can consider the inverse of h as a section $s_4 : \mathrm{Im}(h) \to H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$ of h. The fact that $\mathrm{Im}(g)$ is zero and Splitting Lemma yield

$$C_4(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_4(\mathrm{Im}(h)) = s_4(\mathrm{Im}(h)). \tag{3.1.22}$$

Recall that the given basis $\mathbf{h}_4$ of $H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$ is $\mathbf{h}^{H_1(\Sigma_{1,1})\oplus H_1(\Sigma_{1,1})}$. Moreover, in the previous step, we chose the basis $\mathbf{h}^{\mathrm{Im}(h)}$ of $\mathrm{Im}(h)$ as

$$\left\{h((\mathbf{h}_{1,1}^{\Sigma_{1,1}}, 0)), h((0, \mathbf{h}_{1,1}^{\Sigma_{1,1}})), h((\mathbf{h}_{1,2}^{\Sigma_{1,1}}, 0)), h((0, \mathbf{h}_{1,2}^{\Sigma_{1,1}}))\right\}.$$

It follows from equation (3.1.22) that $s_4(\mathbf{h}^{\mathrm{Im}(h)}) = \mathbf{h}^{H_1(\Sigma_{1,1})\oplus H_1(\Sigma_{1,1})}$ is the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. Hence, we have

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{3.1.23}$$

Now, we consider the space $C_5(\mathcal{H}_*) = H_1(\gamma)$ in the short-exact sequence (3.1.5). Using the fact that $B_5(\mathcal{H}_*)$, $B_4(\mathcal{H}_*)$ equal to $\mathrm{Im}(f)$, $\mathrm{Im}(g)$, respectively, we get

$$0 \to \mathrm{Im}(f) \hookrightarrow C_5(\mathcal{H}_*) \xrightarrow{g} \mathrm{Im}(g) \to 0. \tag{3.1.24}$$

Since $\mathrm{Im}(g)$ is zero, the zero map $s_5 : \mathrm{Im}(h) \longrightarrow H_1(\gamma)$ can be considered as a section of g. From Splitting Lemma it follows that

$$C_5(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_5(\mathrm{Im}(g)) = \mathrm{Im}(f). \tag{3.1.25}$$

The given basis $\mathbf{h}_5$ of $H_1(\gamma)$ is $\mathbf{h}_1^\gamma$. By equation (3.1.25), we choose the basis $\mathbf{h}^{\mathrm{Im}(f)}$ of $\mathrm{Im}(f)$ as $\mathbf{h}_1^\gamma$, which is also the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Thus, we obtain

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{3.1.26}$$

Finally, considering the space $C_6(\mathcal{H}_*) = H_2(\Sigma_{2,0})$ in the sequence (3.1.5) and using the fact that $B_6(\mathcal{H}_*)$, $B_5(\mathcal{H}_*)$ equal to zero, $\mathrm{Im}(f)$, respectively, we get

$$0 \to 0 \hookrightarrow C_6 \xrightarrow{f} \mathrm{Im}(f) \to 0. \tag{3.1.27}$$

For $\mathrm{Im}(f)$ being isomorphic to $H_2(\Sigma_{2,0})$, we consider the inverse of f as section $s_6 : \mathrm{Im}(f) \to H_2(\Sigma_{2,0})$ of f. Splitting Lemma results

$$C_6(\mathcal{H}_*) = 0 \oplus s_6(\mathrm{Im}(f)) = s_6(\mathrm{Im}(f)). \tag{3.1.28}$$

From equation (3.1.28) it follows that $f^{-1}(\mathbf{h}^{\mathrm{Im}(f)})$ is the obtained basis $\mathbf{h}'_6$ of $C_6(\mathcal{H}_*)$. If we take the basis $\mathbf{h}_6$, namely $\mathbf{h}_2^{\Sigma_{2,0}}$, of $H_2(\Sigma_{2,0})$ as $f^{-1}(\mathbf{h}^{\mathrm{Im}(f)})$, then we get

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{3.1.29}$$

Equations (3.1.9), (3.1.14), (3.1.17), (3.1.20), (3.1.23), (3.1.26), and (3.1.29) yield

$$\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^6, \{0\}_0^6) = \prod_{p=0}^{6} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{3.1.30}$$

Clearly, the natural bases are compatible in the sequence (3.1.1). Then, Theorem 2.0.2 and (3.1.30) yield us

$$\mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_i^{\Sigma_{1,1}}\}_0^1)^2 = \mathbb{T}(\gamma_1, \{\mathbf{h}_i^\gamma\}_0^1)\, \mathbb{T}(\Sigma_{2,0}, \{\mathbf{h}_i^{\Sigma_{2,0}}\}_0^2). \tag{3.1.31}$$

From Remark 2.0.8 and equation (3.1.31), it follows that

$$|\mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_i^{\Sigma_{1,1}}\}_0^1)| = \sqrt{|\mathbb{T}(\Sigma_{2,0}, \{\mathbf{h}_i^{\Sigma_{2,0}}\}_0^2)|}. \tag{3.1.32}$$

Theorem 2.0.6 and equation (3.1.32) finish the proof of Theorem 3.1.1. □

Remark 3.1.2. *Suppose that* $\Sigma_{1,1}$, $\Sigma_{2,0}$, $\mathbf{h}_i^{\Sigma_{1,1}}$, $\mathbf{h}_i^{S_j}$, *and* $\mathbf{h}_i^{\Sigma_{2,0}}$ *are all as in Theorem 3.1.1. By Poincaré Duallity, Theorem 3.1.1, and [23, Theorem 4.1], we have*

$$\left|\mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_i^{\Sigma_{1,1}}\}_0^1)\right| = \sqrt{\left|\frac{\det \Delta_{0,2}(\Sigma_{2,0})}{\det \wp(\mathbf{h}^1_{\Sigma_{2,0}}, \Gamma)}\right|}.$$

Here, $\det \Delta_{0,2}(\Sigma_{2,0})$ *is the determinant of matrix of the intersection pairing* $(\cdot,\cdot)_{0,2} : H_0(\Sigma_{2,0}) \times H_2(\Sigma_{2,0}) \to \mathbb{R}$ *in the bases* $\mathbf{h}_0^{\Sigma_{2,0}}$ *and* $\mathbf{h}_2^{\Sigma_{2,0}}$, $\mathbf{h}^1_{\Sigma_{2,0}} = \{\omega_i\}_1^4$ *is the Poincaré dual basis of* $H^1(\Sigma_{2,0})$ *corresponding to the basis* $\mathbf{h}_1^{\Sigma_{2,0}}$ *of* $H_1(\Sigma_{2,0})$, $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ *is a canonical basis for* $H_1(\Sigma_{2,0})$, *i.e.* $i = 1, 2$, Γ_i *intersects* Γ_{i+2} *once positively and does not intersect others, and* $\wp(\mathbf{h}^1, \Gamma) = \left[\int_{\Gamma_i} \omega_j\right]$ *is the period matrix of* $\mathbf{h}^1_{\Sigma_{2,0}}$ *with respect to* Γ.

3.2. R-torsion of orientable surface $\Sigma_{1,n}, n \geq 2$

Proposition 3.2.1. *Let* $\Sigma_{1,n}$ *be an orientable surface of genus* 1 *with boundary circles* $S_1, \ldots, S_n$. *For* $i = 1, \ldots, n$, *let* $\mathbb{D}_i$ *denote the closed disk with boundary* S_i. *Consider the surface* $\Sigma_{1,n-1}$ *obtained by gluing the surfaces* $\Sigma_{1,n}$ *and* $\mathbb{D}_1$ *along the common boundary circle* S_1 *(see, Fig. 2). Consider also the associated short-exact sequence of chain complexes*

$$0 \to C_*(S_1) \longrightarrow C_*(\Sigma_{1,n}) \oplus C_*(\mathbb{D}_1) \longrightarrow C_*(\Sigma_{1,n-1}) \to 0, \tag{3.2.1}$$

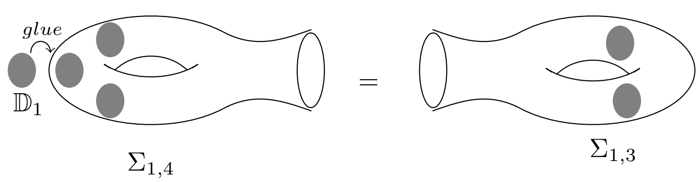

Figure 2: Orientable surface $\Sigma_{1,3}$ is obtained by gluing $\Sigma_{1,4}$ and $\mathbb{D}_1$ along the common boundary circle S_1.

and the long-exact sequence

$$\begin{aligned}\mathcal{H}_* : 0 \; &\to H_1(S_1) \xrightarrow{f} H_1(\Sigma_{1,n}) \xrightarrow{g} H_1(\Sigma_{1,n-1}) \xrightarrow{h} H_0(S_1)\\ &\xrightarrow{i} H_0(\Sigma_{1,n}) \oplus H_0(\mathbb{D}_1) \xrightarrow{j} H_0(\Sigma_{1,n-1}) \xrightarrow{k} 0\end{aligned}$$

obtained by the Snake Lemma for (3.2.1). Let $\mathbf{h}_\nu^{\Sigma_{1,n}}$ *be a basis of* $H_\nu(\Sigma_{1,n})$ *and* $\mathbf{h}_0^{\mathbb{D}_1}$ *be an arbitrary basis of* $H_0(\mathbb{D}_1)$, $\nu = 0, 1$. *Then, for* $\nu = 0, 1$ *there exist bases* $\mathbf{h}_\nu^{\Sigma_{1,n-1}}$ *and* $\mathbf{h}_\nu^{S_1}$ *of* $H_\nu(\Sigma_{1,n-1})$ *and* $H_\nu(S_1)$, *respectively so that R-torsion of* $\mathcal{H}_*$ *in these bases is* 1 *and the following formula is valid*

$$\mathbb{T}(\Sigma_{1,n}, \{\mathbf{h}_\nu^{\Sigma_{1,n}}\}_0^1) = \mathbb{T}(\Sigma_{1,n-1}, \{\mathbf{h}_\nu^{\Sigma_{1,n-1}}\}_0^1)\; \mathbb{T}(S_1, \{\mathbf{h}_\nu^{S_1}\}_0^1)\mathbb{T}(\mathbb{D}_1, \{\mathbf{h}_0^{\mathbb{D}_1}\})^{-1}.$$

Proof. Using the exactness of the sequence $\mathcal{H}_*$ and the First Isomorphism Theorem, we get $\mathrm{Im}(h) = 0$, $\mathrm{Im}(k) = H_0(\Sigma_{1,n-1})$, and the isomorphisms $\mathrm{Im}(f) \cong H_1(S_1)$, $\mathrm{Im}(i) \cong H_0(S_1)$.

For $p = 0, \ldots, 5$, we denote the vector spaces in long-exact sequence $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$ and consider the short-exact sequence

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \twoheadrightarrow B_{p-1}(\mathcal{H}_*) \to 0. \tag{3.2.2}$$

For each p, let us consider the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$. Then, we obtain

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{3.2.3}$$

We first consider the vector space $C_0(\mathcal{H}_*) = H_0(\Sigma_{1,n-1})$ in (3.2.3). Since $\mathrm{Im}\, k$ is zero, we have

$$C_0(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_0(\mathrm{Im}(k)) = \mathrm{Im}(j). \tag{3.2.4}$$

As $\mathrm{Im}(j)$ is a 1−dimensional subspace of $H_0(\Sigma_{1,n}) \oplus H_0(\mathbb{D}_1)$, there is a non-zero vector (a_{11}, a_{12}) in the plane such that $\{a_{11}\mathbf{h}_0^{\Sigma_{1,n}} + a_{12}\mathbf{h}_0^{\mathbb{D}_1}\}$ is the basis $\mathbf{h}^{\mathrm{Im}(j)}$ of $\mathrm{Im}(j)$. From equation (3.2.4) it follows that $\mathbf{h}^{\mathrm{Im}(j)}$ is the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Since $\mathrm{Im}(j)$ is equal to $C_0(\mathcal{H}_*)$, we can choose the beginning basis $\mathbf{h}_0$ (namely, $\mathbf{h}_0^{\Sigma_{1,n-1}}$) of $C_0(\mathcal{H}_*)$ as $\mathbf{h}^{\mathrm{Im}(j)}$. Thus, we get

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{3.2.5}$$

Considering (3.2.3) for $C_1(\mathcal{H}_*) = H_0(\Sigma_{1,n}) \oplus H_0(\mathbb{D}_1)$, we have

$$C_1(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_1(\mathrm{Im}(j)). \tag{3.2.6}$$

Recall that in the previous step we chose the basis of $\mathrm{Im}(j)$ as $\mathbf{h}^{\mathrm{Im}(j)}$. For $\mathrm{Im}(i)$ being a $1-$dimensional subspace of $C_1(\mathcal{H}_*)$, there are non-zero numbers b_{11}, b_{12} such that $\{b_{11}\mathbf{h}_0^{\Sigma_{1,n}} + b_{12}\mathbf{h}_0^{\mathbb{D}_1}\}$ is a basis of $\mathrm{Im}(i)$. Let $\mathbf{h}^{\mathrm{Im}(i)}$ be the basis $\{T_1[b_{11}\mathbf{h}_0^{\Sigma_{1,n}} + b_{12}\mathbf{h}_0^{\mathbb{D}_1}]\}$ of $\mathrm{Im}(i)$. Here, T_1 is a non-zero constant which will be determined later.

On the other hand, $s_1(\mathrm{Im}(j))$ is also a $1-$dimensional subspace of $C_1(\mathcal{H}_*)$. Thus, there is a non-zero vector (b_{21}, b_{22}) in the plane such that the following equality holds

$$s_1(\mathbf{h}^{\mathrm{Im}(j)}) = b_{21}\mathbf{h}_0^{\Sigma_{1,n}} + b_{22}\mathbf{h}_0^{\mathbb{D}_1}.$$

Clearly, the determinant of the matrix $B = [b_{ij}]$ is non-zero. Taking T_1 as $1/\det B$, then from equation (3.2.6) it follows that $\{\mathbf{h}^{\mathrm{Im}(i)}, s_1(\mathbf{h}^{\mathrm{Im}(j)})\}$ is the obtained basis $\mathbf{h}_1'$ of $C_1(\mathcal{H}_*)$. Since the beginning basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$ is $\{\mathbf{h}_0^{\Sigma_{1,n}}, \mathbf{h}_0^{\mathbb{D}_1}\}$, we obtain

$$[\mathbf{h}_1', \mathbf{h}_1] = T_1 \det B = 1. \tag{3.2.7}$$

Next, let us consider the space $C_2(\mathcal{H}_*) = H_0(S_1)$ in (3.2.3). Using the fact that $\mathrm{Im}\, h$ is zero, we have

$$C_2(\mathcal{H}_*) = \mathrm{Im}(h) \oplus s_2(\mathrm{Im}(i)) = s_2(\mathrm{Im}(i)). \tag{3.2.8}$$

Recall that $\mathbf{h}^{\mathrm{Im}(i)}$ was chosen in the previous step. It follows from equation (3.2.8) that $s_2(\mathbf{h}^{\mathrm{Im}(i)})$ is the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. Since $C_2(\mathcal{H}_*)$ is equal to $s_2(\mathrm{Im}(i))$, let the beginning basis $\mathbf{h}_2$ (namely, $\mathbf{h}_0^{S_1}$) of $C_2(\mathcal{H}_*)$ be $s_2(\mathbf{h}^{\mathrm{Im}(i)})$. Hence, we obtain

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{3.2.9}$$

We now consider the case of $C_3(\mathcal{H}_*) = H_1(\Sigma_{1,n-1})$ in (3.2.3). Because $\mathrm{Im}(h)$ is zero, we have the following equality

$$C_3(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_3(\mathrm{Im}(h)) = \mathrm{Im}(g). \tag{3.2.10}$$

$\mathrm{Im}(g)$ is an $n-$dimensional subspace of the $(n+1)-$dimensional space $H_1(\Sigma_{1,n})$ with the given basis $\mathbf{h}_1^{\Sigma_{1,n}}$ as $\left\{\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_\mu\right\}_{\mu=1}^{n+1}$. From this there are non-zero vectors $(c_{\nu,1}, \dots, c_{\nu,n+1})$, $\nu = 1, \dots, n$ such that

$$\mathbf{h}^{\mathrm{Im}(g)} = \left\{\sum_{\mu=1}^{n+1} c_{\nu,\mu}\, g\left(\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_\mu\right)\right\}_{\nu=1}^{n}$$

is the basis of $\mathrm{Im}(g)$. By equation (3.2.10), $\mathbf{h}^{\mathrm{Im}(g)}$ becomes the obtained basis $\mathbf{h}'_3$ of $C_3(\mathcal{H}_*)$. Moreover, for $C_3(\mathcal{H}_*)$ being equal to $\mathrm{Im}(g)$, let the beginning basis $\mathbf{h}_3$ (namely, $\mathbf{h}_1^{\Sigma_{1,n-1}}$) of $C_3(\mathcal{H}_*)$ be $\mathbf{h}^{\mathrm{Im}(g)}$. Therefore, we have

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1. \tag{3.2.11}$$

We now consider (3.2.3) for $C_4(\mathcal{H}_*) = H_1(\Sigma_{1,n})$. Then, we get

$$C_4(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_4(\mathrm{Im}(g)). \tag{3.2.12}$$

For $\mathrm{Im}(f)$ being a $1-$dimensional subspace of $C_4(\mathcal{H}_*)$, there is a non-zero vector $(d_{1,1}, \ldots, d_{1,n+1})$ such that $\left\{d_{1,1}\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_1 + \cdots + d_{1,n+1}\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_{n+1}\right\}$ is a basis of $\mathrm{Im}(f)$. Let

$$\mathbf{h}^{\mathrm{Im}(f)} = \left\{T_2\left[d_{1,1}\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_1 + \cdots + d_{1,n+1}\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_{n+1}\right]\right\}$$

be the basis of $\mathrm{Im}(f)$, where T_2 is a non-zero constant to be chosen later.

Since $s_4(\mathrm{Im}(g))$ is an $n-$dimensional subspace of $(n+1)-$dimensional space $C_4(\mathcal{H}_*)$, there are non-zero vectors $(d_{\nu,1}, \ldots, d_{\nu,n+1})$, $\nu = 2, \ldots, n+1$ such that the following equality holds

$$s_4(\mathbf{h}^{\mathrm{Im}(g)}) = \left\{\sum_{\mu=1}^{n+1} d_{\nu,\mu}\left(\mathbf{h}_1^{\Sigma_{1,n}}\right)_\mu\right\}_{\nu=2}^{n+1}.$$

Clearly, the determinant of the matrix $D = [d_{ij}]$ is non-zero. If we take T_2 as $1/\det D$, then by equation (3.2.12) we have that $\{\mathbf{h}^{\mathrm{Im}(f)}, s_4(\mathbf{h}^{\mathrm{Im}(g)})\}$ is the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. For the beginning basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$ being $\mathbf{h}_1^{\Sigma_{1,n}}$, we get

$$[\mathbf{h}'_4, \mathbf{h}_4] = \mathrm{T}_2(\det D) = 1. \tag{3.2.13}$$

Finally, let us consider the case of $C_5(\mathcal{H}_*) = H_1(S_1)$ in (3.2.3). Since $B_5(\mathcal{H}_*)$ is zero, the following equality holds

$$C_5(\mathcal{H}_*) = B_5(\mathcal{H}_*) \oplus s_5(\mathrm{Im}(f)) = s_5(\mathrm{Im}(f)). \tag{3.2.14}$$

Recall that the basis $\mathbf{h}^{\mathrm{Im}(f)}$ was chosen for $\mathrm{Im}(f)$ in the previous step. By equation (3.2.14), $s_5(\mathbf{h}^{\mathrm{Im}(f)})$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. From the fact that $C_5(\mathcal{H}_*)$ is $s_5(\mathrm{Im}(f))$, it follows that we can take the beginning basis $\mathbf{h}_5$ (namely, $\mathbf{h}_0^{S_1}$) of $C_5(\mathcal{H}_*)$ as $s_5(\mathbf{h}^{\mathrm{Im}(f)})$. Thus, the following equality holds

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{3.2.15}$$

Combining equations (3.2.5), (3.2.7), (3.2.9), (3.2.11), (3.2.13), and (3.2.15), we have

$$\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^5, \{0\}_0^5) = \prod_{p=0}^{5} [\mathbf{h}'{}_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{3.2.16}$$

The compatibility of the natural bases in the short-exact sequence (3.2.1), Theorem 2.0.2, Lemma 2.0.3, and equation (3.2.16) finish the proof of Proposition 3.2.1. □

Using the arguments in Proposition 3.2.1 inductively, we have the following result.

Proposition 3.2.2. *Let $\Sigma_{1,n}$ be an orientable surface of genus 1 with $n \geq 2$ boundary circles $S_1, \dots, S_n$. For $i = 1, \dots, n$, let $\mathbb{D}_i$ denote the closed disks with boundary S_i. For $i = 1, \dots, n-1$, let $\Sigma_{1,n-i}$ be the surface obtained from $\Sigma_{1,n}$ by gluing $\mathbb{D}_1, \dots, \mathbb{D}_i$ along $S_1, \dots, S_i$. Consider the surface $\Sigma_{1,n-i}$ obtained by gluing the surfaces $\Sigma_{1,n-i+1}$ and $\mathbb{D}_i$ along the common boundary circle S_i, $i = 1, \dots, n-1$. Let*

$$0 \to C_*(S_i) \longrightarrow C_*(\Sigma_{1,n-i+1}) \oplus C_*(\mathbb{D}_i) \longrightarrow C_*(\Sigma_{1,n-i}) \to 0$$

be the asociated natural short-exact sequence of chain complexes and ${\mathcal{H}_*}^i$ be the corresponding long-exact sequence obtained by the Snake Lemma. Let $\mathbf{h}_\nu^{\Sigma_{1,n}}$ be a basis of $H_\nu(\Sigma_{1,n})$ and $\mathbf{h}_0^{\mathbb{D}_i}$ be an arbitrary basis of $H_0(\mathbb{D}_i)$, $\nu = 0, 1$, $i = 1, \dots, n-1$. Then, there exist bases respectively $\mathbf{h}_\nu^{\Sigma_{1,1}}$ and $\mathbf{h}_\nu^{S_i}$ of $H_\nu(\Sigma_{1,1})$ and $H_\nu(S_i)$, $\nu = 0, 1$, $i = 1, \dots, n-1$ such that R-torsion of each ${\mathcal{H}_*}^i$ in the coresponding bases is 1 and the following formula is valid

$$\mathbb{T}(\Sigma_{1,n}, \{\mathbf{h}_\nu^{\Sigma_{1,n}}\}_0^1) = \mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_\nu^{\Sigma_{1,1}}\}_0^1) \prod_{i=1}^{n-1} [\mathbb{T}(S_i, \{\mathbf{h}_\nu^{S_i}\}_0^1) \mathbb{T}(\mathbb{D}_i, \{\mathbf{h}_0^{\mathbb{D}_i}\})^{-1}].$$

Combining Remark 2.0.8 and Proposition 3.2.2, we obtain

Proposition 3.2.3. *Let $\Sigma_{1,n}, S_i, \mathbb{D}_i, \Sigma_{1,n-i}$, $\mathcal{H}_*^i$, $\mathbf{h}_\nu^{\Sigma_{1,n}}$, $\mathbf{h}_0^{\mathbb{D}_i}$, $\mathbf{h}_\nu^{\Sigma_{1,1}}, \mathbf{h}_\nu^{S_i}$ be as in Proposition 3.2.2. Then, the following formula holds*

$$|\mathbb{T}(\Sigma_{1,n}, \{\mathbf{h}_\nu^{\Sigma_{1,n}}\}_0^1)| = |\mathbb{T}(\Sigma_{1,1}, \{\mathbf{h}_\nu^{\Sigma_{1,1}}\}_0^1)| \prod_{p=1}^{n-1} |\mathbb{T}(\mathbb{D}_i, \{\mathbf{h}_0^{\mathbb{D}_i}\})|^{-1}.$$

Remark 3.2.4. *It should be mentioned that following the arguments in Proposition 3.2.1, one has similar result for the sphere $\Sigma_{0,k}, k \geq 1$ with boundary circles $S_1, \ldots, S_k$. To be more precise, let $\mathbb{D}_i$ denote the closed disks with boundary S_i, $i = 1, \ldots, k$. Consider the surface $\Sigma_{0,k-1}$ obtained by gluing surfaces $\Sigma_{0,k}$ and $\mathbb{D}_1$ along the common boundary circle S_1. Let*

$$0 \to C_*(S_1) \longrightarrow C_*(\Sigma_{0,k}) \oplus C_*(\mathbb{D}_1) \longrightarrow C_*(\Sigma_{0,k-1}) \to 0$$

be the natural short-exact sequence of chain complexes.

Let us first consider the case $k \geq 2$. The associated long-exact sequence obtained by the Snake Lemma is

$$\begin{aligned} \mathcal{H}_* : 0 \ &\to H_1(S_1) \xrightarrow{f} H_1(\Sigma_{0,k}) \xrightarrow{g} H_1(\Sigma_{0,k-1}) \xrightarrow{h} H_0(S_1) \\ &\xrightarrow{i} H_0(\Sigma_{0,k}) \oplus H_0(\mathbb{D}_1) \xrightarrow{j} H_0(\Sigma_{0,k-1}) \xrightarrow{k} 0. \end{aligned}$$

If for $\nu = 0, 1$, $\mathbf{h}_\nu^{\Sigma_{0,k}}$ is a basis of $H_\nu(\Sigma_{0,k})$ and $\mathbf{h}_0^{\mathbb{D}_1}$ is an arbitrary basis of $H_0(\mathbb{D}_1)$, then there are bases respectively $\mathbf{h}_\nu^{\Sigma_{0,k-1}}$ and $\mathbf{h}_\nu^{S_1}$ of $H_\nu(\Sigma_{0,k-1})$ and $H_\nu(S_1)$, $\nu = 0, 1$ so that R-torsion of $\mathcal{H}_*$ in these bases equals to 1 and the following formula holds

$$\mathbb{T}(\Sigma_{0,k}, \{\mathbf{h}_\nu^{\Sigma_{0,k}}\}_0^1) = \mathbb{T}(\Sigma_{0,k-1}, \{\mathbf{h}_\nu^{\Sigma_{0,k-1}}\}_0^1)\ \mathbb{T}(S_1, \{\mathbf{h}_\nu^{S_1}\}_0^1)\mathbb{T}(\mathbb{D}_1, \{\mathbf{h}_0^{\mathbb{D}_1}\})^{-1}.$$

Let us consider the case $k = 1$. The corresponding long-exact sequence $\mathcal{H}_*$ is

$$0 \to H_2(\Sigma_{0,0}) \xrightarrow{f} H_1(S_1) \xrightarrow{0} H_0(S_1) \xrightarrow{i} H_0(\Sigma_{0,1}) \oplus H_0(\mathbb{D}_1) \xrightarrow{j} H_0(\Sigma_{0,0}) \xrightarrow{k} 0.$$

Let $\mathbf{h}_0^{\Sigma_{0,1}}$ be a basis of $H_0(\Sigma_{0,1})$. Let $\mathbf{h}_0^{\mathbb{D}_1}$ and $\mathbf{h}_1^{S_1}$ be arbitrary bases of $H_0(\mathbb{D}_1)$ and $H_1(S_1)$, respectively. Then there exist respectively bases $\mathbf{h}_0^{S_1}$, $\mathbf{h}_\nu^{\Sigma_{0,0}}$ of $H_0(S_1)$, $H_\nu(\Sigma_{0,0})$, $\nu = 0, 2$ such that R-torsion of $\mathcal{H}_*$ in these bases equals to 1 and the following formula is valid

$$\begin{aligned} \mathbb{T}(\Sigma_{0,1}, \{\mathbf{h}_0^{\Sigma_{0,1}}\}) &= \mathbb{T}(\Sigma_{0,0}, \{\mathbf{h}_0^{\Sigma_{0,0}}, 0, \mathbf{h}_2^{\Sigma_{0,0}}\})\ \mathbb{T}(S_1, \{\mathbf{h}_\nu^{S_1}\}_0^1) \\ &\quad \times \mathbb{T}(\mathbb{D}_1, \{\mathbf{h}_0^{\mathbb{D}_1}\})^{-1}. \end{aligned} \tag{3.2.17}$$

Equation (3.2.17) suggests a formula for R-torsion of closed disk $\mathbb{D}$. More precisely, by the fact that R-torsion of a chain complex C_* of length m can be considered as an element of the dual of the one dimensional vector space $\otimes_{p=0}^m(\det(H_p(C)))^{(-1)^p}$ [19, Theorem 2.0.4.], we have $\mathbb{T}(\mathbb{D}_1)$ is a non-zero *linear functional on the one dimensional real vector space* $H_0(\mathbb{D}_1)$*. Thus, considering the basis* $\mathbf{h}_0^{\mathbb{D}_1}$ *of* $H_0(\mathbb{D}_1)$ *so that* $\mathbb{T}(\mathbb{D}_1, \{\mathbf{h}_0^{\mathbb{D}_1}\}) = 1$ *and using equation (3.2.17), one has the following formula for R-torsion of closed disk*

$$\mathbb{T}(\Sigma_{0,1}, \{\mathbf{h}_0^{\Sigma_{0,1}}\}) = \mathbb{T}(\Sigma_{0,0}, \{\mathbf{h}_0^{\Sigma_{0,0}}, 0, \mathbf{h}_2^{\Sigma_{0,0}}\})\, \mathbb{T}(S_1, \{\mathbf{h}_\nu^{S_1}\}_0^1). \tag{3.2.18}$$

Moreover, from Remark 2.0.8 and Theorem 2.0.6 it follows that

$$\left|\mathbb{T}(\Sigma_{0,1}, \{\mathbf{h}_\nu^{\Sigma_{0,1}}\}_0^1)\right| = \left|\left(\mathbf{h}_0^{\Sigma_{0,0}}, \mathbf{h}_2^{\Sigma_{0,0}}\right)_{0,2}\right|, \tag{3.2.19}$$

where $(\cdot,\cdot)_{0,2} : H_0(\Sigma_{0,0}) \times H_2(\Sigma_{0,0}) \to \mathbb{R}$ *is the intersection pairing of sphere* $\Sigma_{0,0}$.

Note that equation (3.2.19) suggests a formula for computing R-torsion of closed disk $\Sigma_{0,1}$ *in terms of R-torsion of sphere* $\Sigma_{0,0}$.

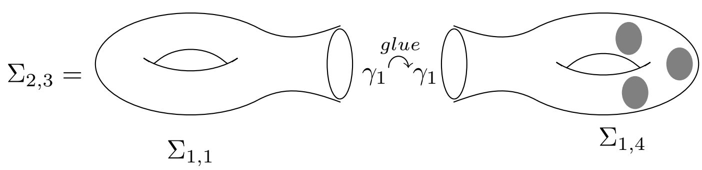

Figure 3: Orientable surface $\Sigma_{2,3}$ is obtained by gluing $\Sigma_{1,1}$ and $\Sigma_{1,4}$ along common boundary circle γ_1.

The following result proves a formula for R-torsion of $\Sigma_{g,n}$ in terms of R-torsion of the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,n+1}$ and also circle. More precisely,

Proposition 3.2.5. *Let* $g \geq 2$ *and* $n \geq 1$*. Consider the surface* $\Sigma_{g,n}$ *obtained by gluing the surfaces* $\Sigma_{g-1,1}$ *and* $\Sigma_{1,n+1}$ *along the common boundary circle* γ_1 *(see, Fig. 3). Consider also the associated short-exact sequence of chain complexes*

$$0 \to C_*(\gamma_1) \longrightarrow C_*(\Sigma_{g-1,1}) \oplus C_*(\Sigma_{1,n+1}) \longrightarrow C_*(\Sigma_{g,n}) \to 0, \tag{3.2.20}$$

and the long-exact sequence

$$\begin{aligned}\mathcal{H}_* : 0 \;&\to H_1(\gamma_1) \xrightarrow{f} H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,n+1}) \xrightarrow{g} H_1(\Sigma_{g,n}) \\ &\xrightarrow{h} H_0(\gamma_1) \xrightarrow{i} H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,n+1}) \xrightarrow{j} H_0(\Sigma_{g,n}) \xrightarrow{k} 0\end{aligned}$$

obtained by the Snake Lemma for (3.2.20). Let $\mathbf{h}_\nu^{\Sigma_{g,n}}$ *be a basis of* $H_\nu(\Sigma_{g,n})$*,* $\nu = 0, 1$*. Let* $\mathbf{h}_\nu^{\gamma_1}$ *be an arbitrary basis of* $H_\nu(\gamma_1)$*,* $\nu = 0, 1$*. Then, there exist* *bases* $\mathbf{h}_\nu^{\Sigma_{g-1,1}}$ *and* $\mathbf{h}_\nu^{\Sigma_{1,n+1}}$ *of* $H_\nu(\Sigma_{g-1,1})$ *and* $H_\nu(\Sigma_{1,n+1})$, $\nu = 0, 1$, *respectively such that R-torsion of* $\mathcal{H}_*$ *in the corresponding bases is* 1 *and the following formula holds*

$$\begin{aligned}\mathbb{T}(\Sigma_{g,n}, \{\mathbf{h}_\nu^{\Sigma_{g,n}}\}_0^1) &= \mathbb{T}(\Sigma_{g-1,1}, \{\mathbf{h}_\nu^{\Sigma_{g-1,1}}\}_0^1)\, \mathbb{T}(\Sigma_{1,n+1}, \{\mathbf{h}_\nu^{\Sigma_{1,n+1}}\}_0^1) \\ &\quad \times \mathbb{T}(\gamma_1, \{\mathbf{h}_\nu^{\gamma_1}\}_0^1)^{-1}.\end{aligned}$$

Proof. First, we denote the vector spaces in $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$, $p = 0, \ldots, 5$. For each p, the exactness of $\mathcal{H}_*$ yields the following short-exact sequence

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \twoheadrightarrow B_{p-1}(\mathcal{H}_*) \to 0.$$

For all p, considering the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*)) \subseteq C_p(\mathcal{H}_*)$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$, we obtain

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{3.2.21}$$

Let us consider the space $C_0(\mathcal{H}_*) = H_0(\Sigma_{g,n})$ in (3.2.21). From the fact that $\mathrm{Im}(k)$ is equal to zero it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_0(\mathrm{Im}(k)) = \mathrm{Im}(j). \tag{3.2.22}$$

Let us choose the basis of $\mathrm{Im}\, j$ as $\mathbf{h}_0^{\Sigma_{g,n}}$. From equation (3.2.22) it follows that the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$ becomes $\mathbf{h}_0^{\Sigma_{g,n}}$. Since the given basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$ is also $\mathbf{h}_0^{\Sigma_{g,n}}$, we have

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{3.2.23}$$

Next consider $C_1(\mathcal{H}_*) = H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,n+1})$ in (3.2.21), we get

$$C_1(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_1(\mathrm{Im}(j)). \tag{3.2.24}$$

As i is injective, let $i(\mathbf{h}_0^{S_1})$ be the basis of $\mathrm{Im}(i)$. In the previous step, we chose $\mathbf{h}_0^{\Sigma_{g,n}}$ as the basis of $\mathrm{Im}(j)$. Thus, by equation (3.2.24), the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$ becomes $\{i(\mathbf{h}_0^{S_1}), s_1(\mathbf{h}_0^{\Sigma_{g,n}})\}$.

$H_0(\Sigma_{g-1,1})$ and $H_0(\Sigma_{1,n+1})$ are both $1-$dimensional subspaces of the $2-$dimensional space $C_1(\mathcal{H}_*)$. Thus, there exist non-zero vectors $(a_{\nu 1}, a_{\nu 2})$, $\nu = 1, 2$ such that $\{a_{11} i(\mathbf{h}_0^{\gamma_1}) + a_{12} s_1(\mathbf{h}_0^{\Sigma_{g,n}})\}$ is a basis of $H_0(\Sigma_{g-1,1})$ and $\{a_{21} i(\mathbf{h}_0^{\gamma_1}) + a_{22} s_1(\mathbf{h}_0^{\Sigma_{g,n}})\}$ is a basis of $H_0(\Sigma_{1,n+1})$. Clearly, the 2×2 matrix

$A = [a_{\nu\mu}]$ is invertible. Let $\mathbf{h}_0^{\Sigma_{g-1,1}}$ denote the basis $\{(\det A)^{-1}[a_{11}i(\mathbf{h}_0^{\gamma_1}) + a_{12}s_1(\mathbf{h}_0^{\Sigma_{g,n}})]\}$ of $H_0(\Sigma_{g-1,1})$ and $\mathbf{h}_0^{\Sigma_{1,n+1}}$ denote the basis $\{a_{21}i(\mathbf{h}_0^{\gamma_1}) + a_{22}s_1(\mathbf{h}_0^{\Sigma_{g,n}})\}$ of $H_0(\Sigma_{1,n+1})$. Considering $\{\mathbf{h}_0^{\Sigma_{g-1,n}}, \mathbf{h}_0^{\Sigma_{1,n+1}}\}$ as the beginning basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$, we have

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{3.2.25}$$

Now, consider (3.2.21) for the space $C_2(\mathcal{H}_*) = H_0(\gamma_1)$. For h being the zero map, we get

$$C_2(\mathcal{H}_*) = \mathrm{Im}(h) \oplus s_2(\mathrm{Im}(i)) = s_2(\mathrm{Im}(i)). \tag{3.2.26}$$

Recall that the basis of $\mathrm{Im}(i)$ was chosen previously as $i(\mathbf{h}_0^{\gamma_1})$. From this and equation (3.2.26) it follows that the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$ becomes $\mathbf{h}_0^{\gamma_1}$. From the fact that the beginning basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$ is $\mathbf{h}_0^{\gamma_1}$ it follows

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{3.2.27}$$

Let us consider $C_3(\mathcal{H}_*) = H_1(\Sigma_{g,n})$ in (3.2.21). Obviously, we have

$$C_3(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_3(\mathrm{Im}(h)) = \mathrm{Im}(g). \tag{3.2.28}$$

Let us choose the basis of $\mathrm{Im}(g)$ as $\mathbf{h}_1^{\Sigma_{g,n}} = \{(\mathbf{h}_1^{\Sigma_{g,n}})_\nu\}_{\nu=1}^{2g+n-1}$. By equation (3.2.28), we get the obtained basis $\mathbf{h}_3'$ of $C_3(\mathcal{H}_*)$ as $\mathbf{h}_1^{\Sigma_{g,n}}$. The fact that the beginning basis $\mathbf{h}_3$ of $C_3(\mathcal{H}_*)$ is also $\mathbf{h}_1^{\Sigma_{g,n}}$ yields

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{3.2.29}$$

Considering the space $C_4(\mathcal{H}_*) = H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,n+1})$ in (3.2.21), we have

$$C_4(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_4(\mathrm{Im}(g)). \tag{3.2.30}$$

As f is injective, we can take the basis of $\mathrm{Im}(f)$ as $f(\mathbf{h}_1^{\gamma_1})$. In the previous step, we chose the basis of $\mathrm{Im}(g)$ as $\mathbf{h}_1^{\Sigma_{g,n}}$. Then, from (3.2.30) it follows that the obtained basis $\mathbf{h}_4'$ of $C_4(\mathcal{H}_*)$ becomes $\{f(\mathbf{h}_1^{\gamma_1}), s_4(\mathbf{h}_1^{\Sigma_{g,n}})\}$.

Since $H_1(\Sigma_{g-1,1})$ and $H_1(\Sigma_{1,n+1})$ are respectively $(2g-2)$ and $(n+2)-$dimensional subspaces of the $(2g+n)-$dimensional space $C_4(\mathcal{H}_*)$, there are non-zero vectors $(b_{\nu 1}, \ldots, b_{\nu(2g+n)})$, $\nu = 1, \ldots, 2g+n$ such that

$$\left\{ \sum_{\mu=1}^{2g+n-1} b_{\nu\mu} s_4(\mathbf{h}_{1\mu}^{\Sigma_{g,n}}) + b_{\nu(2g+n)} f(\mathbf{h}_1^{\gamma_1}) \right\}_{\nu=1}^{2+n}$$

is a basis of $H_1(\Sigma_{1,n+1})$ and

$$\left\{\sum_{\mu=1}^{2g+n-1} b_{\nu\mu} s_4(\mathbf{h}_{1\mu}^{\Sigma_{g,n}}) + b_{\nu(2g+n)} f(\mathbf{h}_1^{\gamma_1})\right\}_{\nu=n+3}^{2g+n}$$

is a basis of $H_1(\Sigma_{g-1,1})$. Moreover, the $(2g+n)\times(2g+n)$ matrix $B = [b_{\nu\mu}]$ has non-zero determinant. Let us choose the basis $\mathbf{h}_1^{\Sigma_{1,n+1}}$ of $H_1(\Sigma_{1,n+1}^1)$ as

$$\left\{(\det B)^{-1} \sum_{\mu=1}^{2g+n-1} [b_{1\mu} s_4(\mathbf{h}_{1\mu}^{\Sigma_{g,n}}) + b_{1(2g+n)} f(\mathbf{h}_1^{S_1})],\right.$$
$$\left.\left\{\sum_{\mu=1}^{2g+n-1} b_{\nu\mu} s_4(\mathbf{h}_{1\mu}^{\Sigma_{g,n}}) + b_{\nu(2g+n)} f(\mathbf{h}_1^{\gamma_1})\right\}_{\nu=2}^{2+n}\right\},$$

and let the basis $\mathbf{h}_1^{\Sigma_{g-1,1}}$ of $H_1(\Sigma_{g-1,1})$ be

$$\left\{\sum_{\mu=1}^{2g+n-1} b_{\nu\mu} s_4(\mathbf{h}_{\nu\mu}^{\Sigma_{g,n}}) + b_{\nu(2g+n)} f(\mathbf{h}_1^{\gamma_1})\right\}_{\nu=n+3}^{2g+n}.$$

If we consider $\{\mathbf{h}_1^{\Sigma_{g-1,1}}, \mathbf{h}_1^{\Sigma_{1,n+1}}\}$ as the beginning basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$, then we have

$$[\mathbf{h}_4', \mathbf{h}_4] = 1. \tag{3.2.31}$$

Finally, we consider (3.2.21) for the space $C_5(\mathcal{H}_*) = H_1(\gamma_1)$. The fact that $B_5(\mathcal{H}_*)$ equals to zero gives us the following equality

$$C_5(\mathcal{H}_*) = B_5(\mathcal{H}_*) \oplus s_5(\mathrm{Im}(f)) = s_5(\mathrm{Im}(f)). \tag{3.2.32}$$

In the previous step, $f(\mathbf{h}_1^{\gamma_1})$ was chosen as the basis of $\mathrm{Im}(f)$. By equation (3.2.32), the obtained basis $\mathbf{h}_5'$ of $C_5(\mathcal{H}_*)$ is $\mathbf{h}_1^{\gamma_1}$. As the beginning basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$ is also $\mathbf{h}_1^{\gamma_1}$, we get

$$[\mathbf{h}_5', \mathbf{h}_5] = 1. \tag{3.2.33}$$

Combining equations (3.2.23), (3.2.25), (3.2.27), (3.2.29), (3.2.31), (3.2.33), we obtain

$$\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_0^5, \{0\}_0^5) = \prod_{p=0}^{5} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{3.2.34}$$

Compatibility of the natural bases in the short-exact sequence (3.2.20), Theorem 2.0.2, and equation (3.2.34) end the proof of Proposition 3.2.5. □

By Proposition 3.2.2 and Proposition 3.2.5, we have

Theorem 3.2.6. *Let $\Sigma_{g,n}, g \geq 2, n \geq 1$ be an orientable surface with boundary circles $S_1,\dots,S_n$. Consider $\Sigma_{g,n}$ as the connected sum $\Sigma_{1,0}\#\cdots\#\Sigma_{1,0}\#\Sigma_{1,n}$ (see, Fig. 1). From left to right let $\gamma_1,\dots,\gamma_{g-1}$ be the circles obtained by the connected sum operation. For $i\in\{1,\dots,g-1\}$ and $j\in\{1,\dots,n\}$, $\mathbb{D}_{\gamma_i}$, $\mathbb{D}_{S_j}$ be the closed disk with boundary circle γ_i, S_j, respectively. For $i\in\{1,g-1\}$, let $\Sigma_{1,1}^{\gamma_i}$ be the torus with boundary circle γ_i and for $i\in\{1,\dots,g-2\}$, let $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$ be the torus with boundary circles γ_i,γ_{i+1}. Assume $\mathbf{h}_\nu^{\Sigma_{g,n}}$ is a basis of $H_\nu(\Sigma_{g,n})$, $\nu=0,1$. For $i\in\{1,\dots,g-1\}$, $\nu\in\{0,1\}$, assume $\mathbf{h}_\nu^{\gamma_i}$ is an arbitrary basis of $H_\nu(\gamma_i)$. Moreover, for $j\in\{1,\dots,n\}$, $k\in\{2,\dots,g-1\}$, assume $\mathbf{h}_0^{\mathbb{D}_{S_j}}$ and $\mathbf{h}_0^{\mathbb{D}_{\gamma_k}}$ are respectively bases of $H_0(\mathbb{D}_{S_j})$ and $H_0(\mathbb{D}_{\gamma_k})$. Then, for $\nu=0,1$, there exist bases $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}$, $(\mathbf{h}_\nu^{\gamma_i})'$, $i=2,\dots,g-1$, $\mathbf{h}_\nu^{S_k}$, $k=1,\dots,n$ so that the following formula is valid*

$$\begin{aligned}\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_\nu^{\Sigma_{g,n}}\}_0^1) &= \mathbb{T}(\Sigma_{1,1}^{\gamma_1},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}\}_0^1)\,\mathbb{T}(\Sigma_{1,1}^{\gamma_{g-1}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}\}_0^1)\\ &\times\prod_{i=2}^{g-1}\mathbb{T}(\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}\}_0^1)\prod_{j=1}^{g-1}\mathbb{T}(\gamma_j,\{\mathbf{h}_\nu^{\gamma_j}\}_0^1)^{-1}\\ &\times\prod_{i=2}^{g-1}[\mathbb{T}(\gamma_i,\{(\mathbf{h}_\nu^{\gamma_i})'\}_0^1)\mathbb{T}(\mathbb{D}_{\gamma_i},\{\mathbf{h}_0^{\mathbb{D}_{\gamma_i}}\})^{-1}]\\ &\times\prod_{k=1}^{n}[\mathbb{T}(S_k,\{\mathbf{h}_\nu^{S_k}\}_0^1)\mathbb{T}(\mathbb{D}_{S_k},\{\mathbf{h}_0^{\mathbb{D}_{S_k}}\})^{-1}].\end{aligned}$$

Here, $\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}$ is the torus with boundary circle γ_{i-1} which is obtained by gluing $\Sigma_{1,2}^{\gamma_{i-1},\gamma_i}$ and the closed disk $\mathbb{D}_{\gamma_i}$ along the common boundary circle γ_i. $(\mathbf{h}_\nu^{\gamma_i})'$ is the basis of γ_i by considering $\Sigma_{1,2}^{\gamma_{i-1},\gamma_i}=\Sigma_{1,1}^{\gamma_{i-1},\hat{\gamma}_i}\cup_{\gamma_i}\mathbb{D}_{\gamma_i}$ and applying Proposition 3.2.2.

From Remark 2.0.8, Remark 3.2.4, and Theorem 3.2.6 it follows that

Theorem 3.2.7. *Let $\Sigma_{g,n}$, S_j, γ_i, $\mathbb{D}_{\gamma_i}$, $\mathbb{D}_{S_j}$, $\Sigma_{1,1}^{\gamma_i}$, $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$, $\mathbf{h}_\nu^{\Sigma_{g,n}}$, $\mathbf{h}_\nu^{\gamma_i}$, $\mathbf{h}_0^{\mathbb{D}_{S_j}}$, $\mathbf{h}_0^{\mathbb{D}_{\gamma_k}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}$, $(\mathbf{h}_\nu^{\gamma_j})'$, $\mathbf{h}_\nu^{S_k}$ be as in Theorem 3.2.6. Then, we* *have*

$$\begin{aligned}|\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_\nu^{\Sigma_{g,n}}\}_0^1)| &= |\mathbb{T}(\Sigma_{1,1}^{\gamma_1},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}\}_0^1)||\mathbb{T}(\Sigma_{1,1}^{\gamma_{g-1}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}\}_0^1)| \\ &\quad\times\prod_{i=2}^{g-1}|\mathbb{T}(\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}\}_0^1)| \\ &\quad\times\prod_{i=2}^{g-1}|\mathbb{T}(\mathbb{D}_{\gamma_i},\{\mathbf{h}_0^{\mathbb{D}_{\gamma_i}}\})|^{-1}\prod_{k=1}^{n}|\mathbb{T}(\mathbb{D}_{S_k},\{\mathbf{h}_0^{\mathbb{D}_{S_k}}\})|^{-1}.\end{aligned}$$

Using the same arguments in Proposition 3.2.5, we obtain

Theorem 3.2.8. *Let $\Sigma_{g,0}, g\geq 2$ be a closed orientable surface. From left to right let $\gamma_1,\ldots,\gamma_{g-1}$ be the circles obtained by the connected sum operation for $\Sigma_{g,0}$ (See Fig 1). Consider the surface $\Sigma_{g,0}$ obtained by gluing the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$ along the common boundary circle γ_{g-1}. Let*

$$0\to C_*(\gamma_{g-1})\to C_*(\Sigma_{g-1,1})\oplus C_*(\Sigma_{1,1})\to C_*(\Sigma_{g,0})\to 0$$

be the associated short-exact sequence of chain complexes and let

$$\begin{aligned}\mathcal{H}_*:0 &\to H_2(\Sigma_{g,0})\xrightarrow{\delta}H_1(\gamma_1)\xrightarrow{f}H_1(\Sigma_{g-1,1})\oplus H_1(\Sigma_{1,1})\xrightarrow{g}H_1(\Sigma_{g,0}) \\ &\xrightarrow{h}H_0(\gamma_1)\xrightarrow{i}H_0(\Sigma_{g-1,1})\oplus H_0(\Sigma_{1,1})\xrightarrow{j}H_0(\Sigma_{g,0})\xrightarrow{k}0\end{aligned}$$

be the corresponding long-exact sequence obtained by the Snake Lemma, where the connecting map δ is an isomorphism. Let $\mathbf{h}_\nu^{\Sigma_{g,0}}$ be a basis of $H_\nu(\Sigma_{g,0})$, $\nu=0,1,2$. Let $\mathbf{h}_1^{\gamma_{g-1}}=\delta(\mathbf{h}_2^{\Sigma_{g,0}})$ be the basis of $H_1(\gamma_{g-1})$ and $\mathbf{h}_0^{\gamma_{g-1}}$ be an arbitrary basis of $H_0(\gamma_{g-1})$. Then, there are bases $\mathbf{h}_\nu^{\Sigma_{g-1,1}}$ and $\mathbf{h}_\nu^{\Sigma_{1,1}}$ of $H_\nu(\Sigma_{g-1,1})$ and $H_\nu(\Sigma_{1,1})$, $\nu=0,1$, respectively such that R-torsion of $\mathcal{H}_*$ in the corresponding bases is 1 and the following formula holds

$$\begin{aligned}\mathbb{T}(\Sigma_{g,0},\{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_0^2) &= \mathbb{T}(\Sigma_{g-1,1},\{\mathbf{h}_\nu^{\Sigma_{g-1,1}}\}_0^1)\,\mathbb{T}(\Sigma_{1,1},\{\mathbf{h}_\nu^{\Sigma_{1,1}}\}_0^1) \\ &\quad\times\mathbb{T}(\gamma_{g-1},\{\mathbf{h}_\nu^{\gamma_{g-1}}\}_0^1)^{-1}.\end{aligned}$$

Combining Theorem 3.2.6 and Theorem 3.2.8, we have the following result.

Theorem 3.2.9. *Let $\Sigma_{g,0}$, γ_i, and δ be as in Theorem 3.2.8. Let $\mathbb{D}_{\gamma_i}$ be the closed disk with boundary circle γ_i, $i=1,\ldots,g-1$. For $i\in\{1,g-1\}$, let $\Sigma_{1,1}^{\gamma_i}$ be the torus with boundary circle γ_i and for $i\in\{1,\ldots,g-2\}$, let $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$ be the torus with boundary circles γ_i,γ_{i+1}. Assume $\mathbf{h}_\nu^{\Sigma_{g,0}}$ is a basis of $H_\nu(\Sigma_{g,0})$,*

$\nu=0,1,2$. *For* $i\in\{1,\dots,g-1\}$, $\nu\in\{0,1\}$, *assume* $\mathbf{h}_\nu^{\gamma_i}$ *is an arbitrary basis of* $H_\nu(\gamma_i)$ *such that* $\mathbf{h}_1^{\gamma_{g-1}}=\delta(\mathbf{h}_2^{\Sigma_{g,0}})$. *Assume also that for* $k\in\{1,\dots,g-1\}$, $\mathbf{h}_0^{\mathbb{D}_{\gamma_k}}$ *is a basis of* $H_0(\mathbb{D}_{\gamma_k})$. *Then, for* $\nu=0,1$, *there are bases* $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}$, $(\mathbf{h}_\nu^{\gamma_i})'$, $i=2,\dots,g-1$ *so that the following formula is valid*

$$\begin{aligned}\mathbb{T}(\Sigma_{g,0},\{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_0^1) &= \mathbb{T}(\Sigma_{1,1}^{\gamma_1},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}\}_0^1)\mathbb{T}(\Sigma_{1,1}^{\gamma_{g-1}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}\}_0^1)\\ &\times\prod_{i=2}^{g-1}\mathbb{T}(\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}\}_0^1)\prod_{j=1}^{g-1}\mathbb{T}(\gamma_j,\{\mathbf{h}_\nu^{\gamma_j}\}_0^1)^{-1}\\ &\times\prod_{i=2}^{g-1}[\mathbb{T}(\gamma_i,\{(\mathbf{h}_\nu^{\gamma_i})'\}_0^1)\mathbb{T}(\mathbb{D}_{\gamma_i},\{\mathbf{h}_0^{\mathbb{D}_{\gamma_i}}\})^{-1}].\end{aligned}$$

Here, $\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}$ *is the torus with boundary circle* γ_{i-1} *which is obtained by gluing* $\Sigma_{1,1}^{\gamma_{i-1},\gamma_i}$ *and the closed disk* $\mathbb{D}_{\gamma_i}$ *along the common boundary circle* γ_i.

By Remark 2.0.8, Remark 3.2.4, and Theorem 3.2.9, we have the following result

Theorem 3.2.10. *Let* $\Sigma_{g,0}$, $\Sigma_{1,1}^{\gamma_{g-1}}$, $\Sigma_{1,1}^{\gamma_1}$, $\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}$, $\mathbb{D}_{\gamma_i}$, $\mathbf{h}_\nu^{\Sigma_{g,0}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}$, $\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}$, $\mathbf{h}_0^{\mathbb{D}_{\gamma_i}}$ *be as in Theorem 3.2.9. Then, the following formula holds*

$$\begin{aligned}|\mathbb{T}(\Sigma_{g,0},\{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_0^1)| &= |\mathbb{T}(\Sigma_{1,1}^{\gamma_1},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_1}}\}_0^1)|\,|\mathbb{T}(\Sigma_{1,1}^{\gamma_{g-1}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{g-1}}}\}_0^1)|\\ &\times\prod_{i=2}^{g-1}|\mathbb{T}(\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}},\{\mathbf{h}_\nu^{\Sigma_{1,1}^{\gamma_{i-1},\widehat{\gamma_i}}}\}_0^1)|\prod_{i=2}^{g-1}|\mathbb{T}(\mathbb{D}_{\gamma_i},\{\mathbf{h}_0^{\mathbb{D}_{\gamma_i}}\})|^{-1}.\end{aligned}$$

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