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Higher order terms in asymptotic expansion of colored Jones polynomials

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Abstract: $SL(2,\mathbb{C})$ Chern-Simons theory provide several methods to calculate the higher order terms in asymptotic expansion of colored Jones polynomial from the view of A-polynomial and noncommutative A-polynomial. First, we present one of those algorithms explicitly. Then through the detailed calculations, we conjecture that the Melvin-Morton-Rozansky (MMR) expansion of colored Jones polynomial is consistent with the asymptotic expansion of colored Jones polynomial in abelian branch of A-polynomial studied in this article.
Keywords: Colored Jones polynomial, asymptotic expansion, volume conjecture, A-polynomial, non-commutative A-polynomial, AJ conjecture.

1. Introduction

1.1.

Let $J_N(\mathcal{K};q)$ be the *normalized colored Jones polynomial* of a knot $\mathcal{K}$ colored by the N-dimensional irreducible representation of $SU(2)$. Thus, $J_N(\text{unknot};q) = 1$, $J_1(\mathcal{K};q) = 1$ for all $\mathcal{K}$ and $J_2(\mathcal{K};q)$ is the Jones polynomial of $\mathcal{K}$. $J_N(\mathcal{K};q)$ is an important quantum invariant in knot theory. People want to find the geometric information from $J_N(\mathcal{K};q)$. More precisely, let $q = e^{\frac{2\pi i}{k}}$ and consider the following limit,

$$k,\ N \to \infty, \quad u = \pi i \frac{N}{k} \quad \text{fixed}. \tag{1}$$

The question is what can we get from the following limit

$$\lim_{N\to\infty} J_N(\mathcal{K}; e^{\frac{2u}{N}}). \tag{2}$$

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The first progress in this direction is the volume conjecture. Let us briefly review it. R.M. Kashaev [29, 30] defined a knot invariant associated with the quantum dilogarithm and an integer N, denoted by $\langle\mathcal{K}\rangle_N$. He conjectured that for any hyperbolic knot $\mathcal{K}$ [31], when $N\to\infty$,

$$|\langle\mathcal{K}\rangle|_N \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}Vol(M_{\mathcal{K}})\right) \tag{3}$$

where $M_{\mathcal{K}}$ is equal to the knot complement $S^3\setminus\mathcal{K}$ and $Vol(M_{\mathcal{K}})$ is the hyperbolic volume of $M_{\mathcal{K}}$. Then, in [34], H. Murakami and J. Murakami proved that for any knot $\mathcal{K}$,

$$\langle\mathcal{K}\rangle_N = J_N(\mathcal{K}; e^{\frac{2\pi i}{N}}).$$

Moreover, they generalized the volume definition at the right hand side of (3) to simplicial volume of any knot complement $M_{\mathcal{K}}$. Now, the standard volume conjecture can be formulated as follow [34]: For a knot $\mathcal{K}$, then

$$|J_N(\mathcal{K}; e^{\frac{2\pi i}{N}})| \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}Vol(M_{\mathcal{K}})\right)$$

where $Vol(M_{\mathcal{K}})$ is the simplicial volume of knot complement $M_{\mathcal{K}}=S^3\setminus\mathcal{K}$. In particular, when $\mathcal{K}$ is a hyperbolic knot, $Vol(M_{\mathcal{K}})$ is the hyperbolic volume of $M_{\mathcal{K}}$.

We remark that the original volume conjecture was actually proposed for link $\mathcal{L}$ [31], but in this paper, we only consider the case of knot $\mathcal{K}$. It is also possible to remove the absolute value to study the complexified volume conjecture [35]: For any hyperbolic knot $\mathcal{K}$,

$$J_N(\mathcal{K}; e^{\frac{2\pi i}{N}}) \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}\left(Vol(M_{\mathcal{K}})+iCS(M_{\mathcal{K}})\right)\right) \tag{4}$$

where $CS(M_{\mathcal{K}})$ is the Chern-Simons invariant of $M_{\mathcal{K}}$ [8]. Furthermore, a u-parameterized version of complexified volume conjecture for any hyperbolic knot $\mathcal{K}$ was firstly proposed in [25]:

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{k}{\pi i}S_0(u)\right) \tag{5}$$

where $S_0(u)$ is a geometric invariant related to the u-deformation volume of $M_{\mathcal{K}}$ [49]. In fact, formula (5) is a generalization of (4) for u near the point πi in $\mathbb{C}$. Moreover, the expansion form of (5) has been further extended to higher order terms by S. Gukov and H. Murakami [26]. We refer to [38] for a nice introduction of the volume conjecture.

It is also interesting to consider the situation when u near the point 0. Another expansion form of colored Jones polynomial which was referred as Melvin-Morton-Rozansky (MMR) conjecture was proposed in [33] and generalized by [45]. Later, the MMR conjecture was proved by D. Bar-Natan and S. Garoufalidis in [5]. Recently, S. Garoufalidis and T. T. Q. Le obtained the following analytic version of MMR expansion: For every knot $\mathcal{K}$, there exists a neighborhood $\mathcal{O} \subset \mathbb{C}$ at $u = 0$, such that for any $u \in \mathcal{O}$, we have

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \sum_{d=0}^{\infty} \frac{P_{\mathcal{K},d}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})^{2d+1}} \left(\frac{2u}{N}\right)^d, \tag{6}$$

where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of $\mathcal{K}$, and $\{P_{\mathcal{K},d}(t), d \geq 0\}$ is a sequence of Laurent polynomials with $P_{\mathcal{K},0}(t) = 1$.

1.2.

Now we focus on the general expansion form of the limit (2), it was conjectured in [25] that the asymptotic expansion of $J_N(\mathcal{K}; q)$ at the limit $N \to \infty,\ q \to 1$ was equal to the $SL(2, \mathbb{C})$ Chern-Simons partition $Z(M_{\mathcal{K}})$ function up to a certain normalization. Based on the standard perturbative Chern-Simons theory [1, 6, 2], the general perturbative computations of $Z(M_{\mathcal{K}})$ were explored in [12, 13]. Therefore, motivated by the conjectured intimate relation between the colored Jones polynomial and Chern-Simons partition, it is rational to consider the higher order expansion of colored Jones polynomial [26]. If we introduce the quantum parameter $\hbar$ as $\hbar = \frac{i\pi}{k}$. The two parameters (k, N) in colored Jones polynomial are changed to two parameters $(\hbar, u)$ by formula (1). Then the general asymptotic expansion of colored Jones polynomial has the following form [12, 11],

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0(u)}{\hbar} - \frac{\delta_{\mathcal{K}}(u)}{2}\log\hbar + \sum_{n=1}^{\infty} S_n(u)\hbar^{n-1}\right). \tag{7}$$

In this paper, we propose that the two expansion formulas (5) and (6) can be unified from the view of A-polynomial and noncommutative A-polynomial of a knot $\mathcal{K}$. In order to determine every terms $S_n(u)$ appearing at the right side of (7), one needs to solve the following equation with initial value $S_{\text{Initial}}(u)$:

$$\begin{cases} \hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}; q) J_N(\mathcal{K}; e^{\frac{2u}{N}}) = 0 \\ S_0(u) = S_{\text{Initial}}(u) \end{cases} \tag{8}$$

where the initial value $S_{\text{Initial}}(u)$ is determined by the solution of the equation $A_{\mathcal{K}}(e^v, e^u) = 0$ up to a constant, where $A_{\mathcal{K}}(l, m)$ is the A-polynomial of $\mathcal{K}$ and $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}; q)$ is an operator defined from the noncommutative A-polynomial of $\mathcal{K}$ which will be introduced in Section 2. First, the work [12] implies

Conjecture 1.1. *i) There exists a solution of equation $A_{\mathcal{K}}(e^v, e^u) = 0$ called* ***geometric branch*** *of A-polynomial: $v = v^G(u)$. In this branch, we have a neighborhood $\mathcal{O}^G \subset \mathbb{C}$ at $u = \pi i$, such that for any $u \in \mathcal{O}^G$,*

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0^G(u)}{\hbar} - \frac{3}{2}\log\hbar + \sum_{n=1}^{\infty} S_n^G(u)\hbar^{n-1}\right), \tag{9}$$

with $\frac{dS_0^G(u)}{du} = v^G(u)$, $S_0^G(u)$ is related to the u-deformed volume of $M_{\mathcal{K}}$ by Gukov's conjecture formula (5), and $S_1^G(u) = \frac{1}{2}\log\frac{iT_{\mathcal{K}}(u)}{4\pi}$ [26]. Moreover, every $S_n^G(u)$ for $n \geq 2$ can be obtained by the algorithm described in Section 2.

ii) By the properties of A-polynomial, we know that there is an ***abelian branch*** *which corresponding to the branch $l = 1$ of $A_{\mathcal{K}}(l, m) = 0$. In this branch, we have a neighborhood $\mathcal{O}^A \subset \mathbb{C}$ of 0, such that for any $u \in \mathcal{O}^A$,*

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0^A(u)}{\hbar} + \sum_{n=1}^{\infty} S_n^A(u)\hbar^{n-1}\right), \tag{10}$$

with $S_0^A(u) = 0$ and $S_1^A(u) = \log\frac{1}{\Delta_{\mathcal{K}}(2u)}$, where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of knot $\mathcal{K}$. Moreover, every $S_n^A(u)$ for $n \geq 2$ can also be obtained by the same algorithm.

Through the detailed calculations carried out by using the algorithm illustrated in Section 2, we propose that

Conjecture 1.2. *The expansion formula (10) is consistent with the analytic version of MMR expansion (6).*

In fact, there exists a sequence of Laurent polynomials $\{Q_{\mathcal{K},n}(t)\}$ such that for $n \geq 2$, we have

$$S_n^A(u) = \frac{Q_{\mathcal{K}}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})}.$$

By the consistence of (10) and (6), if we let $C_{\mathcal{K},d}(u) = \frac{2^d P_{\mathcal{K},d}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})^{2d+1}}$, then

$$C_{\mathcal{K},d}(u) = \exp(S_1^A(u)) \sum_{\mu \mapsto d} \frac{\prod_{i=1}^{l(\mu)} S_{\mu_i+1}^A(u)}{|Aut(\mu)|},$$

where μ is the partition of d with length $l(\mu)$. In other words, Conjecture 1.2 provides a method to compute every $P_{\mathcal{K},d}(e^{2u})$ appears in the analytic version of MMR expansion. The concrete calculations will be carried out in Section 3.

Remark 1.3. The algorithm mentioned above is extracted from the physicists' works on perturbative computation of $SL(2,\mathbb{C})$ Chern-Simons theory [12, 13]. We note that, by the definitions in their works, the colored Jones polynomial $J_N(\mathcal{K}; e^{\frac{2\pi i}{k}})$ and $Z(M_{\mathcal{K}}; u, \hbar)$ are only difference with a normalization $\frac{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$. Thus, we are able to apply their method to study the colored Jones polynomial $J_N(\mathcal{K}; e^{\frac{2\pi i}{k}})$ with a slight modification. Besides the geometric and abelian branches, they also introduced the conjugate branch of A-polynomial. See [12] for more details.

The rest of this paper is organized as follows: In Section 2, we review the definitions of A-polynomial, non-commutative A-polynomial, AJ conjecture for colored Jones polynomial and their recent progresses. Then, we illustrate the quantization algorithm to compute $S_n(u)$ which was introduced in [12] to study the pertubative computation of $SL(2,\mathbb{C})$ Chern-Simons theory. In Section 3, we give some examples to illustrate the calculations of the higher order terms by this quantization algorithm. More precisely, we have calculated the following examples:

i) Figure-8 knot 4_1 in both geometric and abelian branches which has been computed in [12, 13] under the context of $SL(2,\mathbb{C})$ Chern-Simons theory with three different methods.

ii) Twist knots 5_2 and 6_1 in abelian branch which support our Conjecture 1.2.

In the final Section 4, we discuss the possible generalizations and related works.

2. A-polynomial, noncommutative A-polynomial and the quantization algorithm

2.1. A-polynomial $A_{\mathcal{K}}(l,m)$ of a knot $\mathcal{K}$

Let us start with the review of definition of A-polynomial of a knot $\mathcal{K}$ in S^3 [7]. Denoted by $R(M) = Hom(\pi_1(M), SL(2,\mathbb{C}))$ the set of all homomorphisms ρ from $\pi_1(M)$ to $SL(2,\mathbb{C})$ where $M = S^3 \setminus \mathcal{K}$. Let $R_U(M)$ be the subset of $R(M)$ consisting of a representation ρ such that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matrices for a fixed meridian μ and longitude λ of $\mathcal{K}$. Then one can define a projection $\xi = (\xi_\lambda, \xi_\mu) : R_U(M) \to \mathbb{C}^2$ by $\xi(\rho) = (l, m)$ for $\rho \in R_U(M)$ with

$$\rho(\lambda) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix}, \ \rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}.$$

The Zariski closure of $\xi(R_U(M))$ is an algebraic variety in $\mathbb{C}^2$ and each of its irreducible components is a curve, which is defined by zeros of polynomial with integer coefficients in l and m. Then the product of those defining polynomials is defined as the **A-polynomial** of knot $\mathcal{K}$. Note that the A-polynomial of $\mathcal{K}$ has a factor $l-1$, which corresponds to abelian representations which related to the Alexander polynomial of $\mathcal{K}$. Thus someone defines the A-polynomial $A_{\mathcal{K}}(l, m)$ as the original A-polynomial divided by $l-1$. The A-polynomial reflects the geometric properties of the knot $\mathcal{K}$. More algebraic properties of A-polynomial are listed in [25].

Many A-polynomials of knots have been computed by now. Here we give the A-polynomials for two types of knots. For a (p, q)-torus knot $\mathcal{K}_{p,q}$, the A-polynomial is given by:

$$A_{\mathcal{K}_{p,q}}(l, m) = 1 + lm^{pq}.$$

Denote by $\mathcal{K}_p$, $p \in \mathbb{Z}$ the p-twist knot, its A-polynomial was computed in [28].

When $p \neq -1, 0, 1, 2$, $A_{\mathcal{K}_p}(l, m)$ is given recursively by

$$A_{\mathcal{K}_p}(l, m) = \begin{cases} cA_{\mathcal{K}_{p-1}}(l, m) - dA_{\mathcal{K}_{p-2}}(l, m), & p > 0, \\ cA_{\mathcal{K}_{p+1}}(l, m) - dA_{\mathcal{K}_{p+2}}(l, m), & p < 0. \end{cases} \tag{11}$$

where

$$\begin{aligned} c &= -l + l^2 + 2lm^2 + m^4 + 2lm^4 + l^2m^4 + 2lm^6 + m^8 - lm^8, \\ d &= m^4(l + m^2)^4, \end{aligned}$$

and with the initial conditions

$$\begin{aligned} A_{\mathcal{K}_2}(l, m) &= -l^2 + l^3 + 2l^2m^2 + lm^4 + 2l^2m^4 - lm^6 - l^2m^8 \\ &\quad + 2lm^{10} + l^2m^{10} + 2lm^{12} + m^{14} - lm^{14}, \\ A_{\mathcal{K}_1}(l, m) &= l + m^6, \\ A_{\mathcal{K}_0}(l, m) &= 1, \end{aligned}$$

$$A_{\mathcal{K}_{-1}}(l,m) = -l + lm^2 + m^4 + 2lm^4 + l^2m^4 + lm^6 - lm^8.$$

For example, by the recursion (11), we have

$$\begin{aligned} A_{\mathcal{K}_{-2}}(l,m) &= l^2 - l^3 - 3l^2m^2 + l^3m^2 - 2lm^4 - l^2m^4 + 3lm^6 + 3l^2m^6 \\ &+ m^8 + 3lm^8 + 6l^2m^8 + 3l^3m^8 + l^4m^8 + 3l^2m^10 + 3l^3m^{10} \\ &- l^2m^{12} - 2l^3m^{12} + lm^{14} - 3l^2m^{14} - lm^{16} + l^2m^{16}. \end{aligned}$$

Remark 2.1. The twist knots $\mathcal{K}_p$ for $p \in \mathbb{Z}$ include some basic knots from Rolfsen's table:

$$\begin{aligned} &\mathcal{K}_1 = 3_1, \mathcal{K}_2 = 5_2, \mathcal{K}_3 = 7_2, \mathcal{K}_4 = 9_2, \\ &\mathcal{K}_{-1} = 4_1, \mathcal{K}_{-2} = 6_1, \mathcal{K}_{-3} = 8_1, \mathcal{K}_{-4} = 10_1. \end{aligned}$$

Recently, S. Garoufalidis and T. Mattman [19] gave a recursion formula for the A-polynomial of the $(-2, 3, n)$ Pretzel knots.

2.2. Noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E, Q; q)$

The colored Jones polynomial $J_N(\mathcal{K}; q)$ has many beautiful structures. It was shown by S. Garoufalidis and TTQ Le [16, 22] that the colored Jones function is q-holonomic, i.e. it satisfies a nontrivial linear recursion relation with appropriate coefficients. With such holonomicity, they introduce a geometric invariant of a knot: the characteristic variety which is an affine 1-dimensional variety in $\mathbb{C}^2$. By comparing the character variety of $SL(2,\mathbb{C})$ representations in the case of the trefoil and figure-eight knots, they stated a conjecture that these two varieties must be equal [16, 22]. They also define the noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E, Q; q)$ for a knot $\mathcal{K}$ which is the unique monic, linear, minimal order q-difference equation satisfied by the sequence of colored Jones polynomials $\{J_N(\mathcal{K}; q)\}$. Considering two operators E and Q acting on the Jones polynomial $J_N(\mathcal{K}; q)$ by

$$(QJ_{\mathcal{K}})(N) = q^N J_N(\mathcal{K}; q), (EJ_{\mathcal{K}})(N) = J_{N+1}(\mathcal{K}; q). \tag{12}$$

It is easy to see that $EQ = qQE$.

Then $\hat{A}_{\mathcal{K}}(E, Q; q)$ controls the recursion structure of colored Jones polynomial

$$\hat{A}_{\mathcal{K}}(E, Q; q) J_N(\mathcal{K}; q) = 0. \tag{13}$$

Note that $\hat{A}_{\mathcal{K}}(E,Q;q)$ can be written as the form

$$\hat{A}_{\mathcal{K}}(E,Q;q)=\sum_{k\geq 0}a_k(Q;q)E^k \tag{14}$$

with $a_k(Q;q)\in\mathbb{Z}[q,Q]$. Then S. Garoufalidis conjectured that

Conjecture 2.2 (AJ Conjecture)**.** *For every knot $\mathcal{K}$ in S^3, $A_{\mathcal{K}}(l,m)=\epsilon\hat{A}_{\mathcal{K}}(l,m^2;q)$, where ϵ is the evaluation map at $q=1$.*

In order to prove the AJ conjecture, a natural way is to compute the non-commutative A-polynomial. So far, we have known its explicit formula for torus knot in [24], figure-eight knot 4_1 in [16], and 2-bridge knots [32]. Moreover, Takata found out an explicit inhomogeneous q-difference equations for knots 5_2 and 6_1 with degree 5 and 6 respectively [47]. But it is not the really non-commutative A-polynomial in the sense of our definition. Then S. Garoufalidis and X. Sun [20, 21] gave an explicit formula for non-commutative A-polynomial of twist knots $\mathcal{K}_p$ for $p=-8,..,11$. Recently, S. Garoufalidis and C. Koutschan [18] obtained the non-commutative A-polynomial the for Pretzel knot $(-2,3,3+2p)$ for $p=-5,..,5$ by using the method of guessing.

Let us briefly describe the philosophy to calculate the noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E,Q;q)$ of knot $\mathcal{K}$.

For a generic planar projection of a knot $\mathcal{K}$, S. Garoufalidis and T.T.Q. Le proved that the colored Jones polynomial of a knot $\mathcal{K}$ can be written as a multisum [22]

$$J_N(\mathcal{K};q)=\sum_{k_1,..,k_r}^{\infty}F(N,k_1,..,k_r), \tag{15}$$

where $F(N,k_1,..,k_r)$ is a proper q-hypergeometric function and for a fixed $N\in\mathbb{Z}^+$, only finitely many terms are nonzero. Because $F(N,k_1,..,k_r)$ is a proper q-hypergeometric function, one can use the algorithm invented by Wilf-Zeilberger [44, 52] (the WZ algorithm), also called creative telescoping method, to produce the noncommutative operator eliminate $J_N(\mathcal{K};q)$. See [42, 43] for a mathematica implementation of WZ-algorithm. We will give some examples to demonstrate how to use this computer program to derive the noncommutative A-polynomial in next section.

2.3. The algorithm to compute the asymptotic expansion of $J_N(\mathcal{K};q)$

Let $A_{\mathcal{K}}(l,m)$ be the A-polynomial of a knot $\mathcal{K}$. Define the operator $\hat{l}$ and $\hat{m}$ such that

$$\hat{l}=E,\ \hat{m}^2=Q. \tag{16}$$

Then by (13), we known that $\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2;q)J_N(\mathcal{K};q)=0$.

Recall the parameters we have described in Section 1

$$\hbar=\frac{\pi i}{k},\ u=\frac{\pi i N}{k},\ q=e^{\frac{2\pi i}{k}}.$$

Then $q=e^{2\hbar}$, the operator $\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2;q)$ annihilates

$$J(\mathcal{K};\hbar,u):=J_N(\mathcal{K};e^{2\hbar})$$

i.e. we have the equation

$$\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2;q)J(\mathcal{K};\hbar,u)=0, \tag{17}$$

and by (12) and (16), the action of the operators $\hat{l},\hat{m}$ is

$$\hat{m}J(\mathcal{K};\hbar,u)=e^{u}J(\mathcal{K};\hbar,u),\quad \hat{l}J(\mathcal{K};\hbar,u)=J(\mathcal{K};\hbar,u+\hbar). \tag{18}$$

It is clear that $\hat{l}\hat{m}=q^{\frac{1}{2}}\hat{m}\hat{l}$. As in (14), we expand $\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2;q)$ as follows

$$\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2;q)=\sum_{j=0}^{d}a_j(\hat{m},\hbar)\hat{l}^j. \tag{19}$$

Then we obtain

$$\sum_{j=0}^{d}a_j(\hat{m},\hbar)J(\mathcal{K};\hbar,u+j\hbar)=0. \tag{20}$$

By the formula (7), one can assume that at large N,

$$J(\mathcal{K};\hbar,u)=\exp\left(\frac{S_0(u)}{\hbar}-\frac{\delta_{\mathcal{K}}(u)}{2}\log\hbar+\sum_{n=1}^{\infty}S_n(u)\hbar^{n-1}\right).$$

Therefore, from the above restriction equation for $J(\mathcal{K};q,u)$, one can obtain a sequence of expansion coefficients $\{S_n(u)\}$ recursively by solving the equation (17) for a given initial value $S_0(u)$ [12]. In the following, we show how to get the recursion formula for $S_n(u)$ step by step.

Equation (20) is equivalent to

$$
\begin{aligned}
(21)\quad 0 &= \sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(\frac{1}{\hbar}S_0(u+j\hbar)-\frac{3}{2}\cdot\log\hbar+\sum_{n\geq 0}\hbar^n S_{n+1}(u+j\hbar)\right)\\
&= \exp\left(-\frac{\delta_{\mathcal{K}}(u)}{2}\cdot\log\hbar\right)\sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(\sum_{n\geq -1}\hbar^n S_{n+1}(u+j\hbar)\right).
\end{aligned}
$$

By Taylor expansion

$$
\begin{aligned}
\sum_{n\geq -1}\hbar^n S_{n+1}(u+j\hbar) &= \sum_{n\geq -1}\sum_{k\geq 0}\frac{S_{n+1}^{(k)}(u)}{k!}j^k\hbar^{k+n}\\
&= \sum_{t\geq -1}\sum_{r=-1}^{t}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t\\
&= \sum_{t\geq -1}S_{t+1}(u)\hbar^t+\sum_{t\geq 0}\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t\\
&= \sum_{t\geq -1}S_{t+1}(u)\hbar^t+jS_0'(u)+\sum_{t\geq 1}\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t.
\end{aligned}
$$

It follows that

$$
\sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(jS_0'(u)+\sum_{t\geq 1}B_t(u,j)\hbar^t\right)=0,
$$

where we have defined

$$
B_t(u,j)=\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}.
$$

Furthermore, one can expand $a_j(\hat{m},\hbar)$ and $\exp\left(\sum_{t\geq 1}B_t(u,j)\hbar^t\right)$ as

$$
a_j(\hat{m},\hbar)=\sum_{p\geq 0}a_{i,p}(\hat{m})\hbar^p
$$

and

$$\exp\left(\sum_{t\geq 1} B_t(u,j)\hbar^t\right) = 1 + \sum_{n\geq 1} \frac{\left(\sum_{t\geq 1} B_t(u,j)\hbar^t\right)^n}{n!}$$
$$= \sum_{\mu\in\mathcal{P}} \frac{B_\mu(u,j)}{|Aut(\mu)|}\hbar^{|\mu|}$$

where, $\mathcal{P} = \cup_{n\geq 1}\mathcal{P}_n \cup \{\emptyset\}$, and $\mathcal{P}_n$ is the set of all partitions of integer $n \in \mathbb{Z}^+$, and we denote $B_\mu(u,j) = B_{\mu_1}(u,j)\cdots B_{\mu_{l(\mu)}}(u,j)$ and $B_\emptyset(u,j) = 1$.

Then formula (21) is equal to

$$\sum_{j=0}^{d} e^{jS_0'(u)} \left(\sum_{p\geq 0}\sum_{\mu\in\mathcal{P}} a_{j,p}(\hat{m})\frac{B_\mu(u,j)}{|Aut(\mu)|}\hbar^{|\mu|+p}\right) = 0. \tag{22}$$

By the action of $\hat{m}$ defined in (18), one can replace the $\hat{m}$ with e^u in $a_{j,p}(\hat{m})$. As a series of $\hbar$, all the coefficients of left hand side of (22) must be zero. The constant term gives

$$\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) = 0. \tag{23}$$

which in fact is the A-polynomial.

When $n = |\mu| + p > 0$, one can solve the n-th equation obtained from the coefficient of $\hbar^n$ in equation (22) and get

(24)

$$S_n'(u) = -\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left(\sum_{p=1}^{n} a_{j,p}(e^u) \sum_{\mu\in\mathcal{P}_{n-p}\cup\{\emptyset\}} \frac{B_\mu(u,j)}{|Aut(\mu)|}\right.$$
$$\left. + a_{j,0}(e^u)\sum_{\mu\in\mathcal{P}\setminus\{(n)\}} \frac{B_\mu(u,j)}{|Aut(\mu)|} + a_{j,0}(e^u)\sum_{r=-1}^{n-2} \frac{S_{r+1}^{(n-r)}(u)}{(n-r)!} j^{n-r}\right).$$

Example 2.3. When $n = 1$ and 2, we have

$$S_1'(u) = -\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left(a_{j,1}(e^u) + a_{j,0}(e^u)\frac{S_0''(u)}{2} j^2\right)$$

$$S_2'(u) = -\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left[a_{j,1}(e^u) \left(\frac{S_0''(u)}{2} j^2 + S_1'(u) j \right) \right.$$
$$\left. + a_{j,2}(e^u) + a_{j,0}(e^u) \left(\frac{1}{2} \left(\frac{S_0''(u)}{2} j^2 + S_1'(u) j \right)^2 + \frac{S_0'''(u)}{6} j^3 + \frac{S_1''(u)}{2} j^2 \right) \right]$$

The above formula (24) gives a recursion relation for $S_n(u)$. In other words, if one knows the initial value $S_0(u)$, then all the coefficients $S_n(u)$ are determined uniquely. How to choose $S_0(u)$ depends on the choice of the branch of A-polynomial as described in Section 1, i.e. in the geometric branch: choosing $S_0(u) = S_0^G(u)$; and in abelian branch: choosing $S_0(u) = S_0^A(u) = 0$.

Remark 2.4. By AJ-conjecture, the classical limit $q \to 1$ of noncommutative A-polynomial is the A-polynomial. Thus, the noncommutative A-polynomial can be considered as the quantization of A-polynomial. So the above method to compute $S_n(u)$ can be called quantization algorithm. All the information of colored Jones polynomial are implied in a hierarchy of equations (23), (24). The equation (23) is the A-polynomial if we let $l = e^{S_0'(u)}$. A-polynomial reflects some geometric information of the knot complement $M_{\mathcal{K}}$. Finding the geometric meaning of the generic equations (24) will be interesting.

Remark 2.5. The above quantization algorithm was introduced in [12] to study the $SL(2,\mathbb{C})$ Chern-Simons partition function of $M_{\mathcal{K}}$. They assumed that the colored Jones polynomial $J_N(\mathcal{K}; e^{\frac{2\pi i}{k}})$ and Chern-Simons partition $Z(M_{\mathcal{K}}; u, \hbar)$ are only difference with a normalization $\frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. So in order to get the quantization operator $\tilde{A}(\tilde{l}, \tilde{m})$ of $Z(M_{\mathcal{K}}; u; \hbar)$ such that

$$\tilde{A}(\tilde{l}, \tilde{m}) Z(M_{\mathcal{K}}; u; \hbar) = 0,$$

one only needs to do some modifications on operator $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q)$. See [12, 13] for detail discussion.

3. Examples and calculations

Example 3.1. When $p = -1$, $\mathcal{K}_{-1} = 4_1$, this example was first calculated in [12]. But as a basic example to illustrate the application of the above algorithm, we still recalculate here with a slight difference; see Remark 3.2.

Step 1. Finding the noncommutative A-polynomial of 4_1.

We download Paule and Riese's qZeil.m and qMultiSum.m package [42] and run them in Mathematica 7.0.

In[1]:= << c:/qZeil.m
q-Zeilberger Package by Axel Riese–@RISC Linz -V 2.42 (02/18/05)
In[2]:= << c:/qMultiSum.m
qMultiSum Package by Axel Riese–@RISC Linz -V 2.52 (30-Jul-2010)
In[3]:=summandfigure8 = $q^{nk}qfac[q^{-n-1}, q^{-1}, k]qfac[q^{-n+1}, q, k]$
Out[3]:=$q^{kn}qPochhammer[q^{-1-n}, 1/q, k]qPochhammer[q^{1-n}, q, k]$
In[4]:=qZeil[summandfigure8, k, 0, Infinity, n, 2]
Out[4]:=

$$\mathrm{SUM[n]} == \frac{q^{-1-n}(q+q^n)(-q+q^{2n})}{-1+q^n} - \frac{(1-q^{-2+n})(1-q^{-1+2n})\mathrm{SUM[-2 + n]}}{(1-q^n)(1-q^{-3+2n})} + \frac{q^{-2-2n}(1-q^{-1+n})^2(1+q^{-1+n})(q^4+q^{4n}-q^{3+n}-q^{1+2n}-q^{3+2n}-q^{1+3n})\mathrm{SUM[-1 + n]}}{(1-q^n)(1-q^{-3+2n})}$$

This is a second-order inhomogeneous recursion relation, we convert it into a third-order homogeneous recursion relation:
In[5]:=MakeHomRec [%, SUM[n]];
Converting to forward shifts:
In[6]:=Rec41=ForwardShifts[%]

$$\begin{aligned}
\mathrm{Out[6]} :=& q^{5+n}(q-q^{3+n})(q^3-q^{3+n})(q+q^{3+n})(q-q^{6+2n})(q^3-q^{6+2n})\mathrm{SUM[n]}\\
&- q^{-5-n}(q-q^{3+n})(q^2-q^{3+n})(q^2+q^{3+n})(q-q^{6+2n})(q^3-q^{6+2n})\\
&\times(q^8-2q^{9+n}+q^{10+n}-q^{9+2n}+q^{10+2n}-q^{11+2n}+q^{10+3n}\\
&-2q^{11+3n}+q^{12+4n})\mathrm{SUM[1 + n]}+q^{-4-n}(q-q^{3+n})^2(q+q^{3+n})\\
&\times(q^3-q^{6+2n})(q^5-q^{6+2n})(q^4+q^{5+n}-2q^{6+n}-q^{7+2n}+q^{8+2n}\\
&-q^{9+2n}-2q^{10+3n}+q^{11+3n}+q^{12+4n})\ \mathrm{SUM[2 + n]}+q^{4+n}\\
&\times(q-q^{3+n})(-1+q^{3+n})(q^2+q^{3+n})(q^3-q^{6+2n})(q^5-q^{6+2n})\\
&\times\mathrm{SUM[3 + n]} == 0
\end{aligned}$$

Converting it to operator form:
In[7]:=F=ToqHyper[Rec41[[1]] - rec41[[2]]]/.$\{\mathrm{SUM[N]} \to 1, \mathrm{SUM[N}\ q^c_.] :> X^c\}/.\mathrm{N} \to Q$

$$\begin{aligned}
\mathrm{Out[7]} :=& q^5Q(q-q^3Q)(q^3-q^3Q)(q+q^3Q)(q-q^6Q^2)(q^3-q^6Q^2)\\
&-\frac{1}{q^5Q}(q-q^3Q)(q^2-q^3Q)(q^2+q^3Q)(q-q^6Q^2)(q^3-q^6Q^2)\\
&\times(q^8-2q^9Q+q^{10}Q-q^9Q^2+q^{10}Q^2-q^{11}Q^2+q^{10}Q^3-2q^{11}Q^3\\
&+q^{12}Q^4)X+\frac{1}{q^4Q}(q-q^3Q)^2(q+q^3Q)(q^3-q^6Q^2)(q^5-q^6Q^2)
\end{aligned}$$

$$\times (q^4 + q^5Q - 2q^6Q - q^7Q^2 + q^8Q^2 - q^9Q^2 - 2q^{10}Q^3 + q^{11}Q^3 + q^{12}Q^4)X^2 + q^4Q(q - q^3Q)(-1 + q^3Q)(q^2 + q^3Q)(q^3 - q^6Q^2) \times (q^5 - q^6Q^2)X^3$$

Then F is the non-commutative A-polynomial of 4_1 if we replace X by E.

Step 2. Finding the operator $\hat{A}_{4_1}(\hat{l}, \hat{m}; q) = \sum_{j=0}^{d} a_j(\hat{m}, \hbar)\hat{l}^j$.

Substituting Q and X by m^2 and l respectively in F, we get

$$\hat{A}_{4_1}(\hat{l}, \hat{m}) = \sum_{j=0}^{3} a_j(\hat{m}, \hbar)\hat{l}^j$$

where

$$\hat{a}_0(\hat{m}, q) = \hat{m}^2q^5(q - \hat{m}^2q^3)(q^3 - \hat{m}^2q^3)(q + \hat{m}^2q^3)(q - \hat{m}^4q^6)(q^3 - \hat{m}^4q^6)$$

$$\hat{a}_1(\hat{m}, q) = \frac{1}{\hat{m}^2q^5}(q - \hat{m}^2q^3)(q^2 - \hat{m}^2q^3)(q^2 + \hat{m}^2q^3)(q - \hat{m}^4q^6)(q^3 - \hat{m}^4q^6) \times (q^8 - 2\hat{m}^2q^9 - \hat{m}^4q^9 + \hat{m}^2q^{10} + \hat{m}^4q^{10} + \hat{m}^6q^{10} - \hat{m}^4q^{11} - 2\hat{m}^6q^{11} + \hat{m}^8q^{12})$$

$$\hat{a}_2(\hat{m}, q) = \frac{1}{\hat{m}^2q^4}(q - \hat{m}^2q^3)^2(q + \hat{m}^2q^3)(q^3 - \hat{m}^4q^6)(q^5 - \hat{m}^4q^6) \times (q^4 + \hat{m}^2q^5 - 2\hat{m}^2q^6 - \hat{m}^4q^7 + \hat{m}^4q^8 - \hat{m}^4q^9 - 2\hat{m}^6q^{10} + \hat{m}^6q^{11} + \hat{m}^8q^{12})$$

$$\hat{a}_3(\hat{m}, q) = \hat{m}^2q^4(q - \hat{m}^2q^3)(-1 + \hat{m}^2q^3)(q^2 + \hat{m}^2q^3)(q^3 - \hat{m}^4q^6)(q^5 - \hat{m}^4q^6)$$

Step 3 Choosing the different branches.

The A-polynomial of 4_1 is

$$A_{4_1}(l, m) = (-1 + l)(l - lm^2 - m^4 - 2lm^4 - l^2m^4 - lm^6 + lm^8)$$

Solving this equation, we obtain the three branches: $l_A = 1$ is called the abelian branch and $l_G = -\frac{-1+m^2+2m^4+m^6-m^8+(-1+m^4)\sqrt{1-2m^2-m^4-2m^6+m^8}}{2m^4}$ is the geometric branch. The third one is the conjugate of l_G called conjugate branch which have the intimate relation with geometric branch discussed in [12].

Step 4 Calculating the expansion coefficients $S_n(u)$ in different branches by formula (24).

Abelian branch expansion: taking the initial value $S_0^A(u) = \log l_A = 0$, then

$$\begin{aligned}
S_1^A(u) &= \log \frac{1}{\Delta_{4_1}(m^2)};\\
S_2^A(u) &= \text{constant};\\
S_3^A(u) &= \frac{4(m^{-2} - 1 + m^2)}{\Delta_{4_1}(m^2)^3};\\
S_4^A(u) &= \text{constant};\\
&\dots
\end{aligned}$$

where $\Delta_{4_1}(t) = \frac{1}{t} + t - 3$ is the Alexander polynomial of 4_1. The above results match the Conjecture 1.2.

Geometric branch expansion: the initial value $S_0^G(u) = \frac{i}{2}Vol(4_1) + \int_{i\pi}^{u} v_G(u)du - 2\pi^2$ [12].

$$\begin{aligned}
S_1^G(u) &= 2\log(m) - \log(-1+m^2) - \frac{1}{4}\log(1 - 2m^2 - m^4 - 2m^6 + m^8);\\
S_2^G(u) &= \frac{1 - m^2 - 2m^4 + 15m^6 - 2m^8 - m^{10} + m^{12}}{12(-1 + 2m^2 + m^4 + 2m^6 - m^8)^{\frac{3}{2}}};\\
S_3^G(u) &= -\frac{2m^6(-1 + m^2 + 2m^4 - 5m^6 + 2m^8 + m^{10} - m^{12})}{(1 - 2m^2 - m^4 - 2m^6 + m^8)^3};\\
S_4^G(u) &= \frac{m^2}{90(1 - 2m^2 - m^4 - 2m^6 + m^8)^{\frac{9}{2}}}(1 - 4m^2 - 128m^4 + 36m^6\\
&+ 1074m^8 - 5630m^{10} + 5782m^{12} + 7484m^{14} - 18311m^{16} + 7484m^{18}\\
&+ 5782m^{20} - 5630m^{22} + 1074m^{24} + 36m^{26} - 128m^{28} - 4m^{30} + m^{32});\\
&\dots
\end{aligned}$$

If we use the Ray-Singer torsion of 4_1 [26, 11]

$$T_{4_1}(u) = \frac{4\pi^2 m^2}{\sqrt{-1 + 2m^2 + m^4 + 2m^6 - m^8}},$$

we may conjecture that $S_n(u)$ has the form

$$S_n(u) = \left(\frac{T_{4_1}(u)}{4\pi^2}\right)^{3n-3} G_n(m) \text{ for } n \geq 2,$$

where $\{G_n(m)\}$ is a sequence of Laurent polynomial of m.

Remark 3.2. [12] has calculated the perturbative expansion for $Z\left(M_{4_1};u;\hbar\right)$ assume that $Z(M_{4_1};u;\hbar)=\frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}J(\mathcal{K};u,\hbar)$. By this relation, we should make a modification for $\hat{a}_j(\hat{m},q)$,

$$\hat{a}_j(\hat{m},q)\to\frac{\hat{a}_j(\hat{m},q)}{m^2q^{j/2}-q^{-j/2}}.$$

With these new $\hat{a}(\hat{m},q)$, they calculated the $S_n(u)$ for $Z(M_{4_1};u,\hbar)$ up to $n=8$.

Now, we give more new examples calculated in abelian branch expansion by using the above method.

Example 3.3. When $p=2$, the twist knot $\mathcal{K}_2=5_2$. Setting the initial value $S_0^A(u)=0$, we get

$$S_1^A(u)=\log\left(\frac{1}{\Delta_{5_2}(m^2)}\right);$$

$$S_2^A(u)=\frac{-4(m^{-2}+m^2)+13}{2\Delta_{5_2}(m^2)^2};$$

$$S_3^A(u)=-\frac{-32+104m^2+200m^4-607m^6+200m^8+104m^{10}-32m^{12}}{8m^6\Delta_{5_2}(m^2)^4};$$

$$S_4(u)=-\frac{1}{24m^{10}\Delta_{5_2}(m^2)^6}(320-752m^2-3808m^4+3052m^6+39692m^8$$
$$-78163m^{10}+39692m^{12}+3052m^{14}-3808m^{16}-752m^{18}+320m^{20})$$
$$\cdots$$

where $\Delta_{5_2}(t)=2(t^{-1}+t)-3$ is the Alexander polynomial of 5_2. It is easy to see these results match the Conjecture 1.2.

Example 3.4. When $p=-2$, the twist knot $\mathcal{K}_{-2}=6_1$. Setting the initial value $S_0^A(u)=0$, we obtain

$$S_1^A(u)=\log\left(\frac{1}{\Delta_{6_1}(m^2)}\right);$$

$$S_2^A(u)=\frac{-4m^2+7m^4-4m^6}{2m^4\Delta_{6_1}(m^2)^2};$$

$$S_3^A(u)=-\frac{32-504m^2+1656m^4-2303m^6+1656m^8-504m^{10}+32m^{12}}{8m^6\Delta_{6_1}(m^2)^4};$$

$$S_4(u)=-\frac{1}{24m^{10}\Delta_{6_1}(m^2)^6}(320-2512m^2+23968m^4-103404m^6$$

$$+ 225900m^8 - 288925m^{10} + 225900m^{12} - 103404m^{14} + 23968m^{16}$$
$$- 2512m^{18} + 320m^{20}) \cdots$$

where $\Delta_{6_1}(t) = 2(t^{-1}+t)-5$ is the Alexander polynomial of 6_1. These results match the Conjecture 1.2.

Remark 3.5. In the above two examples, we have used the non-commutative A-polynomial for twist knot $\mathcal{K}_p$ obtained by S. Garoufalidis and X. Sun [20, 21]. In fact, they have calculated all the non-commutative A-polynomial of $\mathcal{K}_p$ for $p = -8, .., 11$. So we can compute more examples by using their results.

4. Conclusions and discussions

In this paper, following [12], we present an algorithm to calculate the higher ordered terms in asymptotic expansion of colored Jones polynomial from the view of A-polynomial and noncommutative A-polynomial. In the large N limit, the colored Jones polynomial $J_N(\mathcal{K}; e^{2u})$ has the following expansion form

$$J(\mathcal{K}; \hbar, u) = \exp\left(\frac{S_0(u)}{\hbar} - \frac{3}{2}\log\hbar + \sum_{n=1}^{\infty} S_n(u)\hbar^{n-1}\right). \tag{25}$$

In order to determine every terms $S_n(u)$ appearing at the left hand side of (25), we need to solve the following equation with initial value $S_{\text{Initial}}(u)$:

$$\begin{cases} \hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}; q)J(\mathcal{K}; \hbar, u) = 0 \\ S_0(u) = S_{\text{Initial}}(u) \end{cases} \tag{26}$$

Up to a constant, the initial value $S_{\text{Initial}}(u)$ is determined by the solution of the equation $A_{\mathcal{K}}(e^v, e^u) = 0$, where $A_{\mathcal{K}}(l, m)$ is the A-polynomial of knot $\mathcal{K}$. More precisely, if we assume $A_{\mathcal{K}}(l, m) = (l-1)f_d(l, m)$, where $f_d(l, m) = \sum_{i=1}^{d} a_i(m)l^i$. For a given m, the equation $f_d(l, m) = 0$ has d solutions in $\mathbb{C}$ denoted by $l = l^\alpha(m)$, $\alpha = 1, .., d$. Thus $A_{\mathcal{K}}(l, m) = 0$ has $d+1$ branches: abelian branch $l^A = 1$, and $l^\alpha(m)$, for $\alpha = 1, .., d$. There are some symmetries between these different branches $\alpha = 1, .., d$. See [12] for discussions from Chern-Simon theory.

One of the most interesting branch is called geometric branch denoted by $l^G(m)$ which is relevant with the hyperbolic volume of knot complement $M_{\mathcal{K}}$. In this geometric branch, the initial value is $\frac{dS_{\text{Initial}}(u)}{du} = v^G(u)$ and $S_{\text{Initial}}(u)$ is the complexified volume of $M_{\mathcal{K}}$ parameterized by u. Then all the terms

$S_n^G(u)$ can be solved from the recursive relation (24). Moreover, $S_1^G(u)$ has the geometric interpretation $S_1^G(u) = \frac{1}{2}\log\frac{iT_{\mathcal{K}}(u)}{4\pi}$ [26], where $T_{\mathcal{K}}(u)$ denotes the u-deformed torsion of $M_{\mathcal{K}}$. But what's the geometric mean of $S_n^G(u)$ for $n \geq 2$ is still unknown.

In the abelian branch $l_A = 1$, the initial value is $\frac{dS_{\text{Initial}}(u)}{du} = v^A = 0$. So $S_{\text{Initial}}(u)$ is a constant. One can also get every $S_n^A(u)$ by formula (24). Moreover, the first term is $S_1^A(u) = \log\frac{1}{\Delta_{\mathcal{K}}(2u)}$, where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of $\mathcal{K}$ which has the geometric meaning, but the geometric interpretation is still unclear for $S_n^A(u), n \geq 2$. We found that the higher order term $S_n^A(u)$ is in consistence with the Melvin-Morton-Rozansky expansion for colored Jones polynomial [33, 11, 45, 46, 22], so we proposed the Conjecture 1.2.

Colored Jones polynomial is just a special quantum link invariant. One of generalization is the categorified link invariant, i.e. superpolynomial. During the past several years, a lot of works, such as [14, 15, 41] is devoted to investigating the asymptotic expansion of the superpolynomial. Another generalization is the colored HOMFLYPT invariants which is an important quantum invariant in large N duality of topological string and Chern-Simons theory [55, 9]. Aganagic and Vafa [3] predicted that colored HOMFLYPT invariant carried the similar A-polynomial, i.e. Q-deformed A-polynomial, whose existence was then explained in [17]. From the view of representation theory, colored Jones polynomials is the $SU(2)$ quantum invariant, it is natural to consider the higher rank $SU(n)$ quantum invariant. In [10], we have proposed the volume conjecture for $SU(n)$ quantum invariant. Therefore, one can also ask how to investigate the higher order terms in asymptotic expansion of the $SU(n)$-invariants.

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