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On the existence of solution for degenerate parabolic equations with singular terms

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Abstract: We are interested in results concerning the solutions to the parabolic problems whose simplest model is the following:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u \ (:= \operatorname{div}(|\nabla u|^{p-2}\nabla u)) + B\frac{|\nabla u|^p}{u} = f & \text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \end{cases}$$

where $T > 0$, $N \geq 2$, $B > 0$, u_0 is a positive function in $L^\infty(\Omega)$ bounded away from zero and f is a nonnegative function that belongs to some Lebesgue space.

Keywords: Nonlinear parabolic equations, singular parabolic equations, Sobolev space.

1. Introduction

In this paper, we are going to study the following parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(t,x,\nabla u) + H(t,x,u,\nabla u) = f & \text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, $T > 0$, $N \geq 2$, $2 < p < +\infty$, $u_0 \in L^\infty(\Omega)$ and $0 \leq f$ in $L^r(0,T;L^q(\Omega))$, with $\frac{p}{r} + \frac{N}{q} < p$, $r \geq \frac{p}{p-1}$, $q > 1$. Such equation arises in the theory of non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [5, 10] for detailed discussion.

In the elliptic case, Consider the equation

$$\begin{cases} -\Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

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here $m > 0$, $B > 0$ and f is a nonnegative (not identically zero) function in $L^1(\Omega)$. The problem is obviously singular as we ask the solution to vanish at the boundary of Ω. In [1] the existence of a finite energy (i.e., in $H_0^1(\Omega)$) solution to problem (1.2) has been proved if $m < 2$ and for data f locally bounded away from zero. The case of a possibly degenerate datum f has been also considered. If $m < 1$ the existence of a solution in $H_0^1(\Omega)$ was proved in [2] for general nonnegative (not identically zero) data, while the case $m = 1$ was faced in [13] provided B was small enough. Problems as in (1.2) with possibly changing-sign data have also been considered in [8] in the case $m < 1$ (see also [9] for further considerations concerning the strongly singular case).

In the parabolic case, problems as

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial t} - \Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{in} & (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} & \Omega, \\ u(t,x) = 0 & \text{on} & (0,T)\times\partial\Omega, \end{array}\right. \tag{1.3}$$

have been considered in the case $p = 2$ and $m < 1$ (see [14]). If $m = 1$ singular problems as (1.3) have been considered in [19, 20] for smooth strictly positive data, while degenerate problems (i.e. $p > 2$) were studied in [21] in the one dimensional case and in [7], the authors study the existence of solutions for a general class of singular homogeneous ($f = 0$).

The aim of this paper is to study existence of solutions for a class of singular non-homogeneous ($f \neq 0$) parabolic problems as (1.3) in the limit case $m = 1$. We will mainly be concerned with the case $p > 2$.

The paper is structured as follows: in the next section 2 we set the main assumptions, we state our main result, and we introduce some preliminary tools. Section 3 is devoted to prove existence of the main result.

2. Basic assumptions and main result

From now on, we will set $Q = (0,T)\times\Omega$. Let us spend a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C, C' and C'' to denote any positive constant (possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (f, N, Ω, T, B, α, p, β, and so on...). But they will never depend on the indexes of the sequences we will often introduce. For the sake of simplicity we will often use the simplified notation

$$\int_Q f = \int_Q f(t,x)dtdx$$

when referring to integrals when no ambiguity on the variable of integration is possible. For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) = \max(-k, \min(s, k)) \text{ and } G_k(s) = s - T_k(s) = (|s| - k)^+ \text{sign}(s).$$

Let us state our main assumptions. Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $T > 0$. The function

$$0 \leq f \in L^r(0, T; L^q(\Omega)), \tag{2.1}$$

with $\frac{p}{r} + \frac{N}{q} < p, \quad r \geq p', \quad q > 1$, satisfies

$$m_\omega(f) = \text{ess inf } \{f(x, t) : x \in \omega, t \in (0, T)\} > 0, \forall \omega \subset\subset \Omega.$$

Moreover, we consider an initial datum $u_0(x)$ which is a function in $L^\infty(\Omega)$ such that $u_0 \geq c > 0$ almost everywhere on Ω, and we suppose that

$$m_\omega(u_0) = \text{ess inf } \{u_0(x) : x \in \omega\} > 0, \forall \omega \subset\subset \Omega. \tag{2.2}$$

Let $a : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory vector-valued function such that

$$a(t, x, \xi)\xi \geq \alpha|\xi|^p, \tag{2.3}$$

$$|a(x, s, \xi)| \leq \beta|\xi|^{p-1}, \tag{2.4}$$

$$(a(t, x, \xi) - a(t, x, \eta))(\xi - \eta) > 0, \tag{2.5}$$

for a.e. $(t, x) \in (0, T) \times \Omega$, for every $\xi \neq \eta \in \mathbb{R}^N$ and α, β are positive constants and $p > 2$.

Furthermore, let $H(t, x, s, \xi) : (0, T) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that

$$0 \leq H(t, x, s, \xi) \leq B\frac{|\xi|^p}{s}, \tag{2.6}$$

for a.e. $(t, x) \in (0, T) \times \Omega$, for every $s > 0, \xi \in \mathbb{R}^N$, and B is a positive constant.

Consider problem (1.1). Here is the meaning of weak solution for such a problem.

Definition 2.1. A weak solution to problem (1.1) is a function $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C(0,T; L^1(\Omega))$ such that for every $\omega \subset\subset \Omega$ there exists c_ω such that $u \geq c_\omega > 0$ in $(0,T) \times \omega$; furthermore, we have that

$$-\int_Q \langle u, \varphi' \rangle - \int_\Omega u_0 \varphi(0) + \int_Q a(t,x,\nabla u)\nabla\varphi + \int_Q H(t,x,u,\nabla u)\varphi = \int_Q f\varphi,$$

for every $\varphi \in C_c^1([0,T) \times \Omega)$, that is, for every C^1 function which vanishes in a neighborhood of $\{T\} \times \Omega$ and of $(0,T) \times \partial\Omega$.

Note that under assumption (2.6), the function $H(t,x,u,\nabla u)$ belongs to $L^1_{loc}(Q)$ thanks to the property of local positivity required on u.

Our main result is the following:

Theorem 2.1. *Assume that* (2.1)–(2.6) *hold. If $p > 2$ there exists a weak solution to problem* (1.1).

Before the proof we recall some technical tools we will use. The first one is a well known consequence of the Gagliardo-Nirenberg inequality which is valid on any cylinder of the type $Q = (0,T) \times \Omega$ with bounded Ω (see for instance [16], Lecture II).

Lemma 2.2. *Let $v \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^\beta(\Omega))$, with $p \geq 1$, $\beta \geq 1$. Then $v \in L^\sigma(Q)$ with $\sigma = p\frac{N+\beta}{N}$ and*

$$\int_Q |v|^\sigma \leq C \|v\|_{L^\infty(0,T;L^\beta(\Omega))}^{\frac{\beta p}{N}} \int_Q |\nabla v|^p. \tag{2.7}$$

Finally, we will need the following local version of a lemma by Stampacchia (see [18]).

Lemma 2.3. *Let $\omega(h,r)$ be a function defined on $[0,+\infty) \times [0,1]$, which is nonincreasing in h and nondecreasing in r; suppose that there exist constants $k_0 \geq 0$, $M, \rho, \sigma > 0$, and $\eta > 1$ such that*

$$\omega(h,r) \leq \frac{M\omega(k,R)^\eta}{(h-k)^\rho (R-r)^\sigma},$$

for all $h > k \geq k_0$ and $0 \leq r < R \leq 1$. Then, for every r in $(0,1)$, there exists $d > 0$, given by

$$d^\rho = \frac{M 2^{\frac{\eta(\rho+\sigma)}{\eta-1}} \omega(k_0,1)^{\eta-1}}{(1-r)^\sigma}$$

such that

$$\omega(d,r)=0.$$

Finally, let us state the following result that will be very useful in the sequel; its proof relies on an easy application of Egorov theorem.

Lemma 2.4. *Let $Q=\Omega\times]0,T[$, where Ω is an open and bounded subset of $\mathbb{R}^N$ and $T>0$. Let ρ_ε be a sequence of $L^1(Q)$ functions that converges to ρ weakly in $L^1(Q)$, and let σ_ε be a sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to σ almost everywhere on Q. Then*

$$\lim_{\varepsilon\to 0}\int_Q \rho_\varepsilon\sigma_\varepsilon dxdt=\int_Q \rho\sigma dxdt$$

3. Proof of main result

Our strategy in order to prove Theorem 2.1 will rely on an approximation argument. The next subsection will introduce our approximating problems.

3.1. The approximating problems

We consider the approximating problems

$$(3.1)\quad \begin{cases} \frac{\partial u_n}{\partial t}-\operatorname{div} a(t,x,\nabla u_n)+H(t,x,u_n,\nabla u_n)=f & \text{in} \quad Q,\\ u_n(0,x)=u_0(x)+\frac{1}{n} & \text{in} \quad \Omega,\\ u_n(t,x)=\frac{1}{n} & \text{on} \quad (0,T)\times\partial\Omega.\end{cases}$$

A weak solution to this problem is a function u_n such that $u_n\geq\frac{1}{n}$ a.e. in Q.

$u_n-\frac{1}{n}\in L^p([0,T];W_0^{1,p}(\Omega))\cap C(0,T;L^1(\Omega))$ and $\frac{\partial u_n}{\partial t}\in L^1(\Omega)+L^{p'}([0,T];W^{-1,p'}(\Omega))$ and such that

$$(3.2)\qquad \int_0^T\langle\frac{\partial u_n}{\partial t},v\rangle+\int_Q a(t,x,\nabla u_n)\nabla v+\int_Q H(t,x,u_n,\nabla u_n)v=\int_Q fv$$

for any $v\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$. A nonnegative weak solution u_n to problem (3.1) does exist. In fact, problem (3.1) is equivalent to

$$(3.3)\quad \begin{cases} \frac{\partial v_n}{\partial t}-\operatorname{div} a(t,x,\nabla v_n)+H(t,x,v_n+\frac{1}{n},\nabla v_n)=f & \text{in} \quad Q,\\ v_n(0,x)=u_0(x) & \text{in} \quad \Omega,\\ v_n(t,x)=0 & \text{on} \quad (0,T)\times\partial\Omega.\end{cases}$$

where $v_n = u_n - \frac{1}{n}$.

To prove that a solution of (3.3) exists, we first extend $H(t,x,s,\xi)$ to be zero for $s \leq 0$, and, for $\varepsilon > 0$, we consider the problem

$$(3.4)\qquad \begin{cases} \frac{\partial v_{n,\varepsilon}}{\partial t} - \operatorname{div} a(t,x,\nabla v_{n,\varepsilon}) + \frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon} H(t,x,v_{n,\varepsilon} + \frac{1}{n}, \nabla v_{n,\varepsilon}) = f \\ \quad \text{in} \quad Q, \\ v_{n,\varepsilon}(0,x) = u_0(x) \quad \text{in} \quad \Omega, \\ v_{n,\varepsilon}(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \end{cases}$$

A nonnegative solution $v_{n,\varepsilon}$ to problem (3.4) exists by the results proven in [6]. Then, if we take a sequence of values $\varepsilon \downarrow 0$, one can prove that the sequence $\{v_{n,\varepsilon}\}_\varepsilon$ converges strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ to some function v_n. Then one can pass to the limit for $\varepsilon \downarrow 0$ in the first two terms of (3.4), in the sense of distributions. As far as the third term is concerned, we observe that, on the set $\{v_n > 0\}$, the function $\frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon}$ converges a.e. to 1, while on the set $\{v_n = 0\}$ (where we cannot identify the limit of $\frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon}$, but where $\nabla v_n = 0$ a.e. by Stampacchia's result contained in [18]) the term $H(t,x,v_{n,\varepsilon}+\frac{1}{n},\nabla v_{n,\varepsilon})$ converges to $H(t,x,v_n + \frac{1}{n}, \nabla v_{n,})$ a.e., which is zero a.e. on this set, since $H(t,x,\frac{1}{n},0) = 0$ a.e. by assumption (2.6). Therefore, v_n is a weak solution of (3.3).

3.2. A priori estimates

A standard argument allows us to show that some basic estimates on the approximating solutions hold. We collect them in the following:

Lemma 3.1. *Let $p \geq 2$, and let u_n be a solution to problem (3.1). Then,*

$$(3.5)\qquad \|u_n\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C,\ \|u_n\|_{L^\infty(Q))} \leq C \ \text{and} \int_Q H(t,x,u_n,\nabla u_n) \leq C.$$

Moreover, there exists a function u in $L^p(0,T;W_0^{1,p}(\Omega))$ such that (up to subsequences) $u_n - \frac{1}{n}$ converges to u weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and a.e. on Q. Finally,

$$\nabla u_n \to \nabla u, \ \text{a.e. on } Q.$$

Proof. The proof of the first two estimates is quite standard and can be deduced for instance as in [4] (see also [3]) using the fact that the lower order term is positive. In order to get (3.5) one can use $\frac{1}{\varepsilon}T_\varepsilon(u_n - \frac{1}{n})$ as test function in (3.1). Integrating by parts, dropping nonnegative terms and letting ε go to zero one gets, by Fatou's lemma

$$\int_Q H(t,x,u_n,\nabla u_n) \leq \int_\Omega (u_0(x) + \frac{1}{n}) + \int_Q f,$$

which implies (3.5). The almost everywhere convergence of the gradients of u_n is a consequence of (3.5) and of a result in [[3], Theorem 3.3.]. $\square$

3.3. Concluding the proof of Theorem 2.1

We wish to pass to the limit in the weak formulation of (3.1).

Claim 1: A key tool in order to pass to the limit will be the following one.

Lemma 3.2. *Let $p > 2$, and let u_n be a weak solution of problem (3.1). Then, for any $\omega \subset\subset \Omega$, there exists a constant c_ω such that*

$$u_n \geq c_\omega > 0, \text{ in } (0,T) \times \omega, \text{ for every } n \text{ in } \mathbb{N}.$$

In order to simplify notations, we henceforth write $a(\nabla u_n)$ instead of $a(t,x,\nabla u_n)$ and $H(u_n,\nabla u_n)$ instead of $H(t,x,u_n,\nabla u_n)$.

Proof. We divide the proof into three steps.

Step 1. There is no loss of generality in assuming that the constant B which appears in (2.6) satisfies $B > \max(\alpha, p-1)$.

We use $v = -u_n^{-B}\psi$ in (3.2), for any nonnegative $\psi(t,x) \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0,T) \times \partial\Omega$, in order to obtain

$$\begin{aligned} -\int_0^T \langle \frac{\partial u_n}{\partial t}, u_n^{-B}\psi\rangle + B\int_Q a(\nabla u_n)\nabla u_n u_n^{-B-1}\psi + \int_Q a(\nabla u_n)\nabla\psi u_n^{-B} \\ -\int_Q H(u_n,\nabla u_n)u_n^{-B}\psi \\ = -\int_Q f u_n^{-B}\psi \leq 0, \end{aligned}$$

from which taking into account the assumptions (2.3), (2.6) and $B > \alpha$, one obtains

$$-\int_0^T \langle \frac{\partial u_n}{\partial t}, u_n^{-B}\psi\rangle + \int_Q a(\nabla u_n)\nabla\psi u_n^{-B} \leq 0.$$

Therefore, if we set $u_n = \frac{B+1-p}{p-1}\omega_n^{-\frac{p-1}{B+1-p}}$ and $\gamma = \frac{(p-1)(B-1)}{B+1-p}$, we have

$$\int_0^T \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1}\psi\rangle + C\int_Q a(-\omega_n^{\frac{-B}{B+1-p}}\nabla\omega_n)\nabla\psi\omega_n^{\frac{B(p-1)}{B+1-p}} \;\; \leq \;\; 0, \tag{3.6}$$

for every nonnegative $\psi(t,x) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0,T)\times\partial\Omega$ and for some positive constant C depending only on B and p. Observe that $B > p-1$ and $p > 2$ imply that $\gamma > p-1 > 1$. Moreover, observe that

$$\omega_n(t,x) = cn^{\frac{B+1-p}{p-1}} \text{ on } (0,T)\times\partial\Omega, \qquad \omega_n(0,x) = c(u_0 + \frac{1}{n})^{-\frac{B+1-p}{p-1}} = \omega_{0n}$$

for some positive constant c. In particular, since u_0 is bounded away from zero, then ω_{0n} is bounded in $L^\infty(\Omega)$ and the values of ω_n blow up on the boundary as n goes to infinity. We look for an a priori local bound on the L^∞ norm of ω_n.

Step 2. Local L^∞ bound for ω_n.

Without loss of generality we assume that $0 \in \Omega$; we will prove that the bound holds true in a ball B_ρ centered at zero of radius ρ with $0 < r < \rho < R \leq 1$, then a standard covering argument will allow us to conclude.

We fix $k > \|\omega_{0n}\|_{L^\infty(B_R)}$ and we define the sets

$$A_{K,\rho}(t) = \{x \in B_\rho : \omega_n(t,x) > k\},$$
$$A_{K,\rho} = \{(t,x) \in (0,T)\times B_\rho : \omega_n(t,x) > k\}$$

We consider a cut-off function $\varphi(x) \in C_c^\infty(B_R)$ such that

$$0 \leq \varphi(x) \leq 1, \qquad \varphi(x) \equiv 1 \text{ in } B_r, \qquad |\nabla\varphi| \leq \frac{C}{R-r},$$

and use $\psi(t,x) = G_k(\omega_n(t,x))\varphi^\delta(x)$ as test function in (3.6) where $\delta = \frac{p(\gamma+1)}{\gamma-p+1}$. Integrating between 0 and $t < T$, and using the assumptions (2.3) and (2.4), we obtain

$$\begin{aligned}&\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1}G_k(\omega_n)\varphi^\delta\rangle + \int_0^t\int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^p\varphi^\delta \leq \\ &\qquad \frac{C}{R-r}\int_0^t\int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^{p-1}\varphi^{\delta-1}G_k(\omega_n).\end{aligned} \tag{3.7}$$

We use Young's inequality in order to absorb the term $|\nabla G_k(\omega_n)|^{p-1}$ in the right hand side; using the definition of δ, we get

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^p \varphi^\delta \leq \frac{C}{(R-r)^p} \int_{A_{k,R}} G_k(\omega_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.$$

Notice that, since $0 \leq \varphi \leq 1$,

$$\begin{aligned} |\nabla(G_k(\omega_n)\varphi^\delta)|^p &\leq C(|\nabla G_k(\omega_n)|^p \varphi^{\delta p} + \tfrac{G_k(\omega_n)^p \varphi^{\delta p - p}}{(R-r)^p}) \\ &\leq C(|\nabla G_k(\omega_n)|^p \varphi^{\delta} + \tfrac{G_k(\omega_n)^p \varphi^{\delta - p}}{(R-r)^p}) \end{aligned}$$

so that, observing that $\delta - p = \frac{p^2}{\gamma-p+1}$, we finally get

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)\varphi^\delta|^p \leq \frac{C}{(R-r)^p} \int_{A_{k,R}} G_k(\omega_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.$$

Since $\delta > p-1$, we can use again Young's inequality to have

(3.8)
$$\begin{aligned} &\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)\varphi^\delta|^p \\ &\quad \leq \frac{1}{2(\gamma+1)T} \int_{A_{k,R}} G_k(\omega_n)^{\gamma+1}\varphi^\delta + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}} |A_{k,R}| \\ &\quad \leq \frac{1}{2(\gamma+1)} \sup_t \int_{A_{k,R}(t)} G_k(\omega_n)^{\gamma+1}\varphi^\delta \, dx + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}} |A_{k,R}|. \end{aligned}$$

Now we deal with the time derivative part. Let us define, for $s \geq 0$,

$$\Psi_k(s) = \int_0^s G_k(\sigma)\sigma^{\gamma-1} \, d\sigma$$

Then it is easy to check that

$$\Psi_k(s) \geq \frac{1}{\gamma+1} G_k(s)^{\gamma+1}$$

that $k > \|\omega_0\|_{L^\infty(B_R)}$, we obtain

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle = \int_{B_R} \Psi_k(\omega_n(t,x))\varphi^\delta \geq \frac{1}{\gamma+1}\int_{B_R} G_k(\omega_n)^{\gamma+1}\varphi^\delta.$$

Therefore we can use (3.8) in order to deduce

$$\begin{aligned} &\frac{1}{\gamma+1}\int_{B_R} G_k(\omega_n)^{\gamma+1}\varphi^\delta \\ &\quad \leq \frac{1}{2(\gamma+1)}\sup_t \int_{A_{k,R}} G_k(\omega_n)^{\gamma+1}\varphi^\delta\, dx + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}}|A_{k,R}|. \end{aligned}$$

and we can take the spermium over $t \in (0,T)$ on the left in order to get

$$\sup_t \int_{A_{k,R}(t)} G_k(\omega_n)^{\gamma+1}\varphi^\delta\, dx \leq \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}}|A_{k,R}|.$$

Gathering together all these facts, and using again that $\varphi^\delta \geq \varphi^{\delta(\gamma+1)}$, we end up with the following estimate

$$\sup_t \int_{A_{k,R(t)}} G_k(\omega_n\varphi^\delta)^{\gamma+1}\, dx + \int_{A_{k,R}} |\nabla G_k(\omega_n)\varphi^\delta|^p\, dx \leq \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}}|A_{k,R}|.$$

We are now in the position to apply the Gagliardo-Nirenberg inequality (2.7) to the function $G_k(\omega_n)\varphi^\delta$, with $\beta = \gamma+1$; recalling that $\varphi \equiv 1$ on B_r, we obtain

$$\int_{A_{k,R}} G_k(\omega_n)^{p\frac{N+\gamma+1}{N}}\, dx \leq \frac{C}{(R-r)^{\frac{(\gamma+1)p(N+p)}{(\gamma+1-p)N}}}|A_{k,R}|^{1+\frac{p}{N}}.$$

Stampacchia's procedure is now quite standard. For $h > k$, one obtains

$$\int_{A_{k,R}} G_k(\omega_n)^{p\frac{N+\gamma+1}{N}}\, dx \geq \int_{A_{h,r}} G_h(\omega_n)^{p\frac{N+\gamma+1}{N}}\, dx \geq (h-k)^{p\frac{N+\gamma+1}{N}}|A_{h,r}|,$$

that is,

$$|A_{h,r}| \leq \frac{C|A_{k,R}|^{1+\frac{p}{N}}}{(h-k)^{p\frac{N+\gamma+1}{N}}(R-r)^{\frac{(\gamma+1)p(N+p)}{(\gamma+1-p)N}}}.$$

Therefore, if we choose $\omega(h,r) = |A_{h,r}|$, we can apply Lemma 2.3 in order to deduce that, for every fixed $\rho \in (r,R)$, $|A_{h,r}| = 0$ if h is larger than some constant C_ρ. It follows that

$$\omega_n \leq C_\rho, \text{ a.e. on } (0,T)\times B_\rho, \text{ for every } n. \tag{3.9}$$

Step 3. End of the proof. Recalling that $B > p-1$ and the definition of ω_n we use (3.9) to have, a.e. on $(0,T)\times B_\rho$

$$u_n = \frac{B+1-p}{p-1}\omega_n^{-\frac{p-1}{B+1-p}} \geq \frac{B+1-p}{p-1}C_\rho^{-\frac{p-1}{B+1-p}} = c_\rho > 0.$$

As we said, by means of a standard covering procedure this estimate can be proven to hold on any set of the form $(0,T)\times\omega$, with $\omega \subset\subset \Omega$. □

Claim 2: Passing to the limit. In order to pass to the limit as n tends to infinity, we need the following result.

Proposition 3.3. *We have*

$$u_n \longrightarrow u \text{ strongly in } L^p(0,T;W^{1,p}(\omega)),$$

for every open set $\omega \subset\subset \Omega$.

Proof. The sequence $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$. Using the Aubin-Simon compactness argument (see Corollary 4 in [17]), we deduce that, up to a subsequence,

$$u_n \longrightarrow u \text{ in } L^p(Q),$$

for some u in $L^p(0,T;W^{1,p}(\Omega))\cap L^\infty(Q)$. We will prove that, for every open set $\omega \subset\subset \Omega$,

$$u_n \longrightarrow u \text{ in } L^p(0,T;W^{1,p}(\omega)), \tag{3.10}$$

We now introduce a classical regularization u_ν of the function u with respect to time (see [11]). For every $\nu \in \mathbb{N}$, we define u_ν as the solution of the Cauchy problem

$$\begin{cases} \frac{1}{\nu}\frac{\partial u_\nu}{\partial t} + u_\nu = u \\ u_\nu(0) = u_{0,\nu} \end{cases}$$

where $u_{0,\nu} \in W_0^{1,p}(\Omega))\cap L^\infty(\Omega)$, satisfies $u_{0,\nu} \longrightarrow u_0$ strongly in $L^1(\Omega)$, $*-$weakly in $L^\infty(\Omega)$ and $\lim_{\nu\to+\infty}\frac{1}{\nu}\|u_{0,\nu}\|_{W_0^{1,p}(\Omega)} = 0$.

Then one has (see [11]):

$$u_\nu \in L^p(0,T;W_0^{1,p}(\Omega)),\ \frac{\partial u_\nu}{\partial t} \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ and } \|u_\nu\|_{L^\infty(Q)} \leq \|u\|_{L^\infty(Q)},$$

and as ν tends to infinity,

$$u_\nu \longrightarrow u \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.11}$$

Let $\varphi_\lambda(s) = se^{\lambda s^2}$ (with λ to be chosen later). We will denote by $\varepsilon(\nu, n)$ any quantity such that

$$\lim_{\nu\to+\infty} \limsup_{n\to\infty} |\varepsilon(\nu, n)| = 0.$$

For $0 \leq \phi \in C_c^\infty(\Omega)$, we have that

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - u_\nu)\phi\rangle \geq \varepsilon(\nu, n). \tag{3.12}$$

Now, using (3.12) and $\varphi_\lambda(u_n - u_\nu)\phi$ as test function in (3.2), we obtain

$$\begin{aligned}\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi + \int_Q a(\nabla u_n)\nabla\phi\varphi_\lambda(u_n - u_\nu)\\ + \int_Q H(u_n, \nabla u_n)\varphi_\lambda(u_n - u_\nu)\phi\\ \leq \int_Q f\varphi_\lambda(u_n - u_\nu)\phi - \varepsilon(\nu, n).\end{aligned}$$

Moreover, if $\omega \subset\subset \Omega$ is such that supp $\phi \subset \omega$ since $u_n \to u$ weakly in $L^p(0, T; W_0^{1,p}(\Omega))$ and $\varphi_\lambda(u_n - u_\nu)\phi$ converges to $\varphi_\lambda(u - u_\nu)\phi$ $*$-weakly in $L^\infty(Q)$, so that, by Lemma 2.4, we have

$$\int_Q f\varphi_\lambda(u_n - u_\nu)\phi - \int_Q a(\nabla u_n)\nabla\phi\varphi_\lambda(u_n - u_\nu) = \varepsilon(\nu, n).$$

If c_ω is the constant given by Lemma 3.2, we have, recalling that supp $\phi \subset \omega$

$$\begin{aligned}\Big|\int_Q H(u_n, \nabla u_n)\varphi_\lambda(u_n - u_\nu)\phi\Big| &\leq B\int_{\omega\times(0,T)} \frac{|\nabla u_n|^p}{u_n}|\varphi_\lambda(u_n - u_\nu)|\phi\\ &\leq \frac{B}{c_\omega}\int_Q |\nabla u_n|^p|\varphi_\lambda(u_n - u_\nu)|\phi.\end{aligned}$$

Thus,

$$\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi - \frac{B}{c_\omega}\int_Q |\nabla u_n|^p|\varphi_\lambda(u_n - u_\nu)|\phi \leq \varepsilon(\nu, n). \tag{3.13}$$

Then we can write

$$\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi$$

$$
\begin{aligned}
&= \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi \\
&\quad + \int_Q a(\nabla u_\nu)\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi \\
&= \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\varphi'_\lambda(u_n - u_\nu)\phi + \varepsilon(\nu, n).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_Q |\nabla u_n|^p |\varphi_\lambda(u_n - u_\nu)|\phi \\
&\quad \le \alpha^{-1} \int_Q a(\nabla u_n)\nabla u_n |\varphi_\lambda(u_n - u_\nu)|\phi \\
&\quad = \alpha^{-1} \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi \\
&\qquad + \int_Q a(\nabla u_\nu)\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi + \int_Q a(\nabla u_n)\nabla u_\nu |\varphi_\lambda(u_n - u_\nu)|\phi \\
&\quad = \alpha^{-1} \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi + \varepsilon(\nu, n).
\end{aligned}
$$

Therefore, from (3.13) we obtain

$$
\begin{aligned}
&\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)[\varphi'_\lambda(u_n - u_\nu) - \frac{B}{\alpha c_\omega}|\varphi_\lambda(u_n - u_\nu)|]\phi \\
&\quad \le \varepsilon(\nu, n).
\end{aligned}
$$

Choosing λ large enough so that $\varphi'_\lambda(s) - \frac{B}{\alpha c_\omega}|\varphi_\lambda(s)| \ge \frac{1}{2}$ for every $s \in \mathbb{R}$, we deduce that

$$
\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\phi \le \varepsilon(\nu, n).
$$

From here it is standard (see for example [15]) to prove that $u_n - u_\nu$ tends to zero strongly in $L^p(0, T; W^{1,p}(\omega))$. Recalling (3.11), we thus have that (3.10) holds. □

Using Proposition 3.3 we can prove the (local) strong convergence of the lower order terms.

Lemma 3.4. *We have*

$$
H(u_n, \nabla u_n) \to H(u, \nabla u) \textit{ locally strongly in } L^1(Q).
$$

Proof. Gathering together the results of Proposition 3.3, Lemma 3.1, and Lemma 3.2, we can apply Lebesgue's dominated convergence theorem to prove that

$$\frac{|\nabla u_n|^p}{u_n} \to \frac{|\nabla u|^p}{u}, \text{ locally strongly in } L^1(Q).$$

It is then straightforward to conclude using (2.6) and Vitali's theorem. □

Thanks to all the results proved so far we can pass to the limit in (3.1), to have that u is a solution of (1.1) in the sense of Definition 2.1, thus concluding the proof of Theorem 2.1.

Remark 3.1. The hypothesis $p > 2$ seems necessary to obtain the boundlessness of Lemma 3.2, we can not say if we can do without it or not.

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