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Cartan motion groups and dual topology

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To the memory of Majdi Ben Halima

Abstract: Let G be a connected reductive Lie group with Lie algebra $\mathfrak{g}$, and let $\overline{G}$ be the analytic subgroup corresponding to $[\mathfrak{g},\mathfrak{g}]$. Assume $\overline{G}$ has finite center. Let K be a maximal compact subgroup of G and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ be the corresponding Cartan decomposition. Then K acts on $\mathfrak{s}$ by the adjoint representation $(k.X=Ad_K(k)X)$. The Cartan motion group H (associated to G) is the semidirect product $H=K\ltimes\mathfrak{s}$. In this paper, we prove that the unitary dual $\widehat{H}$ of H is homeomorphic to the space $\mathfrak{h}^{\ddagger}/H$ of all admissible coadjoint orbits of H.

Keywords: Lie groups, semidirect product, unitary representations, coadjoint orbits, symplectic induction.

1. Introduction

Let G be a locally compact group. We denote by $\widehat{G}$ the unitary dual of G. It well-known that $\widehat{G}$ equipped with the Fell topology (see [8]). The first representation-theoretic question concerning the group G is the parametrezation of the set $\widehat{G}$. In the setting of Lie group with Lie algebra $\mathfrak{g}$, the investigation of the relationship between $\widehat{G}$ and the space $\mathfrak{g}^*/G$ of G-coadjoint orbits turns out to be a deep mathematical problem. Its well-known that for a simply connected nilpotent Lie group or, more generally, for an exponential solvable Lie group, the unitary dual $\widehat{G}$ is homeomorphic to the orbit space $\mathfrak{g}^*/G$ (see [18]).

Let now G be a connected reductive Lie group with Lie algebra $\mathfrak{g}$ and let $\overline{G}$ be the analytic subgroup corresponding to $[\mathfrak{g},\mathfrak{g}]$. Assume $\overline{G}$ has finite center. Let K be a maximal compact subgroup of G and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ be the corresponding Cartan decomposition. Then K acts on $\mathfrak{s}$ by the adjoint representation $(k.X=Ad_K(k)X)$. The Cartan motion group H (associated to G) is the semidirect product $H=K\ltimes\mathfrak{s}$.

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Let H_ψ be the stabilizer in H of a linear form $\psi \in \mathfrak{h}^*$ ($\mathfrak{h} := Lie(H)$). Then ψ is called admissible if there exists a unitary character χ of the identity component of H_ψ such that $d\chi = i\psi|_{\mathfrak{h}_\psi}$. By $\mathfrak{h}^\ddagger$, we mean the set of all admissible linear forms on $\mathfrak{h}$. For $\psi \in \mathfrak{h}^\ddagger$, one can construct an irreducible unitary representation π_ψ by holomorphic induction. According to Lipsman (see [19]), every unitary irreducible representation of H arises in this manner. Thus we obtain a map from the set $\mathfrak{h}^\ddagger$ onto the unitary dual $\widehat{H}$. By observing that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' lie in the same H-orbit, we get finally a bijection between the space $\mathfrak{h}^\ddagger/H$ of admissible coadjoint orbits and the unitary dual $\widehat{H}$. The natural question arises of whether this bijection is a homeomorphism. In the present paper, we give an affirmative answer to this question in the case of the Cartan motion groups. This result is a generalization of analogous results in the case of the Euclidean motion group (see, [7]) and in the case of a class of Cartan motion groups associated to a compact Riemannian symmetric pair (G, K) (where the pair (G, K)) has rank one (see, [3]). Note that in our proof we use a different method than the one used in ([3, 7, 20, 21]).

Our paper is organized as follows. Section 2 introduces the Cartan motion groups and reviews some results about the parameterization of the nonunitary dual of the Cartan motion group H and of the Fell topology on it. Mackey's theory of unitary induction is ideally suited to Cartan motion groups, making the computation of the unitary dual easy. In the last section, the convergence in the quotient space $\mathfrak{h}^\ddagger/H$ is studied and the main result of this work is derived (Theorem 3.5)

2. Preliminaries and some results

Let G be a locally compact, separable topological group and let K be a compact subgroup of G. Let $\widehat{K} \ni \tau : K \longrightarrow \mathcal{L}(E_\tau)$ be an irreducible representation of K. If $\pi : G \longrightarrow \mathcal{L}(E_\pi)$ is a representation on the complete, locally convex, topological vector space E_π, let

$$[\pi|_K : \tau] = dimHom_K(E_\tau, E_\pi)$$

the multiplicity of τ in $\pi|_K$.

As it turn out, all of the representations considered in this paper are admissible, that is $[\pi|_K : \tau] \leq Md_\tau$ (where d_τ is the degree of τ) for some constant $M > 0$. Note that this holds for Cartan motion groups (for details, see [10]).

Definition 2.1. Let $\pi_1 : G \longrightarrow \mathcal{L}(E_1)$ and $\pi_2 : G \longrightarrow \mathcal{L}(E_2)$ be admissible representations. A Naimark intertwining operator $Q : E_1 \longrightarrow E_2$ is a linear operator with dense domain and closed graph such that if m is a compactly supported measure on G, then the domain and range of Q are stable under $\pi_1(m)$ and $\pi_2(m)$ respectively, and $Q\pi_1(m) = \pi_2(m)Q$. If in addition Q is one-to-one and has dense range, we say that π_1 and π_2 are Naimark equivalent and write $[\pi_1] = [\pi_2]$.

Let $[G]$ denote the set of Naimark equivalence classes of topologically completely irreducible (abbreviated TCI) representations.

Next, we recall the definition of the Fell topology on $[G]$. For $[\pi] \in [G]$, let $\mathcal{A}(\pi)$ be the space of functions of the form

$$\alpha(x) = tr(T\pi(x))$$

where $T \in \mathcal{L}(E)$ is left and right K-finite. Then a net $[\pi_i]$ converges to $[\pi]$ in $[G]$ if and only if for all $\alpha \in \mathcal{A}(\pi)$ there exist $\alpha_i \in \mathcal{A}(\pi_i)$ such that $\alpha_i(f)$ converges to $\alpha(f)$ for all $f \in C_c(G)$. The fact that π is TCI, then it is the same to say there exist $\alpha \in \mathcal{A}(\pi), \alpha \neq 0$, and $\alpha_i \in \mathcal{A}(\pi_i)$ such that α_i converges to α uniformly on compacta. Recall that $\widehat{G} \subseteq [G]$ (see [10]) and that the Fell topology on $\widehat{G}$ agrees with the hull-kernel topology.

We return to our context, and let G be a connected reductive Lie group, let $\mathfrak{g}$ be its Lie algebra and let $\overline{G}$ be the analytic subgroup corresponding to $[\mathfrak{g}, \mathfrak{g}]$. Assume $\overline{G}$ has finite center. Let K be a maximal compact subgroup of G and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the corresponding Cartan decomposition. Then K acts on $\mathfrak{s}$ by the adjoint representation $(k.X = Ad_K(k)X)$. The Cartan motion group H (associated to G) is the semidirect product $H = K \ltimes \mathfrak{s}$. The multiplication in this group is given by

$$(k_1, X_1)(k_2, X_2) = (k_1k_2, Ad_K(k_2^{-1})X_1 + X_2).$$

Fix a Cartan-Killing form B which is positive definite on $\mathfrak{s}$ and negative definite on $\mathfrak{k}$. Use B to identify elements of $\mathfrak{h} := Lie(H)$ with elements of its dual vector space $\mathfrak{h}^*$. The adjoint representations of H and its Lie algebra $\mathfrak{h}$ are given respectively by the following equalities

$$\begin{aligned}(k, X).(U, Y) &= (Ad_K(k)U, Ad_K(k)Y - [Ad_K(k)U, X]),\\ (U, X).(V, Y) &= ([U, V], [U, Y] - [V, X])\end{aligned}$$

for all $k \in K$, all $U, V \in \mathfrak{k}$ and all $X, Y \in \mathfrak{s}$. Under the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$, we can write the coadjoint representation of H as follows

$$(k, X).(f, \Lambda) = (Ad_K^*(k)f + Ad_K^*(k)\Lambda \odot X, Ad_K^*(k)\Lambda),\ \ (k, X) \in H, (f, \Lambda) \in \mathfrak{s}^*$$

where $\Lambda \odot X \in \mathfrak{k}^*$ (is a linear form on $\mathfrak{k}$) defined by

$$\Lambda \odot X(A) = \Lambda(ad(A)(X)) = -(ad^*(A)\Lambda)(X), \forall A \in \mathfrak{k}, \Lambda \in \mathfrak{s}^*, X \in \mathfrak{s}$$

(where ad and ad^* denote respectively the adjoint and the coadjoint representation of $\mathfrak{k}$). Note that the map $\odot : \mathfrak{s}^* \times \mathfrak{s} \longrightarrow \mathfrak{k}^*$ defined by

$$(\Lambda \odot X)(A) = \Lambda(ad(A)X),\ X \in \mathfrak{s},\ A \in \mathfrak{k}$$

satisfies a fundamental equivariance property:

$$Ad_K^*(k)(\Lambda \odot X) \quad = \quad Ad_K^*(k)\Lambda) \odot Ad_K(k)X,\ k \in K.$$

Therefore, the coadjoint orbit of H passing through $(f, \Lambda) \in \mathfrak{h}^*$ is given by

$$\mathcal{O}^H_{(f,\Lambda)} = \Big\{ \Big(Ad_K^*(k)f + Ad_K^*(k)\Lambda \odot X, Ad_K^*(k)\Lambda \Big), k \in K, X \in \mathfrak{s} \Big\}.$$

For $\Lambda \in \mathfrak{s}^*$, we define $K_\Lambda := \{k \in K;\ Ad_K^*(k)\Lambda = \Lambda\}$ the isotropy subgroup of Λ in K and the Lie algebra of K_Λ is given by the vector space $\mathfrak{k}_\Lambda = \{A \in \mathfrak{k};\ ad^*(A)\Lambda = 0\}$. Let $\imath_\Lambda : \mathfrak{k}_\Lambda \hookrightarrow \mathfrak{k}$ be the injection map, then $\imath_\Lambda^* : \mathfrak{k}^* \longrightarrow \mathfrak{k}_\Lambda^*$ is the projection map and we have

$$\mathfrak{k}_\Lambda^\circ \quad = \quad Ker(\imath_\Lambda^*) \tag{2.1}$$

where $\mathfrak{k}_\Lambda^\circ$ is the annihilator of $\mathfrak{k}_\Lambda$. If we define the linear map $h_\Lambda : \mathfrak{k} \longrightarrow \mathfrak{s}^*$ by

$$h_\Lambda(A) := -ad^*(A)\Lambda,\ \forall A \in \mathfrak{k},$$

then we have $\mathfrak{k}_\Lambda = Ker(h_\Lambda)$. The dual $h_\Lambda^* : \mathfrak{s} \longrightarrow \mathfrak{k}^*$ of h_Λ is given by the relation $h_\Lambda^*(X)(A) = h_\Lambda(A)(X) = -(ad^*(A)\Lambda)(X)$, and so $h_\Lambda^*(X) = \Lambda \odot X$, $\forall \Lambda \in \mathfrak{s}^*$, $\forall X \in \mathfrak{s}$ (see [2]).

The following is a useful Lemma from [2], giving a characterization of the annihilator $\mathfrak{k}_\Lambda^\circ$ in terms of the linear map h_Λ.

Lemma 2.2. *We have:*

$$\mathfrak{k}_\Lambda^\circ = Im(h_\Lambda^*).$$

Let $\mathfrak{a}_\mathfrak{s}$ be a Cartan subspace of $\mathfrak{s}$. We denote by $\mathfrak{z}$ the orthogonal complement of $\mathfrak{a}_\mathfrak{s}$ in $\mathfrak{s}$ ($\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{z}$). Let $\Lambda : \mathfrak{a}_\mathfrak{s} \longrightarrow \mathbb{C}$ be a real linear function. Also we denote by Λ the extension of Λ to $\mathfrak{s}$ so that $\mathfrak{z} \subseteq Ker(\Lambda)$, and let $\mathfrak{e} \in \widehat{K_\Lambda}$. We denote by $\pi := \pi_{(\mathfrak{e},\Lambda)}$ the representation of H induced from

$$K_\Lambda \ltimes \mathfrak{s} \longrightarrow \mathcal{L}(E_\mathfrak{e})$$

$$(k, X) \longmapsto e^{i\Lambda(X)}\mathfrak{e}(k).$$

Let

$$W_{\mathfrak{s}} := N/M$$

be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{s}})$, where N is the normalizer of $\mathfrak{a}_{\mathfrak{s}}$ in K and M is the centralizer. In the end of this section, we describe the Fell topology on $[H]$. For $\beta_1, \beta_2 \in \mathfrak{a}_{\mathfrak{s}}^*$ (the dual vector space of $\mathfrak{a}_{\mathfrak{s}}$), define

$$|\beta_1 + \sqrt{-1}\beta_2|^2 = B(\beta_1, \beta_1) + B(\beta_2, \beta_2).$$

Let $\mathcal{F}_c$ be the set of all pairs $(\mathfrak{e}, \Lambda)$ where $\mathfrak{e} \in \widehat{K}_\Lambda$. We take $(\mathfrak{e}, \Lambda) \in \mathcal{F}_c$, if $\varepsilon > 0$ is sufficiently small then $|\Lambda - \Lambda'| < \varepsilon$ implies $K_{\Lambda'} \subseteq K_\Lambda$. So the subset

$$\mathcal{U} := \left\{(\mathfrak{e}', \Lambda') \in \mathcal{F}_c : \, |\Lambda - \Lambda'| < \varepsilon \; and \; [\mathfrak{e}|_{K_{\Lambda'}} : \mathfrak{e}'] > 0\right\}$$

defines a basis for the neighborhoods of $(\mathfrak{e}, \Lambda)$ in the topology we give $\mathcal{F}_c$ (see [10]). Note that $W_{\mathfrak{s}}$ acts on $\mathcal{F}_c$ by

$$w.(\mathfrak{e}, \Lambda) = (w.\mathfrak{e}, w.\Lambda).$$

Let $\mathcal{F}_c/W_{\mathfrak{s}}$ be the quotient space by this action of $W_{\mathfrak{s}}$, equipped with the quotient topology. Then, we have:

Theorem 2.3 ([10])**.** *The following map*

$$\begin{aligned} \mathcal{F}_c/W_{\mathfrak{s}} : &\longrightarrow [H] \\ (\mathfrak{e}, \Lambda) &\longmapsto [\pi_{(\mathfrak{e}, \Lambda)}] \end{aligned}$$

is a homeomorphism of $\mathcal{F}_c/W_{\mathfrak{s}}$ with the above topology onto the Banach dual $[H]$ of H with the Fell topology.

Now, let

$$\mathcal{F} := \left\{(\mathfrak{e}, \Lambda) \in \mathcal{F}_c : \, \Lambda = \sqrt{-1}\beta \; where \; \beta \; is \; real \; valued\right\}.$$

According to [10], then we have the useful Lemma.

Lemma 2.4. *The unitary dual $\widehat{H}$ of H is homeomorphic to $\mathcal{F}/W_{\mathfrak{s}}$.*

3. Cartan motion groups and their coadjoint orbits

We shall freely use the notations of the previous sections. Let Λ as in Lemma 2.4. In the sequel of this paper we assume that the subgroups K_Λ are connected for each $\Lambda \in \mathfrak{s}^*$. Let $\mathfrak{e}_\nu$ be an irreducible representation of K_Λ with highest weight ν. Then the stabilizer H_ψ of $\psi = (\nu, \Lambda)$ in H is given by

$$\begin{aligned} H_\psi &= \Big\{(k, X) \in H;\ (Ad_K^*(k)\nu + Ad_K^*(k)\Lambda \odot X, Ad_K^*(k)\Lambda) = (\nu, \Lambda)\Big\} \\ &= \Big\{(k, X) \in H;\ k \in K_\Lambda, Ad_K^*(k)\nu + \Lambda \odot X = \nu\Big\} \\ &= \Big\{(k, X) \in H;\ k \in K_\Lambda,\ Ad_K^*(k)\nu = \nu\ \ and\ \ \Lambda \odot X = 0\Big\} \\ &= \Big\{(k, X) \in H;\ k \in K_\psi,\ X \in \mathfrak{s}_\psi = (ad^*\mathfrak{k}\Lambda)^\circ\Big\}, \end{aligned}$$

since $i_\Lambda^*(\Lambda \odot X) = 0$ (see Lemma 2.2). Thus, we have $H_\psi = K_\psi \ltimes \mathfrak{s}_\psi$, then ψ is aligned (see [19]). A linear form $\psi = (\nu, \Lambda) \in \mathfrak{h}^*$ is called admissible if ν is a dominant integral weight of K_Λ. We denote by $\mathfrak{h}^\ddagger \subset \mathfrak{h}^*$ the set of all admissible linear forms on $\mathfrak{h}$. The quotient space $\mathfrak{h}^\ddagger/H$ is called the space of admissible coadjoint orbits of H. By definition $\mathfrak{h}^\ddagger/H$ is the union of the sets of all orbits $\mathcal{O}^H_{(\nu,\Lambda)}$.

Let T_K and T_Λ be maximal tori respectively in K and K_Λ such that $T_\Lambda \subset T_K$. Their corresponding Lie algebras are denoted by $\mathfrak{t}_\mathfrak{k}$ and $\mathfrak{t}_\Lambda$. We denote by W_K and W_Λ the Weyl groups of K and K_Λ associated respectively to the tori T_K and T_Λ. Notice that every element $\lambda \in P_K$ takes pure imaginary values on $\mathfrak{t}_\mathfrak{k}$, where P_K is the integral weight lattice of T_K. Hence such an element $\lambda \in P_K$ can be considered as an element of $(i\mathfrak{t}_\mathfrak{k})^*$. Let C_K^+ be a positive Weyl chamber in $(i\mathfrak{t}_\mathfrak{k})^*$, and we define the set P_K^+ of dominant integral weights of T_K by $P_K^+ := P_K \cap C_K^+$. For $\lambda \in P_K^+$, denote by $\mathcal{O}_\lambda^K$ the K-coadjoint orbit passing through the vector $-i\lambda$. It was proved by Kostant in [17], that the projection of $\mathcal{O}_\lambda^K$ on $\mathfrak{t}_\mathfrak{k}^*$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, and that is the convex hull of $-i(W_K.\lambda)$. For the same manner, we fix a positive Weyl chamber C_Λ^+ in $\mathfrak{t}_\Lambda^*$ and we define the set P_Λ^+ of dominant integral weights of T_Λ.

Also we denote by i_Λ^* the $\mathbb{C}$- linear extension of both the natural projection of $\mathfrak{k}^*$ onto $\mathfrak{k}_\Lambda^*$ and the natural projection of $\mathfrak{t}_\mathfrak{k}^*$ onto $\mathfrak{t}_\Lambda^*$. Consider two irreducible representations $\tau_\lambda \in \widehat{K}$ and $\mathfrak{e}_\nu \in \widehat{K_\Lambda}$ with respective highest weights $\lambda \in P_K^+$ and $\nu \in P_\Lambda^+$. We denote by $\mathcal{O}_\lambda^K$ and $\mathcal{O}_\nu^{K_\Lambda}$ the coadjoint orbits of K and K_Λ passing through $-i\lambda$ and $-i\nu$, respectively.

Now, we have the following.

Lemma 3.1. *If the induced representation* $Ind_{K_\Lambda}^{K}(\mathfrak{e}_\nu)$ *contains* τ_λ*, then the orbit* $\mathcal{O}_\nu^{K_\Lambda}$ *occurs in* $i_\Lambda^*(\mathcal{O}_\lambda^K)$*.*

For the proof of this Lemma one can see [11, 12]. According to [1], we have

Lemma 3.2. *If* $\nu = i_\Lambda^*(s.\lambda)$ *with* $s \in W_K$*, then* $\mathfrak{e}_\nu$ *occurs in the restriction representation* $Res_{K_\Lambda}^{K}(\tau_\lambda)$*.*

To study the convergence in the quotient space $\mathfrak{g}^\ddagger/G$, we need to the following result (see [18, P. 135] for the proof).

Lemma 3.3. *Let* G *be a unimodular Lie group with Lie algebra* $\mathfrak{g}$ *and let* $\mathfrak{g}^*$ *be the vector dual space of* $\mathfrak{g}$*. We denote* $\mathfrak{g}^*/G$ *the space of coadjoint orbits and by* $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ *the canonical projection. We equip this space with the quotient topology, i.e., a subset* V *in* $\mathfrak{g}^*/G$ *is open if and only if* $p_G^{-1}(V)$ *is open in* $\mathfrak{g}^*$*. Therefore, a sequence* $(\mathcal{O}_n^G)_n$ *of elements in* $\mathfrak{g}^*/G$ *converges to the orbit* $\mathcal{O}^G$ *in* $\mathfrak{g}^*/G$ *if and only if for any* $l \in \mathcal{O}^G$*, there exist* $l_n \in \mathcal{O}_n^G$*,* $n \in \mathbb{N}$*, such that* $l = \lim_{n\longrightarrow+\infty} l_n$*.*

With the above notations, we can prove the following Theorem.

Theorem 3.4. *We assume that the stabilizer subgroup* K_Λ *(*$\Lambda \in \mathfrak{s}^*$*) is connected. Then The map*

$$\begin{aligned} \mathcal{F}/W_{\mathfrak{s}} &\longrightarrow \mathfrak{h}^\ddagger/H \\ (\mathfrak{e}_\nu, \Lambda) &\longmapsto \mathcal{O}_{(\nu,\Lambda)}^H \end{aligned}$$

defines a homeomorphism of $\mathcal{F}/W_{\mathfrak{s}}$ *with the above topology onto the space of admissible coadjoint orbits with the quotient topology.*

Proof. Lemma 2.4 says that the map in the statement of the Theorem is one-to-one and onto. Let $(\mathfrak{e}_{\nu^i}, \Lambda_i)_{i\in I}$ be a net converging to $(\mathfrak{e}, \Lambda)$ in $\mathcal{F}/W_{\mathfrak{s}}$. Then for each $\varepsilon > 0$ sufficiently small, there exists $i_0 \in I$ such that for all $i \geq i_0$ we have

$$|\Lambda_i - \Lambda| < \varepsilon$$

and

$$[\mathfrak{e}_\nu|_{K_{\Lambda_i}} : \mathfrak{e}_{\nu^i}] > 0. \tag{3.1}$$

The inequality in (3.1), is equivalent to $\mathfrak{e}_\nu \in Res_{K_{\Lambda_i}}^{K_\Lambda}(\mathfrak{e}_{\nu^i})$ for all $i \geq i_0$. Using Lemma 3.1, we obtain

$$\mathcal{O}_{\nu^i}^{K_{\Lambda_i}} \subseteq i_{\Lambda_i}^*(\mathcal{O}_\nu^{K_\Lambda}) \ \forall i \geq i_0.$$

By Lemma 2.2, there exist two nets $(H_i)_i$ in $\mathfrak{s}$ and $(h_i)_i$ in K_Λ, such that

$$\nu^i + \Lambda_i \odot H_i \quad = \quad Ad_K^*(h_i)\nu \quad \forall i \geq i_0. \tag{3.2}$$

Let $k \in K$ and $X \in \mathfrak{s}$. We define two nets $(k_i := kh_i^{-1})_i$ and $(X_i := Ad_K(kh_i^{-1})H_i + X)_i$. We easily see that

$$Ad_K^*(k_i)\Lambda_i \longrightarrow Ad_K^*(k)\Lambda. \tag{3.3}$$

By the equality (3.2), we obtain for each $i \geq i_0$

$$Ad_K^*(k_i)\nu^i + Ad_K^*(k_i)\Lambda_i \odot X_i \quad = \quad Ad_K^*(k)\nu + Ad_K^*(kh_i^{-1})\Lambda_i \odot X.$$

Then we have

$$Ad_K^*(k_i)\nu^i + Ad_K^*(k_i)\Lambda_i \odot X_i \quad \longrightarrow \quad Ad_K^*(k)\nu + Ad_K^*(k)\Lambda \odot X. \tag{3.4}$$

Lemma 3.3 together with (3.3) and (3.4), allows us to conclude that the net $(\mathcal{O}^H_{(\nu^i,\Lambda_i)})_i$ converges to $\mathcal{O}^H_{(\nu,\Lambda)}$ in $\mathfrak{h}^\ddagger/H$.

Conversely, we assume that the net $(\mathcal{O}^H_{(\nu^i,\Lambda_i)})_{i\in I}$ converges to $\mathcal{O}^H_{(\nu,\Lambda)}$ in $\mathfrak{h}^\ddagger/H$. We recall that each coadjoint orbit $\mathcal{O}^H_{(\nu,\Lambda)}$ is always obtained by symplectic induction from the coadjoint orbit $M = \mathcal{O}^{H_\Lambda}_{(\nu,\Lambda)}$ of $H_\Lambda := K_\Lambda \ltimes \mathfrak{s}$ passing through $(\nu, \Lambda) \in \mathfrak{k}_\Lambda^* \oplus \mathfrak{s}^*$ $(\mathfrak{k}_\Lambda \ltimes \mathfrak{s} := Lie(H_\Lambda))$, i.e.,

$$\mathcal{O}^H_{(\nu,\Lambda)} = M_{ind} := J^{-1}_{\widetilde{M}}(0)/H_\Lambda, \tag{3.5}$$

where $J_{\widetilde{M}} : \widetilde{M} = M \times T^*H \longrightarrow \mathfrak{k}_\Lambda^* \ltimes \mathfrak{s}^*$ is the momentum map of $\widetilde{M}$ and the zero level set $J^{-1}_{\widetilde{M}}(0)$ is given by

$$J^{-1}_{\widetilde{M}}(0) = \Big\{\Big((Ad_K^*(k)\nu, \Lambda), g, (Ad_K^*(k)\nu + \Lambda \odot X, \Lambda)\Big),\ k \in K_\Lambda, g \in H, X \in \mathfrak{s}\Big\}.$$

Let φ_M be the action of H_Λ on M, hence H_Λ acts on $\widetilde{M} = M \times T^*H$ by $\varphi_{\widetilde{M}}$ as follows

$$\varphi_{\widetilde{M}}(h)(m, g, f) = \Big(\varphi_M(h)(m), gh^{-1}, Ad_H^*(h)f\Big), \tag{3.6}$$

for all $h \in H_\Lambda, (m, g, f) \in M \times T^*H$. By identifying $\mathfrak{h}^*$ with the left-invariant 1-form on H. Then we can write $T^*H \cong H \times \mathfrak{h}^*$ (for more details one can see [2]).

Now, Lemma 3.3 together with (3.5) and (3.6) say that there exist $k_i, h_i \in K_{\Lambda_i}, X_i, Y_i \in \mathfrak{s}$ and $g_i \in H$ such that the net $(a_i)_i$ defined by

$$\begin{aligned} a_i &= \varphi_{\widetilde{M}}(k_i, X_i)\big((Ad_K^*(h_i)\nu^i, \Lambda_i), g_i, (Ad_K^*(h_i)\nu^i + \Lambda_i \odot Y_i, \Lambda_i)\big) \\ &= \Big(Ad_K^*(k_i h_i)\nu^i + \imath_{\Lambda_i}^*(\Lambda_i \odot X_i), \Lambda_i), g_i(k_i, X_i)^{-1}, \\ &\qquad (Ad_K^*(k_i h_i)\nu^i + Ad_K^*(k_i)(\Lambda_i \odot Y_i) + \Lambda_i \odot X_i, \Lambda_i)\Big) \end{aligned}$$

converges to $\big((\nu, \Lambda), 1_H, (\nu, \Lambda)\big)$. It follows that

$$\Lambda_i \longrightarrow \Lambda \tag{3.7}$$

and

$$Ad_K^*(k_i h_i)\nu^i + \imath_{\Lambda_i}^*(\Lambda_i \odot X_i) \longrightarrow \nu. \tag{3.8}$$

Let $\varepsilon > 0$ sufficiently small, then there exists $i_0 \in I$ such that for all $i \geq i_0$ we have

$$|\Lambda_i - \Lambda| < \varepsilon. \tag{3.9}$$

Then (3.9) implies $K_{\Lambda_i} \subseteq K_\Lambda$. By compactness of the subgroup K_{Λ_i} and without loss of generality, we may assume that the net $(k_i h_i)_i$ converges to an element $k \in K_\Lambda$. Now by observing that $\imath_{\Lambda_i}^*(\Lambda_i \odot X_i) = 0$, hence there exists $i_1 \in I$ such that for all $i \geq i_1$ we have

$$\nu^i = Ad_K^*(k^{-1})\nu. \tag{3.10}$$

Furthermore, we know that there exists $s \in W_{K_\Lambda}$ such that $Ad_K^*(k^{-1})\nu = s.\nu$. We obtain the equality $\nu^i = s.\nu \ \forall i \geq i_1$. Using Lemma 3.2, we conclude that

$$\mathfrak{e}_{\nu^i} \in Res_{K_{\Lambda_i}}^{K_\Lambda}(\mathfrak{e}_\nu) \ \forall i \geq i_1. \tag{3.11}$$

i.e.,

$$[\mathfrak{e}_\nu|_{K_{\Lambda_i}} : \mathfrak{e}_{\nu^i}] > 0 \ \ \forall i \geq i_1. \tag{3.12}$$

We put $i_2 := Max(i_0, i_1)$, then by combining (3.11) and (3.12) we obtain

$$|\Lambda_i - \Lambda| < \varepsilon \ and \ [\mathfrak{e}_\nu|_{K_{\Lambda_i}} : \mathfrak{e}_{\nu^i}] > 0 \ \ \forall i \geq i_2.$$

It follows that the net $(\mathfrak{e}_{\nu^i}, \Lambda_i)_i$ converges to $(\mathfrak{e}_\nu, \Lambda)$ in $\mathcal{F}/W_{\mathfrak{s}}$. □

Combining the results of Lemma 2.4 and Theorem 3.4, we can state the main result of this paper.

Theorem 3.5. *With the hypothesis of connected small groups $K_\Lambda, \Lambda \in \mathfrak{s}^*$, we show that the topological spaces $\widehat{H}$ and $\mathfrak{h}^\ddagger/H$ are homeomorphic.*

Remark 3.6. (1) The special case of Theorem 3.5 where $H := SO(n) \ltimes \mathbb{R}^n (n \geq 2)$ has been proved in [7]. One can see that for each non-zero linear form Λ on $\mathbb{R}^n$, $K_\Lambda := SO(n)_\Lambda = SO(n-1)$ which is connected subgroup of $SO(n)$. The authors method of proof makes essential use of the classical branching rule from $SO(n)$ to $SO(n-1)$.

(2) We note that the Cartan motion groups are a special case of the semidirect product $G := K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional vector space V. The proof of our result (Theorem 3.5) is based to the structure and some properties of the Cartan motion groups (see for example, Lemma 2.4).

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