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# Periods of linear algebraic cycles

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**Abstract:** In this article we use a theorem of Carlson and Griffiths and compute periods of linear algebraic cycles $\mathbb{P}^{\frac{n}{2}}$ inside the Fermat variety of even dimension $n$ and degree $d$. As an application, for examples of $n$ and $d$, we prove that the locus of hypersurfaces containing two linear cycles whose intersection is of low dimension, is a reduced component of the Hodge locus in the underlying parameter space. We also check the same statement for hypersurfaces containing a complete intersection algebraic cycle. Our result confirms the Hodge conjecture for Hodge cycles obtained by the monodromy of the homology class of such algebraic cycles. This is known as the variational Hodge conjecture.

## 1. Introduction

Let us consider the even dimensional Fermat variety

$$X^d_n \subset \mathbb{P}^{n+1}: \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0. \tag{1}$$

It has the following linear algebraic cycles of dimension $\frac{n}{2}$:

$$\mathbb{P}^{\frac{n}{2}}_{a,b}: \begin{cases} x_{b_0} - \zeta_{2d}^{1+2a_1} x_{b_1} = 0, \\ x_{b_2} - \zeta_{2d}^{1+2a_3} x_{b_3} = 0, \\ x_{b_4} - \zeta_{2d}^{1+2a_5} x_{b_5} = 0, \\ \cdots \\ x_{b_n} - \zeta_{2d}^{1+2a_{n+1}} x_{b_{n+1}} = 0, \end{cases} \tag{2}$$

where $\zeta_{2d}$ is a $2d$-primitive root of unity, $b$ is a permutation of $\{0,1,2,\ldots,n+1\}$ and $0 \le a_i \le d-1$ are integers. In order to get distinct cycles we may further assume that $b_0 = 0$ and for $i$ an even number $b_i$ is the smallest number in $\{0,1,\ldots,n+1\}\backslash\{b_0,b_1,b_2,\ldots,b_{i-1}\}$. It is easy to see that the number of such cycles is $1\cdot 3\cdots(n-1)(n+1)d^{\frac{n}{2}+1}$ (for $d=3, n=2$ this is the famous 27 lines in a smooth cubic surface). In this article we use a theorem of Carlson and Griffiths in [CG80] and prove the following:

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**Theorem 1.** *For non-negative integers $i_0, i_1, \cdots, i_{n+1}$ with $\sum_{k=0}^{n+1} i_k = (\frac{n}{2}+1)d-n-2$, we have*

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_{\mathbb{P}^{\frac{n}{2}}_{a,b}} \text{Residue}\left(\frac{x_0^{i_0}x_1^{i_1}\cdots x_{n+1}^{i_{n+1}}\cdot\sum_{i=0}^{n+1}(-1)^i x_i\widehat{dx_i}}{(x_0^d+x_1^d+\cdots+x_{n+1}^d)^{\frac{n}{2}+1}}\right)= \tag{3}$$
$$\begin{cases} \frac{\text{sign}(b)}{d^{\frac{n}{2}+1}\cdot\frac{n}{2}!}\zeta_{2d}{}^{\sum_{e=0}^{\frac{n}{2}}(i_{b_{2e}}+1)\cdot(1+2a_{2e+1})} \\ \quad \textit{if} \quad i_{b_{2e-2}}+i_{b_{2e-1}}=d-2, \quad \forall e=1,...,\frac{n}{2}+1, \\ 0 \quad \textit{otherwise.} \end{cases}$$

*where $\zeta_{2d}=e^{\frac{\pi i}{d}}$ is the $2d$-th primitive root of unity.*

For the residue map see §3. Using Theorem 1 we can prove a stronger version of the variational Hodge conjecture for many algebraic cycles, see [Gro66, page 103]. We content ourselves with the class of examples in Theorem 2. A complete list of cases will appear in another publication. Recall that a stronger version of the variational Hodge conjecture (alternative Hodge conjecture in [Mov17a, §18.2]) holds for an algebraic cycle $Z$ of codimension $\frac{n}{2}$ inside a smooth hypersurface of degree $d$ and dimension $n$, if deformations of $Z$ as an algebraic cycle and Hodge cycle are the same. Let $\mathsf{T}$ be the open subset of $\mathbb{C}[x]_d$ parameterizing smooth hypersurfaces of degree $d$. We use the notation $X_t,\ t\in\mathsf{T}$ and denote by $0\in\mathsf{T}$ the point corresponding to Fermat variety. We also denote by $Z_\infty$ the trivial algebraic cycle in $X$ obtained by intersecting a projective space $\mathbb{P}^{\frac{n}{2}+1}\subset\mathbb{P}^{n+1}$ with $X$. For the definition of a Hodge cycle and Hodge locus see §2. As a corollary of Theorem 1 we get:

**Theorem 2.** *Let $\check{\mathsf{T}}$ be the subvariety of $\mathsf{T}$ parametrizing hypersurfaces containing two linear cycle $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}}\cap\check{\mathbb{P}}^{\frac{n}{2}}=\mathbb{P}^m$. There is a Zariski open (and hence dense) subset $U$ of $\check{\mathsf{T}}$ such that the variational Hodge conjecture is true for $Z:=\mathbb{P}^{\frac{n}{2}}+\check{\mathbb{P}}^{\frac{n}{2}}\in X:=X_t,\ t\in U$ with the triples $(n,d,m)$:*

$$\begin{aligned}
&(2,d,-1),\ 5\le d\le 14,\\
&(4,4,-1),(4,5,-1),(4,6,-1),(4,5,0),(4,6,0),\\
&(6,3,-1),(6,4,-1),(6,4,0),\\
&(8,3,-1),(8,3,0),\\
&(10,3,-1),(10,3,0),(10,3,1),
\end{aligned}$$

*where $\mathbb{P}^{-1}$ means the empty set. In particular, if another algebraic cycle $\check{Z}\subset X$ of dimension $\frac{n}{2}$ in $X$ satisfies $[\check{Z}]=b[Z]+c[Z_\infty]$ in $H_n(X,\mathbb{Q})$ for some $b,c\in\mathbb{Q},\ b\neq 0$, then the pair $(X,\check{Z})$ cannot be deformed to $(X_t,\check{Z}_t)$ with $t\in\mathsf{T}\backslash\check{\mathsf{T}}$.*

For larger $m$'s Theorem 2 fails to be true and this is the main topic of the article [Mov17a, Chapter 18]. The limitation in Theorem 2 is due to the fact that a part of its proof is rank computation of certain matrices, for which we use a computer, and we do not know how to handle it for arbitrary $n$ and $d$. Theorem 2 implies that the parameter space $\check{\mathsf{T}}$ is an irreducible reduced component of the Hodge locus in the parameter space $\mathsf{T}$ of smooth hypersurfaces. Note that for $n = 2$ the hypothesis on $\check{Z}$ is the same as to say that the equality holds in $\mathrm{Pic}(X) \otimes \mathbb{Q}$. By deformation of a pair $(X, Z)$ we mean a proper flat family $g : \mathcal{X} \to (\mathbb{C}, 0)$ with a closed subvariety $\mathcal{Z} \subset \mathcal{X}$ such that $g|_{\mathcal{Z}}$ is flat, $g^{-1}(0) = X$ and $g|_{\mathcal{Z}}^{-1}(0) = Z$.

The Zarski open subset $U$ in Theorem 2 may not contain the Fermat point as our choice of $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$ for Fermat is very special, see (20) and (21). For large degree $d$, all linear cycles of dimension $\frac{n}{2}$ and inside the Fermat variety are of the form (2), see [Mov17a, §17.4] and so in order to have $0 \in U$, we must verify a rank computation in §5 for all possible pairs of such $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$.

S. Bloch in [Blo72] proves variational Hodge conjecture for semi-regular algebraic cycles which is a strong condition on algebraic cycles and it is not at all clear whether it holds in our situation. The only result in this direction is given in [DK16], where the authors prove that any smooth projective variety $Z$ of dimension $\frac{n}{2}$ is a semi-regular sub-variety of a smooth projective hypersurface in $\mathbb{P}^{n+1}$ of large enough degree. We can also prove similar statements as in Theorem 2 for complete intersections algebraic cycles, see §7.

The strategy to prove results similar to Theorem 2 has been explained in the first author's book [Mov17a, Chapters 17, 18]. The main tools are 1. the infinitesimal variation of Hodge structures (IVHS) developed by Carlson, Green, Griffiths and Harris in [CGGH83] 2. A theorem of Carlson and Griffiths in [CG80, page 7] which describes a Cech cohomology description of the Griffiths' basis of the de Rham cohomology of smooth hypersurfaces, and it does not appear in the IVHS formulation (despite the fact that IVHS is originated from this article). 3. the relation between IVHS and the Zariski tangent space of Hodge loci as analytic spaces 4. and finally the computation of periods of linear cycles inside the Fermat variety, see Theorem 1. This is also the heart of our proof of Theorem 2 which has inspired the title of the article. For a full exposition of old and new results on Hodge locus the reader is referred to Voisin's article [Voi13].

## 2. Infinitesimal variation of Hodge structures

Let $X \to \mathsf{T}$ be a family of smooth complex projective varieties, where $\mathsf{T}$ is irreducible and smooth. The main ingredient of the infinitesimal variation of

Hodge structures (IVHS) at $0 \in \mathsf{T}$ is the bilinear map

$$\mathbf{T}_0\mathsf{T} \times H^{\frac{n}{2}-1}(X_0, \Omega_{X_0}^{\frac{n}{2}+1}) \to H^{\frac{n}{2}}(X_0, \Omega_{X_0}^{\frac{n}{2}}) \tag{4}$$

where $\mathbf{T_0}\mathsf{T}$ is the tangent space of $\mathsf{T}$ at 0. This gives us Voisin's ${}^0\bar{\nabla}$ map:

$${}^0\bar{\nabla} : H^{\frac{n}{2}}(X_0, \Omega_{X_0}^{\frac{n}{2}})^\vee \to \mathrm{Hom}\left(\mathbf{T}_0\mathsf{T}, H^{\frac{n}{2}-1}(X_0, \Omega_{X_0}^{\frac{n}{2}+1})^\vee\right), \tag{5}$$

where $\vee$ denotes the dual of a vector space. A cycle $\delta_0 \in H_n(X_0, \mathbb{Q})$ satisfying

$$\int_{\delta_0} \omega = 0, \quad \forall \omega \in F^{\frac{n}{2}+1} H^n_{\mathrm{dR}}(X_0)$$

is called a Hodge cycle. For a Hodge cycle $\delta_0$, the integrations

$$\int_{\delta_0} \omega, \quad \omega \in H^{\frac{n}{2}}(X_0, \Omega_{X_0}^{\frac{n}{2}}) \cong \frac{F^{\frac{n}{2}} H^n_{\mathrm{dR}}(X_0)}{F^{\frac{n}{2}+1} H^n_{\mathrm{dR}}(X_0)}, \quad \delta_0 \in H_n(X_0, \mathbb{Q}), \tag{6}$$

are well-defined and so we get $\delta_0^{\mathsf{pd}} \in H^{\frac{n}{2}}(X_0, \Omega_{X_0}^{\frac{n}{2}})^\vee$. Moreover, $\ker({}^0\bar{\nabla}\delta_0^{\mathsf{pd}})$ is the Zariski tangent space of the analytic space $V_{\delta_0}$ with

$$\mathcal{O}_{V_{\delta_0}} := \mathcal{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle, \tag{7}$$

at 0, where $\omega_1, \omega_2, \cdots, \omega_a$ are sections of the cohomology bundle $H^n_{\mathrm{dR}}(X_t)$, $t \in (\mathsf{T}, 0)$ such that they form a basis of $F^{\frac{n}{2}+1} H^n_{\mathrm{dR}}(X_t)$, and $\delta_t \in H_n(X_t, \mathbb{Q})$ is the monodromy/parallel transport of $\delta_0$ to $X_t$, see [Voi03, §5.3.2]. The analytic space $V_{\delta_0}$ is called the Hodge locus passing through 0 and corresponding to $\delta_0$. It might be non-reduced, see for instance [Voi03, Exercise 2, page 154]. For the full family of smooth hypersurfaces and $Z_\infty$ as in Introduction, we have identifications

$$\mathbf{T}_0\mathsf{T} \cong \mathbb{C}[x]_d \tag{8}$$

$$H^k(X_t, \Omega_{X_t}^{n-k}) \cong (\mathbb{C}[x]/J)_{(k+1)d-n-2}, \quad k = 0, 1, \ldots, n, \quad k \neq \frac{n}{2} \tag{9}$$

where $J := \mathrm{jacob}(f_t)$ is the Jacobian ideal of the equation $f_t$ of $X_t$. For $k = \frac{n}{2}$, $(\mathbb{C}[X]/J)_{(k+1)d-n-2}$ is identified with the codimension one subspace $\ker[Z_\infty]^{\mathsf{pd}}$ of $H^k(X_t, \Omega_{X_t}^{n-k})$, which is called the primitive part and it is in the image of (4). After these identifications, (4) is induced by the multiplication of polynomials.

## 3. Carlson-Griffiths theorem

There can be many ways to compute hypercohomology groups. In this section in order to compute integrals (6) we use a theorem of Carlson and Griffiths which gives a description of the algebraic de Rham cohomology of hypersurfaces using Čech cohomology. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d$ given by $f = 0$. Recall that for a monomial $x^i = x_0^{i_0} x_1^{i_1} \cdots x_{n+1}^{i_{n+1}}$ of degree $(k+1)d - n - 2$

$$\omega_i = \text{Residue}\left(\frac{x^i \cdot \Omega}{f^{k+1}}\right) \in H^n_{\text{dR}}(X).$$

where $\Omega := \sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i}$, $\widehat{dx_i} = dx_0 \wedge dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$ and Residue : $H^{n+1}(\mathbb{P}^{n+1} - X) \to H^{n+2}(\mathbb{P}^{n+1}, X) \cong H^n(X)$ is the composition of the coboundary map with the Leray-Thom-Gysin isomorphism, see [Mov17a, Chapter 4]. We say that $\omega_i$ has adjoint level $k$. Carlson and Griffiths in [CG80] found an explicit expression for these forms in the algebraic de Rham cohomology of $X$ relative to the Jacobian covering $\mathcal{J}_X$ of $\mathbb{P}^{n+1}$:

$$\mathcal{J}_X := \{U_j, \quad j = 0, 1, 2, \cdots, n+1\}, \quad U_j := \left\{\frac{\partial f}{\partial x_j} \neq 0\right\}.$$

Since $X$ is smooth, this is a covering of $\mathbb{P}^{n+1}$ and hence $X$ itself. For a vector field $Z$ in $\mathbb{C}^{n+2}$, let $\iota_Z$ denote the contraction of differential forms along $Z$ and for a multi-index $j = (j_0, ..., j_l)$ with $|j| := l$ let

$$\Omega_j := \iota_{\frac{\partial}{\partial x_l}} \circ \iota_{\frac{\partial}{\partial x_{l-1}}} \circ \cdots \circ \iota_{\frac{\partial}{\partial x_0}} \Omega \tag{10}$$

$$f_j := \frac{\partial f}{\partial x_{j_0}} \cdot \frac{\partial f}{\partial x_{j_1}} \cdots \frac{\partial f}{\partial x_{j_l}}. \tag{11}$$

**Theorem 3** (Carlson-Griffiths, [CG80], page 7)**.** *Let $\omega_i$ be a differential form of adjoint level $k$. Then, in $F^{n-k}/F^{n-k+1} \cong H^k(X, \Omega_X^{n-k})$, it is represented by the cocycle*

$$(\omega_i)^{n-k,k} = \frac{(-1)^{n(k+1)}}{k!} \left\{\frac{x^i \Omega_j}{f_j}\right\}_{|j|=k} \tag{12}$$

*with respect to the Jacobian covering.*

For the constant term in (12) see [CG80, page 12]. In order to be able to compute the integrals of the present text explicitly and without any constant ambiguity, see Theorem 1, we will need the following integration formula:

$$\int_{\mathbb{P}^{n+1}} \frac{\sum_{i=0}^{n+1}(-1)^i x_i \widehat{dx_i}}{x_0 x_1 \cdots x_{n+1}} = (-1)^{\binom{n+2}{2}} (2\pi\sqrt{-1})^{n+1}. \tag{13}$$

The integrand induces an element in the top algebraic de Rham cohomology $H^{2(n+1)}_{\mathrm{dR}}(\mathbb{P}^{n+1})$ and we have to use a canonical isomorphism between algebraic de Rham and usual de Rham cohomology in order to write it as a $C^\infty$ $2(n+1)$ differential form. Since this will not play any role in the proof of Theorem 4 we skip its proof.

## 4. Proof of Theorem 1

We prove Theorem 1 in the case, where $b$ is the identity and all $a_i$'s are zero. In this case we simply write $\mathbb{P}^{\frac{n}{2}} = \mathbb{P}^{\frac{n}{2}}_{a,b}$. Since $\mathbb{P}^{\frac{n}{2}}_{a,b}$'s are obtained by acting the automorphism group of the Fermat variety on a single linear cycle, the general formula easily follows. Let $\phi : \mathbb{P}^{\frac{n}{2}}_{(y_1:\cdots:y_{\frac{n}{2}+1})} \to \mathbb{P}^{n+1}_{(x_0:\cdots:x_{n+1})}$ be the immersion with the image $\mathbb{P}^{\frac{n}{2}}$ given by

$$\phi[y_1 : \cdots : y_{\frac{n}{2}+1}] = [\zeta_{2d} y_1 : y_1 : \cdots : \zeta_{2d} y_{\frac{n}{2}+1} : y_{\frac{n}{2}+1}].$$

We know from Carlson-Griffiths Theorem that

$$(\omega_i)^{\frac{n}{2},\frac{n}{2}} = \frac{1}{\frac{n}{2}!} \left\{ \frac{x^i \Omega_j}{d^{\frac{n}{2}+1}(x_{j_0} x_{j_1} \cdots x_{j_{\frac{n}{2}}})^{d-1}} \right\}_{|j|=\frac{n}{2}} \in H^{\frac{n}{2}}(\mathcal{U}, \Omega^{\frac{n}{2}}), \tag{14}$$

where $\mathcal{U} = \mathcal{J}_{X^d_n}$ is the standard covering of $\mathbb{P}^{n+1}$ and for simplicity we have written $\Omega^k = \Omega^k_{X^d_n}$. Therefore

$$\phi^* \omega_i = \frac{1}{d^{\frac{n}{2}+1} \cdot \frac{n}{2}!} \left\{ \frac{\zeta_{2d}^{i_0+i_2+\cdots+i_n} y^{i'} \phi^* \Omega_j}{\phi^*(x_{j_0} x_{j_1} \cdots x_{j_{\frac{n}{2}+1}})^{d-1}} \right\}_{|j|=\frac{n}{2}} \in H^{\frac{n}{2}}(\phi^{-1}(\mathcal{U}), \Omega^{\frac{n}{2}}), \tag{15}$$

where $i' = (i_0 + i_1, i_2 + i_3, \cdots, i_n + i_{n+1})$ and $\phi^{-1}(\mathcal{U})$ is the open covering of $\mathbb{P}^{\frac{n}{2}}$ given by the pre-images of the standard covering of $\mathbb{P}^{n+1}$. Note that this covering has repeated open sets. Since for $\{k_1, k_2, \cdots, k_{\frac{n}{2}}\} \subset \{1, 2, \cdots, \frac{n}{2}+1\}$

with $k_1 < k_2 < \cdots < k_{\frac{n}{2}}$ we have

$$(\phi^*\Omega_j)(\frac{\partial}{\partial y_{k_1}}, \cdots, \frac{\partial}{\partial y_{k_{\frac{n}{2}}}}) =$$
$$\Omega(\frac{\partial}{\partial x_{j_0}}, \cdots, \frac{\partial}{\partial x_{j_{\frac{n}{2}}}}, \zeta_{2d}\frac{\partial}{\partial x_{2k_1-2}} + \frac{\partial}{\partial x_{2k_1-1}}, \cdots, \zeta_{2d}\frac{\partial}{\partial x_{2k_{\frac{n}{2}}-2}} + \frac{\partial}{\partial x_{2k_{\frac{n}{2}}-1}}),$$

it follows that if $\#(j \cap \{2l-2, 2l-1\}) = 2$ for some $l \in \{1, 2, \cdots, \frac{n}{2}+1\}$, then $\phi^*\Omega_j = 0$. By abuse of notation here we have used $j$ for the set of its entries. On the other hand, if

$$\#(j \cap \{2l-2, 2l-1\}) = 1, \quad \forall l \in \{1, 2, \cdots, \frac{n}{2}+1\} \tag{16}$$

then

$$\phi^*\Omega_j(\frac{\partial}{\partial y_{k_1}}, \cdots, \frac{\partial}{\partial y_{k_{\frac{n}{2}}}}) = \zeta_{2d}^{j_{\text{odd}}}(-1)^{k+\binom{\frac{n}{2}}{2}+j_{\text{odd}}} y_k,$$

where $k$ is the missing element, that is, $\{k_1, \cdots, k_{\frac{n}{2}}, k\} = \{1, \cdots, \frac{n}{2}+1\}$ and $j_{\text{odd}} := \#\{0 \le i \le \frac{n}{2}, \ \ j_i \text{ is odd }\}$. Hence

$$\phi^*\Omega_j = (-\zeta_{2d})^{j_{\text{odd}}}(-1)^{\binom{\frac{n}{2}}{2}+1}\Omega', \qquad \text{where} \qquad \Omega' := \sum_{k=1}^{\frac{n}{2}+1}(-1)^{k-1}y_k\widehat{dy_k}. \tag{17}$$

Since for such $j$ we have $\phi^*(x_{j_0}\cdots x_{j_{\frac{n}{2}}})^{d-1} = \zeta_{2d}^{(d-1)(\frac{n}{2}+1-j_{\text{odd}})}(y_1\cdots y_{\frac{n}{2}+1})^{d-1}$, replacing (17) in (15) we get

$$\phi^*\omega_i = \frac{(-1)^{\binom{\frac{n}{2}+1}{2}}\zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n}y^{i'}\Omega'}{d^{\frac{n}{2}+1}\cdot\frac{n}{2}!(y_1\cdots y_{\frac{n}{2}+1})^{d-1}} \in H^{\frac{n}{2}}(\mathcal{U}', \Omega^{\frac{n}{2}}), \tag{18}$$

where $\mathcal{U}'$ is the standard covering of $\mathbb{P}^{\frac{n}{2}}$. The form (18) is exact except for the cases in which $i'_l = d-2, \forall l \in \{1, \cdots, \frac{n}{2}+1\}$. The result follows from the fact that the volume form $\frac{\Omega'}{y_1\cdots y_{\frac{n}{2}+1}}$ integrates $(-1)^{\binom{\frac{n}{2}+1}{2}}(2\pi\sqrt{-1})^{\frac{n}{2}}$ over $\mathbb{P}^{\frac{n}{2}}$.

## 5. An elementary linear algebra problem

The remaining piece in the proof of Theorem 2 is the following. For $N = d, \frac{n}{2}d - n - 2$ and $(\frac{n}{2}+1)d - n - 2$ let

(19)
$$I_N := \left\{(i_0, i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+2} \middle| 0 \le i_e \le d-2, \ \ i_0 + i_1 + \cdots + i_{n+1} = N\right\}$$

We fix two linear cycles

$$\mathbb{P}^{\frac{n}{2}} = \mathbb{P}^{\frac{n}{2}}_{a,b} \text{ with } a = (0,0,\cdots,0),\ b = (0,1,\cdots,n+1) \tag{20}$$

$$\check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^{\frac{n}{2}}_{a,b} \text{ with } a = (\underbrace{0,0,\cdots,0}_{m+1 \text{ times}},1,1,\cdots,1),\ b = (0,1,\cdots,n+1) \tag{21}$$

and for $i \in I_{(\frac{n}{2}+1)d-n-2}$ we define the number

$$\mathsf{p}_i := \int_{\mathbb{P}^{\frac{n}{2}}} \omega_i + \int_{\check{\mathbb{P}}^{\frac{n}{2}}} \omega_i \tag{22}$$

where $\omega_i$ is the differential form inside the integral in Theorem 1. For any other $i$ which is not in the set $I_{(\frac{n}{2}+1)d-n-2}$, $\mathsf{p}_i$ by definition is zero. Let $[\mathsf{p}_{i+j}]$ be the matrix whose rows and columns are indexed by $i \in I_{\frac{n}{2}d-n-2}$ and $j \in I_d$, respectively, and in its $(i,j)$ entry we have $\mathsf{p}_{i+j}$. For a sequence of natural numbers $\underline{a} = (a_1,\ldots,a_s)$ let us define

(23)
$$\mathsf{C}_{\underline{a}} = \binom{n+1+d}{n+1} - \sum_{k=1}^{s}(-1)^{k-1} \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_k}\le d} \binom{n+1+d-a_{i_1}-a_{i_2}-\cdots-a_{i_k}}{n+1},$$

where the second sum runs through all $k$ elements (without order) of $a_i$, $i = 1,2,\ldots,s$. By abuse of notation we write $a^b := \underbrace{a,a,\cdots,a}_{b \text{ times}}$.

**Proposition 1.** *For the triples $(n,d,m)$ in Theorem 2 we have*

$$\operatorname{rank}([\mathsf{p}_{i+j}]) = 2\mathsf{C}_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}} - \mathsf{C}_{1^{n-m+1},(d-1)^{m+1}}. \tag{24}$$

*Proof.* We verify Proposition 1 by a computer. For this the reader may download `foliation.lib`[1] from the the first author's web page, run SINGULAR (see [GPS01]), and type

```
LIB "foliation.lib";
Example SumTwoLinearCycle;
```

[1] http://w3.impa.br/~hossein/foliation-allversions/foliation.lib.

Modifying $n, d, m$ arguments in the example session of the procedure `SumTwo-LinearCycle` one gets all the cases in Theorem 2. The procedures `Periods-LinearCycle, Matrixpij, CodComIntZar` of the library `foliation.lib` are used for this verification. □

## 6. IVHS, periods and the proof of Theorem 2

Let us consider the family of hypersurface $X_t$ in the usual projective space $\mathbb{P}^{n+1}$ given by the homogeneous polynomial:

$$f_t := x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_j t_j x^j = 0, \tag{25}$$
$$t = (t_j)_{j\in I} \in (\mathsf{T}, 0),$$

where $x^j$ runs through $j \in I_d$. In a Zariski neighborhood of the Fermat variety, and up to linear transformations of $\mathbb{P}^{n+1}$, every hypersurface can be written in this format. In other words, the parameter space in (25) is transversal to the $\mathrm{PGL}(n+2)$-orbits near 0 of $\mathsf{T}$ in the introduction and its projection in $\mathsf{T}/\mathrm{PGL}(n+2)$ is etale near 0. We choose basis $x^i \in I_d$, $x^i, \ i \in I_{\frac{n}{2}d-n-2}, x^i, \ i \in I_{(\frac{n}{2}+1)d-n-2}$ for $\mathbf{T_0}\mathsf{T}$, $H^{\frac{n}{2}-1}(X_0, \Omega_{X_0}^{\frac{n}{2}+1})$ and $H^{\frac{n}{2}}(X_0, \Omega_{X_0}^{\frac{n}{2}})$, respectively. For a Hodge cycle $\delta_0 \in H_n(X_0, \mathbb{Q})$, we write ${}^0\bar{\nabla}\delta_0^{\mathsf{pd}}$ in the above basis and we get the matrix $[\mathsf{p}_{i+j}]$, where $\mathsf{p}_i := \int_{\delta_0} \omega_i$ are the periods of $\delta_0$. This matrix has been computed for the first time in [Mov17b]. For $\delta_0 := [Z_0], \quad Z_0 := \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$, Theorem 1 gives us an explicit formula for the periods $\mathsf{p}_i$ in (22). Using Koszul complex one can easily see that the right hand side of (24) is the codimension of $\check{\mathsf{T}}$ in $\mathsf{T}$, see [Mov17a, §17.9]. Knowing that $\ker[\mathsf{p}_{i+j}]$ is the Zariski tangent space of the analytic space $V_{[Z_0]}$ and the local branch of $(\check{\mathsf{T}}, 0)$ corresponding to deformations of $Z_0$ is inside the underlying analytic variety of $V_{[Z_0]}$, Proposition 1 implies that $V_{[Z_0]}$ is smooth and reduced and its underlying analytic variety is an open subset of the algebraic variety $\check{\mathsf{T}}$. Therefore, the restriction on $n$ and $d$ in our main theorem comes from the fact that we can prove Proposition 1 for the special cases of $(n, d, m)$ announced in Theorem 4.

Since $\omega_i, \ i \in I_{\frac{n}{2}(d-2)-n-2}$ form a basis of the primitive part of $F^{\frac{n}{2}}/F^{\frac{n}{2}+1}$ of $H^n_{\mathrm{dR}}(X)$, all the periods of $Z_\infty$ are zero. This implies that for two Hodge cycles $\delta_0, \tilde{\delta}_0 \in H_n(X_0, \mathbb{Q})$ such that $\delta_0 = b\tilde{\delta}_0 + c[Z_\infty] = 0$ for some $b, c \in \mathbb{Q}, \ b \neq 0$, we have $V_{\delta_0} = V_{\tilde{\delta}_0}$. For $\delta_0 = [\check{Z}]$ and $\tilde{\delta}_0 = [Z]$, this implies the second part in Theorem 4.

## 7. Complete intersection algebraic cycles

Let $\mathbb{C}[x]_d = \mathbb{C}[x_0, x_1, \cdots, x_{n+1}]_d$ be the set of homogeneous polynomials of degree $d$ in $n+2$ variables. Assume that $n \geq 2$ is even and $f \in \mathbb{C}[x]_d$ is of the following format:

$$f = f_1 f_{\frac{n}{2}+2} + f_2 f_{\frac{n}{2}+3} + \cdots + f_{\frac{n}{2}+1} f_{n+2}, \;\; f_i \in \mathbb{C}[x]_{d_i}, \;\; f_{\frac{n}{2}+1+i} \in \mathbb{C}[x]_{d-d_i}, \tag{26}$$

where $1 \leq d_i < d, \; i = 1, 2, \ldots, \frac{n}{2}+1$ is a sequence of natural numbers. Let $X \subset \mathbb{P}^{n+1}$ be the hypersurface given by $f = 0$ and $Z \subset X$ be the algebraic cycle given by $f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0$. We call $Z$ a complete intersection algebraic cycle in $X$. The Fermat variety has many of such algebraic cycles. Let $\mathsf{T}$ be the open subset of $\mathbb{C}[x]_d$ parameterizing smooth hypersurfaces of degree $d$ and $\mathsf{T}_{\underline{d}} \subset \mathsf{T}$ be its subset parameterizing those with (26). We use the notation $X_t, \; t \in \mathsf{T}$ and denote by $0 \in \mathsf{T}$ the point corresponding to Fermat variety. As another corollary of Theorem 1 we get:

**Theorem 4.** *Let consider the following cases:*

1. $d \geq 2 + \frac{4}{n}$ *and* $d_1 = d_2 = \cdots = d_{\frac{n}{2}+1} = 1$,
2. $n = 2, 4 \leq d \leq 15$,
3. $n = 4$ *and* $3 \leq d \leq 6$,
4. $n = 6$ *and* $3 \leq d \leq 4$.

*In all these cases, except the first one, all possible $\underline{d}$ is considered. There is a Zariski open (and hence dense) subset $U$ of $\mathsf{T}_{\underline{d}}$ such that for all $t \in U$ and a complete intersection algebraic cycle $Z \subset X := X_t$ as above, deformations of $Z$ as an algebraic cycle and Hodge cycle are the same.*

The property in Theorem 4 is actually verified for the Fermat hypersurface with one of its complete intersection algebraic cycles. Actually, for the first case in Theorem 4 we prove that the local analytic branches of $\mathsf{T}_{\underline{d}}$ near the Fermat point are smooth and reduced. For the rest we prove this property at least for one branch.

When the first draft of this article was written, we got to know the preprint [Dan14, Theorem 1.1] in which the author states Theorem 4 for arbitrary $d$. The exposition in this article can be improved, for instance the assumption $d > deg(Z)$ in the statement of Theorem 1.1 can be removed. The main ingredient in this theoretical proof is Macaulay's theorem which is missing in our computational proof. We highlight that the advantage of our computational proof is that it works for other algebraic cycles which are not complete intersections, see Theorem 2, whereas the proof in [Dan14] only works for complete intersections. The disadvantage is that one has to work with special values of $d$ and $n$ and it proves Theorem 4 for hypersurfaces in a Zariski open subset of $\mathsf{T}_{\underline{d}}$. We note that the main result in [Otw03] implies Theorem 4 for very large degrees, however, the lower bound in this article is not explicit and cannot be applied for a given degree.

For $n = 2$ the Hodge locus is also called Noether-Lefschetz locus, and for $d_1 = d_2 = 1$ one can even say more, that is namely, $\mathsf{T}_{1,1}$ is the only component of the Noether-Lefschetz locus with codimension $d-3$, see [Voi88, Gre89]. For a similar statement for the case $n = 2, d_1 = 1, d_2 = 2$ see [Voi89]. We do not deal with this issue in this article. The first case in Theorem 4 is proved in [Mov17b] and we give a new proof of this. The limitation in other cases is due to the fact that a part of the proof of Theorem 4, see Conjecture 1 below, is an elementary problem in linear algebra, for which we use a computer, and apart from the first case, we do not know how to solve it in general.

The proof of Theorem 4 is similar to Theorem 2. Proposition 1 is replaced with the following. Let

(27)
$$\check{I}_{(\frac{n}{2}+1)d-n-2} := \left\{ i \in I_{(\frac{n}{2}+1)d-n-2} \middle| i_{2l-2} + i_{2l-1} = d-2, \forall l = 1, \cdots, \frac{n}{2}+1 \right\}.$$

Let also $B_1, B_2, \cdots, B_{\frac{n}{2}+1}$ be subsets of $\{\zeta \in \mathbb{C} | \zeta^d + 1 = 0\}$ with cardinalities $d_1, d_2, \ldots, d_{\frac{n}{2}+1}$, respectively. For $i \in \check{I}_{(\frac{n}{2}+1)d-n-2}$ we define the number

$$\mathsf{p}_i := \prod_{k=0}^{\frac{n}{2}} \sum_{\zeta \in B_{k+1}} \zeta^{i_{2k}+1}. \tag{28}$$

For any other $i$ which is not in the set $\check{I}_{(\frac{n}{2}+1)d-n-2}$, $\mathsf{p}_i$ by definition is zero.

**Conjecture 1.** *We have*

$$\operatorname{rank}([\mathsf{p}_{i+j}]) = \mathsf{C}_{d_1,d_2,\ldots,d_{\frac{n}{2}+1},d-d_1,d-d_2,\ldots,d-d_{\frac{n}{2}+1}} \tag{29}$$

*where the number in the right hand side is defined in* (23).

We can verify Conjecture 1 by a computer for $n$ and $d$ given in item 2 of Theorem 4. The only theoretical proof that we have is the following.

**Proposition 2.** *For the case $d_1 = d_2 = \cdots = d_{\frac{n}{2}+1} = 1$ we have*

$$\operatorname{rank}[\mathsf{p}_{i+j}] = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2.$$

*Proof.* Let

$$A := \{i \in I_{\frac{n}{2}d-n-2} | i_0 = i_2 = \cdots = i_n = 0\},$$
$$B := \{j \in I_d | j_0 = j_2 = \cdots = j_n = 0\}.$$

Consider the map $\phi : B \to A$ given by $\phi(j)_{2l-2} = 0$, $\phi(j)_{2l-1} = d - 2 - j_{2l-1}$, for $l = 1, \cdots, \frac{n}{2} + 1$. It is easy to see that $\phi$ is a bijection and

$$\#A = \#B = \binom{\frac{n}{2} + d}{d} - (\frac{n}{2} + 1)^2.$$

We claim that the rows $\mathsf{p}_{i+\bullet}, i \in A$ form a base for the image of $[\mathsf{p}_{i+j}]$. Indeed, since for $(i, j) \in A \times B$

$$\mathsf{p}_{i+j} = \begin{cases} 1 & \text{if } i = \phi(j), \\ 0 & \text{otherwise,} \end{cases}$$

it follows that these rows are linearly independent. To see that they generate the image, it is enough to show that they generate all the rows. For $i \in I_{\frac{n}{2}d-n-2}$ if $i_{2l-2} + i_{2l-1} > d - 2$ for some $l \in \{1, \cdots, \frac{n}{2} + 1\}$, then $\mathsf{p}_{i+\bullet} = 0$. If not then there exists a unique $j \in B$ such that $i + j \in \check{I}_{(\frac{n}{2}+1)d-n-2}$. In fact $j_{2l-2} = 0$, $j_{2l-1} = d - 2 - i_{2l-2} - i_{2l-1}$, for $l = 1, \cdots, \frac{n}{2} + 1$. We claim that

$$\mathsf{p}_{i+\bullet} = \zeta_{2d}^{i_0+i_2+\cdots+i_n} \cdot \mathsf{p}_{\phi(j)+\bullet}.$$

For $h \in I_d$ with $\mathsf{p}_{\phi(j)+h} = 0$ we have $\phi(j) + h \notin \check{I}_{(\frac{n}{2}+1)d-n-2}$ and so there exists $l \in \{1, \cdots, \frac{n}{2} + 1\}$ such that

$$\phi(j)_{2l-2} + \phi(j)_{2l-1} + h_{2l-2} + h_{2l-1} > d - 2.$$

Since $\phi(j)_{2l-2} + \phi(j)_{2l-1} = i_{2l-2} + i_{2l-1}$, it follows that $\mathsf{p}_{i+h} = 0$. On the other hand, for $h \in I_d$ with $\phi(j) + h \in \check{I}_{(\frac{n}{2}+1)d-n-2}$, we have $i + h \in \check{I}_{(\frac{n}{2}+1)d-n-2}$ and

$$\mathsf{p}_{i+h} = \zeta_{2d}^{(i_0+h_0)+\cdots+(i_n+h_n)} = \zeta_{2d}^{i_0+\cdots+i_n} \cdot \zeta_{2d}^{h_0+\cdots+h_n} = \zeta_{2d}^{i_0+\cdots+i_n} \cdot \mathsf{p}_{\phi(j)+h}. \qquad \square$$

Let

$$(30) \quad Z_0 : \prod_{\zeta \in B_1} (x_0 - \zeta x_1) = \prod_{\zeta \in B_2} (x_2 - \zeta x_3) = \cdots = \prod_{\zeta \in B_{\frac{n}{2}+1}} (x_n - \zeta x_{n+1}) = 0,$$

where $B_i$'s are as in §5. For $\delta_0 := [Z_0]$ Theorem 1 implies that up to multiplication by a constant which does not depend on $i$ we have $\int_Z \omega_i = \mathsf{p}_i$, where $\mathsf{p}_i$

is defined in (28). Using Koszul complex one can easily see that the left right hand side of (29) is the codimension of $\mathsf{T}_{\underline{d}}$ in $\mathsf{T}$, see [Mov17a, Chapter 17]. The rest of the argument is similar to the proof of Theorem 2. Note that the restriction on $n$ and $d$ in our main theorem comes from the fact that we can prove Conjecture 1 for the special cases of $n$ and $d$ announced in Theorem 4.

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