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Vanishing theorems for (k, s)-positive vector bundles on weakly pseudoconvex Kähler manifolds

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Abstract: We will use analytic methods in this paper, based on L^2-methods for the $\overline{\partial}$-equation, to obtain some new vanishing theorems for holomorphic vector bundles of (k, s)-positivity on weakly pseudoconvex Kähler manifolds which generalize those obtained by Demailly for s-positive vector bundles in his sense.

1. Introduction

The subject of cohomology vanishing theorems for holomorphic vector bundles on complex manifolds occupies a role of central importance in several complex variables and algebraic geometry. The theory formally began in 1953 with the Kodaira vanishing theorem, and its roots can be traced back to Riemann and Roch for the case of curves and to Picard for surfaces.

In this paper, we will investigate vanishing theorems from the viewpoint of complex differential geometry. Similarly, there are many excellent researches on vanishing theorems in recent years, such as articles [14] by Liu-Sun-Yang and [15] by Liu-Yang. Our original motivation was to understand the Demailly s-positivity (see [6]), which is equivalent to the Griffiths positivity when $s = 1$ and to the Nakano positivity when s attains its maximum. The Demailly s-positivity is a concept which is difficult to understand and

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to check. In [1], Yang introduce the notion of (k, s)-positivity from the viewpoint of the theory of Hermitian forms in linear algebra. When $k = 0$, it is equivalent to Demailly's s-positivity; if $s = 1$ and in the case of line bundle, it is just k-positivity used implicitly by Gigante and Girbau in [7, 8].

We will use the analytic methods, based on the theory of harmonic integrals or the L^2-method for the $\overline{\partial}$-equation, to study vanishing theorems for holomorphic vector bundles of (k, s)-positivity on weakly pseudoconvex Kähler manifolds.

The main result in this paper is the following Theorem 1.1,

Theorem 1.1. *Let (X, ω) be a weakly pseudoconvex Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E >_{(k,s)} 0$. Then*

$$H^{n,q}(X, E) = 0$$

for any $q > k$ and $s \geq min\{n - q + 1, r\}$.

If $k = 0$, the above theorem was firstly proved by Demailly in [2]. In addition, when Kähler manifold is compact, the result was proved by Yang in [1].

The following results follow immediately from Theorem 1.1:

Corollary 1.2. *Let (X, ω) be a compact Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E >_{(k,s)} 0$. Then for any nef line bundle F on X, we have*

$$H^{n,q}(X, E \otimes F) = 0$$

for any $q > k$ and $s \geq min\{n - q + 1, r\}$.

As applications of Theorem 1.1, we get the following results.

Theorem 1.3. *Let E be a Griffiths k-positive Hermitian holomorphic vector bundle of rank $r \geq 2$ on a weakly pseudoconvex Kähler manifold of dimension n. Then for any integer $s \geq 1$, we have:*

(1) $H^q(X, K_X \otimes E \otimes detE) = 0$ for $q > k$;

(2) $H^q(X, K_X \otimes E^* \otimes (detE)^s) = 0$ for $q > k$ and $s \geq min\{n - q + 1, r\}$.

When $k = 0$, the above theorem covers the following corollary which was proved by Demailly:

Corollary 1.4 ([2], 346)**.** *Let X be a compact n-dimensional complex manifold, E a vector bundle of rank $r \geq 2$ and $m \geq 1$ an integer. Then*

(a) $E >_{Grif} 0 \Rightarrow H^q(X, K_X \otimes E \otimes detE) = 0$ for $q \geq 1$;

(b) $E >_{Grif} 0 \Rightarrow H^q(X, K_X \otimes E^* \otimes (detE)^m) = 0$ for $q \geq 1$ and $m \geq min\{n - q + 1, r\}$.

2. Preliminaries

In this section we will collect some fundamental material.

Let X be a complex manifolds of dimension n and E a holomorphic vector bundle of rank r over X. Now suppose that X is equipped with a Hermitian metric g, and E is any complex vector bundle with a Hermitian metric h and a Hermitian connection D, which splits in a unique way as a sum of a (1,0)- and (0,1)-connection: $D = D^{'} + D^{''}$. If E is a holomorphic vector bundle, there is a unique Hermitian connection, called the Chern connection such that $D^{''} = \overline{\partial}$ and hence $D^{'2} = D^{''2} = 0$. Thus the curvature form $\Theta_h(E) = D^{'}D^{''} + D^{''}D^{'}$ is an $End(E)$-valued(1,1)-form and $i\Theta_h(E)$ is called the Chern curvature form of E. Let $\{z^j\}$ be the local holomorphic coordinate of X, $\{e_\alpha\}$ an orthonormal frame and $\{e^\alpha\}$ the dual orthonormal frame of E. Let $(g_{j\overline{k}})$ and$(h_{\alpha\overline{\beta}})$ be the Hermitian metrics on X and on E respectively, and their inverses denoted respectively by $(g^{i\overline{j}})$ and $(h^{\alpha\overline{\beta}})$. Then we can write the Kähler form and the curvature form respectively as

$$\omega = ig_{j\overline{k}}dz^j \wedge d\overline{z}^k,\ i\Theta_h(E) = R^{\alpha}_{\beta j\overline{k}}dz^j \wedge d\overline{z}^k \otimes e_\alpha \otimes e^\beta, \tag{2.1}$$

where

$$R^{\alpha}_{\beta j\overline{k}} = -h^{\alpha\overline{\gamma}}\partial_j\overline{\partial}_k h_{\beta\overline{\gamma}} + h^{\alpha\overline{\gamma}}h^{\lambda\overline{\mu}}\partial_j h_{\beta\overline{\mu}}\overline{\partial}_k h_{\lambda\overline{\gamma}}. \tag{2.2}$$

For $u = u^{\alpha}_{J,\overline{K}}dz^J \wedge dz^{\overline{K}} \otimes e_\alpha \in \Omega^{p,q}_X(E)$, we have the following formula:

$$\begin{aligned}\langle[i\Theta_h(E),\Lambda]u,u\rangle_\omega =& \frac{1}{(p-1)!q!}R_{\alpha\overline{\beta}j\overline{k}}g^{l\overline{k}}u^{\alpha}_{lR_{p-1},\overline{K}_q}\overline{u^{\beta,\overline{j}\overline{R}_{p-1},K_q}}\\ &+\frac{1}{(p)!(q-1)!}R_{\alpha\overline{\beta}j\overline{k}}g^{j\overline{l}}u^{\alpha}_{J_p,\overline{l}\overline{S}_{q-1}}\overline{u^{\beta,\overline{J}_p,kS_{q-1}}}\\ &-\frac{1}{(p)!q!}R_{\alpha\overline{\beta}j\overline{k}}g^{j\overline{k}}u^{\alpha}_{J_p,\overline{K}_q}\overline{u^{\beta}_{\overline{J}_p,K_q}}.\end{aligned}$$

If we choose normal metrics both on the base X and the fiber E with $g_{j\overline{k}} = \delta_{j\overline{k}}$ and $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$, then we have the following simpler expression:

$$\begin{aligned}\langle[i\Theta_h(E),\Lambda]u,u\rangle_\omega =& R_{\alpha\overline{\beta}j\overline{k}}u^{\alpha}_{kR_{p-1},\overline{K}_q}\overline{u^{\beta}_{\overline{j}\overline{R}_{p-1},K_q}}\\ &+R_{\alpha\overline{\beta}j\overline{k}}u^{\alpha}_{J_p,\overline{j}\overline{S}_{q-1}}\overline{u^{\beta}_{\overline{J}_p,kS_{q-1}}}\\ &-R_{\alpha\overline{\beta}j\overline{j}}u^{\alpha}_{J_p,\overline{K}_q}\overline{u^{\beta}_{\overline{J}_p,K_q}}\end{aligned} \tag{2.3}$$

where the summation is over all indices $1 \leq j, k \leq n, 1 \leq \alpha, \beta \leq r$ and over all multi-indices $J_p, K_q, R_{p-1}, S_{q-1}$ of an increasing order with $|J_p| = p, |K_q = q|, |R_{p-1}| = p-1$ and $|S_{q-1}| = q-1$. However, it is still very hard to decide when the expression (2.3) is positive, except in some special cases.

A first tractable case is that when $p = n$. Then in the first summation of (2.3) we must have $j = k$ and $R_{n-1} = \{1, \cdots, n\} \setminus \{j\}$. So the first summation cancels the last summation.

In [6] Demailly introduced the notion of s-positivity for any integer $1 \leq s \leq r$ for a vector bundle E of rank r. We cited it as the Demailly s-positivity. In particular, the Demailly one-positivity is just the Griffths-positivity and the Demailly s-positivity for $s \geq min\{r, n\}$ is exactly the Nakano-positivity.

Definition 2.1 ([6])**.** A tensor $u \in TX \otimes E$ is called rank s if s is the smallest non-negative interger such that u can be written as

$$u = \sum_{j=1}^{s} \xi^j \otimes \upsilon^j, \xi^j \in TX, \upsilon^j \in E.$$

E is said to be Demailly s-positive if $i\Theta_h(E)(u, u) > 0$ for any nonzero $u \in TX \otimes E$ of rank$\leq s$. From this definition, the set of tensors whose rank is no more than s, do not form a linear subspace since they are not closed under addition. To capture the essence of it, we adopt the following formulation, which uses the theory of Hermitian form and is a more general definition.

Definition 2.2 ([1])**.** A holomorphic vector bundle E of rank r with Hermitian metric h on a complex manifold X of complex dimension n is called (k, s)-positive for $1 \leq s \leq r$, if the following holds for any $x \in X$: For any nonzero s-tuple vectors $\upsilon^j \in V, 1 \leq j \leq s$, where $V = E_x$ (respectively, $T_x X$), the Hermitian form

$$Q_x(\bullet, \bullet) = i\Theta_h(E)(\sum_{j=1}^{s} \cdot \otimes \upsilon^j, \sum_{j=1}^{s} \cdot \otimes \upsilon^j), \bullet \in W^{\oplus s}, \cdot \in W$$

defined on $W^{\oplus}$ is semipositive and the dimension of its kernel is at most k, where $W = T_x X$ (respectively, E_x). In this case we write as

$$i\Theta_h(E) >_{(k,s)} 0, E >_{(k,s)} 0.$$

Clearly $(0, s)$-positivity is equivalent to Demailly s-positivity and Nakano-posivity is equivalent to $(0, s)$-positivity if $s \geq min\{n, r\}$. The $(0, 1)$-positivity is equivalent to Griffiths-positivity. For general integer k, the $(k, 1)$-positivity is a semipositive version of Griffiths-positivity. A holomorphic vector bundle E of arbitrary rank is called *Griffiths k-positive* if it is $(k,1)$-positive.

Of course one such example is Grassmannian which was proved by Yang in [1]:

Example 2.3. Let V be a complex vector space of dimension n and let $Gr(V,d)$ denote the Grassmannian of subspaces W of V of codimension d. Yang explained in detail that the holomorphic tangent bundle of $Gr(V,d)$ is $((d-1)(n-d-1),1)$-positive. Similar calculations indicate that the holomorphic tangent bundle of complex projective space $\mathbb{P}^{n-1}$ is Nakano-semipositive and $((n-1)(s-1),s)$-positive for any $1 \leq s \leq n-1$.

Definition 2.4 ([2])**.** A complex manifold M is said to be *weakly pseudoconvex* if there exists an exhaustion function $\psi \in C^\infty(M,\mathbb{R})$ such that $i\partial\overline{\partial}\psi \geq 0$ on X, i.e. ψ is plurisubharmonic (psh for short).

Remark 2.5. It is obvious that for domains $\Omega \subset \mathbb{C}^n$, the above weak pseudoconvexity notion is equivalent to pseudoconvexity (see [2]). Note that every compact complex manifold is also weakly pseudoconvex (take $\psi \equiv 0$).

Definition 2.6. Let (M,ω) be a compact Kähler manifold. A line bundle L over M is said to be *nef*, if for any $\varepsilon > 0$, there exists a (smooth) hermitian metric h^L_ε on L such that the curvature $\sqrt{-1}\Theta(h^L_\varepsilon)$ of h^L_ε satisfies $\sqrt{-1}\Theta(h^L_\varepsilon) \geq -\varepsilon\omega$.

3. Proof of Theorem 1.1, Corollary 1.2 and Theorem 1.3

In this section, we prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. We denote X a weakly pseudoconvex manifold of dimension n. In particular, we may choose X to be a compact complex manifold. Assume that E (resp. L) is a hermitian vector (resp. line) bundle over X of rank r. We need the following Theorem 3.1, which plays a key role in proving all kinds of vanishing theorems in non compact cases.

Theorem 3.1 ([3, 5])**.** *Let (M,ω) be a Kähler manifold. Here M is not necessarily compact, but we assume that the geodesic distance δ_ω is complete (i.e ω is complete) on M. Let E be a hermitian vector bundle of rank r over M, and assume that the curvature operator $A = A^{p,q}_{E,h,\omega} = [\sqrt{-1}\Theta(E,h),\Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^*M \otimes E$, $q \geq 1$. Then for any form $g \in L^2(M,\Lambda^{p,q}T^*M \otimes E)$ satisfying $\overline{\partial}g = 0$ and $\int_M (A^{-1}g,g)dV_\omega < +\infty$, there exists $f \in L^2(M,\Lambda^{p,q-1}T^*M \otimes E)$ such that $\overline{\partial}f = g$ and*

$$\int_M |f|^2 dV_\omega \leq \int_M (A^{-1}g,g)dV_\omega.$$

This famous result is essentially due to Hörmander in [5] and Andreotti-Vesentini in [3]. Here we use the version suitable for our purpose as stated in [4].

Recall from (2.3) we have

$$\langle [i\Theta_h(E),\Lambda]u,u\rangle_\omega = \sum_{S_{q-1},\alpha,\beta,j,k} R_{\alpha\overline{\beta}j\overline{k}} u^{\alpha}_{\overline{j}S_{q-1}} \overline{u^{\beta}_{kS_{q-1}}}$$

for any (n,q)-form $u=\sum u^{\alpha}_{\overline{K}} dz^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^K\otimes e^{\alpha}$ with value in E. Since $u^{\alpha}_{\overline{j}S_{q-1}}=0$ for $j\in\overline{S}_{q-1}$, so the rank (in the sense of Demailly) of $(u^{\alpha}_{\overline{j}})_{\overline{j},\alpha}$ is in fact no more than $\min\{n-q+1,r\}$. Therefore we obtain:

Lemma 3.2 ([2], 341)**.** *Assume that E is Demailly s-positive. Then the Hermitian operator $[i\Theta(E),\Lambda]$ is positive definite on $\Lambda^{n,q}T^*X\otimes E$ for $q\geq 1$ and $m\geq min\{n-q+1,r\}$.*

Now we are ready to prove the main result in this paper:

Theorem 3.3 (=Theorem 1.1)**.** *Let (X,ω) be a weakly pseudoconvex Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E>_{(k,s)}0$. Then*

$$H^{n,q}(X,E)=0$$

for any $q>k$ and $s\geq min\{n-q+1,r\}$.

Proof of Theorem 3.3. We may assume $k<n$. since every weakly pseudoconvex Kähler manifold (X,ω) carries a complete Kähler $\widehat{\omega}$ (see [2], page 372), we will discuss the following proof in the sense of $\widehat{\omega}$.

For any E-valued (n,q)-form

$$u=\sum u^{\alpha}_{\overline{K}} dz^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^K\otimes e^{\alpha}\in\Omega^{n,q}(E)$$

by Definition 2.2, we could diagonalize the Hermitian form $i\Theta_h(E)(u,u)$ at $x\in X$ since it is semipositive. Therefore we could write

$$\begin{aligned}\langle [i\Theta_h(E),\Lambda]u,u\rangle_{\widehat{\omega}} &= \sum_{S_{q-1},\alpha,\beta,j,k} R_{\alpha\overline{\beta}j\overline{k}}(x) u^{\alpha}_{\overline{j}S_{q-1}} \overline{u^{\beta}_{kS_{q-1}}}\\ &= \sum_{S_{q-1},\alpha,j} \lambda^{\alpha}_j(x) u^{\alpha}_{\overline{j}S_{q-1}} \overline{u^{\alpha}_{jS_{q-1}}}\end{aligned}$$

Where the eigenvalues $(\lambda_j^\alpha)_{1\le j\le n,1\le\alpha\le r}$ are non-negative. For fixed α, without loss of generality, assume that $\lambda_1^\alpha \le \lambda_2^\alpha \le \cdots \le \lambda_n^\alpha$ with $\lambda_{k+1}^\alpha > 0$ (by Definition 2.2, the dimension of kernel is at most k). Put $\lambda(x) = min\{\lambda_{k+1}^\alpha | 1 \le \alpha \le r\}$, Then λ is a positive number. If $q > k$, then

$$\begin{aligned}\langle[i\Theta_h(E),\Lambda]u,u\rangle_{\widehat{\omega}} &= \sum_{j,\overline{S}_{q-1}}\sum_{\alpha}\lambda_j^\alpha|u^\alpha_{\overline{jS_{q-1}}}|^2\\ &= \sum_{k}\sum_{\alpha}\sum_{j\in K}\lambda_j^\alpha|u^\alpha_{\overline{K}}|^2\\ &\ge \sum_{K}\sum_{\alpha}(\lambda_1^\alpha+\lambda_2^\alpha+\cdots+\lambda_q^\alpha)|u^\alpha_{\overline{K}}|^2\\ &\ge \lambda(x)(\sum_{K}\sum_{\alpha}|u^\alpha_{\overline{K}}|^2)\\ &= \lambda(x)|u|^2_{\widehat{\omega}}\end{aligned}$$

Thus when $q > k$, then for any $x \in X$, $[i\Theta_h(E),\Lambda]$ is positive-definite on E-valued (n,q)-forms.

Assume that the Hermitian operator $A_{E,\widehat{w}} = [i\Theta_h(E),\Lambda]$. Let $\psi \in C^\infty(X,R)$ be a exhaustive plurisubharmonic function. For any convex increasing function $\chi \in C^\infty(R,R)$, we denote by E_χ the holomorphic vector bundle E together with the modified metric $|u|^2_\chi = |u|^2 exp(-\chi\circ\psi(x))$, $u \in E_x$. We get

$$i\Theta(E_\chi) = i\Theta(E) + id^{'}d^{''}(\chi\circ\psi)\otimes Id_E$$

Since we have

$$\begin{aligned}&id^{'}d^{''}(\chi\circ\psi) = i(\chi^{'}\circ\psi)d^{'}d^{''}\psi + i(\chi^{''}\circ\psi)d^{'}\psi\wedge d^{''}\psi\\ &\chi^{'}\circ\psi \ge 0, \chi^{''}\circ\psi\ge 0, d^{'}d^{''}\psi\ge 0, d^{'}\psi\wedge d^{''}\psi \ge 0.\end{aligned}$$

Therefore $i\Theta(E_\chi) - i\Theta(E) >_{(k,s)} 0$. Thus $A_{E_\chi,\widehat{\omega}} \ge A_{E,\widehat{\omega}} > 0$ in bidegree(n,q). For $g \in C^\infty_{n,q}(X,E)$ with $D^{''}g = 0$. The integrals

$$\begin{aligned}&\int_X \langle A^{-1}_{E_\chi,\widehat{\omega}}g,g\rangle_\chi dV_{\widehat{\omega}} = \int_X \langle A^{-1}_{E_\chi\widehat{\omega}}g,g\rangle e^{-\chi\circ\psi}dV_{\widehat{\omega}}\\ &\le \int_X \langle A^{-1}_{E,\widehat{\omega}}g,g\rangle e^{-\chi\circ\psi}dV_{\widehat{\omega}} < +\infty, \int_X |g|^2 e^{-\chi\circ\psi}dV_{\widehat{\omega}} < +\infty\end{aligned}$$

become convergent if χ grow fast enough. In fact, since ψ is an exhaustion function, for every $c \in R$, the sublevel set $X_c = \{x \in M; \psi(x) < c\}$ is relatively compact in X, we can assume $\psi \geq 0$. we can take χ, such that

$$\begin{aligned}\int_X \langle A^{-1}_{E,\widehat{\omega}} g, g\rangle e^{-\chi\circ\psi} dV_{\widehat{\omega}} &= \lim_{c\to+\infty} \int_{X_c} \langle A^{-1}_{E,\widehat{\omega}} g, g\rangle e^{-\chi\circ\psi} dV_{\widehat{\omega}} \\ &= \lim_{n\to+\infty} \sum_{k=0}^{n} \int_{X\{k\leq\psi\leq k+1\}} \langle A^{-1}_{E,\widehat{\omega}} g, g\rangle e^{-\chi\circ\psi} dV_{\widehat{\omega}} \\ &\leq \lim_{n\to+\infty} \sum_{k=0}^{n} e^{-\chi(k)} \int_{X(k\leq\psi\leq k+1)} \langle A^{-1} g, g\rangle dV_{\widehat{\omega}} \\ &\leq \lim_{n\to+\infty} \sum_{k=0}^{n} 3^{-k} = \frac{3}{2} < +\infty\end{aligned}$$

We cen get $g \in L^2_{n,q}(X, E_\chi, \widehat{\omega})$.

In conclusion, we have proved that for any $g \in C^\infty_{n,q}(X, E)$ with $D''g = 0$, there exists a convex increasing function χ such that:

(1) $A_{E_\chi,\widehat{\omega}} > 0$ on $C^\infty_{n,q}(X, E)$

(2) $g \in L^2_{n,q}(X, E_\chi, \widehat{\omega})$

By Theorem 3.1, we can find $f \in L^2_{n,q-1}(X, E_\chi, \widehat{\omega})$ such that $D''f = g$. By a remark in [2], page 372, we conclude that f can be chosen smooth. So $H^{n,q}(X, E) = 0$ for $q > k$ and $s \geq min\{n - q + 1, r\}$. □

Especially, we can obtain the following result.

Corollary 3.4. *(1) Let (X, ω) be a weakly pseudoconvex Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E >_{(k,r)} 0$. Then*

$$H^{n,q}(X, E) = 0, \ \forall\, q > k$$

(2) Let (X, ω) be a weakly pseudoconvex Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E >_{(k,s)} 0$. Then

$$H^{n,n}(X, E) = 0$$

If X is a compact complex manifold, then we can also deal with some nef cases.

Corollary 3.5 (=Corollary 1.2)**.** *Let (X, ω) be a compact Kähler manifold of dimension n and E a hermitian holomorphic vector bundle of rank r on X such that $E >_{(k,s)} 0$. Then for any nef line bundle L over X, we have*

$$H^{n,q}(X, E \otimes L) = 0$$

for any $q > k$ and $s \geq min\{n-q+1, r\}$.

Proof of Corollary 3.5. Similarly, for any $E \otimes L$-valued (n,q)-form u, we can get

$$\langle [i\Theta(E), \Lambda]u, u\rangle \geq \lambda(x)|u|^2_\omega, \ \lambda(x) > 0$$

Since X is compact, there exists a positive constant λ_0, such that $\lambda_0 = \liminf_{x\in X}\{\lambda(x)\} > 0$. By definition of nef, we can take $\epsilon = -\frac{\lambda_0}{2q}$, there is a metric h_ϵ on L, such that $i\Theta_{L,h_\varepsilon} \geq \epsilon\omega = -\frac{\lambda_0}{2q}\omega$.

Note that $i\Theta(E\otimes L) = i\Theta(E) + Id_E \otimes i\Theta(L) \geq i\Theta(E) + \epsilon\omega\otimes Id_E$, Clearly, we have

$$\begin{aligned}\langle [i\Theta(E\otimes L), \Lambda]u, u\rangle_\omega &\geq \langle [i\Theta(E), \Lambda]u, u\rangle_\omega + \langle [\varepsilon\omega \otimes Id_E, \Lambda]u, u\rangle_\omega \\ &\geq \lambda_0|u|^2_\omega + \varepsilon q|u|^2_\omega = \frac{\lambda_0}{2}\end{aligned}$$

Thus $[i\Theta(E\otimes L), \Lambda]$ acting on $\Lambda^{n,q}T^*X \otimes E \otimes L$ is positive definite. So we have $H^{n,q}(X, E\otimes L) = 0$, for any $q > k$ and $s \geq min\{n-q+1, r\}$. □

To show Theorem 1.3, we need give the following propositions and corollary.

Proposition 3.6 (Yang, [1])**.** *Let E be a rank r holomorphic vector bundle with Hermitian metric h on a complex manifold X of dimension n. If E is Griffiths k-positive, then for any integer $1 \leq s \leq min\{r, n\}$,*

$$i\Theta_h(E) + Tr_E(i\Theta_h(E)) \otimes h >_{(k,s)} 0.$$

Proposition 3.7 (Yang, [1])**.** *Let E be a holomrphic vector bundle of rank $r \geq 2$ with Hermitian metric h on a complex manifold X of dimension n. If E is Griffiths k-positive, then for any integer $1 \leq s \leq max\{r, n\}$, we have*

$$sTr_E(i\Theta_h(E)) \otimes h - i\Theta_h(E) >_{(k,s)} 0.$$

Corollary 3.8. *Let E be a Griffith k-positive Hermitian holomorphic vector bundle of rank $r \geq 2$. Then for any integer $m \geq 1$,*

$$E^* \otimes (detE)^s >_{(k,s)} 0.$$

Proof. Apply Proposition 3.7 to $E^* \otimes TX$ and note that

$i\Theta(E^*\otimes(detE)^s) = s(i\Theta(detE))\otimes h + i\Theta(E^*) = sTr_E(i\Theta(E))\otimes h - i\Theta(E)^t$ and $Tr_E(i\Theta(E)) = Tr_E(i\Theta(E)^t)$. □

Theorem 3.3 in combination with Proposition 3.6 and Corollary 3.8 immediately imply the following consequences:

Theorem 3.9 (=Theorem 1.3)**.** *Let E be a Griffiths k-positive Hermitian holomorphic vector bundle of rank $r \geq 2$ on a weakly pseudoconvex Kähler manifold of dimension n. Then for any integer $s \geq 1$, we have:*

(1) $H^q(X, K_X \otimes E \otimes detE) = 0$ for $q > k$;

(2) $H^q(X, K_X \otimes E^* \otimes (detE)^s) = 0$ for $q > k$ and $s \geq min\{n - q + 1, r\}$.

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References

[1] Q. Yang, (k, s)-positivity and vanishing theorems for compact Kähler manifolds. Internat. J. Math, **22**(4) (2011), 545–576. MR2794461

[2] J.-P. Demailly, *Complex analytic and algebraic geometry.* Book online http://www-fourier.ujf-grenoble.fr/~demailly/books.html.

[3] A. Andreotti, E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds.* Publications Mathématiques de l'IHéS, **25** (1965), 81–130. MR0175148

[4] J.-P. Demailly, *L2 vanishing theorems for positive line bundles and adjunction theory. Transcendental methods in algebraic geometry.* Springer, Berlin, Heidelberg, 1996: 1–97. MR1603616

[5] L. Hörmander, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator.* Acta Mathematica, **113**(1) (1965), 89–152. MR0179443

[6] J.-P. Demailly, Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète, Ann.Sci.École Norm.Sup. **15** (1982), 457–511. MR0690650

[7] G. Gigante, Vector bundles with semidefinite curvature and cohomology vanishing theorems, Adv. Math. **41** (1981), 40–56. MR0625333

[8] J. Girbau, Sur le théorème de Le Potier d'annulation de la cohomology, C. R. Acad. Sci. Paris **283** (1976), 355–358. MR0430333

[9] P. Griffiths, J. Harris, *Principles of algebraic geometry.* John Wiley & Sons, New York, 2014. MR0507725

[10] P. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*. Global Analysis (papers in honor of K. Kodaira). Princeton Univ. Press, Princeton, 1969: 181–251. MR0258070

[11] S. Kobayashi, *Differential geometry of complex vector bundles*. Princeton University Press, Princeton, 2014. MR3643615

[12] J. A. Morrow, K. Kodaira, *Complex manifolds*. American Mathematical Soc., Providence, 1971. MR0302937

[13] J.-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry, in *Vanishing Theorems and Effective Results in Algebraic Geometry*. The Abdus Salam I.C.T.P., 2001: 1–149. MR1919457

[14] K.-F. Liu, X.-F. Sun and X.-K. Yang, *Positivity and vanishing theorems for ample vector bundles*. J. Algebraic Geom. **22** (2013), 303–331. MR3019451

[15] K.-F. Liu, X.-K. Yang, *Effective vanishing theorems for ample and globally generated vector bundles*. Comm. Anal. Geom. **23**(4) (2015), 797–818. MR3385779

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