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Asymptotic behavior of solutions to general hyperbolic-parabolic systems of balance laws in multi-space dimensions*

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Abstract: We study time asymptotic behavior of solutions for a general system of hyperbolic-parabolic balance laws in m space dimensions, $m \geq 2$. The system has physical viscosity matrices. Besides, there is a lower order term to account for relaxation, damping or chemical reaction. The viscosity matrices and the Jacobian matrix of the lower order term are rank deficient. We study Cauchy problem around a constant equilibrium state. Under a set of reasonable assumptions, existence of solution global in time has been established recently, and L^p decay rates ($p \geq 2$) of the solution to the constant equilibrium state have been obtained. In this paper we further study the large time behavior of the solution. We show that it is time-asymptotically approximated by the solution of the corresponding linear system with the same initial data. For $p \geq 2$, optimal L^p convergence rates to the asymptotic solution are obtained. These rates are faster by $(t+1)^{-1/2}$ (or $(t+1)^{-1/2}\ln(t+2)$ if $m = 2$) when comparing to the convergence rates to the constant equilibrium state. Our result is general and applies to physical models such as gas flows with translational and vibrational non-equilibrium. Our result is new even for the special case of hyperbolic balance laws.

1. Introduction

We study the Cauchy problem of a general class of partial differential equations in the following form:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w) w_{x_k}\right]_{x_j} + r(w), \tag{1.1}$$

$$w(x,0) = w_0(x), \tag{1.2}$$

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where $w, f_j, r, w_0 \in \mathbb{R}^n$ and $B_{jk} \in \mathbb{R}^{n\times n}$. The unknown function $w = w(x,t)$ depends on the space variable $x = (x_1, \dots, x_m)^t \in \mathbb{R}^m$ and the time variable $t \in \mathbb{R}^+$. It represents physical densities such as mass density, momentum density, etc. The functions f_j are flux functions, while r describes external forces, relaxation, chemical reactions, and so forth. The matrices B_{jk} are known as viscosity matrices, representing viscosity, heat conduction, species diffusion, etc. All these are known functions of w. System (1.1) arises from continuum mechanics for the balance of quantities such as mass, momentum and energy of a flow. These balance laws are hyperbolic-parabolic since the flux functions satisfy an entropy condition hence the corresponding inviscid system is completely hyperbolic [2], and the viscosity matrices are rank deficient as dictated by physics.

A special case of (1.1) is the system of hyperbolic-parabolic conservation laws obtained by setting $r = 0$:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w) w_{x_k}\right]_{x_j}. \tag{1.3}$$

An important example of (1.3) is the familiar Navier-Stokes equations for compressible flows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u u^t) + \nabla p = \operatorname{div}(2\mu \mathscr{P}) + \nabla(\mu' \operatorname{div} u), \\ (\rho E)_t + \operatorname{div}(\rho E u + p u) = \operatorname{div}[2\mu \mathscr{P} u + \mu'(\operatorname{div} u) u + \kappa \nabla T], \end{cases} \tag{1.4}$$

where ρ is the gas density, $u = (u_1, \dots, u_m)^t \in \mathbb{R}^m$ $(m \geq 1)$ the velocity, p the pressure, $E = e + |u|^2/2$ the total energy, with the internal energy e, and T the temperature. The strain rate tensor $\mathscr{P} \in \mathbb{R}^{m\times m}$ has entries

$$\mathscr{P}_{jk} = \frac{1}{2}(u_{jx_k} + u_{kx_j}), \quad 1 \leq j,k \leq m.$$

The dissipation parameters are the shear viscosity coefficient μ, the second viscosity coefficient μ', and the thermal conductivity κ. The thermodynamic equation reads

$$T ds = de + p dv, \quad v = 1/\rho, \tag{1.5}$$

where v is the specific volume, and s is the entropy. Equation (1.5) implies that two of the thermodynamic variables are independent, and the others, including the dissipation parameters, can be regarded as known functions of them. With the m components of the velocity, (1.4) is a system of $m+2$ equations for $m+2$ unknowns.

Another special case of (1.1) is the system of hyperbolic balance laws with vanished viscosity matrices:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = r(w). \tag{1.6}$$

Physical examples of (1.6) include Euler equations with damping for inviscid, compressible, isentropic/isothermal flows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u u^t) + \nabla p = -\rho u, \end{cases} \tag{1.7}$$

which describes the motion of a gas through a porous medium. The system has $m+1$ equations for $m+1$ unknowns, due to the fact that the gas is isentropic/isothermal hence only one thermodynamic variable is independent.

An important physical example of (1.1) that includes both viscosity matrices and the lower order term is the system describing the motion of a polyatomic gas with an internal structure and under translational non-equilibrium. The translational non-equilibrium induces dissipation mechanism such as viscosity, heat conduction and species diffusion, represented by the second derivatives. On the other hand, the internal structure, which is not in dynamic equilibrium, gives rise to relaxation or similar mechanism in the form of lower order terms.

For simplicity we consider the case of one non-equilibrium internal mode, the vibrational mode. The system then reads

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u u^t) + \nabla p = \operatorname{div}(2\mu \mathscr{P}) + \nabla(\mu' \operatorname{div} u), \\ (\rho E)_t + \operatorname{div}(\rho E u + p u) = \operatorname{div}[2\mu \mathscr{P} u + \mu'(\operatorname{div} u) u + \kappa \nabla T_1 + \nu \rho \nabla e_2], \\ (\rho e_2)_t + \operatorname{div}(\rho e_2 u) = \operatorname{div}(\nu \rho \nabla e_2) + \rho \dfrac{e_2^E - e_2}{\tau}, \end{cases} \tag{1.8}$$

see [9, 1]. Here in addition to the notations from (1.4), there are a few new ones. The internal structure requires us to use two sets of thermodynamic variables: one for the total of the translational mode and the internal modes other than the vibrational mode (such as the rotational mode), and one for the non-equilibrium vibrational mode. We use subscript "1" for the first set and "2" for the second one, respectively. In such notations, the total energy is

$$E = e + \frac{1}{2}|u|^2, \qquad e = e_1 + e_2,$$

where e_2 is the vibrational energy, and e_1 is the rest of internal energy, which includes that from the translational mode of gas molecules. The last equation in (1.8) describes the relaxation of e_2 towards its local equilibrium value e_2^E in the time scale τ, called the relaxation time. Both e_2^E and τ are known functions of the first set of thermodynamic variables. We note that in the energy equation of (1.8), T_1 denotes the translational temperature, which is the common temperature of the translational mode and rotational mode, etc (but not the vibrational mode). We also note that there is a new dissipation parameter ν, which is the self-diffusion coefficient.

The two sets of thermodynamic variables obey different thermodynamic equations:

$$T_1 ds_1 = de_1 + pdv, \quad T_2 ds_2 = de_2, \tag{1.9}$$

where s_1 and s_2 are the translational and vibrational entropies, respectively, and T_2 is the vibrational temperature. We note that the first equation is alike to (1.5), while the second one is volume independent. Equation (1.9) implies that two variables are needed for mode 1 and one for mode 2. With the m components of u, (1.8) is a system of $m + 3$ equations for $m + 3$ unknowns.

In a recent paper [11], the author has proposed a set of structural conditions for (1.1). Under those conditions, the global existence of solution to (1.1), (1.2) is established for all space dimensions when the initial datum w_0 is a small perturbation of a constant equilibrium state $\bar{w}$. In another paper [12], optimal L^p decay rates of w to $\bar{w}$ have been obtained under the same set of conditions for $m \geq 2$ and $p \geq 2$. In this paper we continue to discuss time asymptotic behavior of the solution. We show that under the same structural conditions, the solution of (1.1), (1.2) is time-asymptotically approximated by the solution of the corresponding linear system with the same initial datum, in L^p sense ($p \geq 2$) and for $m \geq 2$.

We comment that in one space dimension ($m = 1$), it is not true that the solution of (1.1) can be approximated by that of the linear system. Indeed, the leading term of the asymptotic solution comes from the nonlinear coupling of different characteristic families in the flux function. This leads to the diffusion waves [5]. Detailed discussion and pointwise estimates in one space dimension for the special case (1.3) are given in [6]. Similar results for the special case (1.6) can be found in [13]. Parallel study for the general system (1.1) is left to the future.

We now focus on multi-space dimensions $m \geq 2$. We start with the basic structural conditions proposed in [11] for (1.1). Consider a neighborhood $\mathbb{O}$ of a constant equilibrium state $\bar{w}$ such that $r(\bar{w}) = 0$. We define the equilibrium manifold $\mathbb{E}$ in $\mathbb{O}$ as

$$\mathbb{E} = \{w \in \mathbb{O} \,|\, r(w) = 0\}. \tag{1.10}$$

The functions $f_j(w)$, $B_{jk}(w)$ and $r(w)$ are assumed to be smooth in $\mathbb{O}$. In the following we use f_j' to denote the Jacobian matrix of f_j with respect to w, etc.

Assumption 1.1. *1. There exists a strictly convex entropy function η, which is a scalar function of w in $\mathbb{O}$, satisfying the following properties.*

(i) $\eta'' f_j'$, $1 \leq j \leq m$, are symmetric in $\mathbb{O}$, where η'' is the Hessian of η with respect to w.

(ii) In $\mathbb{O}$, $(\eta'' B_{jk})^t = \eta'' B_{kj}$, $1 \leq j,k \leq m$, and $\eta'' \sum_{j,k=1}^m B_{jk}\xi_k\xi_j$ is symmetric, semi-positive definite for all unit vectors $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$.

(iii) On $\mathbb{E}$, $\eta'' r'$ is symmetric, semi-negative definite.

2. Equation (1.1) has n_1 conservation laws. That is, there is a partition $n = n_1 + n_2$, $n_1, n_2 \geq 0$, such that

$$r(w) = \begin{pmatrix} 0_{n_1\times 1} \\ r_2(w) \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.11}$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, and $(r_2)_{w_2} \in \mathbb{R}^{n_2\times n_2}$ is nonsingular. Here $(r_2)_{w_2}$ denotes the Jacobian matrix of r_2 with respect to w_2, etc.

3. There is a diffeomorphism $\varphi \to w$ from an open set $\tilde{\mathbb{O}} \subset \mathbb{R}^n$ to $\mathbb{O}$ and a constant orthogonal matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^t B_{jk}(w(\varphi)) w_\varphi(\varphi) P = \begin{pmatrix} 0_{n_3\times n_3} & 0_{n_3\times n_4} \\ 0_{n_4\times n_3} & B_{jk}^* \end{pmatrix}, \qquad 1 \leq j,k \leq m. \tag{1.12}$$

Here $n_3, n_4 \geq 0$ are two constants such that $n_3 + n_4 = n$, and $\sum_{j,k=1}^m B_{jk}^*\xi_k\xi_j \in \mathbb{R}^{n_4\times n_4}$ is nonsingular (if $n_4 > 0$) for all $\varphi \in \tilde{\mathbb{O}}$ and all $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$.

4. [8] For $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$ let

$$A(\xi) = \sum_{j=1}^m f_j'(\bar{w})\xi_j, \qquad B(\xi) = \sum_{j,k=1}^m B_{jk}(\bar{w})\xi_k\xi_j. \tag{1.13}$$

Let $\mathbb{N}_1$ be the null space of $B(\xi)$ and $\mathbb{N}_2$ be the null space of $r'(\bar{w})$. Then for each ξ, $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of $A(\xi)$.

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to x:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{R}^m)}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^m)}. \tag{1.14}$$

With φ and P given in condition 3 of Assumption 1.1, we define

$$\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \equiv P^t \varphi(w), \tag{1.15}$$

where $\tilde{w}_1 \in \mathbb{R}^{n_3}$ and $\tilde{w}_2 \in \mathbb{R}^{n_4}$. For a positive integer l we use D_x^l to denote partial derivatives $(\partial/\partial x)^\alpha$ with a multi-index α such that $|\alpha| = l$. For $l = 1$ we simply write D_x. We cite the following results on the global existence and L^p decay rates from the author's previous work.

Theorem 1.2. *[11] Let $\bar{w}$ be a constant equilibrium state of (1.1), Assumption 1.1 be satisfied, $s > \frac{m}{2} + 1$ ($m \geq 1$) be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \bar{w}\|_s \leq \varepsilon$, the Cauchy problem (1.1), (1.2) has a unique global solution w. The solution satisfies $w - \bar{w} \in C([0,\infty); H^s(\mathbb{R}^m))$, $D_x w \in L^2([0,\infty); H^{s-1}(\mathbb{R}^m))$, $D_x\tilde{w}_2(w) \in L^2([0,\infty); H^s(\mathbb{R}^m))$, $r(w) \in L^2([0,\infty); H^s(\mathbb{R}^m))$, and*

$$\begin{aligned} \sup_{t\geq 0} \|w - \bar{w}\|_s^2(t) + \int_0^\infty (\|D_x w\|_{s-1}^2 + \|D_x \tilde{w}_2(w)\|_s^2 + \|r_2(w)\|_s^2)(t)\,dt \\ \leq C\|w_0 - \bar{w}\|_s^2, \end{aligned} \tag{1.16}$$

where $C > 0$ is a constant.

Theorem 1.3. *[12] Let $\bar{w}$ be a constant equilibrium state of (1.1), and Assumption 1.1 be true. Let $m \geq 2$, $s > \frac{m}{2} + 1$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the solution of (1.1), (1.2) given in Theorem 1.2 has the following estimates for $t \geq 0$:*

$$\|D_x^l (w - \bar{w})\|(t) \leq C\delta_0 (t+1)^{-\frac{m}{4} - \frac{l}{2}} \tag{1.17}$$

for $0 \leq l \leq s-2$, and

$$\|D_x^l r_2(w)\|(t) \leq C\delta_0 (t+1)^{-\frac{m}{4} - \frac{l+1}{2}} \tag{1.18}$$

for $0 \leq l \leq s-4$. Here $C > 0$ in (1.17) and (1.18) is a constant.

Recall Gagliardo-Nirenberg inequality [7]: There is a constant $C > 0$ such that for $g \in H^k(\mathbb{R}^m)$,

$$\|D_x^l g\|_{L^p} \leq C\|D_x^k g\|^{\theta}\|g\|^{1-\theta}, \tag{1.19}$$

where $0 \leq l \leq k$, $p \in [2, \infty]$, and $\theta = [l + m(1/2 - 1/p)]/k \leq 1$ ($\theta < 1$ if $p = \infty$). Applying (1.19) to $g = w - \bar{w}$ with $k = s - 2$, and to $g = r_2(w)$ with $k = s - 4$, we have the following corollary of Theorem 1.3:

Corollary 1.4. *[12] Under the assumptions and notations of Theorem 1.3, the solution of (1.1), (1.2) has the following L^p estimates with $p \geq 2$: For $t \geq 0$,*

$$\|D_x^l(w - \bar{w})\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l}{2}} \tag{1.20}$$

for $0 \leq l \leq s - 2 - m(1/2 - 1/p)$, and

$$\|D_x^l r_2(w)\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}} \tag{1.21}$$

for $0 \leq l \leq s-4-m(1/2-1/p)$. If $p = \infty$, we further require $l \neq s-2-m/2$ for (1.20), and $l \neq s - 4 - m/2$ for (1.21). Here $C > 0$ is a constant.

The time asymptotic solution w^* of (1.1), (1.2) is the solution of the corresponding linear system with the same initial data:

$$w_t^* + \sum_{j=1}^{m} f_j'(\bar{w})w_{x_j}^* = \sum_{j,k=1}^{m} B_{jk}(\bar{w})w_{x_k x_j}^* + r'(\bar{w})(w^* - \bar{w}), \tag{1.22}$$

$$w^*(x, 0) = w_0(x). \tag{1.23}$$

The main result of this paper is the following theorem:

Theorem 1.5. *Under the assumptions and notations of Theorem 1.3, the solution of (1.1), (1.2) is time-asymptotically approximated by the solution of (1.22), (1.23), with the following L^p estimates for $p \geq 2$:*

$$\begin{aligned} &\|D_x^l(w - w^*)\|_{L^p}(t) \\ &\quad\leq C\delta_0^2(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}} \begin{cases} [1 + \ln(t+1)] & \text{if } m = 2 \\ 1 & \text{if } m > 2 \end{cases}, \quad t \geq 0, \end{aligned} \tag{1.24}$$

where $0 \leq l \leq s - 2 - m(1/2 - 1/p)$, ($l \neq s - 2 - m/2$ if $p = \infty$,) and $C > 0$ is a constant.

Remark 1.6. Comparing (1.24) with (1.20), not only the convergence rate is faster by $(t+1)^{-\frac{1}{2}}$ (or by $(t+1)^{-\frac{1}{2}}[1+\ln(t+1)]$ if $m=2$), the amplitude of the error is in the order of δ_0^2, smaller than δ_0. □

For the special case of (1.3), Assumption 1.1 and Theorem 1.5 are simplified to the following:

Assumption 1.7. *1. There exists a strictly convex entropy function η of w such that in $\mathbb{O}$, $\eta'' f_j'$ are symmetric for $1 \le j \le m$, $(\eta'' B_{jk})^t = \eta'' B_{kj}$ for $1 \le j,k \le m$, and $\eta'' \sum_{j,k=1}^m B_{jk}\xi_k\xi_j$ is symmetric, semi-positive definite for all $\xi = (\xi_1,\dots,\xi_m)^t \in \mathbb{S}^{m-1}$.*

2. There is a diffeomorphism $\varphi \to w$ from an open set $\tilde{\mathbb{O}} \subset \mathbb{R}^n$ to $\mathbb{O}$, such that

$$B_{jk}(w(\varphi))w_\varphi(\varphi) = \operatorname{diag}(0_{n_3\times n_3}, B_{jk}^*), \qquad 1 \le j,k \le m,$$

where $B_{jk}^* \in \mathbb{R}^{n_4\times n_4}$, n_3 and $n_4 = n - n_3 > 0$ are constants, and $\sum_{j,k=1}^m B_{jk}^*\xi_k\xi_j$ is nonsingular for all $\varphi \in \tilde{\mathbb{O}}$ and $\xi = (\xi_1,\dots,\xi_m)^t \in \mathbb{S}^{m-1}$.

3. For each $\xi = (\xi_1,\dots,\xi_m)^t \in \mathbb{S}^{m-1}$, let $A(\xi) = \sum_{j=1}^m f_j'(\bar{w})\xi_j$ and $B(\xi) = \sum_{j,k=1}^m B_{jk}(\bar{w})\xi_k\xi_j$. Then the null space of $B(\xi)$ contains no eigenvectors of $A(\xi)$.

Theorem 1.8. *Let $\bar{w}$ be a constant state and Assumption 1.7 be true. Let $m \ge 2$, $s > \frac{m}{2}+1$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m)\cap L^1(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0-\bar{w}\|_s + \|w_0-\bar{w}\|_{L^1} \le \varepsilon$, the Cauchy problem (1.3), (1.2) has a unique solution for $t \ge 0$, satisfying*

$$\|D_x^l(w-\bar{w})\|_{L^p}(t) \le C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l}{2}}, \tag{1.25}$$

(1.26)

$$\|D_x^l(w-w^*)\|_{L^p}(t) \le C\delta_0^2(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}}\begin{cases}[1+\ln(t+1)] & \text{if } m=2\\ 1 & \text{if } m>2\end{cases},$$

where $p \ge 2$, $0 \le l \le s-2-m(1/2-1/p)$, ($l \ne s-2-m/2$ if $p=\infty$,) $C>0$ is a constant, and w^* satisfies

$$\begin{cases} w_t^* + \displaystyle\sum_{j=1}^m f_j'(\bar{w})w_{x_j}^* = \sum_{j,k=1}^m B_{jk}(\bar{w})w_{x_kx_j}^*,\\ w^*(x,0) = w_0(x).\end{cases}$$

We note that in Assumption 1.7, the orthogonal matrix P in condition 3 of Assumption 1.1 has been taken as the identity here without loss of generality. This is because there is no intertwining of second derivatives and the lower order term. Besides, since there is no contribution from the lower order term to dissipation, we need to explicitly set $n_4 > 0$. Equation (1.25) is a consequence of Corollary 1.4, listed for comparison with (1.26).

Similarly, for the special case of (1.6), Assumption 1.1 and Theorem 1.5 are reduced to Assumption 1.9 and Theorem 1.10 as follows.

Assumption 1.9. *1. There exists a strictly convex entropy function η of w in $\mathbb{O}$ such that $\eta'' f_j'$, $1 \le j \le m$, are symmetric in $\mathbb{O}$, and $\eta'' r'$ is symmetric, semi-negative definite on $\mathbb{E}$.*

2. Equation (1.6) has n_1 conservation laws, i.e., there is a partition $n = n_1 + n_2$, $n_1, n_2 > 0$, such that

$$r(w) = \begin{pmatrix} 0_{n_1\times 1} \\ r_2(w) \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, and $(r_2)_{w_2}$ is nonsingular.

3. The null space of $r'(\bar{w})$ contains no eigenvectors of $A(\xi) = \sum_{j=1}^m f_j'(\bar{w})\xi_j$ for all $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$.

Theorem 1.10. *Let $\bar{w}$ be a constant equilibrium state and Assumption 1.9 be true. Let $m \ge 2$, $s > \frac{m}{2}+1$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \le \varepsilon$, the Cauchy problem (1.6), (1.2) has a unique solution for $t \ge 0$, satisfying*

$$\|D_x^l(w-\bar{w})\|_{L^p}(t) \le C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l}{2}} \tag{1.27}$$

(1.28)

$$\|D_x^l(w-w^*)\|_{L^p}(t) \le C\delta_0^2(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}} \begin{cases} [1+\ln(t+1)] & \text{if } m=2 \\ 1 & \text{if } m>2 \end{cases},$$

where $p \ge 2$, $0 \le l \le s-1-m(1/2-1/p)$, ($l \ne s-1-m/2$ if $p=\infty$,) $C > 0$ is a constant, and w^* is the solution to

$$\begin{cases} w_t^* + \displaystyle\sum_{j=1}^m f_j'(\bar{w}) w_{x_j}^* = r'(\bar{w})(w^*-\bar{w}), \\ w^*(x,0) = w_0(x). \end{cases}$$

Similar to Assumption 1.7, we have set $n_2 > 0$ explicitly in Assumption 1.9 since there is no dissipation from viscosity. We also set $n_1 > 0$ due to the fact that the case $n_1 = 0$ leads to better decay rates while physical models dictate $n_1 > 0$. In the conclusions of Theorem 1.10 we allow $0 \le l \le s - 1 - m(1/2 - 1/p)$ rather than $0 \le l \le s - 2 - m(1/2 - 1/p)$. This is due to the lack of second derivatives in (1.6). The justification for such better regularity has been done in [12] for (1.27), and will be done in Section 4 for (1.28).

For the special case of hyperbolic-parabolic conservation laws (1.3), a result similar to Theorem 1.8 has been obtained in [4], see Theorem 4.5 therein. On the other hand, for the special case of hyperbolic balance laws (1.6), something similar to (1.28) has not been proved. Comparing with (1.3), the main difficulty of (1.1), or even of (1.6), lies on the fact that the system is one of balance laws rather than conservation laws. This makes it impossible to obtain better rates in the time asymptotic study by simple integration by parts when applying Duhamel's principle. Indeed, we need to use the special structure of r, (1.11) in Assumption 1.1, to obtain the extra rate needed. In the case of one space dimension, at least for the special case (1.6), this is done by spectral analysis using Kato's perturbation theory [3], see [13] for details. The theory, however, does not apply to multi-space dimensions in a straightforward manner. Our new approach in this paper is a bootstrap strategy: We start with an estimate from energy method that applies to the general case (obtained in [12]). Then in the Fourier space we consider the perturbation around the origin. The perturbation is expressed via Duhamel's principle in terms of solutions to linear systems (rather than in terms of spectrum). After iteration, the special structure (1.11) is incorporated in, and the needed accuracy is obtained, see Section 3 for details.

As applications we comment that under physical assumptions, the Navier-Stokes equations (1.4) satisfy Assumption 1.7, hence Theorem 1.8 applies. Similarly, Assumption 1.9 is satisfied by the Euler equations with damping (1.7), and Theorem 1.10 applies. As an important physical model, the system of vibrational non-equilibrium flow (1.8) contains both viscosity and relaxation, hence fits the general framework (1.1). Under physical assumptions, it has been shown that Assumption 1.1 is satisfied [11]. Therefore, Theorem 1.5 applies. Below we give a precise description of this application.

Based on the relation among thermodynamic variables for this model, as discussed earlier, we introduce the following notations:

$$p = p(v, e_1) = \tilde{p}(v, T_1), \quad T_1 = T_1(v, e_1), \quad e_2 = \omega(T_2). \tag{1.29}$$

Note that a state is an equilibrium state if and only if $T_2 = T_1$. Thus the equilibrium manifold is

$$\mathbb{E} = \{T_2 = T_1\} \cap \mathbb{O}, \tag{1.30}$$

and e_2 satisfies

$$e_2^E = \omega(T_1), \quad \bar{e}_2 = \omega(\bar{T}_1), \tag{1.31}$$

where $\bar{e}_2$ is the value of e_2 at the constant equilibrium state $\bar{w}$, etc. Without loss of generality, we take $\bar{u} = 0 \in \mathbb{R}^m$. Noting

$$w = (\rho, \rho u^t, \rho E, \rho e_2)^t = (\rho, \rho u^t, \rho(e + \frac{1}{2}|u|^2), \rho e_2)^t, \tag{1.32}$$

we have

$$\bar{w} = (\bar{\rho}, 0_{1\times m}, \bar{\rho}\bar{e}, \bar{\rho}\bar{e}_2)^t. \tag{1.33}$$

The physical assumptions to be imposed on (1.8) are

$$\begin{aligned} \tilde{p}_v = \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \quad T_{1e_1} = \frac{\partial}{\partial e_1} T_1(v, e_1) > 0, \\ p_{e_1} = \frac{\partial}{\partial e_1} p(v, e_1) \neq 0, \quad \omega'(T) > 0. \end{aligned} \tag{1.34}$$

The following proposition has been obtained in [11].

Proposition 1.11. *[11] Let (1.34) be true, and the dissipation parameters in (1.8) at $\bar{w}$ satisfy*

$$\bar{\kappa} > 0, \quad \bar{\nu} \geq 0, \quad \bar{\mu} > 0, \quad 2\bar{\mu} + \bar{\mu}' > 0. \tag{1.35}$$

Then (1.8) satisfies Assumption 1.1 in a small neighborhood $\mathbb{O}$ of $\bar{w}$.

The time asymptotic solution $w^* = (\rho^*, \rho^* u^{*t}, \rho^* E^*, \rho^* e_2^*)^t = (\rho^*, \rho^* u^{*t}, \rho^*(e^* + \frac{1}{2}|u^*|^2), \rho^* e_2^*)^t$ satisfies the linear system

$$\begin{cases} \rho_t^* + \bar{\rho}\,\mathrm{div}\, u^* = 0, \\ \bar{\rho} u_t^* + \nabla p^* = 2\bar{\mu}\,\mathrm{div}\,\mathscr{P}^* + \bar{\mu}'\nabla(\mathrm{div}\, u^*), \\ \bar{\rho} E_t^* + \bar{p}\,\mathrm{div}\, u^* = \bar{\kappa}\triangle T_1^* + \bar{\nu}\bar{\rho}\triangle e_2^*, \\ \bar{\rho} e_{2t}^* = \bar{\nu}\bar{\rho}\triangle e_2^* + r_2'(\bar{w})(w^* - \bar{w}), \end{cases} \tag{1.36}$$

where $r_2(w) = \rho(e_2^E - e_2)/\tau$. Here we use the accent "$*$" to denote a function evaluated at the state w^*, e.g., $p^* = p(1/\rho^*, e_1^*)$, etc. Similarly, we use the bar accent to denote a function evaluated at the constant equilibrium state $\bar{w}$.

The explicit expression of $r_2'(\bar{w})(w^* - \bar{w})$ is

$$-\frac{\omega'(\bar{T}_1)}{\bar{\tau}}\{[\bar{v}\bar{T}_{1v} + \bar{e}_1\bar{T}_{1e_1} - \frac{\omega(\bar{T}_1)}{\omega'(\bar{T}_1)}](\rho^* - \bar{\rho}) - \bar{T}_{1e_1}(\rho^* E^* - \bar{\rho}\bar{e}) + [\bar{T}_{1e_1} + \frac{1}{\omega'(\bar{T}_1)}][\rho^* e_2^* - \bar{\rho}\omega(\bar{T}_1)]\}, \tag{1.37}$$

see [11] for $r'(\bar{w})$.

Proposition 1.11 implies that Theorem 1.2, Corollary 1.4 and Theorem 1.5 all apply. In particular, we have the following:

Theorem 1.12. *Let $\bar{\rho}, \bar{e}_1 > 0$ be constants, $\bar{T}_1 = T_1(1/\bar{\rho}, \bar{e}_1)$, $\bar{e}_2 = \omega(\bar{T}_1)$ and $\bar{e} = \bar{e}_1 + \bar{e}_2$. Let (1.34) and (1.35) be true, $s > \frac{m}{2} + 1$ be an integer with $m \geq 2$, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ for*

$$w_0(x) \equiv (\rho_0, \rho_0 u_0^t, \rho_0 E_0, \rho_0 e_{20})^t(x) = (\rho_0, \rho_0 u_0^t, \rho_0(e_0 + \frac{1}{2}|u_0|^2), \rho_0 e_{20})^t(x)$$

and $\bar{w}$ given in (1.33), respectively. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the Cauchy problem (1.8), (1.2) has a unique solution for $t \geq 0$. The solution satisfies the following L^p properties:

$$\|D_x^l(\rho - \bar{\rho}, \rho u^t, \rho E - \bar{\rho}\bar{e}, \rho e_2 - \bar{\rho}\bar{e}_2)\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l}{2}}, \tag{1.38}$$

$$\begin{aligned} &\|D_x^l(\rho - \rho^*, \rho u^t - \rho^* u^{*t}, \rho E - \rho^* E^*, \rho e_2 - \rho^* e_2^*)\|_{L^p}(t) \\ &\quad \leq C\delta_0^2(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}} \begin{cases} [1 + \ln(t+1)] & \text{if } m = 2 \\ 1 & \text{if } m > 2 \end{cases}. \end{aligned} \tag{1.39}$$

Here $p \geq 2$, $0 \leq l \leq s - 2 - m(1/2 - 1/p)$, ($l \neq s - 2 - m/2$ if $p = \infty$,) C is a constant, and w^* is the solution of (1.36) with initial condition $w^*(x, 0) = w_0(x)$.

To finish this section we outline the plan of the paper: Section 2 is for preliminaries, Section 3 is for key estimates of linear systems, and Section 4 is for the proof of Theorem 1.5.

Throughout this paper, we use C to denote a universal positive constant. We use the bar accent for the value of a variable taken at the constant equilibrium state $\bar{w}$, e.g., $\bar{\varphi} \equiv \varphi(\bar{w})$, etc. We also use the accent "*" to denote variables at the state w^*, say, $\varphi^* \equiv \varphi(w^*)$.

2. Preliminaries

From Theorem 1.2, by observing the integral in (1.16) we conclude that $\tilde{w}_2$ is the part of better regularity, while r_2 is the part with faster decay rate in the solution. To separate the solution according to time decay rates we introduce a new variable ψ using the notations in (1.11):

$$\psi = \psi(w) = \begin{pmatrix} w_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (w), \tag{2.1}$$

where $\psi_1 = w_1 \in \mathbb{R}^{n_1}$ and $\psi_2 = r_2 \in \mathbb{R}^{n_2}$. Under condition 2 of Assumption 1.1, ψ is a diffeomorphism, with the Jacobian matrices

$$\psi_w = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ (r_2)_{w_1} & (r_2)_{w_2} \end{pmatrix}, \quad w_\psi = \psi_w^{-1} = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ -(r_2)_{w_2}^{-1}(r_2)_{w_1} & (r_2)_{w_2}^{-1} \end{pmatrix}. \tag{2.2}$$

Consider the perturbation $\tilde{\psi}$ of $\bar{\psi}$:

$$\tilde{\psi} = \psi - \bar{\psi} = \begin{pmatrix} w_1 - \bar{w}_1 \\ r_2(w) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}, \tag{2.3}$$

noting $r_2(\bar{w}) = 0$. Multiplying (1.1) from the left by ψ_w we obtain the equation for $\tilde{\psi}$:

$$\tilde{\psi}_t + \sum_{j=1}^{m} \psi_w f_j' w_\psi \tilde{\psi}_{x_j} = \sum_{j,k=1}^{m} \psi_w \left[B_{jk} w_\psi \tilde{\psi}_{x_k}\right]_{x_j} + \psi_w r. \tag{2.4}$$

We linearize (2.4) and symmetrize it by multiplying from the left a constant matrix $\tilde{A}_0$ to be defined in (2.6). These give us the following equation equivalent to (1.1):

$$\tilde{A}_0 \tilde{\psi}_t + \sum_{j=1}^{m} \tilde{A}_j \tilde{\psi}_{x_j} = \sum_{j,k=1}^{m} \tilde{B}_{jk} \tilde{\psi}_{x_k x_j} + \tilde{L}\tilde{\psi} + \tilde{R}, \tag{2.5}$$

where

$$\begin{aligned} \tilde{A}_0 = (w_\psi^t \eta'' w_\psi)(\bar{w}), \quad & \tilde{A}_j = (w_\psi^t \eta'' f_j' w_\psi)(\bar{w}), \\ \tilde{B}_{jk} = (w_\psi^t \eta'' B_{jk} w_\psi)(\bar{w}), \quad & \tilde{L} = (w_\psi^t \eta'' r' w_\psi)(\bar{w}). \end{aligned} \tag{2.6}$$

To express $\tilde{R}$ we introduce the following notations using the $n = n_1 + n_2$ partition for vectors in $\mathbb{R}^n$:

$$f_j = \begin{pmatrix} f_{j1} \\ f_{j2} \end{pmatrix}, \tag{2.7}$$

$$\begin{aligned} \tilde{f}_{j1} &= -[f_{j1}(w) - f_{j1}(\bar{w}) - (f'_{j1} w_\psi)(\bar{w})\tilde{\psi}] \\ &= -[f_{j1}(w) - f_{j1}(\bar{w}) - \overline{(f_{j1})_\psi}\tilde{\psi}] = O(1)|\tilde{\psi}|^2, \end{aligned} \tag{2.8}$$

$$\sum_{k=1}^m [B_{jk}(w)w_{x_k} - (B_{jk}w_\psi)(\bar{w})\psi_{x_k}] \equiv \begin{pmatrix} b_{j1} \\ b_{j2} \end{pmatrix}, \tag{2.9}$$

$$\sum_{j=1}^m [(\psi_w f'_j w_\psi)(\bar{w}) - (\psi_w f'_j w_\psi)(w)]\psi_{x_j} \equiv \begin{pmatrix} R_{11} \\ R_{12} \end{pmatrix}, \tag{2.10}$$

$$\sum_{j,k=1}^m \left\{\psi_w(w)[B_{jk}(w)w_{x_k}]_{x_j} - (\psi_w B_{jk} w_\psi)(\bar{w})\psi_{x_k x_j}\right\} \equiv \begin{pmatrix} R_{21} \\ R_{22} \end{pmatrix}, \tag{2.11}$$

$$R_{32} \equiv [(r_2)_{w_2}(w) - (r_2)_{w_2}(\bar{w})]r_2(w) = O(1)|w - \bar{w}|^2, \tag{2.12}$$

where $f_{j1}, \tilde{f}_{j1}, b_{j1}, R_{11}, R_{21} \in \mathbb{R}^{n_1}$ and $f_{j2}, b_{j2}, R_{12}, R_{22}, R_{32} \in \mathbb{R}^{n_2}$. Applying (2.2), we express $\tilde{R}$ in (2.5) as

$$\tilde{R} = \tilde{A}_0 \sum_{j=1}^m \begin{pmatrix} \tilde{f}_{j1} + b_{j1} \\ 0_{n_2 \times 1} \end{pmatrix}_{x_j} + \tilde{A}_0 \begin{pmatrix} 0_{n_1 \times 1} \\ R_{12} + R_{22} + R_{32} \end{pmatrix}. \tag{2.13}$$

The corresponding homogeneous equation of (2.5) is a linear system for $\tilde{\psi}$ with constant coefficients. The coefficient matrices possess nice properties, which we cite from Lemma 2.8 of [11]:

Lemma 2.1. *[11] Under conditions 1, 2 and 4 of Assumption 1.1, the coefficient matrices of (2.5) have the following properties.*

(i) *$\tilde{A}_0$ and $\tilde{L}$ are real, symmetric. $\tilde{A}_0$ is positive definite while $\tilde{L}$ is semi-negative definite.*

(ii) *For $\xi = (\xi_1, \dots \xi_m)^t \in \mathbb{S}^{m-1}$, let*

$$\tilde{A}(\xi) = \sum_{j=1}^m \tilde{A}_j \xi_j, \quad \tilde{B}(\xi) = \sum_{j,k=1}^m \tilde{B}_{jk}\xi_k\xi_j. \tag{2.14}$$

Then $\tilde{A}(\xi)$ is real, symmetric, and $\tilde{B}(\xi)$ is real, symmetric and semi-positive definite. They satisfy $\tilde{A}(-\xi) = -\tilde{A}(\xi)$ and $\tilde{B}(-\xi) = \tilde{B}(\xi)$.

(iii) *If* $\zeta \in \mathbb{R}^n \setminus \{0\}$ *and* $\tilde{B}(\xi)\zeta = \tilde{L}\zeta = 0$ *for some* $\xi \in \mathbb{S}^{m-1}$, *then* $\lambda \tilde{A}_0 \zeta + \tilde{A}(\xi)\zeta \neq 0$ *for any* $\lambda \in \mathbb{R}$.

We may use Fourier transform to study a linear system with constant coefficients. We use the hat accent to denote the transform with respect to the space variable. Thus

$$\begin{aligned} \hat{\tilde{\psi}}(\xi,t) &= \int_{\mathbb{R}^m} \tilde{\psi}(x,t) e^{-ix\cdot\xi}\, dx, \\ \tilde{\psi}(x,t) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \hat{\tilde{\psi}}(\xi,t) e^{ix\cdot\xi}\, d\xi. \end{aligned} \tag{2.15}$$

Taking Fourier transform of (2.5) and with the notations in (2.14), we have

$$\tilde{A}_0 \hat{\tilde{\psi}}_t + [i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2 \tilde{B}(\tilde{\xi}) - \tilde{L}]\hat{\tilde{\psi}} = \hat{\tilde{R}}, \tag{2.16}$$

where $\tilde{\xi} = \xi/|\xi|$.

In the analysis of next two sections we need some details of the coefficient matrices. Here we cite results from [10, 11]:

Lemma 2.2. *[10] If* $(\eta'' r')(\bar{w})$ *is symmetric then*

$$[\eta_{w_2 w_2}(r_2)_{w_1}](\bar{w}) = [(r_2)^t_{w_2} \eta_{w_1 w_2}](\bar{w}), \tag{2.17}$$
$$[\eta_{w_2 w_2}(r_2)_{w_2}](\bar{w}) = [(r_2)^t_{w_2} \eta_{w_2 w_2}](\bar{w}). \tag{2.18}$$

If $(\eta'' r')(\bar{w})$ *is semi-negative definite as well then* $[\eta_{w_2 w_2}(r_2)_{w_2}](\bar{w})$ *is semi-negative definite.*

Lemma 2.3. *[10, 11] If* $(\eta'' r')(\bar{w})$ *is symmetric then*

$$\tilde{A}_0 = \mathrm{diag}(A_{01}, A_{0,2}), \tag{2.19}$$
$$A_{01} = [\eta_{w_1 w_1} - \eta_{w_2 w_1}(r_2)^{-1}_{w_2}(r_2)_{w_1}](\bar{w}) \in \mathbb{R}^{n_1 \times n_1}, \tag{2.20}$$
$$A_{02} = \{[(r_2)^{-1}_{w_2}]^t \eta_{w_2 w_2}(r_2)^{-1}_{w_2}\}(\bar{w}) \in \mathbb{R}^{n_2 \times n_2}, \tag{2.21}$$
$$\tilde{L} = \mathrm{diag}(0_{n_1 \times n_1}, [\eta_{w_2 w_2}(r_2)^{-1}_{w_2}](\bar{w})). \tag{2.22}$$

In the study of the nonlinear source $\tilde{R}$ in (2.16), we need the following from condition 3 of Assumption 1.1:

(2.23)

$$B_{jk}(w) w_{x_k} = P[P^t B_{jk}(w) w_\varphi P \tilde{w}_{x_k}] = P \begin{pmatrix} 0_{n_3 \times 1} \\ B^*_{jk}(w)\tilde{w}_{2x_k} \end{pmatrix}, \quad 1 \le j,k \le m,$$

where we have used notations in (1.15).

Our study also needs some tools from analysis. They are Moser-type calculus inequalities, which can be obtained from Gagliardo-Nirenberg inequality. They can be found in papers studying energy estimates in multi-space dimensions, e.g. [4]. Here we follow the formulation in [10] to summarize them as Lemmas 2.4-2.6.

Lemma 2.4. *[7] If $w \in H^s(\mathbb{R}^m)$ with $s > m/2$ then*

$$\|w\|_{L^\infty} \le C\|D_x^s w\|^\alpha \|w\|^{1-\alpha} \le C\|w\|_s, \tag{2.24}$$

where $\alpha = m/(2s) < 1$ and $C > 0$ is a constant depending only on m and s.

Lemma 2.5. *Let g be a smooth function of w in a neighborhood of $\bar{w}$. If $w - \bar{w} \in H^s(\mathbb{R}^m)$ with $\|w - \bar{w}\|_s \le \varepsilon$ and $s > m/2$, then*

$$\|D_x^l g\| \le C\|D_x^l w\|, \quad 1 \le l \le s, \tag{2.25}$$

where $C > 0$ is a constant depending only on m, s and ε.

Lemma 2.6. *If $D_x g, \tilde{g} \in H^{l-1}(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ then*

$$\|D_x^l(g\tilde{g}) - gD_x^l\tilde{g}\| \le C(\|D_x g\|_{L^\infty}\|D_x^{l-1}\tilde{g}\| + \|D_x^l g\|\|\tilde{g}\|_{L^\infty}), \tag{2.26}$$

where $C > 0$ is a constant depending only on m and l.

To finish this section we cite Theorem 4.1 in [12]. It gives L^2 decay rates of the solution to (1.1), (1.2), obtained by weighted energy method. Although the rates are not optimal, the theorem supplements Theorem 1.3 in providing rates for $D_x^{s-1}w$ and other higher derivatives, not available from Theorem 1.3.

Theorem 2.7. *[12] Let $\bar{w}$ be a constant equilibrium state of (1.1), and Assumption 1.1 be true. Let $m \ge 2$, $s > \frac{m}{2} + 1$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \bar{w}\|_s \le \varepsilon$, the solution of (1.1), (1.2) given in Theorem 1.2 has the following estimates:*

$$\|D_x^l(w - \bar{w})\|(t) \le C\|w_0 - \bar{w}\|_s(t+1)^{-\frac{l}{2}}, \quad t \ge 0, \quad 0 \le l \le s, \tag{2.27}$$

$$\begin{aligned} &\int_0^\infty \sum_{l=0}^{s-1}(t+1)^l\|D_x^{l+1}w\|_{s-l-1}^2(t)\,dt \\ &\qquad + \int_0^\infty \sum_{l=0}^{s}(t+1)^l(\|D_x^{l+1}\tilde{w}_2\|_{s-l}^2 + \|D_x^l r_2(w)\|_{s-l}^2)(t)\,dt \\ &\le C\|w_0 - \bar{w}\|_s^2. \end{aligned} \tag{2.28}$$

3. Estimates for linear system

Motivated by (2.16), the equation for $\tilde{\psi}$, and the specific form of $\tilde{R}$ in (2.13), we study the following linear system with particular source terms in the Fourier space:

$$\tilde{A}_0\hat{\psi}_t + [i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2\tilde{B}(\tilde{\xi}) - \tilde{L}]\hat{\psi} = i\hat{H}\xi + \hat{h}, \tag{3.1}$$

where $\tilde{A}_0, \tilde{A}, \tilde{B}$ and $\tilde{L}$ are defined in (2.6) and (2.14), $\tilde{\xi} = \xi/|\xi| \in \mathbb{S}^{m-1}$, $H = H(x,t) \in \mathbb{R}^{n\times m}$, $h = h(x,t) \in \mathbb{R}^n$, and

$$h = \begin{pmatrix} 0_{n_1\times 1} \\ h_2 \end{pmatrix}. \tag{3.2}$$

The goal of this section is to obtain an optimal estimate of the solution to (3.1). We start with a lemma from [12], obtained by energy method:

Lemma 3.1. *[12] Under conditions 1, 2 and 4 of Assumption 1.1, the solution of (3.1) with h satisfying (3.2) has the following estimate: For $\xi \in \mathbb{R}^m$, $t \geq 0$,*

$$\begin{aligned} |\hat{\psi}(\xi,t)|^2 \leq{}& Ce^{-2c_1\varrho(|\xi|)t}|\hat{\psi}(\xi,0)|^2 \\ &+ C\int_0^t e^{-2c_1\varrho(|\xi|)(t-\tau)}[(1+|\xi|^2)|\hat{H}(\xi,\tau)|^2 + |\hat{h}_2(\xi,\tau)|^2]\,d\tau, \end{aligned} \tag{3.3}$$

where C and c_1 are positive constants, and

$$\varrho(r) = \frac{r^2}{1+r^2}. \tag{3.4}$$

The corresponding homogeneous equation of (3.1) is

$$\hat{\psi}_t = \tilde{E}(\xi)\hat{\psi}, \tag{3.5}$$

where

$$\tilde{E}(\xi) = \tilde{A}_0^{-1}[-i|\xi|\tilde{A}(\tilde{\xi}) - |\xi|^2\tilde{B}(\tilde{\xi}) + \tilde{L}]. \tag{3.6}$$

Solving (3.5) we have

$$\hat{\psi}(\xi,t) = e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0). \tag{3.7}$$

Applying Lemma 3.1 to (3.5), from (3.3) and (3.7) we have

$$|e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0)| \le Ce^{-c_1\varrho(|\xi|)t}|\hat{\psi}(\xi,0)|. \tag{3.8}$$

Noting $|\hat{\psi}(\xi,0)|$ is arbitrary, we further have

$$|e^{t\tilde{E}(\xi)}| \le Ce^{-c_1\varrho(|\xi|)t}, \tag{3.9}$$

where C and c_1 are positive constants.

The following lemma is our key estimate, which improves (3.8) for bounded ξ when the initial data have the special structure (3.2).

Lemma 3.2. *Let $h = h(x) \in \mathbb{R}^n$ satisfy (3.2). Under conditions 1, 2 and 4 of Assumption 1.1, for $|\xi| \le 1$, we have*

$$|e^{t\tilde{E}(\xi)}\hat{h}(\xi)| \le C(|\xi|e^{-c_2|\xi|^2t} + e^{-c_3t})|\hat{h}_2(\xi)|, \tag{3.10}$$

where $c_2, c_3 > 0$ are constants.

Proof. Let

$$\hat{\psi}(\xi,t) = e^{t\tilde{E}(\xi)}\hat{h}(\xi).$$

Then $\hat{\psi}$ satisfies

$$\begin{cases} \hat{\psi}_t = \tilde{E}(\xi)\hat{\psi}, \\ \hat{\psi}(\xi,0) = \hat{h}(\xi). \end{cases} \tag{3.11}$$

We write

$$\hat{\psi} = \hat{\psi}^{\mathrm{I}} + \hat{\psi}^{\mathrm{II}}, \tag{3.12}$$

where

$$\begin{aligned} &\hat{\psi}^{\mathrm{I}} = \hat{\psi}^{\mathrm{I}}(\xi,t) = e^{t\tilde{A}_0^{-1}\tilde{L}}\hat{h}(\xi), \\ &\hat{\psi}^{\mathrm{II}} = \hat{\psi}^{\mathrm{II}}(\xi,t) = \hat{\psi} - \hat{\psi}^{\mathrm{I}}. \end{aligned} \tag{3.13}$$

Note that $\hat{\psi}^{\mathrm{I}}$ satisfies

$$\begin{cases} \hat{\psi}^{\mathrm{I}}_t = \tilde{A}_0^{-1}\tilde{L}\hat{\psi}^{\mathrm{I}}, \\ \hat{\psi}^{\mathrm{I}}(\xi,0) = \hat{h}(\xi). \end{cases} \tag{3.14}$$

From (3.6), (3.11), (3.13) and (3.14) we conclude that $\hat{\psi}^{\mathrm{II}}$ satisfies

$$\begin{cases} \hat{\psi}^{\mathrm{II}}_t = \tilde{E}(\xi)\hat{\psi}^{\mathrm{II}} - \tilde{A}_0^{-1}[i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2\tilde{B}(\tilde{\xi})]\hat{\psi}^{\mathrm{I}}, \\ \hat{\psi}^{\mathrm{II}}(\xi,0) = 0. \end{cases} \tag{3.15}$$

To estimate $\hat{\psi}^{\mathrm{I}}$ we consider the coefficient matrix in (3.14). From (2.19), (2.22), (2.21) and (2.18), we have

$$\tilde{A}_0^{-1}\tilde{L} = \mathrm{diag}(0_{n_1\times n_1}, A_{02}^{-1}[\eta_{w_2w_2}(r_2)_{w_2}^{-1}](\bar{w})) = \mathrm{diag}(0_{n_1\times n_1}, (r_2)_{w_2}(\bar{w})). \tag{3.16}$$

From Lemma 2.2, $[\eta_{w_2w_2}(r_2)_{w_2}](\bar{w})$ is symmetric, semi-negative definite. It is nonsingular by conditions 1 and 2 of Assumption 1.1, hence negative definite. Since $\eta_{w_2w_2}(\bar{w})$ is symmetric, positive definite, we may choose $\eta^{\frac{1}{2}}_{w_2w_2}(\bar{w})$ that is symmetric, positive definite, so is $(\eta^{\frac{1}{2}}_{w_2w_2})^{-1}(\bar{w})$. Multiplying $[\eta_{w_2w_2}(r_2)_{w_2}](\bar{w})$ from the left and from the right by $(\eta^{\frac{1}{2}}_{w_2w_2})^{-1}(\bar{w})$, we conclude that $[\eta^{\frac{1}{2}}_{w_2w_2}(r_2)_{w_2}(\eta^{\frac{1}{2}}_{w_2w_2})^{-1}](\bar{w})$ is symmetric, negative definite. Therefore, it has spectral decomposition

$$[\eta^{\frac{1}{2}}_{w_2w_2}(r_2)_{w_2}(\eta^{\frac{1}{2}}_{w_2w_2})^{-1}](\bar{w}) = -\tilde{c}_1Q_1 - \cdots - \tilde{c}_{n_0}Q_{n_0}.$$

Here for some integer $1 \le n_0 \le n_2$, the constants $-\tilde{c}_1, \dots, -\tilde{c}_{n_0}$ are the distinct eigenvalues, with $0 < \tilde{c}_1 < \cdots < \tilde{c}_{n_0}$, and $Q_1, \dots, Q_{n_0}$ are the corresponding eigenprojections. Consequently, (3.16) implies that the spectral decomposition of $\tilde{A}_0^{-1}\tilde{L}$ is

$$\begin{gathered} \tilde{A}_0^{-1}\tilde{L} = -\tilde{c}_1\tilde{Q}_1 - \cdots - \tilde{c}_{n_0}\tilde{Q}_{n_0}, \\ \tilde{Q}_j = \mathrm{diag}(0_{n_1\times n_1}, (\eta^{\frac{1}{2}}_{w_2w_2})^{-1}(\bar{w})Q_j\eta^{\frac{1}{2}}_{w_2w_2}(\bar{w})), \quad 1 \le j \le n_0. \end{gathered} \tag{3.17}$$

To find $\hat{\psi}^{\mathrm{I}}$ we note that (3.17) gives us

$$e^{t\tilde{A}_0^{-1}\tilde{L}} = \mathrm{diag}(I_{n_1\times n_1}, 0_{n_2\times n_2}) + e^{-\tilde{c}_1t}\tilde{Q}_1 + \cdots + e^{-\tilde{c}_{n_0}t}\tilde{Q}_{n_0}. \tag{3.18}$$

Since h satisfies (3.2), from (3.13) and (3.18) we obtain

$$\hat{\psi}^{\mathrm{I}}(\xi,t) = e^{-\tilde{c}_1t}\tilde{Q}_1\hat{h}(\xi) + \cdots + e^{-\tilde{c}_{n_0}t}\tilde{Q}_{n_0}\hat{h}(\xi).$$

This implies

$$|\hat{\psi}^{\mathrm{I}}(\xi,t)| \le Ce^{-\tilde{c}_1t}|\hat{h}(\xi)| = Ce^{-\tilde{c}_1t}|\hat{h}_2(\xi)|. \tag{3.19}$$

To estimate $\hat{\psi}^{\mathrm{II}}$ we apply Duhamel's principle to (3.15), which gives us

$$\hat{\psi}^{\mathrm{II}}(\xi,t) = -\int_0^t e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}[i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2\tilde{B}(\tilde{\xi})]\hat{\psi}^{\mathrm{I}}(\xi,\tau)\,d\tau. \tag{3.20}$$

Applying (3.9) and (3.19) to (3.20), and noting (3.4) and $|\xi| \le 1$, we have

$$\begin{aligned}
|\hat{\psi}^{\mathrm{II}}(\xi,t)| &\le C\int_0^t |e^{(t-\tau)\tilde{E}(\xi)}||\xi||\hat{\psi}^{\mathrm{I}}(\xi,\tau)|\,d\tau \\
&\le C\int_0^t e^{-c_1\varrho(|\xi|)(t-\tau)}|\xi|e^{-\tilde{c}_1\tau}|\hat{h}_2(\xi)|\,d\tau \\
&\le C|\xi||\hat{h}_2(\xi)|\Big(\int_0^{\frac{t}{2}} e^{-c_1\frac{|\xi|^2}{2}\frac{t}{2}}e^{-\tilde{c}_1\tau}\,d\tau + \int_{\frac{t}{2}}^t e^{-\tilde{c}_1\tau}\,d\tau\Big) \\
&\le C|\xi||\hat{h}_2(\xi)|(e^{-c_1|\xi|^2t/4} + e^{-\tilde{c}_1t/2}).
\end{aligned} \tag{3.21}$$

Substituting (3.19) and (3.21) into (3.12) we obtain (3.10), with $c_2 = c_1/4$ and $c_3 = \tilde{c}_1/2$. □

Based on (3.9) and (3.10), and by applying Plancherel's theorem, we have the following optimal L^2 estimate on the solution of (3.1) in the physical space. The estimate allows us to obtain convergence of the solution to the time asymptotic solution. This is to compare with Lemma 3.2 in [12], which is sufficient for the convergence to the constant equilibrium state only.

Lemma 3.3. *Let $\psi(\cdot,0) \in L^1(\mathbb{R}^m)\cap H^l(\mathbb{R}^m)$, $H \in C([0,T];L^1(\mathbb{R}^m))$, $D_x^lH \in C([0,T];L^1(\mathbb{R}^m)\cap H^1(\mathbb{R}^m))$, $h \in C([0,T];L^1(\mathbb{R}^m))$ and $D_x^lh \in C([0,T];L^1(\mathbb{R}^m)\cap L^2(\mathbb{R}^m))$. Also, let*

$$\psi(\cdot,0) = \begin{pmatrix} 0_{n_1\times 1} \\ \psi_2(\cdot,0)\end{pmatrix} \in \mathbb{R}^n. \tag{3.22}$$

Under conditions 1, 2 and 4 of Assumption 1.1, the solution of (3.1) with h satisfying (3.2) has the following estimate in the physical space: For $0\le t\le T$,

$$\begin{aligned}
\|D_x^l\psi\|(t) &\le C[(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\|\psi_2\|_{L^1}(0) + e^{-c_4t}\|D_x^l\psi_2\|(0)] \\
&\quad + C\int_0^{\frac{t}{2}}(t-\tau+1)^{-\frac{m}{4}-\frac{l+1}{2}}(\|H\|_{L^1} + \|h_2\|_{L^1})(\tau)\,d\tau \\
&\quad + C\int_{\frac{t}{2}}^t(t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}(\|D_x^lH\|_{L^1} + \|D_x^lh_2\|_{L^1})(\tau)\,d\tau \\
&\quad + C\int_0^t e^{-c_4(t-\tau)}(\|D_x^{l+1}H\| + \|D_x^lh_2\|)(\tau)\,d\tau,
\end{aligned} \tag{3.23}$$

where C and c_4 are positive constants.

Proof. By Duhamel's principle we express the solution to (3.1) as

$$\hat{\psi}(\xi,t) = e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0) + \int_0^t e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}[i\hat{H}(\xi,\tau)\xi + \hat{h}(\xi,\tau)]\,d\tau.$$

By Plancherel's theorem and triangle inequality, for a multi-index α with $|\alpha| = l$, we have

$$\begin{aligned}(3.24)\qquad \|D_x^\alpha \psi\|(t) = \|(i\xi)^\alpha \hat{\psi}\|(t) &\le \|(i\xi)^\alpha e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0)\| \\ &\quad + \int_0^t \|(i\xi)^\alpha e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}[i\hat{H}(\xi,\tau)\xi + \hat{h}(\xi,\tau)]\|\,d\tau.\end{aligned}$$

Applying (3.10), (3.9) and (3.4) gives us

$$\begin{aligned}\|(i\xi)^\alpha e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0)\|^2 &\le C\int_{|\xi|\le 1} |(i\xi)^\alpha|^2(|\xi|^2 e^{-2c_2|\xi|^2 t} + e^{-2c_3 t})|\hat{\psi}_2(\xi,0)|^2\,d\xi \\ &\quad + C\int_{|\xi|\ge 1} e^{-\frac{2c_1|\xi|^2}{1+|\xi|^2}t}|(i\xi)^\alpha\hat{\psi}(\xi,0)|^2\,d\xi \\ &\le C\Big[\int_{|\xi|\le 1}|\xi|^{2l+2}e^{-2c_2|\xi|^2 t}\,d\xi\|\hat{\psi}_2\|_{L^\infty}^2(0) \\ &\quad + \int_{|\xi|\le 1} e^{-2c_3 t}|(i\xi)^\alpha\hat{\psi}_2(\xi,0)|^2\,d\xi \\ &\quad + \int_{|\xi|\ge 1} e^{-c_1 t}|(i\xi)^\alpha\hat{\psi}_2(\xi,0)|^2\,d\xi\Big] \\ &\le C[(t+1)^{-\frac{m}{2}-l-1}\|\psi_2\|_{L^1}^2(0) + e^{-2c_4 t}\|D_x^\alpha\psi_2\|^2(0)],\end{aligned}$$

with $c_4 = \min\{c_3, c_1/2\}$. Therefore,

$$(3.25)\quad \|(i\xi)^\alpha e^{t\tilde{E}(\xi)}\hat{\psi}(\xi,0)\| \le C[(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\|\psi_2\|_{L^1}(0) + e^{-c_4 t}\|D_x^\alpha\psi_2\|(0)].$$

Similarly, by applying (3.9) we have

$$\begin{aligned}(3.26)\qquad &\|(i\xi)^\alpha e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}i\hat{H}(\xi,\tau)\xi\| \\ &\quad \le C[(t-\tau+1)^{-\frac{m}{4}-\frac{l+1}{2}}\|H\|_{L^1}(\tau) + e^{-c_4(t-\tau)}\|D_x^{l+1}H\|(\tau)],\end{aligned}$$

and

$$(3.27)\qquad \|(i\xi)^\alpha e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}i\hat{H}(\xi,\tau)\xi\|$$

$$\leq C[(t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}\|D_x^\alpha H\|_{L^1}(\tau)+e^{-c_4(t-\tau)}\|D_x^{l+1}H\|(\tau)].$$

From Lemmas 2.1 and 2.3, $\tilde{A}_0^{-1}=\operatorname{diag}(A_{01}^{-1},A_{02}^{-1})$, thus with (3.2) we have

$$\tilde{A}_0^{-1}h=\begin{pmatrix}0_{n_1\times 1}\\ A_{02}^{-1}h_2\end{pmatrix}.$$

Following the argument that leads to (3.25), we have

$$\begin{aligned}\|(i\xi)^\alpha e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}\hat{h}(\xi,\tau)\|\\ \leq C[(t-\tau+1)^{-\frac{m}{4}-\frac{l+1}{2}}\|h_2\|_{L^1}(\tau)+e^{-c_4(t-\tau)}\|D_x^\alpha h_2\|(\tau)],\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}\|(i\xi)^\alpha e^{(t-\tau)\tilde{E}(\xi)}\tilde{A}_0^{-1}\hat{h}(\xi,\tau)\|\\ \leq C[(t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}\|D_x^\alpha h_2\|_{L^1}(\tau)+e^{-c_4(t-\tau)}\|D_x^\alpha h_2\|(\tau)].\end{aligned}\tag{3.29}$$

Finally, we substitute (3.25)-(3.29) into (3.24). This gives us (3.23) □

4. Asymptotic behavior

In this section we prove Theorem 1.5. For this we prove (1.24) with $p=2$ under the assumptions of the theorem. That is, we prove that the solution w of (1.1), (1.2) satisfies

$$\|D_x^l(w-w^*)\|(t)\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\begin{cases}[1+\ln(t+1)] & \text{if } m=2\\ 1 & \text{if } m>2\end{cases}\tag{4.1}$$

for $t\geq 0$, where $0\leq l\leq s-2$, and w^* is the solution of (1.22), (1.23). The case $p>2$ follows from Gagliardo-Nirenberg inequality (1.19).

In Section 2 we have defined the diffeomorphism $w\to\psi$ and the perturbation $\tilde{\psi}$, see (2.1) and (2.3). Equation (2.5) for $\tilde{\psi}$ is equivalent to (1.1), and its Fourier transform is given in (2.16). We now define a similar diffeomorphism for the solution w^* of (1.22), (1.23). We write

$$w^*=\begin{pmatrix}w_1^*\\ w_2^*\end{pmatrix},\qquad r^*(w^*)\equiv r'(\bar{w})(w^*-\bar{w})=\begin{pmatrix}0_{n_1\times 1}\\ r_2^*\end{pmatrix},\tag{4.2}$$

with $w_1^*\in\mathbb{R}^{n_1}$, $w_2^*,r_2^*\in\mathbb{R}^{n_2}$. By direct calculation,

$$r_2^*=r_2^*(w^*)=(r_2)_{w_1}(\bar{w})(w_1^*-\bar{w}_1)+(r_2)_{w_2}(\bar{w})(w_2^*-\bar{w}_2).\tag{4.3}$$

Let

$$\psi^* = \psi^*(w^*) = \begin{pmatrix} w_1^* \\ r_2^*(w^*) \end{pmatrix} = \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix}(w^*), \tag{4.4}$$

where $\psi_1^* = w_1^*$ and $\psi_2^* = r_2^*(w^*)$. By direct calculation and using (2.2) and (4.3), we have

$$\psi^* = \psi_w(\bar{w})w^* - r'(\bar{w})\bar{w}. \tag{4.5}$$

Thus

$$\psi^*_{w^*} = \psi_w(\bar{w}), \qquad w^*_{\psi^*} = (\psi^*_{w^*})^{-1} = w_\psi(\bar{\psi}) = \psi_w^{-1}(\bar{w}). \tag{4.6}$$

It is clear that under condition 2 of Assumption 1.1, ψ^* is a diffeomorphism. Note that from (4.3), (4.4) and (2.1),

$$\bar{\psi}^* \equiv \psi^*(\bar{w}) = \begin{pmatrix} \bar{w}_1 \\ 0_{n_2\times 1} \end{pmatrix} = \bar{\psi}. \tag{4.7}$$

Define the perturbation

$$\tilde{\psi}^* = \psi^* - \bar{\psi}^* = \psi^* - \bar{\psi} = \begin{pmatrix} w_1^* - \bar{w}_1 \\ r_2^*(w^*) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\psi}_1^* \\ \tilde{\psi}_2^* \end{pmatrix}. \tag{4.8}$$

Solving (4.5) for w^* and using (4.8) we have

$$w^* = w_\psi(\bar{\psi})[\tilde{\psi}^* + \bar{\psi} + r'(\bar{w})\bar{w}].$$

From (4.7) and (2.2) and by direct calculation, we find $\bar{\psi}+r'(\bar{w})\bar{w} = \psi_w(\bar{w})\bar{w}$. Therefore,

$$w^* = \bar{w} + w_\psi(\bar{\psi})\tilde{\psi}^*. \tag{4.9}$$

Substituting (4.9) into (1.22) and multiplying the result by $(w_\psi^t\eta'')(\bar{w})$ from the left, with the notations in (2.6), we have

$$\tilde{A}_0\tilde{\psi}^*_t + \sum_{j=1}^m \tilde{A}_j\tilde{\psi}^*_{x_j} = \sum_{j,k=1}^m \tilde{B}_{jk}\tilde{\psi}^*_{x_kx_j} + \tilde{L}\tilde{\psi}^*. \tag{4.10}$$

The Fourier transform of (4.10) is

$$\tilde{A}_0\hat{\tilde{\psi}}^*_t + [i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2\tilde{B}(\tilde{\xi}) - \tilde{L}]\hat{\tilde{\psi}}^* = 0, \tag{4.11}$$

using the notations in (2.14), with $\tilde{\xi} = \xi/|\xi|$.

Now we relate $w - w^*$ to $\tilde{\psi} - \tilde{\psi}^*$. By Taylor expansion of $w(\psi)$ around $\bar{\psi}$ and applying (2.3) and (4.9), we have

$$w - w^* = w_\psi(\bar{\psi})(\tilde{\psi} - \tilde{\psi}^*) + O(1)|\tilde{\psi}|^2. \tag{4.12}$$

Therefore,

$$\|w - w^*\| \le C(\|\tilde{\psi} - \tilde{\psi}^*\| + \|\tilde{\psi}\|_{L^\infty}\|\tilde{\psi}\|). \tag{4.13}$$

Similarly, for $l \ge 1$ we write $D_x^l = D_x^{l-1}D_{x_k}$ for some $1 \le k \le m$. Thus by triangle inequality,

$$\begin{aligned} \|D_x^l w - D_x^l w^*\| &\le \|D_x^{l-1}(w_\psi \psi_{x_k}) - w_\psi D_x^{l-1}\psi_{x_k}\| \\ &\quad + \|[w_\psi(\psi) - w_\psi(\bar{\psi})]D_x^l \psi\| + \|w_\psi(\bar{\psi})(D_x^l\tilde{\psi} - D_x^l\tilde{\psi}^*)\| \\ &\le C[\|D_x^l(\tilde{\psi} - \tilde{\psi}^*)\| + \|\tilde{\psi}\|_{L^\infty}\|D_x^l\tilde{\psi}\|] \\ &\quad + \|D_x^{l-1}(w_\psi \psi_{x_k}) - w_\psi D_x^{l-1}\psi_{x_k}\|. \end{aligned} \tag{4.14}$$

We note that the last term on the right-hand side of (4.14) is zero if $l = 1$. Combining (4.13) and (4.14) and applying (2.25) and (2.26) to the right-hand side give us

$$\begin{aligned} \|D_x^l(w - w^*)\| \le C[&\|D_x^l(\tilde{\psi} - \tilde{\psi}^*)\| + \|w - \bar{w}\|_{L^\infty}\|D_x^l(w - \bar{w})\| \\ &+ \|D_x w\|_{L^\infty}\|D_x^{l-1}w\|] \end{aligned} \tag{4.15}$$

for $0 \le l \le s$, where the last term on the right-hand side exists only when $l \ge 2$.

Applying Lemma 2.4 to $w - \bar{w}$ and $D_x w$, respectively, noting $s - 1 > m/2$, we have

$$\|w - \bar{w}\|_{L^\infty} \le C\|w - \bar{w}\|_{s-1}, \qquad \|D_x w\|_{L^\infty} \le C\|D_x w\|_{s-1}. \tag{4.16}$$

To obtain decay rates we apply Theorem 1.3 to $D_x^l(w - \bar{w})$ for $0 \le l \le s - 2$ and Theorem 2.7 to $D_x^l w$ for $l = s - 1, s$. This gives us

$$\begin{aligned} \|w - \bar{w}\|_{s-1}^2(t) &= \sum_{l=0}^{s-2}\|D_x^l(w - \bar{w})\|^2(t) + \|D_x^{s-1}w\|^2(t) \\ &\le C\delta_0^2(t+1)^{-\frac{m}{2}} + C\|w_0 - \bar{w}\|_s^2(t+1)^{-(s-1)} \le C\delta_0^2(t+1)^{-\frac{m}{2}}, \end{aligned}$$

$$\begin{aligned}\|D_x w\|_{s-1}^2(t) &= \sum_{l=1}^{s-2}\|D_x^l w\|^2(t) + \|D_x^{s-1}w\|^2(t) + \|D_x^s w\|^2(t)\\ &\le C\delta_0^2(t+1)^{-\frac{m}{2}-1} + C\|w_0-\bar{w}\|_s^2(t+1)^{-(s-1)}\end{aligned}$$

for $t \ge 0$. Since $s-1 > m/2$, we have $s-1 \ge m/2+1/2$ if m is odd, and $s-1 \ge m/2+1$ if m is even. Therefore,

$$\tag{4.17} \begin{aligned}&\|w-\bar{w}\|_{s-1}(t) \le C\delta_0(t+1)^{-\frac{m}{4}},\\ &\|D_x w\|_{s-1}(t) \le C\delta_0\begin{cases}(t+1)^{-\frac{m}{4}-\frac{1}{4}} & \text{if } m \text{ is odd}\\ (t+1)^{-\frac{m}{4}-\frac{1}{2}} & \text{if } m \text{ is even}\end{cases} \le C\delta_0(t+1)^{-1}.\end{aligned}$$

With (4.16), (4.17) and Theorem 1.3, we have

$$\tag{4.18} \begin{aligned}&\|w-\bar{w}\|_{L^\infty}(t)\|D_x^l(w-\bar{w})\|(t) \le C\delta_0^2(t+1)^{-\frac{m}{2}-\frac{l}{2}}, \quad 0 \le l \le s-2,\\ &\|D_x w\|_{L^\infty}(t)\|D_x^{l-1}w\|(t) \le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}, \quad 2 \le l \le s-2.\end{aligned}$$

From (4.15) and (4.18), noting $m \ge 2$, to prove (4.1) we only need to prove that for $t \ge 0$,

$$\tag{4.19} \|D_x^l(\tilde{\psi}-\tilde{\psi}^*)\|(t) \le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\begin{cases}[1+\ln(t+1)] & \text{if } m=2\\ 1 & \text{if } m>2\end{cases},$$

where $0 \le l \le s-2$.

To obtain (4.19) we consider the equation satisfied by $\tilde{\psi}-\tilde{\psi}^*$. We subtract (4.11) from (2.16) to arrive at

$$\tag{4.20} \tilde{A}_0(\hat{\tilde{\psi}}-\hat{\tilde{\psi}}^*)_t + [i|\xi|\tilde{A}(\tilde{\xi}) + |\xi|^2\tilde{B}(\tilde{\xi}) - \tilde{L}](\hat{\tilde{\psi}}-\hat{\tilde{\psi}}^*) = \hat{\tilde{R}}.$$

From (2.13) and (2.19) we may write

$$\tag{4.21} \begin{aligned}&\hat{\tilde{R}} = i\hat{H}\xi + \hat{h},\\ &H = \tilde{A}_0\begin{pmatrix}\tilde{f}_{11}+b_{11} & \cdots & \tilde{f}_{m1}+b_{m1}\\ & 0_{n_2\times m} & \end{pmatrix},\\ &h = \tilde{A}_0\begin{pmatrix}0_{n_1\times 1}\\ R_{12}+R_{22}+R_{32}\end{pmatrix} = \begin{pmatrix}0_{n_1\times 1}\\ A_{02}(R_{12}+R_{22}+R_{32})\end{pmatrix} = \begin{pmatrix}0_{n_1\times 1}\\ h_2\end{pmatrix}.\end{aligned}$$

That is, the equation for $\tilde{\psi} - \tilde{\psi}^*$ takes the from (3.1) in the Fourier space, where h satisfies (3.2). For the initial data, from (1.2) and (1.23) we have

$$w(x,0) = w^*(x,0) = w_0(x) = \begin{pmatrix} w_{01} \\ w_{02} \end{pmatrix}(x),$$

with $w_{01} \in \mathbb{R}^{n_1}$ and $w_{02} \in \mathbb{R}^{n_2}$. Therefore, from (2.3) and (4.8) we have

$$(\tilde{\psi} - \tilde{\psi}^*)(x,0) = \begin{pmatrix} 0_{n_1 \times 1} \\ \tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0) \end{pmatrix},$$

which takes the form (3.22). Now we conclude that Lemma 3.3 applies. Applying (3.23) to $\tilde{\psi} - \tilde{\psi}^*$, for $t \geq 0$ we have

(4.22)
$$\begin{aligned} \|D_x^l(\tilde{\psi} - \tilde{\psi}^*)\|(t) \leq C\{&(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\|\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)\|_{L^1} \\ &+ e^{-c_4 t}\|D_x^l[\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)]\|\} \\ &+ C\int_0^{\frac{t}{2}} (t-\tau+1)^{-\frac{m}{4}-\frac{l+1}{2}}(\|H\|_{L^1} + \|h_2\|_{L^1})(\tau)\,d\tau \\ &+ C\int_{\frac{t}{2}}^t (t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}(\|D_x^l H\|_{L^1} + \|D_x^l h_2\|_{L^1})(\tau)\,d\tau \\ &+ C\int_0^t e^{-c_4(t-\tau)}(\|D_x^{l+1}H\| + \|D_x^l h_2\|)(\tau)\,d\tau, \end{aligned}$$

where $c_4 > 0$ is a constant.

From (2.3), (4.8) and (4.3), and by the definition of $\bar{w}$, we have

$$\begin{aligned} \tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0) &= r_2(w_0) - r_2^*(w_0) = r_2(w_0) - r_2(\bar{w}) - (r_2)_w(\bar{w})(w_0 - \bar{w}) \\ &= O(1)|w_0 - \bar{w}|^2. \end{aligned}$$

This gives us

$$\|\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)\|_{L^1} \leq C\|w_0 - \bar{w}\|^2, \tag{4.23}$$

$$\|\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)\| \leq C\|w_0 - \bar{w}\|_{L^\infty}\|w_0 - \bar{w}\| \leq C\|w_0 - \bar{w}\|_{s-1}^2, \tag{4.24}$$

where we have applied (4.15). For $l \geq 1$, we write $D_x^l = D_x^{l-1}D_{x_k}$ for some $1 \leq k \leq m$, hence

$$|D_x^l[\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)]| = |D_x^l r_2(w_0) - (r_2)_w(\bar{w}) D_x^l w_0|$$
$$\leq |D_x^{l-1}[(r_2)_w(w_0) w_{0x_k}] - (r_2)_w(w_0) D_x^{l-1} w_{0x_k}| + C|w_0 - \bar{w}||D_x^l w_0|.$$

Applying (2.25), (2.26) and (4.16) we thus have

(4.25)
$$\|D_x^l[\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)]\| \leq C(\|D_x w_0\|_{L^\infty} \|D_x^{l-1} w_0\| + \|w_0 - \bar{w}\|_{L^\infty} \|D_x^l w_0\|)$$
$$\leq C(\|D_x w_0\|_{s-1} \|D_x^{l-1} w_0\| + \|w_0 - \bar{w}\|_{s-1} \|D_x^l w_0\|),$$

where the first term on the right-hand side exists only when $l \geq 2$. Combining (4.24) and (4.25) gives us

$$\|D_x^l[\tilde{\psi}_2(w_0) - \tilde{\psi}_2^*(w_0)]\| \leq C\|w_0 - \bar{w}\|_s^2, \quad 0 \leq l \leq s. \tag{4.26}$$

Next we estimate the integrals in (4.22). Note that the source terms H and h_2, given in (4.21), (2.8)-(2.12) and (2.21), contain only the solution w to (1.1), (1.2). Similar integrals have been treated in [12] for L^p decay rates of w. Therefore, here we mainly outline the treatment. Readers are referred to [12] for more details.

From (4.21), (2.6), (2.8) and (2.9), we have

$$|H| \leq C \sum_{j=1}^m (|\tilde{f}_{j1}| + |b_{j1}|) \leq C(|\tilde{\psi}|^2 + |w - \bar{w}||D_x \psi|).$$

Thus by (2.3) and (1.17) we have the decay estimate:

$$\|H\|_{L^1}(t) \leq C(\|w - \bar{w}\|^2 + \|w - \bar{w}\|\|D_x w\|)(t) \leq C\delta_0^2 (t+1)^{-\frac{m}{2}}. \tag{4.27}$$

Similarly, (4.21), (2.21), (2.10)-(2.12), (2.25), (1.17) and (2.27) give us

(4.28)
$$\|h_2\|_{L^1}(t) \leq C(\|R_{12}\|_{L^1} + \|R_{22}\|_{L^1} + \|R_{32}\|_{L^1})(t)$$
$$\leq C(\|w - \bar{w}\|\|D_x w\| + \|D_x w\|^2 + \|w - \bar{w}\|\|D_x^2 w\| + \|w - \bar{w}\|^2)(t)$$
$$\leq C\delta_0^2[(t+1)^{-\frac{m}{2}} + (t+1)^{-\frac{m}{4}-1}].$$

Equations (4.27) and (4.28) imply

$$\int_0^{\frac{t}{2}} (t - \tau + 1)^{-\frac{m}{4} - \frac{l+1}{2}} (\|H\|_{L^1} + \|h_2\|_{L^1})(\tau)\, d\tau \tag{4.29}$$

$$\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\int_0^{\frac{t}{2}}[(\tau+1)^{-\frac{m}{2}}+(\tau+1)^{-\frac{m}{4}-1}]\,d\tau$$
$$\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\begin{cases}[1+\ln(t+1)] & \text{if } m=2\\ 1 & \text{if } m>2\end{cases}.$$

For $0\leq l\leq s-2$, from (2.8), (2.25) and (1.17), we have

(4.30)
$$\|D_x^l\tilde{f}_{j1}\|_{L^1}(t)\leq C\sum_{k=0}^{l}\|D_x^k(w-\bar{w})\|(t)\|D_x^{l-k}(w-\bar{w})\|(t)\leq C\delta_0^2(t+1)^{-\frac{m}{2}-\frac{l}{2}}.$$

Similarly, from (2.9)-(2.12) and (4.21) we also have

(4.31)
$$\|D_x^l b_{j1}\|_{L^1}(t)\leq C\sum_{k=0}^{l}\|D_x^k(w-\bar{w})\|(t)\|D_x^{l+1-k}w\|(t)\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}},$$

(4.32)
$$\begin{aligned}\|D_x^l h_2\|_{L^1}(t)&\leq C(\|D_x^l R_{12}\|_{L^1}+\|D_x^l R_{22}\|_{L^1}+\|D_x^l R_{32}\|_{L^1})(t)\\&\leq C\delta_0^2[(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}+(t+1)^{-\frac{m}{2}-\frac{l}{2}}],\end{aligned}$$

where $0\leq l\leq s-2$, and we have used (2.27) for $\|D_x^{l+1}w\|(t)$ and $\|D_x^{l+2}w\|(t)$. Equations (4.21) and (4.30)-(4.32) give us

$$\|D_x^l H\|_{L^1}(t)+\|D_x^l h_2\|_{L^1}(t)\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}},$$

noting $m\geq 2$. Therefore, for $0\leq l\leq s-2$,

(4.33)
$$\int_{\frac{t}{2}}^{t}(t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}(\|D_x^l H\|_{L^1}+\|D_x^l h_2\|_{L^1})(\tau)\,d\tau$$
$$\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\int_{\frac{t}{2}}^{t}(t-\tau+1)^{-\frac{m}{4}-\frac{1}{2}}\,d\tau$$
$$\leq C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\begin{cases}\ln(t+1) & \text{if } m=2\\ 1 & \text{if } m>2\end{cases}.$$

To estimate $\|D_x^{l+1}H\|$ we use (2.26) as well. For instance,

(4.34) $$\|D_x^{l+1}\tilde{f}_{j1}\|(t)=\|D_x^l D_{x_k}\tilde{f}_{j1}\|(t)$$

$$\begin{aligned}
&\le \|D_x^l[(f_{j1})_\psi \psi_{x_k}] - (f_{j1})_\psi D_x^l \psi_{x_k}\|(t) \\
&\quad + \|[(f_{j1})_\psi(w) - (f_{j1})_\psi(\bar{w})] D_x^l \psi_{x_k}\|(t) \\
&\le C(\|D_x w\|_{L^\infty} \|D_x^l w\| + \|w-\bar{w}\|_{L^\infty} \|D_x^{l+1} w\|)(t) \\
&\le C(\|D_x w\|_{s-1} \|D_x^l w\| + \|w-\bar{w}\|_{s-1} \|D_x^{l+1} w\|)(t) \\
&\le C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{l+1}{2}}, \quad 0 \le l \le s-2,
\end{aligned}$$

where the term with $\|D_x^l w\|$ does not exist if $l=0$. Similarly, for $0 \le l \le s-2$,

$$\begin{aligned}
&(4.35) \quad \|D_x^{l+1} b_{j1}\|(t) \le C(\|D_x w\|_{s-1} \|D_x^{l+1} w\| + \|w_0 - \bar{w}\|_{s-1} \|D_x^{l+2} w\|)(t) \\
&\qquad\qquad \le C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{3}{4}-\frac{l}{2}}, \\
&(4.36) \quad \|D_x^l R_{12}\|(t) \le C(\|D_x w\|_{s-1} \|D_x^l w\| + \|w-\bar{w}\|_{s-1} \|D_x^{l+1} w\|)(t) \\
&\qquad\qquad \le C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{l+1}{2}}, \\
&(4.37) \quad \|D_x^l R_{32}\|(t) \le C(\|D_x w\|_{s-1} \|D_x^{l-1} w\| + \|w-\bar{w}\|_{s-1} \|D_x^l (w-\bar{w})\|)(t) \\
&\qquad\qquad \le C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{l+1}{2}},
\end{aligned}$$

where the terms with $\|D_x^l w\|$ does not exist if $l=0$, and the term with $\|D_x^{l-1} w\|$ is omitted if $l \le 1$.

To estimate $\|D_x^l R_{22}\|$ we need (2.23) as follows. From (2.11) and by triangle inequality we have

$$\|D_x^l R_{22}\| \le \sum_{j,k=1}^m \Big\{ \|D_x^l D_{x_j}(\psi_w B_{jk} w_\psi \psi_{x_k}) - (\psi_w B_{jk} w_\psi)(\bar{w}) D_x^l \psi_{x_k x_j}\| \\ + \|D_x^l [(\psi_w)_{x_j} B_{jk} w_{x_k}]\| \Big\}.$$

Here the first term in the braces is similar to $\|D_x^{l+1} b_{j1}\|$, and gives the same estimate. For the second term, we apply (2.23) first, then apply (2.26) repeatedly. This gives us the following upper bound:

$$\begin{aligned}
&C[(\|D_x w\|_{L^\infty}^2 + \|D_x^2 \tilde{w}_2\|_{L^\infty}) \|D_x^l w\| + \|D_x^{l+1} w\| \|D_x w\|_{L^\infty}](t) \\
\le &C[(\|D_x w\|_{s-1}^2 + \|D_x^2 \tilde{w}_2\|_{s-1}) \|D_x^l w\| + \|D_x^{l+1} w\| \|D_x w\|_{s-1}](t) \\
\le &C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{3}{4}-\frac{l}{2}} + C\delta_0 (t+1)^{-\frac{m}{4}-\frac{l}{2}} \|D_x^2 \tilde{w}_2\|_{s-1}(t),
\end{aligned}$$

where the part with $\|D_x^l w\|$ does not exist if $l=0$. Note that it is necessary to apply (2.23) to obtain $\|D_x^2 \tilde{w}_2\|_{s-1}$ since only $\tilde{w}_2$ has better regularity. All together we arrive at

$$(4.38) \quad \|D_x^l R_{22}\|(t) \le C\delta_0^2 (t+1)^{-\frac{m}{4}-\frac{3}{4}-\frac{l}{2}} + C\delta_0 (t+1)^{-\frac{m}{4}-\frac{l}{2}} \|D_x^2 \tilde{w}_2\|_{s-1}(t)$$

for $0 \le l \le s-2$.

Equations (4.21) and (4.34)-(4.38) give us

$$(\|D_x^{l+1}H\|+\|D_x^l h_2\|)(t) \le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}+C\delta_0(t+1)^{-\frac{m}{4}-\frac{l}{2}}\|D_x^2\tilde{w}_2\|_{s-1}(t),$$

which implies

(4.39)

$$\begin{aligned}
&\int_0^t e^{-c_4(t-\tau)}(\|D_x^{l+1}H\|+\|D_x^l h_2\|)(\tau)\,d\tau \\
&\quad\le C\delta_0^2\int_0^t e^{-c_4(t-\tau)}(\tau+1)^{-\frac{m}{4}-\frac{l+1}{2}}\,d\tau \\
&\qquad + C\delta_0\int_0^t e^{-c_4(t-\tau)}(\tau+1)^{-\frac{m}{4}-\frac{l}{2}}\|D_x^2\tilde{w}_2\|_{s-1}(\tau)\,d\tau \\
&\quad\le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}} \\
&\qquad + C\delta_0\Big[\int_0^t e^{-2c_4(t-\tau)}(\tau+1)^{-\frac{m}{2}-l-1}\,d\tau\Big]^{\frac12}\Big[\int_0^t(\tau+1)\|D_x^2\tilde{w}_2\|_{s-1}^2(\tau)\,d\tau\Big]^{\frac12} \\
&\quad\le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}, \quad 0\le l\le s-2,
\end{aligned}$$

where we have applied Cauchy-Schwarz inequality and (2.28).

Now we substitute (4.23), (4.26), (4.29), (4.33) and (4.39) into (4.22). This gives us

$$\|D_x^l(\tilde{\psi}-\tilde{\psi}^*)\|(t) \le C\delta_0^2(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}\begin{cases}[1+\ln(t+1)] & \text{if } m=2\\ 1 & \text{if } m>2\end{cases}$$

for $t\ge 0$, where $0\le l\le s-2$. This is (4.19).

To finish this section, we consider the special case of hyperbolic balance laws (1.6). In this case we relax the restriction on l to $0\le l\le s-1-m(1/2-1/p)$ in Theorem 1.10, or $0\le l\le s-1$ for $p=2$. As commented in Section 1, such a relaxation has been justified for (1.27) in [12]. As a consequence, (4.18) is true for $l=s-1$ as well. This implies that to prove (4.1) (hence (1.28)) for $0\le l\le s-1$, we only need to prove (4.19) for $0\le l\le s-1$. For this, we consider each term on the right-hand side of (4.22). Due to (4.23) and (4.26), we only need to consider $l=s-1$ in the integrals. But similar integrals have been considered in [12] for (1.27), and the justification has been done. The fact behind this is $B_{jk}=0$, or the disappearance of second derivatives, in (1.6). For instance, (4.30) is now true for $0\le l\le s-1$, while $\|D_x^l b_{j1}\|_{L^1}$ in

(4.31) becomes zero. Similarly, $R_{22} = 0$ and (4.32) is true for $0 \le l \le s-1$. These imply (4.33) is true for $0 \le l \le s-1$. The other integral, estimated in (4.39), can be justified similarly for $0 \le l \le s-1$.

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