Notes on super Riemann surfaces and their moduli

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Abstract: These are notes on the theory of super Riemann surfaces and their moduli spaces, aiming to collect results that are useful for a better understanding of superstring perturbation theory in the RNS formalism.

1. Introduction

What are now understood as superstrings – string theories with spacetime supersymmetry – can be formulated in terms of a worldsheet action with worldsheet supersymmetry [1–3], naturally coupled to two-dimensional supergravity [4, 5]. This leads to a description of superstring scattering amplitudes in terms of integration over a suitable class of two-dimensional supergeometries, an early reference being [6]. The relevant supergeometries are best understood as super Riemann surfaces, which were initially defined in [7–10]. The subsequent literature, which includes papers such as [11–15] developing the theory of super Riemann surfaces and papers such as [16–30] with applications to string theory, is too vast to be fully cited here.

The present notes aim to give a relatively understandable account of aspects of super Riemann surface theory that are particularly relevant for superstring perturbation theory. A reconsideration of superstring perturbation theory will appear elsewhere [31]. A companion set of notes gives an introduction to supermanifolds and integration [32], including some concepts used here. The reader will also require some familiarity with ordinary Riemann surfaces and complex manifolds, and some general familiarity with superstring theory. We have tried to structure these notes so that many of the more technical parts can be omitted on first reading.

The material described here is mostly standard. One point on which we differ from some of the literature is that in describing super Riemann surfaces, we assume no relation between holomorphic and antiholomorphic odd variables, since no such relation is natural for superstrings. In particular, in the framework assumed here, one is not allowed to take the complex conjugate of an odd variable. Much of the literature emphasizes the case of a super Riemann surface with a real structure for the odd variables, as is appropriate for the nonsupersymmetric (and tachyonic) Type 0 string theory. On some

points, we have filled in what appear to be missing details in the literature or have attempted to simplify previous arguments.

We introduce super Riemann surfaces from a purely holomorphic point of view in section 2, and from a smooth point of view in section 3. Neveu-Schwarz and Ramond punctures are introduced in section 4. Some examples of low genus are the topic of section 5. In section 6, we describe the behavior of supermoduli space at infinity; this is crucial input for superstring perturbation theory. Up to this point, we consider only oriented super Riemann surfaces without boundary. The generalizations appropriate for open and/or unoriented strings can be found in section 7. In section 8, we describe the super analog of the period matrix of a Riemann surface.

In the bulk of this paper, all super Riemann surfaces are $\mathcal{N} = 1$ super Riemann surfaces, with holomorphic odd dimension 1. This is the relevant case for the usual supersymmetric string theories. However, in section 9, we describe some rather pretty facts [34–37] about $\mathcal{N} = 2$ super Riemann surfaces and generic complex supermanifolds of dimension 1|1. These facts have perhaps not yet been fully incorporated in string theory.

We generally write Σ for a super Riemann surface, Σ_0 for an ordinary Riemann surface, and Σ_{red} for the reduced space of Σ. Also, $\mathcal{M}$ is generally the moduli space of ordinary Riemann surfaces and $\mathfrak{M}$ is the moduli space of super Riemann surfaces. Further details like the genus, the number and types of punctures, or (in the bosonic case) the specification of a spin structure are indicated in an obvious way. Many statements apply either to the ordinary moduli spaces that parametrize smooth Riemann surfaces or super Riemann surfaces, or to the Deligne-Mumford compactifications that parametrize also certain singular surfaces (see section 6). To refer specifically to the compactifications, we write $\widehat{\mathcal{M}}$ or $\widehat{\mathfrak{M}}$. Integration cycles for string perturbation theory are generically called Γ for bosonic string theory and $\mathit{\Gamma}$ for superstring theory; their compactifications are $\widehat{\Gamma}$ and $\widehat{\mathit{\Gamma}}$.

2. Super Riemann surfaces from a holomorphic point of view

2.1. Complex supermanifolds and super Riemann surfaces

We begin by describing[1] a super Riemann surface Σ as a complex superman-ifold. As such Σ has dimension 1|1, so locally it is isomorphic to $\mathbb{C}^{1|1}$ and can be described by bosonic and fermionic local coordinates z and θ. Its tangent

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[1] As noted in the introduction, we consider only $\mathcal{N} = 1$ super Riemann surfaces except in section 9.

bundle $T\Sigma$ and its cotangent bundle $T^*\Sigma$ are both of rank $1|1$. For example, the cotangent bundle has a basis of 1-forms $\mathrm{d}z$ and $\mathrm{d}\theta$; the tangent bundle has the dual basis ∂_z and ∂_θ.

So far, Σ could be any complex supermanifold of dimension $1|1$. To make Σ into a super Riemann surface, we need one more piece of structure, which is a subbundle $\mathcal{D} \subset T\Sigma$ of rank $0|1$, which is required to be "completely nonintegrable" in the sense that if D is a nonzero section of $\mathcal{D}$ (in some open set $U \subset \Sigma$), then $D^2 = \frac{1}{2}\{D, D\}$ is nowhere proportional to D (we describe this by saying that D^2 is nonzero mod D). A typical example is

$$(2.1) \qquad\qquad D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}.$$

For this choice, we see that $D_\theta^2 = \partial/\partial z$, which indeed is nowhere a multiple of D_θ; indeed, D_θ and D_θ^2 are everywhere a basis of $T\Sigma$.

To see that this example is typical, consider a section of $\mathcal{D}$ of the general form

$$(2.2) \qquad\qquad D = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial z}.$$

Here a is an even function and b is an odd function. (In general, a and b depend on z, θ, and possibly on the even and odd moduli of Σ.) In supermanifold theory, to say that a quantity is "nonzero" means that it is nonzero after reducing modulo the odd variables (that is, setting them to zero). The odd function b is certainly proportional to odd variables, so for D to be nonzero means that a is nonzero and thus invertible. The condition that D^2 is not a multiple of D is invariant under $D \to fD$ for any nonzero function f, and by taking $f = a^{-1}$ and redefining b, we can reduce to $D = \partial/\partial\theta + b\partial/\partial z$. We expand b in powers of θ:

$$(2.3) \qquad\qquad D = \frac{\partial}{\partial\theta} + (b_0 + b_1\theta)\frac{\partial}{\partial z}.$$

The condition that D^2 is not a multiple of D implies that b_1 is nonzero, so we can replace θ by $\theta^* = b_1^{-1/2}(b_0 + b_1\theta)$. After also rescaling D by $b_1^{1/2}$, we reduce to $D = \partial_{\theta^*} + \theta^*\partial_z$, showing that locally one can always pick holomorphic coordinates in which D has the form indicated in (2.1). We call these superconformal coordinates. A local superconformal coordinate system $z|\theta$ determines the section D_θ of $\mathcal{D}$ defined in (2.1). (To explain the notation, it can be shown – for instance by using eqn. (2.12) below – that D_θ is completely determined by the choice of θ.)

In many ways, super Riemann surfaces have simpler properties than generic $1|1$ supermanifolds, but describing them explicitly can be more difficult. For example, to describe a $1|1$ supermanifold, one may start with $\mathbb{CP}^{2|1}$, with homogeneous coordinates $z_1 \ldots z_3 | \theta$, and impose a homogeneous equation $G(z_1 \ldots z_3 | \theta) = 0$. But generically in this way one obtains a complex supermanifold that is not a super Riemann surface, since its tangent bundle does not have an appropriate subbundle $\mathcal{D}$. It is actually quite tricky to determine which $1|1$ complex supermanifolds possess the additional structure corresponding to a super Riemann surface. The gluing construction to which we turn presently is one of the few general ways to construct super Riemann surfaces.

2.1.1. Superconformal transformations and primary fields
Let $z|\theta$ be local superconformal coordinates. A vector field $W = \alpha\, \partial_\theta + \beta\, \partial_z$ is said to generate an infinitesimal superconformal transformation or simply to be superconformal if it preserves the subbundle $\mathcal{D} \subset T\Sigma$, which is equivalent to saying that for $D_\theta = \partial_\theta + \theta\partial_z$, $[W, D_\theta\}$ is proportional to D_θ. (The symbol $[\,,\,\}$ denotes a commutator or anticommutator depending on the statistics of the objects in question.) A short computation reveals that a basis of superconformal vector fields is given by the odd vector fields

$$(2.4) \qquad \nu_f = f(z) \left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z} \right),$$

and the even ones

$$(2.5) \qquad V_g = g(z)\frac{\partial}{\partial z} + \frac{\partial_z g(z)}{2}\theta\frac{\partial}{\partial\theta}.$$

f and g are even functions that depend on z and perhaps on moduli, but not on θ. Explicitly,

$$(2.6) \qquad \{\nu_f, D_\theta\} = (\theta\partial_z f)D_\theta, \quad [V_g, D_\theta] = -(\partial_z g/2)D_\theta.$$

Note that $\nu_f^2 = V_g$ with $g = -f^2$. If we set $g(z) = z$, we get the superconformal vector field $z\partial_z + \frac{1}{2}\theta\partial_\theta$, which generates the scaling

$$(2.7) \qquad z \to \lambda z, \quad \theta \to \lambda^{1/2}\theta, \quad \lambda \in \mathbb{C}^*,$$

and we see that θ scales with dimensions of $(\text{length})^{1/2}$ or $(\text{mass})^{-1/2}$, while D_θ scales with mass dimension $1/2$.

If $z|\theta$ and $\widehat{z}|\widehat{\theta}$ are two local superconformal coordinate systems, then $D_\theta = \partial_\theta + \theta\partial_z$ and $D_{\widehat{\theta}} = \partial_{\widehat{\theta}} + \widehat{\theta}\partial_{\widehat{z}}$ are two local nonzero sections of $\mathcal{D}$, and so are related by $D_\theta = F\, D_{\widehat{\theta}}$ for some nonzero function F. Acting with this formula on the function $\widehat{\theta}$, we determine F:

$$(2.8) \qquad\qquad D_\theta = (D_\theta\widehat{\theta})D_{\widehat{\theta}}.$$

On the other hand, for any two coordinate systems $z|\theta$ and $\widehat{z}|\widehat{\theta}$, we can use the chain rule of calculus to compute

$$(2.9) \qquad\qquad D_\theta = (D_\theta\widehat{\theta})D_{\widehat{\theta}} + (D_\theta\widehat{z} - \widehat{\theta}D_\theta\widehat{\theta})\partial_{\widehat{z}},$$

so if $z|\theta$ is a superconformal coordinate system, then the condition that $\widehat{z}|\widehat{\theta}$ is also a superconformal coordinate system can be stated [9]

$$(2.10) \qquad\qquad D_\theta\widehat{z} - \widehat{\theta}D_\theta\widehat{\theta} = 0.$$

It is actually possible from this to deduce a general formula for a superconformal change of coordinates. Let us try [13] a general ansatz

$$
\begin{aligned}
\widehat{z} &= u(z) + \theta\zeta(z) \\
\widehat{\theta} &= \eta(z) + \theta v(z).
\end{aligned}
$$

$$(2.11)$$

Then (2.10) gives $\zeta = v\eta$, $v^2 = u' + \eta\eta'$, so the relation between the two coordinate systems is

$$
\begin{aligned}
\widehat{z} &= u(z) + \theta\eta(z)\sqrt{u'(z)} \\
\widehat{\theta} &= \eta(z) + \theta\sqrt{u'(z) + \eta(z)\eta'(z)}.
\end{aligned}
$$

$$(2.12)$$

In any superconformal coordinate system $z|\theta$, the object D_θ gives a local trivialization of the line bundle $\mathcal{D}$. A global section of $\mathcal{D}$ assigns to each superconformal coordinate system $z|\theta$ a function $A(z|\theta)$ such that the vector field $A(z|\theta)D_\theta$ is independent of the choice of superconformal coordinates. Recalling (2.8), this means that the functions $A(z|\theta)$ and $\widehat{A}(\widehat{z}|\widehat{\theta})$ in two superconformal coordinate systems are related by

$$(2.13) \qquad\qquad A = F^{-1}\widehat{A}, \quad F = D_\theta\widehat{\theta}.$$

A field A with this property is called a superconformal primary field of dimension $-1/2$. (The normalization comes from the fact that D_θ scales with

mass dimension $1/2$.) More generally a section of $\mathcal{D}^n$, for any n, corresponds to a field that transforms under a change of superconformal coordinates as

$$(2.14) \qquad\qquad A = F^{-n}\widehat{A}.$$

Such a field is a superconformal primary of dimension $-n/2$. As an example of this definition, if ϕ is a globally-defined function on Σ, then since

$$(2.15) \qquad\qquad D_\theta\phi = F D_{\widehat{\theta}}\phi,$$

$D_\theta\phi$ is a section of $\mathcal{D}^{-1}$ or in other words a superconformal primary of dimension $1/2$.

If A is a superconformal primary of some dimension h, then locally we can expand

$$(2.16) \qquad\qquad A = a_0 + \theta a_1,$$

where a_0 and a_1 are conformal primaries of dimensions h and $h + 1/2$.

2.1.2. Gluing and split super Riemann surfaces With the help of (2.12), one can construct all possible super Riemann surfaces by gluing together open sets $U_\alpha \subset \mathbb{C}^{1|1}$ by means of superconformal automorphisms. One simply imitates the gluing procedure that builds an ordinary manifold by gluing together small open sets. All super Riemann surfaces can be built in this way. The only problem is that the formulas quickly get complicated.

An important special case of the gluing procedure arises if we glue by superconformal automorphisms with $\eta = 0$. Then (2.12) simplifies and the gluing of superconformal coordinates in open sets U_α and U_β reduces to

$$(2.17) \qquad \begin{aligned} z_\alpha &= u_{\alpha\beta}(z_\beta) \\ \theta_\alpha &= (u'_{\alpha\beta}(z_\beta))^{1/2}\theta_\beta. \end{aligned}$$

We can consistently forget the θ's, because (2.17) says that z_α is a function of z_β only, independent of θ. This would not be true for more general superconformal gluing with $\zeta \neq 0$. When we forget the θ's, the functions $u_{\alpha\beta}(z_\beta)$ are gluing functions that build an ordinary Riemann surface Σ_{red} (known as the reduced space of Σ) by gluing small open sets in $\mathbb{C}$. (To build the super Riemann surface Σ takes a little more information, as we discuss shortly, since the gluing relation for the θ's depends on the signs of the square roots $(u'_{\alpha\beta})^{1/2}$.) Forgetting the θ's amounts to a projection $\pi : \Sigma \to \Sigma_{\mathrm{red}}$. The fibers of π are linear spaces of dimension $0|1$, parametrized by the odd coordinates

θ_α. Thus Σ is the total space of a line bundle $V \to \Sigma_{\mathrm{red}}$, where the fibers of V are fermionic.

To identify V, we observe that the gluing law for the holomorphic differential $\mathrm{d}z_\alpha$ is $\mathrm{d}z_\alpha = u'_{\alpha\beta}(z_\beta)\,\mathrm{d}z_\beta$, so that (2.17) says that θ_α transforms as $(\mathrm{d}z_\alpha)^{1/2}$. Now $\mathrm{d}z_\alpha$ is a local section of a line bundle, the canonical or cotangent bundle[2] K of the ordinary Riemann surface Σ_{red}. So $(\mathrm{d}z_\alpha)^{1/2}$ would be a section of $K^{1/2}$, and thus θ_α transforms as a (fermionic) section of $K^{1/2}$. If a linear function on the fibers of a line bundle V is a section of $K^{1/2}$, we must identify V as the dual line bundle $K^{-1/2}$.

So we conclude that gluing laws of the particular form (2.17) describe a super Riemann surface Σ that is the total space of the line bundle $\Pi K^{-1/2}$ over Σ_{red}. Here the symbol Π simply represents reversal of "parity" or statistics; it is a reminder that the fibers of the fibration $\Sigma \to \Sigma_{\mathrm{red}}$ are fermionic.

In this construction, $K^{1/2}$ may be any line bundle over Σ_{red} whose square is isomorphic to K. If Σ_{red} has genus g (in which case we also say that Σ has genus g), then up to isomorphism, there are $2^{2\mathsf{g}}$ choices of $K^{1/2}$. The choice of $K^{1/2}$ enters in the choices of signs of the square roots $(u'_{\alpha\beta})^{1/2}$ that appear in the superconformal gluing relations (2.17). A choice of $K^{1/2}$ is called a spin structure.

A super Riemann surface Σ with a projection to Σ_{red} as described above is called a split super Riemann surface. The reduced space Σ_{red} is what we will call a spin curve – an ordinary Riemann surface together with a choice of spin structure. Topologically, spin structures are naturally classified as being even or odd depending on whether the space $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ of global holomorphic sections of the line bundle $K^{1/2}$ has even or odd dimension; this is the only property of a spin structure that is invariant under all diffeomorphisms, including those that are not continuously connected to the identity. The moduli space $\mathcal{M}_{\mathrm{spin}}$ of spin curves accordingly has two connected components, which we will call $\mathcal{M}_{\mathrm{spin},\pm}$, parametrizing respectively a Riemann surface with a choice of even or odd spin structure. $\mathcal{M}_{\mathrm{spin},\pm}$ is an unramified cover of $\mathcal{M}$, the moduli space of Riemann surfaces without a choice of spin structure; the degree of the cover is the number of even or odd spin structures, namely $\frac{1}{2}(2^{2\mathsf{g}} \pm 2^{\mathsf{g}})$.

To get $\mathfrak{M}$, the moduli space of super Riemann surfaces, we must allow more general gluing with $\zeta \neq 0$. Since the odd parameters are nilpotent, this does not change the topology of the situation, so again $\mathfrak{M}$ has two connected

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[2]For any space X, we generically write T^*X for the cotangent bundle to X. So we would naturally write $T^*\Sigma_{\mathrm{red}}$ for the cotangent bundle to Σ_{red}. However, the abbreviation K is traditional and convenient.

components $\mathfrak{M}_\pm$. $\mathfrak{M}_\pm$ reduces to $\mathcal{M}_{\mathrm{spin},\pm}$ if we set the odd moduli (and therefore ζ) to zero. So $\mathcal{M}_{\mathrm{spin},\pm}$ is the reduced space of $\mathfrak{M}_\pm$.

2.2. Moduli of super Riemann surfaces

We will now try to understand more systematically the even and odd gluing parameters that arise in building a super Riemann surface Σ.

Recall that Σ is built out of small open sets U_α that are glued together on intersections $U_\alpha \cap U_\beta$. So a first-order deformation of the gluing data is given by a superconformal vector field $\phi_{\alpha\beta}$ defined in each intersection $U_\alpha \cap U_\beta$. The idea is that before gluing U_β to U_α, we transform by $1 + w\phi_{\alpha\beta}$, with w an infinitesimal parameter. The $\phi_{\alpha\beta}$ are subject to a constraint

$$(2.18) \qquad\qquad \phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\gamma\alpha} = 0$$

in each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. This constraint ensures that the gluing data remain consistent after being modified by the $\phi_{\alpha\beta}$. The $\phi_{\alpha\beta}$ are also subject to an equivalence relation

$$(2.19) \qquad\qquad \phi_{\alpha\beta} \cong \phi_{\alpha\beta} + \phi_\alpha - \phi_\beta,$$

where each ϕ_α is a superconformal vector field defined in U_α. The meaning of this equivalence is that Σ is unchanged if we transform each U_α by a symmetry generated by ϕ_α before gluing the U_α together. All this is precisely analogous to the deformation theory of ordinary complex manifolds. The constraint (2.18) means that we should interpret $\phi_{\alpha\beta}$ as a one-cocycle on Σ with values in the sheaf[3] $\mathcal{S}$ of superconformal vector fields on Σ. Modulo the equivalence relation (2.19), such a cocycle determines an element of $H^1(\Sigma, \mathcal{S})$.

Let $T\mathfrak{M}$ be the tangent bundle to $\mathfrak{M}$ and let $T\mathfrak{M}|_\Sigma$ be its fiber at the point in $\mathfrak{M}$ corresponding to Σ. What is explained in the last paragraph implies that $T\mathfrak{M}|_\Sigma = H^1(\Sigma, \mathcal{S})$.

While $H^1(\Sigma, \mathcal{S})$ has this interpretation, the corresponding space of global holomorphic sections $H^0(\Sigma, \mathcal{S})$ has a more elementary interpretation: it classifies superconformal vector fields on Σ that are holomorphic everywhere, and so determine infinitesimal automorphisms of Σ. So $H^0(\Sigma, \mathcal{S})$ is the Lie algebra of the supergroup $\mathcal{G}$ of superconformal automorphisms of Σ. For a super

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[3]This sheaf assigns to any small open set $U \subset \Sigma$ the space of superconformal vector fields on U.

Riemann surface Σ, the higher sheaf cohomology groups $H^k(\Sigma, \mathcal{S})$, $k > 1$, are always zero,[4] so we only have to consider the cases $k = 0, 1$.

In superstring perturbation theory, at least for closed oriented superstrings, one normally encounters only values of the genus g of Σ and the number of punctures such that there are no infinitesimal automorphisms, $H^0(\Sigma, \mathcal{S}) = 0$. For the time being, we consider Riemann surfaces without punctures (they will be incorporated in section 4). With no punctures, just as for ordinary Riemann surfaces, infinitesimal automorphisms are absent precisely for $\mathsf{g} \geq 2$, so the cases $\mathsf{g} = 0, 1$ are exceptional. (For those exceptional cases, the moduli spaces can be described by hand; see section 5.)

We would like to calculate the even and odd dimension of $T\mathfrak{M}|_\Sigma$. For $\mathsf{g} \geq 2$, this dimension does not depend on the odd moduli of Σ, for the following reason. Because the odd moduli are infinitesimal, they could affect the cohomology only by a "jumping" process in which cohomology classes of neighboring degree pair up and disappear under an infinitesimal perturbation. For the present problem, with $\mathsf{g} \geq 2$, since the cohomology is nonzero only in degree 1, jumping cannot occur and the odd moduli do not affect the dimension of the cohomology. It turns out that there is also no jumping for $\mathsf{g} = 0$ or for $\mathsf{g} = 1$ with even spin structure (for the case of $\mathsf{g} = 1$ with odd spin structure, see the end of section 5.2).

So to determine the dimension of $T\mathfrak{M}|_\Sigma$, we can make a convenient choice of the odd moduli. The most convenient choice is to set them to zero, that is, to take Σ to be a split super Riemann surface. In this case, we have a decomposition $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ where $\mathcal{S}_+$ consists of vector fields

$$(2.20) \qquad V_g = g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta$$

with g even, and $\mathcal{S}_-$ consists of vector fields

$$(2.21) \qquad \nu_f = f(z)(\partial_\theta - \theta\partial_z).$$

with f odd. Similarly, for Σ split, there is a natural decomposition $T\mathfrak{M}|_\Sigma = T_+\mathfrak{M}|_\Sigma \oplus T_-\mathfrak{M}|_\Sigma$ in even and odd subspaces, and

$$(2.22) \qquad T_\pm\mathfrak{M}|_\Sigma = H^1(\Sigma, \mathcal{S}_\pm).$$

To evaluate (2.22), we can express the cohomology computation as a computation on the reduced space Σ_{red}. We can associate to V_g the ordinary

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[4]This statement can be deduced from the analogous statement for the ordinary Riemann surface Σ_{red}. As Σ_{red} has complex dimension 1, its cohomology with values in any coherent sheaf vanishes above degree 1.

vector field $g(z)\partial_z$ on Σ_{red}; it is a section of $T\Sigma_{\mathrm{red}}$, the tangent bundle to Σ_{red}. (This is the same as K^{-1}, where K is the canonical bundle of Σ_{red}.) So $\mathcal{S}_+$ is the sheaf of sections of $T\Sigma_{\mathrm{red}}$ and

$$(2.23) \qquad\qquad T_+\mathfrak{M}|_\Sigma = H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}).$$

The right hand side is the tangent space to $\mathcal{M}$, the moduli space of ordinary Riemann surfaces, at the point corresponding to Σ_{red}. For Σ_{red} of genus $g \geq 2$, it has dimension $3g-3$. One route to this result is the Riemann-Roch theorem. For any line bundle $\mathcal{L}$ of degree n, this theorem asserts that

$$(2.24) \qquad \dim H^0(\Sigma_{\mathrm{red}}, \mathcal{L}) - \dim H^1(\Sigma_{\mathrm{red}}, \mathcal{L}) = 1 - g + n.$$

Also, $H^0(\Sigma_{\mathrm{red}}, \mathcal{L}) = 0$ if $n < 0$. For $\mathcal{L} = T\Sigma_{\mathrm{red}}$, we have $n = 2 - 2g$, which is negative for $g \geq 2$, so the Riemann-Roch formula gives $\dim H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}) = 3g - 3$.

Similarly, we can associate to ν_f the object $f(z)\partial_\theta$, which we view as an odd vector field along the fibers of $\Sigma \to \Sigma_{\mathrm{red}}$, or in other words as a section of $K^{-1/2} = T\Sigma_{\mathrm{red}}^{1/2}$. So $\mathcal{S}_-$ is the sheaf of sections of $T\Sigma_{\mathrm{red}}^{1/2}$ and

$$(2.25) \qquad\qquad T_-\mathfrak{M}|_\Sigma = \Pi H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}^{1/2}),$$

where the symbol Π is meant to remind us to view this space as being fermionic. The line bundle $T\Sigma_{\mathrm{red}}^{1/2}$ has degree $1-g$, and for $g \geq 2$, the Riemann-Roch formula gives $\dim H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}^{1/2}) = 2g - 2$.

Thus for $g \geq 2$, the dimension of the moduli space of super Riemann surfaces of genus g with no punctures is

$$(2.26) \qquad\qquad \dim \mathfrak{M}_g = 3g - 3|2g - 2.$$

The general formula that holds for all g includes a contribution in the Riemann-Roch formula from $H^0(\Sigma, \mathcal{S})$, which is the Lie algebra of the supergroup $\mathcal{G}$ of automorphisms of Σ. The general formula is

$$(2.27) \qquad\qquad \dim \mathfrak{M}_g - \dim \mathcal{G} = 3g - 3|2g - 2.$$

For example, for $g = 0$, $\mathfrak{M}_0$ is a point, of dimension $0|0$, while $\mathcal{G}$ is the supergroup $\mathrm{OSp}(1|2)$, of dimension $3|2$.

2.2.1. Odd moduli and spin structures   The fact that the odd moduli take values in $\varPi H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}^{1/2})$ means that once odd moduli are turned on, it does not make sense to claim that two super Riemann surfaces are the same except for a different choice of spin structure. The odd moduli take values in a space whose definition depends on a choice of $T\Sigma_{\mathrm{red}}^{1/2}$ and thus on a choice of spin structure, so odd moduli with different spin structures cannot be compared. This means that what happens in superstring perturbation theory in genus 1 is atypical. In genus 1, with an even spin structure (and no punctures), there are no odd moduli. It therefore makes sense to sum over spin structures before performing any other integration, and this leads to the classic proof of vanishing of the one-loop contribution to the cosmological constant, via an identity whose use in this context goes back to [39]. When odd moduli are present, the closest analog of this procedure is to first integrate over the odd moduli, to reduce an integral over $\mathfrak{M}_{\mathfrak{g}}$ to an integral over $\mathcal{M}_{\mathfrak{g}}$. After this, it makes sense to sum over spin structures before performing the remaining integrals. This can be elegantly done for $\mathfrak{g} = 2$ at least for the vacuum amplitude [21, 40], though as far as is known, there is no such natural procedure in general.

2.2.2. Deformations as a complex supermanifold   In this analysis, we have used a gluing construction to interpret first-order deformations of Σ as a super Riemann surface in terms of the sheaf cohomology group $H^1(\Sigma, \mathcal{S})$.

Alternatively, we could simply deform Σ as a complex supermanifold, not requiring the deformations to preserve a super Riemann surface structure. This is much easier to analyze. In this case, the vector fields $\phi_{\alpha\beta}$ by which we perturb can be any holomorphic sections of the tangent bundle $T\Sigma$, and hence the first-order deformations of Σ as a complex supermanifold are parametrized by $H^1(\Sigma, T\Sigma)$. The same cohomology group parametrizes the first-order deformations of an ordinary complex manifold, for essentially the same reasons.

2.2.3. $\mathfrak{M}_{\mathfrak{g}}$ as an orbifold   The moduli space $\mathcal{M}_{\mathfrak{g}}$ of ordinary Riemann surfaces Σ_0 of genus $\mathfrak{g}$ is not a manifold but an orbifold. This is so because Σ_0 may have automorphisms. Usually in perturbative string theory, one considers a situation in which Σ_0 has no continuous automorphisms (either because $\mathfrak{g} \geq 2$, or because Σ_0 is endowed with a sufficient number of punctures, as described in section 4), but it still may have a finite group of automorphisms.

Orbifold singularities of $\mathcal{M}_{\mathfrak{g}}$ arise when Σ_0 has extra automorphisms compared to a generic surface with the same genus. For example, a generic surface of genus at least 3 has no nontrivial automorphisms at all, so for $\mathfrak{g} \leq 3$, $\mathcal{M}_{\mathfrak{g}}$

has an orbifold singularity whenever Σ_0 has a nontrivial automorphism group. For $\mathfrak{g} = 2$, the generic automorphism group is $\mathbb{Z}_2$, and orbifold singularities occur when it becomes larger.

Super moduli space $\mathfrak{M}_{\mathfrak{g}}$ is similarly an orbifold. Indeed, its orbifold structure is more prominent than that of $\mathcal{M}_{\mathfrak{g}}$, for the following simple reason. Every super Riemann surface Σ is infinitesimally close to a surface with enhanced symmetry. That is so because there is always an enhanced symmetry when we turn off the odd moduli; a split super Riemann surface Σ always has a $\mathbb{Z}_2$ symmetry τ that acts trivially on Σ_{red} and acts as -1 on the fibers of the fibration $\Sigma \to \Sigma_{\mathrm{red}}$. (In eqn. (2.17), this is the symmetry that acts as $\tau : z_\alpha | \theta_\alpha \to z_\alpha | -\theta_\alpha$ for all α.) The odd moduli are odd under this symmetry. So the locus of enhanced symmetries is dense in $\mathfrak{M}_{\mathfrak{g}}$, while in $\mathcal{M}_{\mathfrak{g}}$ it has positive (bosonic) codimension.

In the present notes, we will not go quite deeply enough for the orbifold nature of $\mathfrak{M}_{\mathfrak{g}}$ to play a major role, though the $\mathbb{Z}_2$ automorphism group of a split super Riemann surface will make an occasional appearance.

The fancy way to describe the orbifold nature of $\mathfrak{M}_{\mathfrak{g}}$ is to refer to it as the moduli "stack" of super Riemann surfaces rather than the moduli "space."

2.3. Superconformal vector fields reconsidered

In the definition of a super Riemann surface Σ, we postulated the existence of a subbundle $\mathcal{D} \subset T\Sigma$ of rank $0|1$. As $T\Sigma$ has rank $1|1$, the quotient $T\Sigma/\mathcal{D}$ has rank $1|0$.

On the other hand, we also postulated that if D is a local nonzero section of $\mathcal{D}$, then D^2 is everywhere nonzero mod D. This means that D^2 has an everywhere nonzero projection to $\mathcal{L} = T\Sigma/\mathcal{D}$. The existence of a natural map from a nonzero section D of $\mathcal{D}$ to a nonzero section D^2 of $\mathcal{L}$ (a quadratic map in the sense that if f is a nonzero function, then $(fD)^2 = f^2 D^2 \mod \mathcal{D}$) implies that $\mathcal{L} \cong \mathcal{D}^2$.

Thus the natural exact sequence $0 \to \mathcal{D} \to T\Sigma \to T\Sigma/\mathcal{D} \to 0$ becomes

$$(2.28) \qquad\qquad 0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2 \to 0.$$

We also sometimes need the dual sequence:

$$(2.29) \qquad\qquad 0 \to \mathcal{D}^{-2} \to T^*\Sigma \to \mathcal{D}^{-1} \to 0.$$

Here one must recall that to dualize an exact sequence of vector bundles, one dualizes each bundle involved and reverses the direction of the maps. To get (2.29), we just need to know that the dual of $T\Sigma$ is the cotangent bundle $T^*\Sigma$ and the dual of $\mathcal{D}^h$ is $\mathcal{D}^{-h}$. Just as $T\Sigma$ has a distinguished subbundle

of rank $0|1$, namely $\mathcal{D}$, (2.29) says that $T^*\Sigma$ has a distinguished subbundle of rank $1|0$, isomorphic to $\mathcal{D}^{-2}$. Concretely, what singles out the subbundle $\mathcal{D}^{-2} \subset T^*\Sigma$ is that, under the duality between $T^*\Sigma$ and $T\Sigma$, it is normal to $\mathcal{D} \subset T\Sigma$. Explicitly, in local superconformal coordinates $z|\theta$, the subbundle $\mathcal{D} \subset T\Sigma$ is generated by the vector field $D_\theta = \partial_\theta + \theta \partial_z$, so $\mathcal{D}^{-2} \subset T^*\Sigma$ is generated by the 1-form

$$\varpi = \mathrm{d}z - \theta \mathrm{d}\theta, \tag{2.30}$$

whose contraction with D_θ vanishes.

We would like to give a new description of the space of superconformal vector fields. The formulas (2.4) and (2.5) for superconformal vector fields are clear enough, and not difficult to verify. But they have the peculiar property that the functions $f(z)$ and $g(z)$ depend on z only and not θ. Can we combine f and g to a superfield?

In [11], it is shown how to do this. In terms of a superfield $\mathcal{V}(z|\theta) = g(z) + 2\theta f(z)$, a general superconformal vector field is

$$\mathcal{W} = \mathcal{V}(z,\theta)\partial_z + \frac{1}{2}D_\theta \mathcal{V} \, D_\theta. \tag{2.31}$$

In fact, upon expanding the right hand side in terms of ∂_z and ∂_θ, one finds

$$\mathcal{W} = V_g + \nu_f, \tag{2.32}$$

with V_g and ν_f as defined before.

Since (2.31) has been written in local superconformal coordinates $z|\theta$, the global nature of the superfield $\mathcal{V}(z|\theta)$ is not immediately obvious. To understand it, we observe that a superconformal vector field $\mathcal{W}$ is in particular a vector field, and thus a section of $T\Sigma$. We can project $\mathcal{W}$ from $T\Sigma$ to $T\Sigma/\mathcal{D}$ by dropping the D_θ term. In other words, mod D_θ, $\mathcal{W}$ is equivalent to $\mathcal{V}(z,\theta)\partial_z$, where we view ∂_z as giving a basis for $T\Sigma/\mathcal{D} \cong \mathcal{D}^2$. So globally the superfield $\mathcal{V}$ is a section of $\mathcal{D}^2$, as stated in [11].

The map from superconformal vector fields to sections of $\mathcal{D}^2$ is one-to-one, as is clear from (2.32). This helps in understanding the superconformal ghost fields of string theory. These fields start life as a superconformal vector field with reversed statistics. So they can be combined to a superfield C that is a section of $\Pi\mathcal{D}^2$, where the symbol Π tells us that C has the opposite to usual statistics (its lower component is odd). Locally, we can expand

$$C = c + \theta\gamma, \tag{2.33}$$

where c is an odd section of $T\Sigma_{\mathrm{red}}$, and γ is an even section of $T\Sigma_{\mathrm{red}}^{1/2}$.

The equivalence of $\mathcal{S}$ with the sheaf of sections of $\mathcal{D}^2$ has an application that we will explain later: it leads to a convenient description of the dual space to $H^1(\Sigma, \mathcal{S})$. This equivalence also has one obvious drawback: superconformal vector fields have a natural graded Lie algebra structure, which is not particularly visible in the description via sections of $\mathcal{D}^2$.

2.4. Holomorphic volume forms

Now we will discuss holomorphic volume forms on Σ. Like a holomorphic differential $a(z)\mathrm{d}z$ on an ordinary Riemann surface, a holomorphic volume form is the right object for a contour integral – an integral on a submanifold of Σ of real codimension 1. To introduce volume forms suitable for "bulk" integrals over Σ (analogous to a 2-form $a(\overline{z}, z)\mathrm{d}\overline{z}\wedge\mathrm{d}z$ on an ordinary Riemann surface), we will need the smooth description of super Riemann surfaces that we introduce in section 3.1.

In general, on a complex supermanifold X of dimension $p|q$, one defines a holomorphic line bundle $Ber(X)$ (known as the Berezinian of X) of holomorphic densities on X. The most elementary definition of $Ber(X)$ is as follows (see, for example, sections 3.1 and 5.3.1 of [32], as well as appendix A below). Given any local trivialization of T^*X by basis elements $dz^1 \dots | \dots d\theta^q$ of T^*X, there is a corresponding local trivialization of $Ber(X)$ by a basis element $[dz^1 \dots | \dots d\theta^q]$. Under a change of basis, the symbol $[dz^1 \dots | \dots d\theta^q]$ transforms as one would expect a density to transform (that is, it transforms by the Berezinian – the superanalog of the determinant – of the matrix giving the change of basis). $Ber(X)$ is the analog for complex supermanifolds of what for an ordinary complex manifold is called the determinant line bundle. A holomorphic section of $Ber(X)$ is the superanalog of a holomorphic p-form on an ordinary complex manifold of dimension p.

We claim that in the case of a super Riemann surface Σ, $Ber(\Sigma)$ is naturally isomorphic to $\mathcal{D}^{-1}$. So sections of $\mathcal{D}^{-1}$, which correspond to superconformal primaries of dimension $1/2$, are equivalent to sections of $Ber(\Sigma)$. For example, in discussing eqn. (2.15), we showed that if ϕ is a function on Σ, then there is a section of $\mathcal{D}^{-1}$ that in any local superconformal coordinate system $z|\theta$ can be represented by $D_\theta\phi$. We must therefore have a section of $Ber(\Sigma)$ given by

$$(2.34) \qquad\qquad \sigma = [\mathrm{d}z|\mathrm{d}\theta]\, D_\theta\phi,$$

and in particular we claim that σ does not depend on the choice of the superconformal coordinate system $z|\theta$. Clearly ϕ plays no essential role here. The assertion that $Ber(\Sigma) \cong \mathcal{D}^{-1}$ is equivalent to the statement that the expression $[\mathrm{d}z|\mathrm{d}\theta]\, D_\theta$ is independent of the choice of superconformal coordinates.

Thus if $\widehat{z}|\widehat{\theta}$ is any other superconformal coordinate system, we claim that

$$[\mathrm{d}\widehat{z}|\mathrm{d}\widehat{\theta}]\, D_{\widehat{\theta}} = [\mathrm{d}z|\mathrm{d}\theta]\, D_\theta. \tag{2.35}$$

For a first orientation to this statement, suppose that the two coordinate systems differ by scaling $\widehat{z}|\widehat{\theta} = \lambda z|\lambda^{1/2}\theta$. Then $[\mathrm{d}\widehat{z}|\mathrm{d}\widehat{\theta}] = \lambda^{1/2}[\mathrm{d}z|\mathrm{d}\theta]$ (where a factor of λ comes from scaling of z and a factor of $\lambda^{-1/2}$ from scaling of θ), while $D_{\widehat{\theta}} = \lambda^{-1/2}D_\theta$.

The general proof follows by a direct calculation [23]. By definition of the Berezinian,

$$[\mathrm{d}\widehat{z}|\mathrm{d}\widehat{\theta}] = [\mathrm{d}z|\mathrm{d}\theta]\,\mathrm{Ber}(M), \tag{2.36}$$

where

$$M = \begin{pmatrix} \partial_z\widehat{z} & \partial_z\widehat{\theta} \\ \partial_\theta\widehat{z} & \partial_\theta\widehat{\theta} \end{pmatrix}. \tag{2.37}$$

With the help of (2.10), one may verify that

$$M = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \partial_z\widehat{z} + \widehat{\theta}\partial_z\widehat{\theta} & \partial_z\widehat{\theta} \\ 0 & D_\theta\widehat{\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \widehat{\theta} & 1 \end{pmatrix}, \tag{2.38}$$

from which it follows that

$$\mathrm{Ber}(M) = \frac{\partial_z\widehat{z} + \widehat{\theta}\partial_z\widehat{\theta}}{D_\theta\widehat{\theta}} = D_\theta\widehat{\theta}. \tag{2.39}$$

In the last step, we used the relation

$$\partial_z\widehat{z} + \widehat{\theta}\partial_z\widehat{\theta} - (D_\theta\widehat{\theta})^2 = 0, \tag{2.40}$$

which follows upon applying the operator D_θ to eqn. (2.10). Using this result for $\mathrm{Ber}(M)$ together with (2.8), we find that (2.36) implies the desired result (2.35).

A less computational proof that $Ber(\Sigma) \cong \mathcal{D}^{-1}$ can be found in appendix A.

2.4.1. What is this good for? The most important application of the isomorphism $Ber(\Sigma) \cong \mathcal{D}^{-1}$ will be as a tool in writing Lagrangians. Another important application will involve duality on a super Riemann surface.

But here we will explain an application involving conserved currents and contour integrals. In superconformal field theory, a superconformal primary σ of dimension $1/2$ is the superanalog of a conserved (holomorphic) current. Hence for every codimension 1 cycle γ, there is a conserved charge $q_\gamma = \int_\gamma \sigma$. The statement that q_γ is "conserved" means that it depends only on the homology class of γ. From the point of view of super Riemann surfaces, σ corresponds to a holomorphic section of $\mathcal{D}^{-1}$ and hence of $Ber(\Sigma)$. The fact that such a σ can be integrated over a codimension 1 homology cycle γ is a special case of a general fact about complex supermanifolds. As explained for example in section 5.3.1 of [32], on a complex manifold X of dimension $p|q$, a holomorphic section σ of $Ber(X)$ can be naturally integrated over a cycle $\gamma \subset X$ of real codimension $p|0$, with a result that only depends on the homology class of γ. Moreover, up to homology, a codimension $p|0$ cycle $\gamma \subset X$ is naturally determined by a corresponding ordinary codimension p homology cycle $\gamma_{\mathrm{red}} \subset X_{\mathrm{red}}$. In the super Riemann surface context, that means that an ordinary one-cycle $\gamma_{\mathrm{red}} \subset \Sigma_{\mathrm{red}}$ determines the charge $q_\gamma = \oint_\gamma \sigma$.

An example of a holomorphic section σ of $Ber(X)$ is $\sigma = D_\theta \phi$, where ϕ is a holomorphic function. But in this case, the conserved charges vanish. It is instructive to explain this using the interpretation of $Ber(X)$ as the space of holomorphic integral forms on X of top degree. (See [32] for an explanation of the relevant concepts.) Given local superconformal coordinates $z|\theta$, we can define the section $[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(X)$ and also the vector field D_θ. Each of these separately depends on the choice of coordinates, but as we have seen the product $[\mathrm{d}z|\mathrm{d}\theta]\, D_\theta$ does not. The contraction operator $\mathbf{i}_{D_\theta}$ transforms like D_θ, so $\mathbf{i}_{D_\theta}[\mathrm{d}z|\mathrm{d}\theta]$, which we understand as a holomorphic integral form of codimension 1, does not depend on the choice of coordinates. More generally, given a function ϕ, we can define the codimension 1 holomorphic integral form $\lambda = \phi\, \mathbf{i}_{D_\theta}[\mathrm{d}z|\mathrm{d}\theta]$. Explicitly, the contraction operator is

$$(2.41) \qquad \mathbf{i}_{D_\theta} = \frac{\partial}{\partial \mathrm{d}\theta} + \theta \frac{\partial}{\partial \mathrm{d}z},$$

and as an integral form, $[\mathrm{d}z|\mathrm{d}\theta]$ is naturally written $\delta(\mathrm{d}z)\delta(\mathrm{d}\theta)$, so

$$(2.42) \qquad \lambda = \phi \left(\frac{\partial}{\partial \mathrm{d}\theta} + \theta \frac{\partial}{\partial \mathrm{d}z} \right) \delta(\mathrm{d}z)\delta(\mathrm{d}\theta).$$

The exterior derivative is

$$(2.43) \qquad \mathrm{d} = \mathrm{d}z \frac{\partial}{\partial z} + \mathrm{d}\theta \frac{\partial}{\partial \theta}.$$

Using the fact that $x\delta'(x) = \mp\delta(x)$ for an even or odd variable x, we find

$$(2.44) \qquad d\lambda = -D_\theta\phi\,\delta(dz)\delta(d\theta) = [dz|d\theta]\,D_\theta\phi.$$

So $\oint_\gamma [dz|d\theta]\,D_\theta\phi = \oint_\gamma d\lambda = 0$, by the superspace version of Stokes's theorem. For another explanation of this result, see eqn. (8.12).

3. Super Riemann surfaces from a smooth point of view

So far, we have considered a super Riemann surface purely as a complex supermanifold of dimension $1|1$ with some additional structure. This is not really the right structure for a string worldsheet, as on the string worldsheet, there are both holomorphic and antiholomorphic degrees of freedom. For string theory, we have to know how to go over to a smooth description, in which one can discuss functions on Σ that are not necessarily holomorphic. This will enable us to discuss topics such as fields and Lagrangians on Σ, deformations of Σ as described by fields, Riemannian geometry on Σ, etc.

3.1. String worldsheets and their parameter space

We will adopt the viewpoint of [32], section 5.5. A string worldsheet Σ is a smooth supermanifold that is embedded in a product $\Sigma_L \times \Sigma_R$ of holomorphic Riemann surfaces or super Riemann surfaces. (The notation Σ_L and Σ_R is meant to evoke left- and right-moving degrees of freedom in string theory.) For both heterotic and Type II superstrings, Σ_R is a super Riemann surface. For the heterotic string, Σ_L is an ordinary Riemann surface, but for Type II, Σ_L is another super Riemann surface.

What we will do with Σ is to use it as an integration cycle: we will define the worldsheet action of string theory by integrating a closed form – actually a holomorphic section of $Ber(\Sigma_L \times \Sigma_R)$ – over Σ. For this purpose, small deformations of the embedding of Σ in $\Sigma_L \times \Sigma_R$ do not matter. The basic example is that the reduced spaces of Σ_L and Σ_R are complex conjugates (the complex conjugate of a complex manifold X is the same space with opposite complex structure) and Σ_{red} is the diagonal in $\Sigma_{L,\mathrm{red}} \times \Sigma_{R,\mathrm{red}}$. Then Σ is obtained from Σ_{red} by a slight thickening in the fermionic directions, as explained in [32], section 2.2. This last operation is not completely natural, but it is natural up to homology, which will be good enough.[5] More generally, it suffices if Σ_L is sufficiently close to the complex conjugate of Σ_R and Σ_{red} is close to the diagonal in $\Sigma_{L,\mathrm{red}} \times \Sigma_{R,\mathrm{red}}$. For more on all this, see section 5 of [32].

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[5]If one picks Σ_{red} to be the diagonal in $\Sigma_{L,\mathrm{red}} \times \Sigma_{R,\mathrm{red}}$, then one can restrict the homologies in question to be infinitesimal ones that act trivially on Σ_{red}. We call

The odd dimension of Σ is the same as that of $\Sigma_L \times \Sigma_R$, and its even dimension (as a smooth supermanifold) is 2. For the heterotic string, $\Sigma_L \times \Sigma_R$ is a complex supermanifold of dimension 2|1, so $\Sigma \subset \Sigma_L \times \Sigma_R$ is a smooth supermanifold of dimension 2|1. For Type II, Σ_L and Σ_R are both super Riemann surfaces and Σ is a smooth supermanifold of dimension 2|2.

By definition, a holomorphic function on Σ is a holomorphic function on Σ_R, restricted to Σ. If $z|\theta$ are local superconformal coordinates on Σ_R, then their restrictions to Σ are local holomorphic functions on Σ and locally any holomorphic function on Σ is $f(z|\theta)$. Similarly, an antiholomorphic function on Σ is by definition the restriction to Σ of a holomorphic function on Σ_L. For the heterotic string, if $\widetilde{z}$ is a local holomorphic function on Σ_L, its restriction to Σ is a local antiholomorphic function and locally any antiholomorphic function is a function $g(\widetilde{z})$. Given such local coordinates on Σ_L and Σ_R, with $\widetilde{z}$ sufficiently close[6] to the complex conjugate of z, we call these functions – or more precisely their restrictions to Σ – a standard coordinate system $\widetilde{z};z|\theta$ on Σ. (Our convention will generally be to list antiholomorphic coordinates before holomorphic ones, separating them by the semicolon.) For Type II, we take local superconformal coordinates $\widetilde{z}|\widetilde{\theta}$ and $z|\theta$ on Σ_L and Σ_R respectively, with $\widetilde{z}$ sufficiently close to the complex conjugate of z, to define a standard local coordinate system $\widetilde{z};z|\widetilde{\theta};\theta$.

Since $\Sigma_L \times \Sigma_R$ is a product, its Berezinian is $Ber(\Sigma_L \times \Sigma_R) \cong Ber(\Sigma_L) \otimes Ber(\Sigma_R)$. A holomorphic section σ of $Ber(\Sigma_L \times \Sigma_R)$ can be integrated over Σ, with a result that is invariant under small deformations of Σ within $\Sigma_L \times \Sigma_R$. This is explained in [32], section 5.3.1. One way to understand the statement is to observe that σ can be understood as a codimension 2 integral form on $\Sigma_L \times \Sigma_R$, so it can be integrated over the codimension 2|0 cycle Σ, with a result that only depends on the homology class of Σ. Since the ability to integrate a section of $Ber(\Sigma_L \times \Sigma_R)$ over Σ is important, we will explain it in another way. We consider the heterotic string as an example. As Σ is a smooth supermanifold, it has a Berezinian line bundle $Ber(\Sigma)$ in the smooth sense, whose sections can be integrated; a trivialization of $Ber(\Sigma)$ in any standard coordinate system is given by the symbol $[d\widetilde{z};dz|d\theta]$. $Ber(\Sigma_L)$ and $Ber(\Sigma_R)$ are likewise trivialized by the symbols $[d\widetilde{z}]$ and $[dz|d\theta]$, and so their tensor product is trivialized by the tensor product $[d\widetilde{z}] \otimes [dz|d\theta]$. The definitions

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these fermionic homologies. In any event, Σ is also unique up to isomorphism as a smooth supermanifold and its holomorphic and antiholomorphic structures are both also unique. Only the relation between the holomorphic and antiholomorphic structures depends on how Σ is embedded in $\Sigma_L \times \Sigma_R$.

[6]For example, $\widetilde{z}$ may be the complex conjugate of z if the odd variables including odd moduli are set to zero, though one does not have to limit oneself to this case.

ensure that the map from $Ber(\Sigma_L \times \Sigma_R)$ to $\text{Ber}(\Sigma)$ that takes $[\mathrm{d}\widetilde{z}] \otimes [\mathrm{d}z|\mathrm{d}\theta]$ to $[\mathrm{d}\widetilde{z};\mathrm{d}z|\mathrm{d}\theta]$ does not depend on the choice of coordinates. So we get a natural isomorphism

$$(3.1) \qquad Ber(\Sigma_L \times \Sigma_R)|_\Sigma \cong \text{Ber}(\Sigma),$$

and once again a section of $Ber(\Sigma_L \times \Sigma_R)$ can be integrated over Σ. When focusing on the holomorphic structure of Σ, we sometimes write simply $Ber(\Sigma)$ for $Ber(\Sigma_R)$.

For the heterotic string, Σ_L is an ordinary Riemann surface, so $Ber(\Sigma_L)$ is simply the space of $(1,0)$ forms on Σ_L (these are regarded as $(0,1)$-forms on Σ since a holomorphic function $\widetilde{z}$ on Σ_L is regarded as an antiholomorphic function on Σ). So if ϕ is a function on Σ and we define the operator $\widetilde{\partial}$ by

$$(3.2) \qquad \widetilde{\partial}\phi = \mathrm{d}\widetilde{z}\,\frac{\partial\phi}{\partial\widetilde{z}},$$

then $\widetilde{\partial}\phi$ is a section of $Ber(\Sigma_L)$, pulled back to Σ. This is a more elementary analog of the fact that, because of the isomorphism $Ber(\Sigma_R) \cong \mathcal{D}^{-1}$, the expression

$$(3.3) \qquad [\mathrm{d}z|\mathrm{d}\theta]\,D_\theta\phi$$

makes sense as a section of $Ber(\Sigma_R)$. Multiplying the two constructions, we get a section of $Ber(\Sigma_L) \otimes Ber(\Sigma_R) \cong \text{Ber}(\Sigma)$:

$$(3.4) \qquad \widetilde{\partial}\phi\,[\mathrm{d}z|\mathrm{d}\theta]\,D_\theta\phi = [\mathrm{d}\widetilde{z};\mathrm{d}z|\mathrm{d}\theta]\,\partial_{\widetilde{z}}\phi\,D_\theta\phi.$$

This expression does not depend on the choice of standard local coordinates and can be integrated over Σ. If ϕ is real-analytic, it can be extended to a holomorphic function on $\Sigma_L \times \Sigma_R$, defined in a neighborhood of Σ. Then the expression in 3.4) is a holomorphic section of $Ber(\Sigma_L) \otimes Ber(\Sigma_R)$, defined in that neighborhood. We defer the Type II analog of this to section 3.7.

3.2. Integration cycles of superstring perturbation theory

Having explained what we mean by a superstring worldsheet, we can ask what is the space parametrizing such worldsheets over which we should integrate in superstring perturbation theory. This is more subtle than one might think based on experience with the bosonic string. In bosonic string theory, the parameter space over which one wants to integrate to compute perturbative scattering amplitudes is a canonically defined moduli space of conformal

structures on an ordinary Riemann surface Σ_0. In superstring theory, the worldsheet Σ is only defined up to homology, and there does not seem to be a useful notion of the moduli space of such worldsheets. Instead, one defines the relevant parameter space Γ up to homology by a procedure similar to the one used to define Σ. (In each of the two cases, there are natural choices of Σ_{red} and Γ_{red}, and if one makes those choices, then one can restrict to infinitesimal homologies that act trivially on the reduced spaces.)

We follow the viewpoint of [41], p. 95 (see also [32], section 5.6), and define the desired parameter space as a smooth supermanifold Γ embedded in the moduli space $\mathfrak{M}_L \times \mathfrak{M}_R$ that parametrizes independent deformations of Σ_L and Σ_R. First we define a submanifold Γ_{red} of the reduced space $(\mathfrak{M}_L \times \mathfrak{M}_R)_{\mathrm{red}} = \mathfrak{M}_{L,\mathrm{red}} \times \mathfrak{M}_{R,\mathrm{red}}$ that is characterized by saying that the reduced spaces $\Sigma_{L,\mathrm{red}}$ and $\Sigma_{R,\mathrm{red}}$ are complex conjugate (with no condition on the spin structures). Then we thicken Γ in the fermionic directions, in a way that is unique up to homology, to make a smooth supermanifold embedded in $\mathfrak{M}_L \times \mathfrak{M}_R$ with the same odd dimension as $\mathfrak{M}_L \times \mathfrak{M}_R$. Thus the relation of Γ to $\mathfrak{M}_L \times \mathfrak{M}_R$ is very similar to the relation of Σ to $\Sigma_L \times \Sigma_R$ in section 3.1.

The superstring path integral constructs a section Ξ of $Ber(\mathfrak{M}_L \times \mathfrak{M}_R)$ that is holomorphic in a neighborhood of Γ_{red}. This section can be integrated over Γ by virtue of the same arguments that we used in section 3.1 to show that a section of $Ber(\Sigma_L \times \Sigma_R)$ can be integrated over Σ. And by the same arguments as before, subject to a caveat mentioned shortly, the integral of Ξ over Γ is invariant under small deformations of Γ. Perturbative superstring scattering amplitudes are computed via such integrals.

A caveat is needed because $\mathfrak{M}_L$, $\mathfrak{M}_R$, and Γ are all not compact and the integrals required in superstring perturbation theory have a delicate behavior at infinity. A condition is needed on just how to define Γ at infinity. This is described in section 6.5 of [31].

We have described the construction of Γ in a way that applies uniformly to heterotic and Type II superstrings. For the heterotic string, $\mathfrak{M}_R$ is the moduli space of super Riemann surfaces, with reduced space $\mathcal{M}_{\mathrm{spin}}$, which parametrizes an ordinary Riemann surface $\Sigma_{R,\mathrm{red}}$ with a choice of spin structure. $\mathfrak{M}_L$ parametrizes the ordinary Riemann surface $\Sigma_L = \Sigma_{L,\mathrm{red}}$, so $\mathfrak{M}_L$ and its reduced space are both the moduli space $\mathcal{M}$ of Riemann surfaces. $\Gamma_{\mathrm{red}} \subset \mathcal{M} \times \mathcal{M}_{\mathrm{spin}}$ is defined by the condition that Σ_L is the complex conjugate of $\Sigma_{R,\mathrm{red}}$ (here we place no condition on the spin structure of the latter). This condition means that a point in Γ_{red} is determined by its projection to the second factor of $\mathcal{M} \times \mathcal{M}_{\mathrm{spin}}$ and Γ_{red} is actually a copy of $\mathcal{M}_{\mathrm{spin}}$. The dimension of Γ is $6g - 6|2\mathfrak{g} - 2$.

For Type II, $\mathfrak{M}_L$ and $\mathfrak{M}_R$ are both copies of the moduli space of super Riemann surfaces, and the reduced spaces are both copies of $\mathcal{M}_{\mathrm{spin}}$. $\Gamma_{\mathrm{red}} \subset \mathcal{M}_{\mathrm{spin}} \times \mathcal{M}_{\mathrm{spin}}$ is defined by the condition that Σ_L and Σ_R, ignoring their spin structures, are complex conjugate. The two spin structures can vary independently. The dimension of Γ is $6g - 6|4g - 4$.

Henceforth, until section 3.7, we mostly concentrate on the heterotic string.

3.3. Lagrangians

To formulate the heterotic string on $\mathbb{R}^{10}$, four contributions to the Lagrangian are important. We will now see that they all make sense in the present context.

In writing Lagrangians, we make the following definition

$$(3.5) \qquad \mathcal{D}(\widetilde{z}, z|\theta) = -i[\mathrm{d}\widetilde{z}; \mathrm{d}z|\mathrm{d}\theta]$$

both as an abbreviation and to make it a little easier to compare to standard formulas. This is a superanalog of the following. In the bosonic world, the differential form $\mathrm{d}\overline{z} \wedge \mathrm{d}z$ is imaginary (since if $z = x + iy$, then $\mathrm{d}\overline{z} \wedge \mathrm{d}z = 2i\mathrm{d}x \wedge \mathrm{d}y$), and sometimes one defines the real form $\mathrm{d}^2z = -i\mathrm{d}\overline{z} \wedge \mathrm{d}z$. Our expression $\mathcal{D}(\widetilde{z}, z|\theta)$ is analogous.

The first possible Lagrangian has essentially been described already in eqn. (3.4). A map from the string worldsheet to $\mathbb{R}^{10}$ is described by fields $X^I(\widetilde{z}; z|\theta)$, $I = 1 \ldots 10$. The kinetic energy for these fields is

$$(3.6) \qquad I_X = \frac{1}{2\pi\alpha'} \int \mathcal{D}(\widetilde{z}, z|\theta) \sum_{IJ} \eta_{IJ} \partial_{\widetilde{z}} X^I D_\theta X^J,$$

where η_{IJ} is the metric tensor of $\mathbb{R}^{10}$. This expression does not depend on the choice of local coordinates since as already explained, the integrand is a section of $Ber(\Sigma_L \times \Sigma_R)$. If the functions $X^I(\widetilde{z}; z|\theta)$ are real-analytic, the expression I_X is also invariant under small displacements of Σ in $\Sigma_L \times \Sigma_R$.

To describe the current algebra degrees of freedom of the heterotic string, we will use the description by fermions, since this is convenient for writing an action. We first pick a square root $\mathcal{L}$ of the line bundle $Ber(\Sigma_L) = K_{\Sigma_L}$. Next we introduce 32 fields Λ_a, $a = 1 \ldots 32$, valued in $\Pi\mathcal{L}$ (that is, they are fermionic fields valued in $\mathcal{L}$). These fields are often called current algebra fermions and the choice of $\mathcal{L}$ is called a choice of spin structure for them.[7]

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[7]For brevity we will consider the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string; in constructing the $E_8 \times E_8$ heterotic string, one divides the Λ_a in two sets of 16 with a separate choice of $\mathcal{L}$ for each set.

The expression $D_\theta \Lambda_a$ makes sense as a section of $\mathcal{L} \otimes \mathcal{D}^{-1}$, because the line bundle $\mathcal{L}$ is antiholomorphic – that is, it is a pullback from Σ_L, like its square $Ber(\Sigma_L)$, and can be constructed using gluing functions that are functions of $\widetilde{z}$ only and so commute with D_θ. Accordingly, $\sum_a \Lambda_a D_\theta \Lambda_a$ is a section of $\mathcal{L}^2 \otimes \mathcal{D}^{-1} \cong Ber(\Sigma_L \times \Sigma_R)$, restricted to Σ. So the expression

$$(3.7) \qquad I_\Lambda = \frac{1}{2\pi} \int \mathcal{D}(\widetilde{z}, z|\theta) \sum_a \Lambda_a D_\theta \Lambda_a$$

does not depend on local coordinates and can be integrated.

What remain are the kinetic energies for the ghosts and antighosts. The holomorphic ghosts are a section C of $\Pi \mathcal{S} = \Pi \mathcal{D}^2$, as explained in section 2.3. Here $\mathcal{S} \cong \mathcal{D}^2$ is the sheaf of superconformal vector fields on Σ_R (which we pull back to Σ). As this is a holomorphic line bundle, the expression $\widetilde{\partial} C$ makes sense as a section of $Ber(\Sigma_L) \otimes \Pi \mathcal{D}^2$. The holomorphic antighosts are a section B of $\Pi \mathcal{D}^{-3}$. Hence $B\widetilde{\partial} C$ is a section of $Ber(\Sigma_L) \otimes \mathcal{D}^{-1} \cong Ber(\Sigma_L \times \Sigma_R)$, so it too can be integrated:

$$(3.8) \qquad I_{B,C} = \frac{1}{2\pi} \int \mathcal{D}(\widetilde{z}, z|\theta) \, B\widetilde{\partial} C.$$

Finally, the antiholomorphic antighosts and ghosts are sections $\widetilde{B}$ and $\widetilde{C}$ of $\Pi Ber(\Sigma_L)^2$ and $\Pi Ber(\Sigma_L)^{-1}$, respectively, so $\widetilde{B} D_\theta \widetilde{C}$ is again a section of $Ber(\Sigma_L) \otimes \mathcal{D}^{-1} \cong Ber(\Sigma_L \times \Sigma_R)$, leading to one last term in the action:

$$(3.9) \qquad I_{\widetilde{B},\widetilde{C}} = \frac{1}{2\pi} \int \mathcal{D}(\widetilde{z}, z|\theta) \, \widetilde{B} D_\theta \widetilde{C}.$$

In superstring perturbation theory, it is important to understand the ghost and antighost zero-modes. The C zero-modes obey $\widetilde{\partial} C = 0$, so they are globally-defined superconformal vector fields, with parity reversed. The $\widetilde{C}$ zero-modes are globally-defined antiholomorphic vector fields, also with parity reversed. For this statement, one uses the fact that $D_\theta \widetilde{C} = 0$ implies that also $0 = D_\theta^2 \widetilde{C} = \partial_z \widetilde{C}$. Usually in superstring perturbation theory, there are no global C or $\widetilde{C}$ zero-modes because the genus is too large or there are too many punctures. To understand the B and $\widetilde{B}$ zero-modes, we will need Serre duality (section 3.4).

The local expansions of the fields introduced above are as follows. One has

$$(3.10) \qquad X = x + \theta\psi,$$

where x has antiholomorphic and holomorphic conformal dimension $(0,0)$ and ψ has dimension $(0, 1/2)$;

$$(3.11) \qquad\qquad \Lambda = \lambda + \theta F,$$

where λ has dimension $(1/2, 0)$ and F has dimension $(1/2, 1/2)$;

$$(3.12) \qquad\qquad B = \beta + \theta b, \;\; C = c + \theta \gamma,$$

where β and γ are even fields of respective dimensions $(0, 3/2)$ and $(0, -1/2)$, and b and c are odd fields of respective dimensions $(0, 2)$ and $(0, -1)$; and finally

$$(3.13) \qquad\qquad \widetilde{B} = \widetilde{b} + \theta \widetilde{f}, \;\; \widetilde{C} = \widetilde{c} + \theta \widetilde{g},$$

where $\widetilde{b}$ and $\widetilde{c}$ are odd fields of dimensions $(2, 0)$ and $(-1, 0)$, while $\widetilde{f}$ and $\widetilde{g}$ are even fields of dimensions $(2, 1/2)$ and $(-1, 1/2)$. The fields F, $\widetilde{f}$, and $\widetilde{g}$ are all auxiliary fields that (after performing the integral over θ) appear quadratically in the action, without derivatives. They can hence be "integrated out," and the worldsheet action of the heterotic string is often written in terms of only the remaining fields.

3.4. Cohomology and duality

In the present section, we emphasize the holomorphic structure of Σ and we write $Ber(\Sigma)$ for what we have been calling $Ber(\Sigma_R)$, the Berezinian of Σ in the holomorphic sense. And we interpret a section $g(\widetilde{z}; z|\theta)\mathrm{d}\widetilde{z}$ of $Ber(\Sigma_L)$ as a $(0, 1)$-form on Σ. What can be integrated on Σ is therefore a $(0, 1)$-form with values in $Ber(\Sigma)$. What follows is formulated in terms of objects defined on the smooth supermanifold Σ without regard to the embedding in $\Sigma_L \times \Sigma_R$.

The action (3.8) for the fields B and C would make sense more generally if C is a section of an arbitrary holomorphic line bundle $\mathcal{R}$ and B is a section of $\mathcal{R}^{-1} \otimes Ber(\Sigma)$. The pair of line bundles $\mathcal{R}$ and $\mathcal{R}^{-1} \otimes Ber(\Sigma)$ has a particular significance related to duality.

First let us review how duality works on an ordinary compact Riemann surface Σ_0. What can be integrated over Σ_0 is a $(1, 1)$-form $\alpha = f(\overline{z}, z)\mathrm{d}\overline{z}\,\mathrm{d}z$. From a holomorphic point of view, a $(1, 0)$-form is best understood as a section of $K = T^*\Sigma_0$, the canonical bundle of Σ_0, and a $(1, 1)$-form is best understood as a $(0, 1)$-form with values in K.

Now let u be a $(0, k)$-form with values in some line bundle $\mathcal{R}$ (where $k = 0$ or 1), and let v be a $(0, 1 - k)$-form valued in $\mathcal{R}^{-1} \otimes K$. Then the product uv is naturally a $(0, 1)$-form with values in K, so it can be integrated, to give

$$(3.14) \qquad \Phi(u, v) = \int_{\Sigma_0} uv.$$

Letting $\Omega^k(\Sigma_0, \mathcal{T})$ denote the space of $(0, k)$-forms on Σ_0 valued in any line bundle $\mathcal{T}$, Φ is best understood as a natural duality

$$(3.15) \qquad \Phi : \Omega^k(\Sigma_0, \mathcal{R}) \times \Omega^{1-k}(\Sigma_0, \mathcal{R}^{-1} \otimes K) \to \mathbb{C}.$$

Calling Φ a duality means that it is a nondegenerate pairing that establishes the two vector spaces $\Omega^k(\Sigma_0, \mathcal{R})$ and $\Omega^{1-k}(\Sigma_0, \mathcal{R}^{-1} \otimes K)$ as each other's dual spaces. (Because of the infinite-dimensionality of the spaces involved, there are some technicalities in formulating this precisely, but these need not concern us here.)

To state the implications of this for cohomology, we need to define the Dolbeault cohomology groups $H^k(\Sigma_0, \mathcal{R})$, $k = 0, 1$ of Σ_0 with values in a holomorphic line bundle $\mathcal{R}$. For this, one introduces the operator

$$(3.16) \qquad \overline{\partial} = \mathrm{d}\overline{z} \frac{\partial}{\partial \overline{z}}$$

mapping $(0, 0)$-forms valued in $\mathcal{R}$ to $(0, 1)$-forms valued in $\mathcal{R}$. (We have assumed a local holomorphic trivialization of $\mathcal{R}$ in this simple way of writing the $\overline{\partial}$ operator.) Its kernel – the space of holomorphic sections of $\mathcal{R}$ – is defined to be $H^0(\Sigma_0, \mathcal{R})$. On the other hand, $H^1(\Sigma_0, \mathcal{R})$ is defined to be the cokernel of the operator $\overline{\partial}$ or in other words the space of all $\mathcal{R}$-valued $(0, 1)$-forms modulo those that are $\overline{\partial}$-exact.

Because the duality (3.15) between forms is invariant $u \to u + \overline{\partial}u$ or $v \to v + \overline{\partial}v$ (assuming that Σ_0 is compact[8]) it induces a pairing between cohomology groups:

$$(3.17) \qquad \Psi : H^k(\Sigma_0, \mathcal{R}) \times H^{1-k}(\Sigma_0, \mathcal{R}^{-1} \otimes K) \to \mathbb{C}.$$

For Σ_0 a compact Riemann surface, these cohomology groups are finite-dimensional, and Serre duality is the statement that the pairing Ψ is nondegenerate, establishing a duality between the cohomology groups in question.

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[8]Otherwise one should constrain u (or v) to have compact support and one arrives at a duality between the ordinary $\overline{\partial}$ cohomology groups and the corresponding cohomology groups with compact support. The same statement holds later when we discuss the super case.

All this has an analog on a super Riemann surface Σ. The analog of a $(0, 1)$-form is a 1-form proportional to $\mathrm{d}\widetilde{z}$, and the analog of the $\overline{\partial}$ operator is the operator

$$(3.18) \qquad \widetilde{\partial} = \mathrm{d}\widetilde{z}\frac{\partial}{\partial\widetilde{z}}$$

from sections of a line bundle $\mathcal{T}$ to $(0, 1)$-forms valued in $\mathcal{T}$. Again one defines $H^0(\Sigma, \mathcal{T})$ as the kernel of $\widetilde{\partial}$ and $H^1(\Sigma, \mathcal{T})$ as its cokernel. (One important subtlety is that in general, when one varies the odd moduli of Σ, dimensions of cohomology groups can jump. We will formulate the following only in the absence of such jumping.)

We again define $\Omega^k(\Sigma, \mathcal{T})$, $k = 0, 1$, as the space of $(0, k)$-forms on Σ with values in a holomorphic line bundle $\mathcal{T}$. The product of a $(0, k)$-form valued in $\mathcal{R}$ (a line bundle whose fibers we allow to be either even or odd) with a $(0, 1 - k)$-form valued in $\mathcal{R}^{-1} \otimes Ber(\Sigma)$ is a $(0, 1)$-form valued in $Ber(\Sigma)$. So rather as before we have a natural duality, defined by integration. There are two differences from the bosonic case:

(1) Since $Ber(\Sigma)$ is itself a fermionic line bundle (a typical local section being the odd object $[\mathrm{d}z|\mathrm{d}\theta]$), if one wants an integral $\int uv$ to be bosonic, where u is a section of $\Omega^k(\Sigma, \mathcal{R})$ and v is a section of $\Omega^{1-k}(\Sigma, \mathcal{R}^{-1} \otimes Ber(\Sigma))$, then either u or v must be odd. Accordingly, the duality is usually formulated as a pairing between $\Omega^k(\Sigma, \mathcal{R})$ and $\Pi\Omega^{1-k}(\Sigma, \mathcal{R}^{-1} \otimes Ber(\Sigma))$. (It does not much matter which of the two sides of the pairing one takes Π to act on, since one is free to replace $\mathcal{R}$ by $\Pi\mathcal{R}$.)

(2) The second point is a little more subtle. An ordinary Riemann surface has moduli, but these moduli are bosonic and can be set equal to complex values. That is what we have effectively done in assuming above that the integral $\int_{\Sigma_0} uv$ is $\mathbb{C}$-valued, rather than being a function of the moduli. However, super Riemann surfaces have odd moduli, which cannot be set to nonzero complex values. So unless one wants to simply set the odd moduli to zero, one should work not over $\mathbb{C}$ but over some ring $\mathbb{C}'$ that is generated over $\mathbb{C}$ by odd elements. Then one does not quite set the even and odd moduli of Σ to complex constants; one sets them to even and odd elements of $\mathbb{C}'$. (For a relatively elementary account of introducing a ring such as $\mathbb{C}'$ in the context of field theory, see [43].) In practice, this is somewhat like speaking in prose; if one does what comes naturally, one never has to think about it, though a technical description of what is involved may cause confusion.

With these points understood, the superanalog of the duality (3.15) reads

$$(3.19) \qquad \Phi : \Omega^k(\Sigma, \mathcal{R}) \otimes \Pi\Omega^{1-k}(\Sigma, \mathcal{R}^{-1} \otimes Ber(\Sigma)) \to \mathbb{C}'.$$

Again the definition is simply

$$(3.20) \qquad \Phi(u, v) = \int_\Sigma uv.$$

As in the bosonic case, the fact that $\Phi(u, v)$ is invariant under $u \to u + \widetilde{\partial} u$ or $v \to v + \widetilde{\partial} v$ means that it determines a pairing between cohomology groups:

$$(3.21) \qquad \Psi : H^k(\Sigma, \mathcal{R}) \times \Pi H^{1-k}(\Sigma, \mathcal{R}^{-1} \otimes Ber(\Sigma)) \to \mathbb{C}'.$$

The superanalog of Serre duality is the statement that this is a nondegenerate pairing. If Σ is split, this statement reduces to ordinary Serre duality (separately for the even and odd parts of the cohomology groups). Our no-jumping hypothesis ensures that the nondegeneracy of the pairing persists when infinitesimal fermionic moduli are turned on. We have formulated all this for a holomorphic line bundle $\mathcal{R}$, but all statements have immediate analogs if $\mathcal{R}$ is a holomorphic vector bundle of any rank $a|b$ and $\mathcal{R}^{-1}$ is replaced by the dual bundle $\mathcal{R}^*$.

As an example of the use of this duality, we know that the tangent space to the moduli space $\mathfrak{M}$ of super Riemann surfaces at the point corresponding to Σ is $H^1(\Sigma, \mathcal{S}) = H^1(\Sigma, \mathcal{D}^2)$, where $\mathcal{S} \cong \mathcal{D}^2$ is the sheaf of superconformal vector fields. (A derivation of this fact from the Dolbeault point of view is in section 3.5.) So the cotangent space to $\mathfrak{M}$ is $T^*\mathfrak{M} = H^0(\Sigma, \mathcal{D}^{-2} \otimes Ber(\Sigma)) = H^0(\Sigma, \mathcal{D}^{-3})$, where we use the isomorphism $Ber(\Sigma) \cong \mathcal{D}^{-1}$. This is relevant to superstring perturbation theory, because the zero-modes of the antighost field B are holomorphic sections of $\mathcal{D}^{-3}$ or in other words elements of $H^0(\Sigma, \mathcal{D}^{-3})$.

The analogous statements for ordinary Riemann surfaces are perhaps more familiar. The tangent space to the moduli space $\mathcal{M}$ of ordinary Riemann surfaces at the point corresponding to Σ_0 is $H^1(\Sigma_0, T)$, so the cotangent space to $\mathcal{M}$ at the same point is $H^0(\Sigma_0, T^{-1} \otimes K) = H^0(\Sigma_0, K^2)$. Here we use Serre duality (3.17); $T \cong K^{-1}$ is the tangent bundle of Σ_0. In bosonic string theory, the zero-modes of the antighost field b are elements of $H^0(\Sigma_0, K^2)$.

3.5. Deformation theory

3.5.1. Deformation theory via embedding We now return to the integration cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of superstring perturbation theory, as defined in section 3.2. It is a smooth cs supermanifold (a concept described in [32], section 2.1), meaning that the odd coordinates of Γ have no real structure. Certain general remarks here are applicable to both heterotic and Type II superstrings.

The most natural notion of the tangent or cotangent bundle of a cs supermanifold is the analog of what for an ordinary smooth manifold would be the complexified tangent or cotangent bundle. The reason for this is that if M is a cs manifold, there is no real structure for its odd tangent or cotangent vectors, so it is unnatural to try to impose one for even tangent and cotangent vectors. Instead we simply define the tangent and cotangent bundles TM and T^*M as ($\mathbb{Z}_2$-graded) complex vector bundles.

With this understood, the tangent bundle $T\Gamma$ of Γ is the direct sum $T\mathfrak{M}_L \oplus T\mathfrak{M}_R$ (restricted to Γ), and similarly $T^*\Gamma = T^*\mathfrak{M}_L \oplus T^*\mathfrak{M}_R$. This is an analog of the fact that if Y is an ordinary complex manifold that is the complexification of a middle-dimensional submanifold N, then the complexified tangent space to N is the tangent space to Y, restricted to N. This fact is relevant because we have defined Γ so that its complexification is $\mathfrak{M}_L \times \mathfrak{M}_R$.

We can regard $T^*\mathfrak{M}_R$ as the holomorphic cotangent space to Γ and $T^*\mathfrak{M}_L$ as its antiholomorphic cotangent space. So far our remarks have been general; now we return to the heterotic string. Comparing to the Lagrangian (3.8) for holomorphic ghost and antighost fields, and also recalling the description of $T^*\mathfrak{M}$ from the end of section 3.4, we see now that the space of B zero-modes is $\Pi T^*\mathfrak{M}_R$, the holomorphic cotangent bundle to Γ with parity reversed. Similarly, comparing to the Lagrangian (3.9) for antiholomorphic ghost and antighost fields, and recalling the description of $T^*\mathfrak{M}_L$, we see that the space of $\widetilde{B}$ zero-modes is $\Pi T^*\mathfrak{M}_L$, the antiholomorphic cotangent bundle to Γ with parity reversed.

Of course, we define the tangent and cotangent bundles of the cs supermanifold $\Sigma \subset \Sigma_L \times \Sigma_R$ in the same way, so $T\Sigma = (T\Sigma_L \oplus T\Sigma_R)|_\Sigma$, $T^*\Sigma = (T^*\Sigma_L \oplus T^*\Sigma_R)|_\Sigma$. We abbreviate the summands of $T\Sigma$ as $T_L\Sigma$ and $T_R\Sigma$ and call them the antiholomorphic and holomorphic tangent bundles to Σ. We use analogous notation and terminology for the summands of $T^*\Sigma$. The familiar structures that are present because Σ_R is a super Riemann surface can all be restricted to Σ. For instance, $T_R\Sigma$ has a subbundle $\mathcal{D}$ of rank $0|1$ with an exact sequence $0 \to \mathcal{D} \to T_R\Sigma \to \mathcal{D}^2 \to 0$. The sheaf of sections of $T_R\Sigma$ has a subsheaf $\mathcal{S}$ of superconformal vector fields. $\mathcal{S}$ is a sheaf of graded Lie algebras; if we forget that structure, there is an isomorphism $\mathcal{S} \cong \mathcal{D}^2$.

3.5.2. Deformation theory via fields: review In section 3.3, we described the worldsheet Lagrangian for the heterotic string on the super Riemann surface Σ. This can be the starting point for heterotic string perturbation theory. To develop this perturbation theory, we must understand the deformations of Σ in terms of fields on Σ, rather than via embeddings as in (3.5.1) or cutting and gluing as in section 2.2. This will be our next objective.

(The analog for Type II superstring theory is similar and is briefly described in section 3.7.2.)

First, we recall the appropriate constructions on an ordinary Riemann surface Σ_0, with notation chosen in anticipation of the superstring case (which we study starting in section 3.5.3). The complexified tangent bundle of Σ_0 is $T\Sigma_0 = T_L\Sigma_0 \oplus T_R\Sigma_0$, where $T_R\Sigma_0$ and $T_L\Sigma_0$ are respectively the holomorphic and antiholomorphic subbundles of $T\Sigma_0$. If z and $\tilde{z}$ are local holomorphic and antiholomorphic coordinates on Σ_0, then $T_R\Sigma_0$ and $T_L\Sigma_0$ are generated respectively by ∂_z and by $\partial_{\tilde{z}}$. A function f on Σ_0 is holomorphic if it is annihilated by $\partial_{\tilde{z}}$, and antiholomorphic if it is annihilated by ∂_z. To deform the holomorphic structure of Σ, we perturb $\partial_{\tilde{z}}$ by

$$(3.22) \qquad\qquad \partial_{\tilde{z}} \to \partial'_{\tilde{z}} = \partial_{\tilde{z}} + h^z_{\tilde{z}}\partial_z.$$

An additional deformation $\partial_{\tilde{z}} \to \partial_{\tilde{z}} + u^{\tilde{z}}_{\tilde{z}}\partial_{\tilde{z}}$ is not interesting, since it does not deform the condition for a function to be holomorphic. Multiplying $\partial_{\tilde{z}}$ by a nonzero function does not affect the subbundle $T_L\Sigma_0$ of $T\Sigma_0$ that is generated by $\partial_{\tilde{z}}$, so we allow a gauge transformation

$$(3.23) \qquad\qquad \partial_{\tilde{z}} \to e^{\tilde{\varphi}}\partial_{\tilde{z}},$$

for any function $\tilde{\varphi}$. This gauge-invariance can be used to remove the $u^{\tilde{z}}_{\tilde{z}}$ deformation. Conversely, the holomorphic structure of Σ_0 is deformed by

$$(3.24) \qquad\qquad \partial_z \to \partial'_z = \partial_z + h^{\tilde{z}}_z\partial_{\tilde{z}}.$$

Again, a shift in ∂_z by a multiple of itself is uninteresting, as it does not affect the criterion for a function to be antiholomorphic; we are interested in ∂_z only up to a gauge transformation

$$(3.25) \qquad\qquad \partial_z \to e^{\varphi}\partial_z.$$

In a diffeomorphism-invariant theory, we only want to consider the deformations $h^z_{\tilde{z}}$ and $h^{\tilde{z}}_z$ modulo deformations generated by a vector field

$$(3.26) \qquad\qquad q = q^{\tilde{z}}\partial_{\tilde{z}} + q^z\partial_z.$$

Modulo (3.23) and (3.25), the transformations generated by the commutator of $\partial_{\tilde{z}}$ or ∂_z with q are

$$h^z_{\tilde{z}} \to h^z_{\tilde{z}} + \partial_{\tilde{z}}q^z$$

(3.27)
$$h_z^{\widetilde{z}} \to h_z^{\widetilde{z}} + \partial_z q^{\widetilde{z}}.$$

The field $h_{\widetilde{z}}^z$, modulo this equivalence, defines an element of the holomorphic sheaf cohomology group $H^1(\Sigma_0, T_R\Sigma_0)$, and the field $h_z^{\widetilde{z}}$, modulo this equivalence, defines an element of the analogous antiholomorphic sheaf cohomology group, which we will call $\widetilde{H}^1(\Sigma_0, T_L\Sigma_0)$.

These are the standard answers, although on an ordinary Riemann surface, since it is obvious from the beginning that antiholomorphic deformations are complex conjugates of holomorphic ones, one does not always write out the two cases in such detail. We have presented the analysis this way as preparation for the superstring case, where it is not true that the antiholomorphic deformations are complex conjugates of holomorphic ones.

Notice that, perhaps counterintuitively, a first-order holomorphic deformation is made by deforming the embedding in $T\Sigma_0$ of the antiholomorphic tangent bundle $T_L\Sigma_0$, which is generated by $\partial_{\widetilde{z}}$, without changing $T_R\Sigma_0$, and a first-order antiholomorphic deformation is made by deforming the embedding in $T\Sigma_0$ of the holomorphic tangent bundle $T_R\Sigma_0$, which is generated by ∂_z, without changing $T_L\Sigma_0$. However, the deformation from $\partial_{\widetilde{z}}$ to $\partial_{\widetilde{z}}^\prime$ affects the condition for a section of $T_R\Sigma_0$ to be holomorphic; and similarly an antiholomorphic deformation affects the condition for a section of $T_L\Sigma$ to be antiholomorphic.

Finally, the fields $h_{\widetilde{z}}^z$ and $h_z^{\widetilde{z}}$ can be regarded as either deformations of the metric of Σ_0 modulo Weyl transformations, or equivalently as deformations of its complex structure, which we call J. We will briefly describe the second point of view. J is a linear transformation of $T\Sigma_0$ that obeys $J^2 = -1$ and acts as i and $-i$ on $T_R\Sigma_0$ and $T_L\Sigma_0$, respectively. One can deform Σ_0 as a complex manifold by deforming J. The condition $J^2 = -1$ sets $\delta J_{\widetilde{z}}^{\widetilde{z}} = \delta J_z^z = 0$, so the deformation only involves $\delta J_{\widetilde{z}}^z$ and $\delta J_z^{\widetilde{z}}$, which can be identified with $h_{\widetilde{z}}^z$ and $h_z^{\widetilde{z}}$ in the above formulas. The derivation shows that $\delta J_{\widetilde{z}}^z$ is a deformation of type $(1,0)$ on the space of complex structures on Σ (it represents a first-order deformation of the holomorphic structure of Σ_0) while $\delta J_z^{\widetilde{z}}$ is of type $(0,1)$ (it represents a first-order deformation of the antiholomorphic structure). In complex dimension greater than 1, the deformation δJ would be subject to an integrability condition, but in complex dimension 1, this is trivial. For more on such matters, see section 3.5.4.

3.5.3. Superstring deformation theory via fields The case of deforming a heterotic string worldsheet Σ is similar, with a few inevitable differences. First we restate in terms of objects defined on Σ some concepts that we previously described in terms of the embedding $\Sigma \subset \Sigma_L \times \Sigma_R$. A function $F(\widetilde{z};z|\theta)$

defined in some open set $U \subset \Sigma$ is holomorphic if it is annihilated by vector fields valued in the antiholomorphic tangent bundle $T_L\Sigma$ of Σ and antiholomorphic if it is annihilated by those valued in the holomorphic tangent bundle $T_R\Sigma$. In a standard local coordinate system $\widetilde{z}; z|\theta$, F is holomorphic if it obeys

$$(3.28) \qquad\qquad\qquad\qquad \partial_{\widetilde{z}}F = 0,$$

Similarly the condition for antiholomorphy is

$$(3.29) \qquad\qquad\qquad\qquad D_\theta F = 0.$$

This condition of course implies that F is also annihilated by $\partial_z = D_\theta^2$, so that it is annihilated by all vector fields valued in $T_R\Sigma$.

In deforming a heterotic string worldsheet Σ, there are potentially three types of deformation to consider: *(i)* deformations of the holomorphic structure of Σ, which mean deformations of the embedding of $T_L\Sigma$ in $T\Sigma$; *(ii)* deformations of the antiholomorphic structure of Σ, which mean deformations of the embedding of $T_R\Sigma$ in $T\Sigma$; and *(iii)* deformations of the superconformal structure (rather than the holomorphic or antiholomorphic structure) of Σ – in other words deformations of the embedding of $\mathcal{D}$ in $T_R\Sigma$.

In a diffeomorphism-invariant theory, we are only interested in first-order deformations of Σ modulo those that are generated by a vector field. An arbitrary vector field v on Σ can, of course, be written

$$(3.30) \qquad\qquad\qquad v = w^{\widetilde{z}}\partial_{\widetilde{z}} + w^z\partial_z + w^\theta\partial_\theta,$$

with coefficients $w^{\widetilde{z}}$, w^z, and w^θ that are functions of $\widetilde{z};z|\theta$. However, it turns out that a different expansion is more illuminating:

$$(3.31) \qquad\qquad v = q^{\widetilde{z}}\partial_{\widetilde{z}} + \left(q^z\partial_z + \frac{1}{2}D_\theta q^z D_\theta\right) + q^\theta D_\theta.$$

This way of making the expansion is useful largely because if q^z is holomorphic (annihilated by $\partial_{\widetilde{z}}$) then $q^z\partial_z + \frac{1}{2}D_\theta q^z D_\theta$ is a superconformal vector field (see eqn. (2.31)).

The simplest deformations to analyze are actually those of type *(iii)* in which the superconformal structure is changed. These are deformations in which a generator D_θ of $\mathcal{D}$ is shifted by a section of $T_R\Sigma$ that is not proportional to D_θ (a transformation $D_\theta \to e^f D_\theta$ does not affect the subbundle of $T_R\Sigma$ generated by D_θ). Such a deformation has the form

$$(3.32) \qquad\qquad\qquad D_\theta \to D_\theta + r_\theta^z\partial_z.$$

However, conjugating by a vector field $q^\theta D_\theta$ transforms D_θ by $D_\theta \to D_\theta + [D_\theta, q^\theta D_\theta]$. Modulo a multiple of D_θ, this is $D_\theta \to D_\theta - 2q^\theta \partial_z$. So by taking $q^\theta = -r_\theta^z/2$, we can in a unique way eliminate deformations of type *(iii)* while also eliminating the equivalence by vector fields of the form $q^\theta D_\theta$.

Now we turn to deformations of type *(i)*. To deform the holomorphic structure of Σ, we make a first-order perturbation of the condition of holomorphy, deforming the condition $\partial_{\widetilde{z}} F = 0$ to $\partial'_{\widetilde{z}} F = 0$ with

$$(3.33) \qquad \partial'_{\widetilde{z}} = \partial_{\widetilde{z}} + h_{\widetilde{z}}^z \partial_z + \chi_{\widetilde{z}}^\theta \partial_\theta.$$

As in the bosonic case, there is no point in perturbing $\partial_{\widetilde{z}}$ by an additional term $u_{\widetilde{z}}^{\widetilde{z}} \partial_{\widetilde{z}}$, since this would not affect the condition for a function to be holomorphic. Thus we allow gauge transformations

$$(3.34) \qquad \partial_{\widetilde{z}} \to e^{\widetilde{\varphi}} \partial_{\widetilde{z}}.$$

On the fields $h_{\widetilde{z}}^z$ and $\chi_{\widetilde{z}}^\theta$, we want to impose the equivalence relation of ignoring trivial deformations $[\partial_{\widetilde{z}}, v]$ for any vector field v. It is illuminating to do this first ignoring the condition that the deformation is supposed to preserve the holomorphic superconformal structure of Σ. In this case, one would conveniently use the generic expansion (3.30) of a vector field v to generate gauge invariances

$$(3.35) \qquad h_{\widetilde{z}}^z \to h_{\widetilde{z}}^z + \partial_{\widetilde{z}} w^z, \quad \chi_{\widetilde{z}}^\theta \to \chi_{\widetilde{z}}^\theta + \partial_{\widetilde{z}} w^\theta.$$

(The commutator with v also shifts $\partial_{\widetilde{z}}$ by a term $(\partial_{\widetilde{z}} w^{\widetilde{z}}) \partial_{\widetilde{z}}$, which we remove via (3.23).)

We can make this look more familiar if we introduce a 1-form $\mathrm{d}\widetilde{z}$ and combine the fields $h_{\widetilde{z}}^z$ and $\chi_{\widetilde{z}}^\theta$ to a 1-form with values in $T_R\Sigma$:

$$(3.36) \qquad \mathcal{C} = \mathrm{d}\widetilde{z}\left(h_{\widetilde{z}}^z \partial_z + \chi_{\widetilde{z}}^\theta \partial_\theta \right).$$

Then the equivalence relation (3.35) amounts to

$$(3.37) \qquad \mathcal{C} \to \mathcal{C} + \widetilde{\partial} v',$$

where $\widetilde{\partial} = \mathrm{d}\widetilde{z}\, \partial_{\widetilde{z}}$ was introduced in eqn. (3.18) to define Dolbeault cohomology, and v' is a 0-form with values in $T_R\Sigma$:

$$(3.38) \qquad v' = w^z \partial_z + w^\theta \partial_\theta.$$

The equivalence classes form by definition the Dolbeault cohomology group $H^1(\Sigma, T_R\Sigma)$.

We have arrived from a new vantage point at the result of section 2.2.2: the first-order deformations of Σ as a complex supermanifold are parametrized by $H^1(\Sigma, T_R\Sigma)$. The only difference is that in this derivation, the cohomology appears via the field $\mathcal{C}$, while the previous approach was based on gluing functions and Cech cohomology. The same arguments as on an ordinary complex manifold show that the two types of cohomology are equivalent, but the description by fields is a useful starting point for superstring perturbation theory.

However, so far we did not impose the condition that we want deformations that preserve the fact that from a holomorphic point of view, Σ is a super Riemann surface, rather than a more general complex supermanifold of dimension 1|1. The super Riemann surface structure is defined by the subbundle $\mathcal{D} \subset T_R\Sigma \subset T\Sigma$. So we want to deform Σ preserving the fact that $T_R\Sigma$ has the holomorphic subbundle $\mathcal{D}$. A first-order deformation of the holomorphic structure of Σ will leave fixed $T_R\Sigma$ as a subbundle of $T\Sigma$ (for the same reasons as in the discussion in section 3.5.2 of deformation of an ordinary Riemann surface Σ_0). We look for deformations that also do not alter the embedding of $\mathcal{D}$ in $T_R\Sigma \subset T\Sigma$. (Deformations that do alter the embedding of $\mathcal{D}$ in $T_R\Sigma$ are the type *(iii)* deformations that we already studied above.) However, again as on an ordinary Riemann surface, the first-order holomorphic deformation will modify the condition for a section of $\mathcal{D}$ to be holomorphic. Any section of $\mathcal{D}$ is locally of the form $e^\mu D_\theta$ for some function μ; the condition for this section to be holomorphic after the deformation is that it should commute with the modified operator $\partial'_{\tilde{z}}$:

$$(3.39) \qquad\qquad\qquad [\partial'_{\tilde{z}}, e^\mu D_\theta] = 0.$$

The condition that a function μ should exist locally obeying this equation is that the perturbation in (3.33) should take values not in $T\Sigma$ but in its subsheaf $\mathcal{S}$ of superconformal vector fields. Thus, recalling from eqns. (2.20) and (2.21) the general form of a superconformal vector field, we specialize (3.33) to perturbations of the following kind:

$$(3.40) \qquad \partial'_{\tilde{z}} = \partial_{\tilde{z}} + \left(h^z_{\tilde{z}}(\tilde{z};z)\partial_z + \frac{1}{2}\partial_z h^z_{\tilde{z}}(\tilde{z};z)\theta\partial_\theta \right) + \chi^\theta_{\tilde{z}}(\tilde{z};z)\left(\partial_\theta - \theta\partial_z \right).$$

A key difference from (3.33) is that the fields h and χ defined in this new way do not depend on θ. Precisely for such perturbations, one can locally

find a function μ such that (3.39) holds to first-order in the perturbation.[9] Of course, μ is uniquely determined only modulo the possibility of adding a function that commutes with the unperturbed operator $\partial_{\widetilde{z}}$. What we are constructing is the sheaf of holomorphic sections of $\mathcal{D}$ with its perturbed complex structure, not a particular holomorphic section of this sheaf.

It is again convenient to multiply eqn. (3.40) by the $(0,1)$-form $\mathrm{d}\widetilde{z}$. Then the perturbation of interest is a $(0,1)$-form on Σ valued in $\mathcal{S}$. As before, we need to impose an equivalence relation on this $(0,1)$-form, because perturbations that are generated by vector fields are uninteresting. Now it is best to use the expansion (3.31) of a vector field that is better adapted to the superconformal structure. Vector fields $q^\theta D_\theta$ have already been used to remove deformations of type *(iii)*, and vector fields $q^{\widetilde{z}}\partial_{\widetilde{z}}$ leave $\partial_{\widetilde{z}}$ fixed, modulo a transformation (3.34). Finally, vector fields $q^z\partial_z + \frac{1}{2}D_\theta q^z D_\theta$ generate the expected gauge transformations of $h^z_{\widetilde{z}}$ and $\chi^\theta_{\widetilde{z}}$,

$$(3.41) \qquad h^z_{\widetilde{z}} \to h^z_{\widetilde{z}} + \partial_{\widetilde{z}}a^z, \quad \chi^\theta_{\widetilde{z}} \to \chi^\theta_{\widetilde{z}} + \partial_{\widetilde{z}}\eta^\theta,$$

where $q^z = a^z + 2\theta\eta^\theta$.

The result is that one should classify the perturbations by $h^z_{\widetilde{z}}$ and $\chi^\theta_{\widetilde{z}}$ up to $\widetilde{\partial}$-exact forms. Thus equivalence classes now give a Dolbeault description of $H^1(\Sigma, \mathcal{S})$. We have arrived at the same description as in section 2.2 of the space of first-order deformations of Σ as a super Riemann surface, but now in terms of fields rather than Cech cocycles. The fields appearing in (3.40) have a familiar interpretation: $h^z_{\widetilde{z}}$ is usually called a metric perturbation, and $\chi^\theta_{\widetilde{z}}$ is called a gravitino field.

Now let us perturb the antiholomorphic structure of Σ. As already noted, prior to any perturbation, antiholomorphic functions on Σ are precisely the functions annihilated by D_θ. So now we want to perturb the operator D_θ. The most general perturbation that we have to consider is to replace D_θ by

$$(3.42) \qquad D'_\theta = D_\theta + \tilde{c}^{\widetilde{z}}_\theta \partial_{\widetilde{z}}.$$

There is no point in perturbing D_θ by a multiple of itself, since this does not affect the condition for a function to be antiholomorphic, that is, annihilated by D_θ. For deforming the antiholomorphic structure of Σ, there is also no point in perturbing D_θ by a multiple of $\partial_z = D^2_\theta$, since any function annihilated by D_θ is annihilated by ∂_z. So (3.42) is the most general possible perturbation

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[9]In verifying this, one uses the relations (2.6), which do not require holomorphy of f and g, only the fact that they do not depend on θ.

of the antiholomorphic structure. (A shift of D_θ by a multiple of ∂_z is a type *(iii)* deformation that we have already considered above.) Again, we are only interested in this perturbation modulo perturbations $[D_\theta, v]$ induced by vector fields v on Σ. In perturbing the antiholomorphic structure of Σ, the important vector fields are of the form $v = q^{\widetilde{z}} \partial_{\widetilde{z}}$ (since contributions of other vector fields to $[D_\theta, v]$ are proportional to D_θ or ∂_z), and in the commutator $[D_\theta, v]$, we only care about the term $D_\theta q^{\widetilde{z}} \partial_{\widetilde{z}}$. So the relevant equivalence relation on the field $c_\theta^{\widetilde{z}}$ that appears in (3.42) is

$$(3.43) \qquad\qquad c_\theta^{\widetilde{z}} \cong c_\theta^{\widetilde{z}} + D_\theta q^{\widetilde{z}}.$$

Let us expand these functions in powers of θ,

$$(3.44) \qquad \begin{aligned} c_\theta^{\widetilde{z}}(\widetilde{z};z|\theta) &= e_\theta^{\widetilde{z}}(\widetilde{z};z) + \theta h_z^{\widetilde{z}}(\widetilde{z};z), \\ q^{\widetilde{z}}(\widetilde{z};z|\theta) &= t^{\widetilde{z}}(\widetilde{z};z) + \theta u_\theta^{\widetilde{z}}(\widetilde{z};z). \end{aligned}$$

We find that the gauge-equivalence (3.43) reads

$$(3.45) \qquad \begin{aligned} e_\theta^{\widetilde{z}} &\cong e_\theta^{\widetilde{z}} + u_\theta^{\widetilde{z}} \\ h_z^{\widetilde{z}} &\cong h_z^{\widetilde{z}} + \partial_z t^{\widetilde{z}}. \end{aligned}$$

Clearly we can set $u_\theta^{\widetilde{z}} = -e_\theta^{\widetilde{z}}$, eliminating $e_\theta^{\widetilde{z}}$ and completely fixing the gauge-invariance generated by $u_\theta^{\widetilde{z}}$. But the space of fields $h_z^{\widetilde{z}}$ modulo the equivalence relation in (3.45) is a standard description of a nontrivial sheaf cohomology group – or more exactly it is standard except that the complex structure has been reversed. We will call this group $\widetilde{H}^1(\Sigma, T_L\Sigma)$, the sheaf cohomology of Σ with values in the sheaf of antiholomorphic sections of $T_L\Sigma$. (The tilde is meant as a reminder that this is cohomology with values in a sheaf of antiholomorphic sections.) The field $h_z^{\widetilde{z}}$ is again known as a metric perturbation.

To summarize, the first-order holomorphic deformations of the smooth supermanifold Σ preserving its relevant structures are given by the holomorphic sheaf cohomology $H^1(\Sigma, \mathcal{S})$, where $\mathcal{S} \cong \mathcal{D}^2$ is the sheaf of (holomorphic) superconformal vector fields. And its first-order antiholomorphic deformations are given by the antiholomorphic sheaf cohomology $\widetilde{H}^1(\Sigma, T_L\Sigma)$, the cohomology with values in the sheaf of antiholomorphic vector fields. These are the answers expected from a description by Cech cocycles and gluing, or from the embedding in $\Sigma_L \times \Sigma_R$, but here we have obtained an equivalent description via fields on Σ. This is a useful starting point for superstring perturbation theory.

3.5.4. Complex structures Now we will briefly explain how to state our results in terms of complex structures on Σ and their deformations.

In general, consider a smooth cs supermanifold M of dimension $2p|q$. An almost complex structure is an even endomorphism $\mathcal{J}$ of the complexified tangent bundle TM of M that obeys $\mathcal{J}^2 = -1$. Given an almost complex structure, we define $T_R M$ and $T_L M$ to be the subbundles of TM on which $\mathcal{J}$ acts as i and $-i$, respectively. We require that after reducing modulo the odd variables, $T_L M$ is close to the complex conjugate of $T_R M$. This implies in particular that the even parts of the ranks of $T_L M$ and $T_R M$ are equal (they have ranks $p|a$ and $p|b$ for some a and b with $a + b = q$).

$\mathcal{J}$ is said to be integrable if the sections of $T_L M$ form a Lie algebra and likewise the sections of $T_R M$ form a Lie algebra. Let us first consider the case of an ordinary complex manifold, meaning that M has dimension $2p|0$ for some p. For $p = 1$, integrability of $\mathcal{J}$ is a trivial condition, because $T_L M$ (for example) is generated by a single vector field w (for example $w = \partial_{\widetilde{z}} + h^z_{\widetilde{z}} \partial_z$), and given any one vector field w, the vector fields $e^{\widetilde{\varphi}} w$ always form a Lie algebra. By contrast, for $p > 1$, integrability is a severe constraint; this is because if $T_L M$ is generated by two vector fields w and w', we have to ask whether $[w, w']$ can be expressed as a linear combination of w and w'.

Now let us consider the case that M has dimension $2|1$. Without essential loss of generality, we can consider the case that $T_L M$ has rank $1|0$ and $T_R M$ has rank $1|1$. Integrability of $T_L M$ is now trivial, because $T_L M$ is generated by a single vector field; but integrability of $T_R M$ is non-trivial, since $T_R M$ is generated by two vector fields, namely an odd one and an even one.

However, integrability of $T_R M$ becomes trivial again if we are given that M has a holomorphic superconformal structure, meaning that $T_R M$ has a rank $0|1$ subbundle $\mathcal{D}$ and is generated by D_θ and D_θ^2, for some section (and in fact for any generic section) D_θ of $\mathcal{D}$. This makes integrability of $T_R M$ trivial, because no matter what odd vector field D_θ we choose, the vector fields of the form $e^\varphi D_\theta$ and $e^{\widehat{\varphi}} D_\theta^2$ always form a Lie algebra.

The case described in the last paragraph is of course the case that $M = \Sigma$ is the worldsheet of a heterotic string. Now let us describe deformation theory of Σ in this language. We deform the complex structure $\mathcal{J}$ of Σ under the condition that $\mathcal{J}^2 = -1$ and that Σ has a holomorphic superconformal structure, in other words $T_R \Sigma$ has a distinguished subbundle $\mathcal{D}$ of rank $0|1$. The deformation can be described, just as on an ordinary complex manifold, by a tensor $\delta \mathcal{J}$ that maps $T\Sigma$ to itself. The condition $\mathcal{J}^2 = -1$ means that $\delta \mathcal{J}^{\widetilde{z}}_{\widetilde{z}} = \delta \mathcal{J}^z_z = \delta \mathcal{J}^z_z = \delta \mathcal{J}^\theta_\theta = \delta \mathcal{J}^\theta_\theta = 0$. The superconformal structure determines $\delta \mathcal{J}^\theta_{\widetilde{z}}$ in terms of $\delta \mathcal{J}^z_{\widetilde{z}}$, as exhibited in eqn. (3.40). The fact that

$T_R\Sigma$ is generated by D_θ and $D_\theta^2 = \partial_z$ means that there is no need to deform ∂_z independently of D_θ, so $\delta \mathcal{J}_z^{\widetilde{z}}$ is determined in terms of $\delta \mathcal{J}_\theta^{\widetilde{z}}$. As we have seen in analyzing the deformations of type *(iii)* in section 3.5.3, nothing new comes from varying $\mathcal{D}$ within $T_R\Sigma$.

Though this involves jumping ahead of our story slightly, there is one more important case in which integrability is trivial. This is the case of the worldsheet of a Type II superstring. Here M has dimension $2|2$ and T_LM and T_RM each has rank $1|1$, and each is endowed with a superconformal structure. This means that $T_R\Sigma$ is generated by D_θ and D_θ^2 where D_θ is a section of a distinguished rank $0|1$ subbundle $\mathcal{D} \subset T_RM$, and similarly $T_L\Sigma$ is generated by $D_{\widetilde{\theta}}$ and $D_{\widetilde{\theta}}^2$, where $D_{\widetilde{\theta}}$ is a section of a distinguished rank $0|1$ subbundle $\widetilde{\mathcal{D}} \subset T_LM$. Deformations of the almost complex structure of M that preserve such a structure are automatically integrable, since vector fields $e^\varphi D_\theta$ and $e^{\widehat{\varphi}} D_\theta^2$ (and similarly vector fields $e^{\widetilde{\varphi}} D_{\widetilde{\theta}}$ and $e^{\widehat{\widetilde{\varphi}}} D_{\widetilde{\theta}}^2$) always form a Lie algebra.

3.6. Relation to supergravity

Many approaches to two-dimensional supergravity can be found in the literature, for example [4, 5, 23, 42]. We will aim for a shortcut here, introducing only the necessary definitions and minimizing the number of equations. In contrast to much of the literature, we start by describing the conformally-invariant structure, which after all is the main structure of importance for string theory. Then we describe the super version of a Riemannian metric. This has both intrinsic interest and some applications in string theory.

We focus here on describing the worldsheet of a heterotic string in the language of supergravity. For the Type II analog, see section 3.7.3.

3.6.1. The conformally invariant case   In a standard coordinate system $\widetilde{z};z|\theta$, we can define the 1-forms

$$\widetilde{E}_0 = \mathrm{d}\widetilde{z}$$
$$E_0 = \mathrm{d}z - \theta\mathrm{d}\theta$$
(3.46)
$$F_0 = \mathrm{d}\theta,$$

obeying

$$\mathrm{d}\widetilde{E}_0 = 0$$
$$\mathrm{d}E_0 + F_0 \wedge F_0 = 0$$
(3.47)
$$\mathrm{d}F_0 = 0.$$

To understand the significance of E_0, recall that $T_R^*\Sigma$, which is generated by E_0 and F_0, appears in an exact sequence (2.29):

$$(3.48) \qquad 0 \to \mathcal{D}^{-2} \to T_R^*\Sigma \to \mathcal{D}^{-1} \to 0,$$

where $\mathcal{D}^{-2}$ is generated by $\varpi = \mathrm{d}z - \theta\mathrm{d}\theta$, which we have taken for E_0.

From a conformally invariant point of view, we are interested not in E_0 and $\widetilde{E}_0$, but in the associated holomorphic and superconformal structures. These are invariant under what we might call Weyl transformations. We set

$$(3.49) \qquad \widetilde{E} = e^{\widetilde{\varphi}}\widetilde{E}_0,$$

with an arbitrary function $\widetilde{\varphi}$. So $\widetilde{E}$ obeys

$$(3.50) \qquad \mathrm{d}\widetilde{E} = 0 \quad \mod \ \widetilde{E}.$$

This means that $\mathrm{d}\widetilde{E} = \alpha \wedge \widetilde{E}$ for some 1-form α (in fact $\alpha = \mathrm{d}\widetilde{\varphi}$). Similarly we set

$$(3.51) \qquad E = e^{\varphi}E_0,$$

again with an arbitrary function φ. Now there is an odd 1-form F, unique up to sign (the sign is unique if one asks for F to vary continuously with φ and to equal F_0 at $\varphi = 0$) such that

$$(3.52) \qquad \mathrm{d}E + F \wedge F = 0 \quad \mod \widetilde{E} \wedge E.$$

We leave the reader as an exercise to verify that this is true and to determine F. We remark only that F is a linear combination of E_0 and F_0:

$$(3.53) \qquad F = aF_0 + bE_0, \ \ a \neq 0.$$

From the conditions stated in the last paragraph, one can reconstruct the complex and superconformal structure of Σ, and moreover, these are Weyl-invariant, that is, independent of $\widetilde{\varphi}$ and φ. The decomposition $T^*\Sigma = T_L^*\Sigma \oplus T_R^*\Sigma$ of $T^*\Sigma$ in antiholomorphic and holomorphic summands is given by declaring that $T_L^*\Sigma$ is generated by $\widetilde{E}$, while $T_R^*\Sigma$ is generated by[10] E and F. Finally, the line bundle $\mathcal{D}^{-2} \subset T_R^*\Sigma$ is the subbundle generated by E. (The decomposition $T\Sigma = T_L\Sigma \oplus T_R\Sigma$ is the dual of the decomposition of

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[10]Eqn. (3.53) implies that the linear span of E and F is Weyl-invariant, so that the definition of $T_R^*\Sigma$ is Weyl-invariant.

$T^*\Sigma$, and in particular $\mathcal{D} \subset T\Sigma$ is the subbundle orthogonal to both E and $\widetilde{E}$.) Thus the conditions of the last paragraph capture the entire structure of a heterotic string worldsheet, expressed in supergravity language.

More explicitly, given 1-forms $\widetilde{E}$, E, and F obeying (3.50) and (3.52), one can locally fix the Weyl factors $\widetilde{\varphi}$ and φ so that (3.50) and (3.52) reduce to (3.47), and then one can introduce standard coordinates $\widetilde{z};z|\theta$ so that $\widetilde{E}$, E, and F take the form given in (3.46). So again, eqn. (3.50) and (3.52) contain the full structure of a heterotic string worldsheet.

We have achieved much greater brevity than can sometimes be found in the literature, because we have started with the conformally invariant case. Also, we have presented only a minimum set of necessary equations. For example, we have written no equation for $\mathrm{d}F$; none is needed. This will remain so when we introduce the analog of a Riemannian metric.

3.6.2. More on deformation theory It is interesting to consider deformation theory from this point of view. The analysis will be equivalent to that of section 3.5.3, but in a dual language. We will consider first-order deformations of $\widetilde{E}$ and E, modulo both Weyl rescaling and deformations generated by a vector field

$$(3.54) \qquad v = q^{\widetilde{z}}\partial_{\widetilde{z}} + \left(q^z \partial_z + \frac{1}{2} D_\theta q^z D_\theta \right) + q^\theta D_\theta.$$

We parametrize the vector field as in eqn. (3.31). There is no need to consider deformations of F, since F is uniquely determined (up to sign) in terms of E and $\widetilde{E}$ by eqn. (3.52).

First we consider deformations of E. Deformations of E that are proportional to itself can be removed by a Weyl transformation, and deformations of E that are proportional to F can be removed in a unique fashion using the $q^\theta D_\theta$ term in v. So we can forget about q^θ and consider only deformations of E that are proportional to $\widetilde{E}$:

$$(3.55) \qquad E \to E + h^z_{\widetilde{z}}\widetilde{E}.$$

The notation $h^z_{\widetilde{z}}$ is motivated by the way that this field transforms under a holomorphic reparametrization of z and $\widetilde{z}$. Geometrically, $h^z_{\widetilde{z}}\widetilde{E}$ is a $(0,1)$-form with values in $\mathcal{D}^2$. (Up to a Weyl transformation, E is any section of $\mathcal{D}^{-2}$, and the deformation $E \to E + h^z_{\widetilde{z}}\widetilde{E}$ followed by the projection to the second term, namely $h^z_{\widetilde{z}}\widetilde{E}$, is a map from sections of $\mathcal{D}^{-2}$ to $(0,1)$-forms, or equivalently a $(0,1)$-form with values in $\mathcal{D}^2$.) Often, one omits $\widetilde{E}$ and says informally that

$h_{\widetilde{z}}^z$ is a $(0,1)$-form with values in $\mathcal{D}^2$. $h_{\widetilde{z}}^z$ is subject to the gauge-invariance

$$(3.56) \qquad\qquad h_{\widetilde{z}}^z \to h_{\widetilde{z}}^z + \partial_{\widetilde{z}} q^z,$$

which reflects the transformation of E generated by[11] the vector field v of eqn. (3.54). So the deformations of the holomorphic structure of Σ, modulo trivial ones, are given by $H^1(\Sigma, \mathcal{D}^2)$, a familiar answer.

In deforming $\widetilde{E}$, we can again disregard deformations of $\widetilde{E}$ that are proportional to itself, as these can be removed by a Weyl transformation. So we consider a deformation $\widetilde{E} \to \widetilde{E} + u_{\widetilde{z}}^z E + c_{\widetilde{\theta}}^z F$. However, the requirement that $\mathrm{d}\widetilde{E} = 0 \mod \widetilde{E}$ determines u in terms of c. In standard coordinates $\widetilde{z};z|\theta$ (that is, with $\widetilde{E}$, E, and F as in eqn. (3.46)), the relation is $u_{\widetilde{z}}^z = D_\theta c_{\widetilde{\theta}}^z$. Geometrically, $c_{\widetilde{\theta}}^z$ is a section of $\mathcal{D}^{-1} \otimes T_L\Sigma$. The equivalence relation on $c_{\widetilde{\theta}}^z$ that comes from deformations by the vector field v is

$$(3.57) \qquad\qquad c_{\widetilde{\theta}}^z \to c_{\widetilde{\theta}}^z + D_\theta q^{\widetilde{z}}.$$

This description of antiholomorphic deformations of Σ is familiar from eqn. (3.43), and, as explained in that connection, it amounts to a description of the space of antiholomorphic first-order deformations of Σ via the antiholomorphic sheaf cohomology group $\widetilde{H}^1(\Sigma, T_L\Sigma)$.

3.6.3. The super analog of a Riemannian metric

Our remaining goal along these lines is to describe the super analog of a Riemannian metric. Before considering super Riemann surfaces, let us review Riemannian geometry on an ordinary Riemann surface Σ_0. We write $T_{\mathbb{C}}^*\Sigma_0$ for the complexified cotangent bundle of the smooth two-manifold Σ_0. (We complexify the cotangent bundle of Σ_0 to match the way we defined the cotangent bundle of a smooth cs supermanifold.) To match our notation in the super case, we write $T_{\mathbb{C}}^*\Sigma_0 = T_L^*\Sigma_0 \oplus T_R^*\Sigma_0$, where the two summands are respectively the spaces of $(0,1)$-forms and of $(1,0)$-forms on Σ_0. The additional structure that leads to Riemannian geometry can be formulated in many ways. For example, one can introduce a hermitian metric on the line bundle $T_R^*\Sigma_0$; this is equivalent to a Riemannian metric on Σ_0. In the spirit of supergravity, it is more useful to take for the basic data a "vierbein," which in the present context means locally a complex conjugate pair of nonzero sections $\overline{E}$ and E of $T_L^*\Sigma_0$ and $T_R^*\Sigma_0$, subject to the gauge-invariance

$$(3.58) \qquad\qquad E \to e^{iu}E, \quad \overline{E} \to e^{-iu}\overline{E},$$

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[11] The transformation of a 1-form E generated by a vector field v is in general given by $\mathfrak{L}_v(E)$, where $\mathfrak{L}_v = \mathbf{i}_v\mathrm{d} + \mathrm{d}\mathbf{i}_v$ is the Lie derivative.

with u a real-valued function. The quantity $E \otimes \overline{E}$ is gauge-invariant, and we can think of it as the Riemannian metric on Σ_0. Explicitly, if z is a local complex coordinate on Σ_0, then as E is supposed to be of type $(1,0)$ and $\overline{E}$ is its complex conjugate, we have

$$(3.59) \qquad E = e^{\varphi}\mathrm{d}z, \quad \overline{E} = e^{\overline{\varphi}}\mathrm{d}\overline{z},$$

for some complex-valued function φ. The Riemannian metric is

$$(3.60) \qquad \mathrm{d}s^2 = E \otimes \overline{E} = e^{2\mathrm{Re}\,\varphi}|\mathrm{d}z|^2.$$

With the aid of the gauge-invariance (3.58), we can take φ to be real, if we so desire. In that case, the vierbein reads

$$(3.61) \qquad E = e^{\phi}\,\mathrm{d}z, \quad \overline{E} = e^{\phi}\mathrm{d}\overline{z},$$

with real-valued ϕ. The metric is then $e^{2\phi}|\mathrm{d}z|^2$; the function $e^{2\phi}$ is often called the Weyl factor.

The next step is to define the Levi-Civita connection on $T^*_{\mathbb{C}}\Sigma_0$; it is uniquely characterized by being metric-compatible and torsion-free. It is a connection ω with structure group $U(1)$. We represent ω by a real 1-form on Σ. Under the gauge transformation (3.58), it transforms as

$$(3.62) \qquad \omega \to \omega + \mathrm{d}u.$$

This ensures gauge covariance of the following extension of the exterior derivative:

$$(3.63) \qquad DE = (\mathrm{d} - i\omega)E, \quad D\overline{E} = (\mathrm{d} + i\omega)\overline{E}.$$

The Levi-Civita connection is defined by requiring

$$(3.64) \qquad DE = D\overline{E} = 0.$$

Writing $\omega = \mathrm{d}z\,\omega_z + \mathrm{d}\overline{z}\,\omega_{\overline{z}}$, a very short computation reveals that

$$(3.65) \qquad \omega_{\overline{z}} = -i\partial_{\overline{z}}\varphi, \quad \omega_z = i\partial_z\overline{\varphi}.$$

The sections $\overline{E}$ and E of $T^*_L\Sigma_0$ and $T^*_R\Sigma_0$ cannot be defined globally (unless the Euler characteristic of Σ_0 vanishes). Globally, a better formulation is to think of E as an isomorphism between $T^*_R\Sigma$ and a line bundle $\mathcal{U}$ (on which ω is a connection), while $\overline{E}$ is an isomorphism between the complex

conjugate line bundles. A similar remark applies in the superstring context that we consider next.

Now let Σ be a heterotic string worldsheet. Then $T^*\Sigma = T^*_L\Sigma \oplus T^*_R\Sigma$, where $T^*_L\Sigma$ is of rank $1|0$ and $T^*_R\Sigma$ is of rank $1|1$. Since $T^*_L\Sigma$ has rank $1|0$, it is a direct analog of $T^*_L\Sigma_0$ in the bosonic case. As for $T^*_R\Sigma$, it has rank $1|1$, but it has a subbundle $\mathcal{D}^{-2}$ of rank $1|0$, and this will play the role of $T^*_R\Sigma_0$ in the bosonic case.

We now can introduce what we claim is the appropriate analog on a super Riemann surface of a Riemannian structure on an ordinary Riemann surface. The appropriate structure, locally, is the choice of a nonzero section $\widetilde{E}$ of $T^*_L\Sigma$ and of a nonzero section E of $\mathcal{D}^{-2} \subset T^*_R\Sigma$, subject to a gauge invariance that will be specified shortly. As in section 3.6.1, in standard local coordinates $\widetilde{z};z|\theta$, we have

$$(3.66) \qquad E = e^{\varphi}(\mathrm{d}z - \theta\mathrm{d}\theta) = e^{\varphi}\varpi, \quad \widetilde{E} = e^{\widetilde{\varphi}}\mathrm{d}\widetilde{z}.$$

We also introduce a connection ω on the line bundle $\mathcal{D}^{-2}$. We assume the gauge-invariance

$$(3.67) \qquad E \to e^{iu}E, \quad \widetilde{E} \to e^{-iu}\widetilde{E}, \quad \omega \to \omega + \mathrm{d}u.$$

The gauge-covariant exterior derivatives of E and $\widetilde{E}$ are

$$DE = (\mathrm{d} - i\omega)E$$
$$(3.68) \qquad D\widetilde{E} = (\mathrm{d} + i\omega)\widetilde{E},$$

where d is the ordinary exterior derivative. The strongest reality condition that makes sense for this data is the following. If Σ has been defined so that its reduced space is the diagonal in $\Sigma_{L,\mathrm{red}} \times \Sigma_{R,\mathrm{red}}$, then we can ask that when all odd variables (θ and the odd moduli) vanish, $\widetilde{E}$ is the complex conjugate of E, ω is real, $\widetilde{\varphi}$ is the complex conjugate of φ, and u is also real.

Locally, the gauge-invariance can be used to reduce to the case $\varphi = \phi(\widetilde{z};z) + \theta\chi(\widetilde{z};z)$, $\widetilde{\varphi} = \phi$, where ϕ is a field quite analogous to the one that appears in (3.61), and χ is a fermionic partner of ϕ. With just one even field and one odd one, this is the smallest set of independent fields that one can possibly hope for in a super extension of Riemannian geometry. If the reality condition of the last paragraph is imposed, then ϕ is real when the odd variables vanish.

What remains is to specify the conditions analogous to those that in the ordinary case determine the Levi-Civita connection. These conditions are

extremely simple:

$$(3.69) \qquad D\widetilde{E} = 0, \quad DE \cong 0 \text{ mod } T_R^*\Sigma \otimes T_R^*\Sigma.$$

The condition on E means that the part of DE proportional to $\varpi \wedge d\widetilde{z}$ or $d\theta \wedge d\widetilde{z}$ vanishes, but DE may have terms proportional to $\varpi d\theta$ or $(d\theta)^2$. A very short computation suffices to show that the conditions (3.69) do uniquely determine ω. We start by writing the exterior derivative in a convenient fashion:

$$(3.70) \qquad d = d\widetilde{z}\, \partial_{\widetilde{z}} + dz \partial_z + d\theta\, \partial_\theta = d\widetilde{z}\partial_{\widetilde{z}} + \varpi \partial_z + d\theta D_\theta.$$

(In verifying this, recall that the quantity $d\theta$ is even.) Similarly for the 1-form ω, we write

$$(3.71) \qquad \omega = d\widetilde{z}\omega_{\widetilde{z}} + \varpi\omega_z + d\theta\omega_\theta.$$

Now explicitly we compute

$$(3.72) \qquad D\widetilde{E} = e^{\widetilde{\varphi}}\varpi \wedge d\widetilde{z}(\partial_z\widetilde{\varphi} + i\omega_z) - e^{\widetilde{\varphi}}d\theta \wedge d\widetilde{z}(D_\theta\widetilde{\varphi} + i\omega_\theta).$$

Setting this to zero, we determine part of the connection:

$$(3.73) \qquad \omega_z = i\partial_z\widetilde{\varphi}, \quad \omega_\theta = iD_\theta\widetilde{\varphi}.$$

Similarly, we evaluate DE:

$$(3.74) \quad DE = e^{\varphi}d\widetilde{z} \wedge \varpi(\partial_{\widetilde{z}}\varphi - i\omega_{\widetilde{z}}) + e^{\varphi}\varpi \wedge d\theta(D_\theta\varphi - D_\theta\widetilde{\varphi}) - e^{\varphi}d\theta \wedge d\theta.$$

Setting to zero the coefficient of $d\widetilde{z} \wedge \varpi$ (which is the part not valued in $T_R^*\Sigma \otimes T_R^*\Sigma$), we complete the determination of the connection:

$$(3.75) \qquad \omega_{\widetilde{z}} = -i\partial_{\widetilde{z}}\varphi.$$

In this presentation, there is no need to mention the odd 1-form F of section 3.6.1; it is determined in terms of E by eqn. (3.52). There is a variant of the above construction in which one does include F, thus completing $\widetilde{E}$ and E to a basis $\widetilde{E}, E, F$ of $T^*\Sigma$. Gauge transformations act on F by $F \to e^{iu/2}F$, and the constraint on DE is replaced by $DE + F \wedge F = 0$. There is no need to state an independent condition on DF.

3.7. Analogs for Type II

So far, we have mostly concentrated on the heterotic string. Here we will much more briefly indicate the analogs for Type II superstrings.

The starting point was already described in sections 3.1 and 3.2. A Type II superstring worldsheet Σ is embedded in a product $\Sigma_L \times \Sigma_R$, with Σ_L and Σ_R both super Riemann surfaces. Σ has codimension $2|0$ in $\Sigma_L \times \Sigma_R$. We require that the reduced space $\Sigma_{L,\mathrm{red}}$ is the complex conjugate of $\Sigma_{R,\mathrm{red}}$ (or sufficiently close to this) and that the reduced space of Σ is the diagonal in $\Sigma_{L,\mathrm{red}} \times \Sigma_{R,\mathrm{red}}$ (or, again, sufficiently close to this).

Deformations of $\Sigma_L \times \Sigma_R$ are parametrized by $\mathfrak{M}_L \times \mathfrak{M}_R$, where now $\mathfrak{M}_L$ and $\mathfrak{M}_R$ are both copies of the moduli space of super Riemann surfaces. The integration cycle Γ of Type II superstring perturbation theory is a subsupermanifold $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ that satisfies the following two conditions: (1) its reduced space is the subspace of $\mathfrak{M}_{L,\mathrm{red}} \times \mathfrak{M}_{R,\mathrm{red}}$ characterized by the condition that the reduced spaces $\Sigma_{L,\mathrm{red}}$ and $\Sigma_{R,\mathrm{red}}$, with their spin structures ignored, are complex conjugate; (2) its odd dimension is the odd dimension of $\mathfrak{M}_L \times \mathfrak{M}_R$. These conditions determine Γ up to infinitesimal homology and – together with some discussion of how Γ behaves at infinity – suffice for superstring perturbation theory.

The tangent bundles $T\Sigma$ and $T\Gamma$ have decompositions familiar from section 3.5.1. We have $T\Sigma = T_L\Sigma \oplus T_R\Sigma$, where $T_L\Sigma$ and $T_R\Sigma$ are the restrictions to Σ of $T\Sigma_L$ and $T\Sigma_R$. $T_L\Sigma$ and $T_R\Sigma$ are both of rank $1|1$, with distinguished subbundles $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ inherited from the super Riemann surface structures of Σ_L and Σ_R. These are summarized by exact sequences

$$(3.76) \qquad 0 \to \widetilde{\mathcal{D}} \to T_L\Sigma \to \widetilde{\mathcal{D}}^2 \to 0,$$

and

$$(3.77) \qquad 0 \to \mathcal{D} \to T_R\Sigma \to \mathcal{D}^2 \to 0$$

with the usual properties. Locally, one can choose coordinates in which $\mathcal{D}$ is generated by $D_\theta = \partial_\theta + \theta \partial_z$, and $\widetilde{\mathcal{D}}$ is generated by $\widetilde{D}_{\widetilde{\theta}} = \partial_{\widetilde{\theta}} + \widetilde{\theta} \partial_{\widetilde{z}}$. Given such coordinates, with $\widetilde{z}$ sufficiently close to the complex conjugate of z, we call $\widetilde{z}; z | \widetilde{\theta}; \theta$ a standard local coordinate system. The sheaves of sections of $T_L\Sigma$ and $T_R\Sigma$ have subsheaves $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ of antiholomorphic and holomorphic superconformal vector fields. As sheaves (ignoring their graded Lie algebra structures), $\widetilde{\mathcal{S}}$ is isomorphic to the sheaf of sections of $\widetilde{\mathcal{D}}^2$ and $\mathcal{S}$ to the sheaf of sections of $\mathcal{D}^2$.

The first-order deformations of the holomorphic structure of Σ are inherited from those of Σ_R and are given by the holomorphic sheaf cohomology $H^1(\Sigma, \mathcal{S})$. The first-order deformations of the antiholomorphic structure of Σ are inherited from those of Σ_L and are given by the antiholomorphic sheaf cohomology $\widetilde{H}^1(\Sigma, \widetilde{\mathcal{S}})$. The tangent space to Γ has a decomposition $T\Gamma = \widetilde{H}^1(\Sigma, \widetilde{\mathcal{S}}) \oplus H^1(\Sigma, \mathcal{S})$. The cotangent bundle to Γ has a corresponding description using Serre duality.

The setup is so similar to what it is for the heterotic string that most statements have fairly obvious analogs, so we will be brief in what follows.

3.7.1. Lagrangians   The Berezinian of the product $\Sigma_L \times \Sigma_R$ is a tensor product:

$$(3.78) \qquad Ber(\Sigma_L \times \Sigma_R) = Ber(\Sigma_L) \otimes Ber(\Sigma_R).$$

Because of the exact sequences (3.76) and (3.77), we have $Ber(\Sigma_L) \cong \widetilde{\mathcal{D}}^{-1}$, $Ber(\Sigma_R) \cong \mathcal{D}^{-1}$ (as in section 2.4 or appendix A), so $Ber(\Sigma_L \times \Sigma_R) \cong \widetilde{\mathcal{D}}^{-1} \otimes \mathcal{D}^{-1}$. The Berezinian of the smooth supermanifold Σ is the restriction to Σ of $Ber(\Sigma_L \times \Sigma_R)$, for the same reason as for the heterotic string, so

$$(3.79) \qquad \mathrm{Ber}(\Sigma) \cong \widetilde{\mathcal{D}}^{-1} \otimes \mathcal{D}^{-1}.$$

Just as in section 3.3, this is the information that we need in order to construct Lagrangians. In doing this, we define

$$(3.80) \qquad \mathcal{D}(\widetilde{z}, z | \widetilde{\theta}, \theta) = -i[\mathrm{d}\widetilde{z}; \mathrm{d}z | \mathrm{d}\widetilde{\theta}; \mathrm{d}\theta],$$

by analogy with eqn. (3.5). The usual matter fields of Type II superstrings in $\mathbb{R}^{10}$ are scalar superfields $X^I(\widetilde{z}|\widetilde{\theta}; z|\theta)$. Their kinetic energy is

$$(3.81) \qquad I_X = \frac{1}{2\pi\alpha'} \int \mathcal{D}(\widetilde{z}, z | \widetilde{\theta}, \theta) \sum_{IJ} \eta_{IJ} D_{\widetilde{\theta}} X^I D_\theta X^J.$$

This is well-defined because for a scalar field X, the expression $D_{\widetilde{\theta}} X D_\theta X$ is naturally a section of $\widetilde{\mathcal{D}}^{-1} \otimes \mathcal{D}^{-1} \cong \mathrm{Ber}(\Sigma^\star)$.

Similarly, the holomorphic superghosts are a section C of $\Pi\mathcal{D}^2$, and the corresponding antighosts are a section B of $\Pi\mathcal{D}^{-3}$. The expression $CD_{\widetilde{\theta}} B$ is a section of $\widetilde{\mathcal{D}}^{-1} \otimes \mathcal{D}^{-1}$, so again it can be integrated, giving the kinetic energy for the holomorphic ghost fields:

$$(3.82) \qquad I_{B,C} = \frac{1}{2\pi} \int \mathcal{D}(\widetilde{z}, z | \widetilde{\theta}, \theta) \, B D_{\widetilde{\theta}} C.$$

The antiholomorphic superghosts are a section $\widetilde{C}$ of $\Pi\widetilde{\mathcal{D}}^2$, and the corresponding antighosts are a section $\widetilde{B}$ of $\Pi\widetilde{\mathcal{D}}^{-3}$. In this case the action is

$$(3.83) \qquad I_{\widetilde{B},\widetilde{C}} = \frac{1}{2\pi}\int \mathcal{D}(\widetilde{z},z|\widetilde{\theta},\theta)\, \widetilde{B} D_\theta \widetilde{C}.$$

The C and $\widetilde{C}$ zero-modes are globally-defined holomorphic and antiholomorphic superconformal vector fields, with parity reversed. (In most applications in superstring perturbation theory, there are no such global zero-modes.) The B and $\widetilde{B}$ zero-modes are holomorphic and antiholomorphic cotangent vectors to the integration cycle Γ of superstring perturbation theory.

3.7.2. Deformations from a smooth point of view   To discuss deformations of Σ in terms of fields on Σ, rather than in terms of the embedding in $\Sigma_L \times \Sigma_R$, we start from the fact that holomorphic functions are those that are annihilated by $D_{\widetilde{\theta}}$ (and hence also by $\partial_{\widetilde{z}} = D_{\widetilde{\theta}}^2$). So to deform the holomorphic structure of Σ, we must, rather as in section 3.5, deform the operator $D_{\widetilde{\theta}}$ by adding perturbations proportional to ∂_z and D_θ. To preserve the super Riemann surface structure of the holomorphic variables, the perturbation must take values not in arbitrary sections of $T\Sigma$ but in the subsheaf $\mathcal{S}$. The resulting analysis is similar to that of section 3.5. It gives the expected result that the first-order deformations of the holomorphic structure of Σ are parametrized by $H^1(\Sigma, \mathcal{S})$.

Reciprocally, to deform the antiholomorphic structure of Σ, we deform the D_θ operator by sections of $\widetilde{S}$. An analysis similar to that of section 3.5 shows that first-order deformations of the antiholomorphic structure of Σ are parametrized by $\widetilde{H}^1(\Sigma, \widetilde{S})$.

3.7.3. Supergravity   To describe a Type II worldsheet in the language of supergravity, we proceed as in section 3.6. First we describe the conformally invariant structure.

In standard local coordinates $\widetilde{z};z|\widetilde{\theta};\theta$, the distinguished subbundle $\widetilde{\mathcal{D}}^{-2}$ of $T_L^*\Sigma$ is generated by $\widetilde{\varpi} = \mathrm{d}\widetilde{z} - \widetilde{\theta}\mathrm{d}\widetilde{\theta}$, and the distinguished subbundle $\mathcal{D}^{-2}$ of $T_R^*\Sigma$ is generated by $\varpi = \mathrm{d}z - \theta\mathrm{d}\theta$. We can extend $\widetilde{\varpi}$ to a basis of $T_L^*\Sigma$

$$(3.84) \qquad \begin{aligned} \widetilde{E}_0 &= \widetilde{\varpi} = \mathrm{d}\widetilde{z} - \widetilde{\theta}\mathrm{d}\widetilde{\theta} \\ \widetilde{F}_0 &= \mathrm{d}\widetilde{\theta}, \end{aligned}$$

obeying

$$\mathrm{d}\widetilde{E}_0 + \widetilde{F}_0 \wedge \widetilde{F}_0 = 0$$

(3.85)
$$d\widetilde{F}_0 = 0,$$

and similarly we can extend ϖ to a basis of $T_R^*\Sigma$,

$$E_0 = \varpi = dz - \theta d\theta$$
(3.86)
$$F_0 = d\theta,$$

obeying

$$dE_0 + F_0 \wedge F_0 = 0$$
(3.87)
$$dF_0 = 0.$$

We are really interested in the line bundles $\widetilde{\mathcal{D}}^{-2}$ and $\mathcal{D}^{-2}$ generated by $\widetilde{E}_0$ and E_0, not in $\widetilde{E}_0$ and E_0 themselves. So we allow what one might call Weyl transformations from $\widetilde{E}_0$ and E_0 to

(3.88)
$$\widetilde{E} = e^{\widetilde{\varphi}}\widetilde{E}_0, \quad E = e^{\varphi}E_0,$$

with arbitrary functions $\widetilde{\varphi}$, φ. The conditions obeyed by $\widetilde{E}$ and E are simply that there exist odd 1-forms $\widetilde{F}$ and F (unique up to sign) such that

$$d\widetilde{E} + \widetilde{F} \wedge \widetilde{F} = 0 \quad \mod \ \widetilde{E}$$
(3.89)
$$dE + F \wedge F = 0 \quad \mod \ E.$$

Eqn. (3.89) describes a Type II superstring worldsheet in supergravity language. The antiholomorphic cotangent bundle $T_L^*\Sigma$ is generated by $\widetilde{E}$ and $\widetilde{F}$, while the holomorphic cotangent bundle $T_R^*\Sigma$ is generated by E and F. Their distinguished subbundles $\widetilde{\mathcal{D}}^{-2}$ and $\mathcal{D}^{-2}$ are generated, respectively, by $\widetilde{E}$ and by E.

What we have just described is the Type II analog of what was explained for the heterotic string in section 3.6.1. We leave it to the reader to adapt the reasoning of section 3.6.2 and describe deformation theory from this point of view. Here we will generalize the construction of section 3.6.3 and describe the analog of a Riemannian metric in this context.

To describe a Riemannian metric, we want to give E and $\widetilde{E}$ not up to arbitrary Weyl transformations (3.88) but up to the smaller class of "gauge transformations"

(3.90)
$$E \to e^{iu}E, \quad \widetilde{E} \to e^{-iu}\widetilde{E}.$$

We introduce a gauge connection ω that transforms as

(3.91)
$$\omega \to \omega + \mathrm{d}u.$$

The gauge-covariant exterior derivatives are $DE = (\mathrm{d} - i\omega)E$, $D\widetilde{E} = (\mathrm{d} + i\omega)\widetilde{E}$. Under appropriate conditions (z and $\widetilde{z}$ and likewise E and $\widetilde{E}$ are complex conjugates when odd variables are set to zero) one can require that ω and u are real when odd variables vanish.

The conditions

$$
\begin{aligned}
D\widetilde{E} = 0 \quad &\mathrm{mod}\ \ T_L^*\Sigma \otimes T_L^*\Sigma, \\
DE = 0 \quad &\mathrm{mod}\ \ T_R^*\Sigma \otimes T_R^*\Sigma
\end{aligned}
$$

(3.92)

play the role of eqn. (3.69) and uniquely determine ω. These are the analogs of the conditions that in ordinary Riemannian geometry determine the Levi-Civita connection. As in the remark at the end of section 3.6.3, there is also a variant in which $\widetilde{E}$ is completed to a basis $\widetilde{E}, \widetilde{F}$ of $T_L^*\Sigma$ and E is completed to a basis E, F of $T_R^*\Sigma$, the constraint equations then being $D\widetilde{E} + \widetilde{F} \wedge \widetilde{F} = 0 = DE + F \wedge F$.

4. Punctures

We now return to the purely holomorphic theory of super Riemann surfaces as introduced in section 2. We aim to describe the "punctures" at which external vertex operators can be inserted on a super Riemann surface. (Some of the topics have been treated in [33].)

For our purposes, on an ordinary Riemann surface Σ_0, a "puncture" is the same thing as a marked (or labeled) point; we sometimes use the two terms interchangeably.[12] On a super Riemann surface, there are two kinds of puncture. Bosonic vertex operators are inserted at Neveu-Schwarz (NS) punctures, while fermionic vertex operators are inserted at Ramond (R) punctures.

NS punctures are a fairly obvious idea, quite analogous to punctures on an ordinary Riemann surface. But Ramond punctures are a sufficiently unusual idea that one may ask how one knows that this concept is necessary. One answer is that, pragmatically, fermion vertex operators of superstring theory are spin fields [44] that are naturally inserted at Ramond punctures. Another answer will become clear in section 6. If we do not know about Ramond

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[12]We do not think of removing the "puncture" from Σ_0, except when this is stated. The reason that we use the term "puncture" in this fashion is that in the superstring case, we want a term to apply uniformly for both NS and Ramond insertions. As will become clear, "marked point" would not be a sensible term for what we will call a Ramond puncture.

punctures already, then this notion is forced upon us when we contemplate the compactification of supermoduli space or equivalently the infrared region of superstring theory. One way that a super Riemann surface can degenerate involves the appearance of Ramond punctures, and this makes it inevitable to consider them.

We always consider punctures to be distinguishable, since in string perturbation theory, one will generically insert a different type of vertex operator at each puncture. So for example $\mathcal{M}_{g,n}$ will be the moduli space of genus g surfaces with n labeled punctures.

4.1. Basics

4.1.1. Definitions If a super Riemann surface Σ is parametrized locally by coordinates $z|\theta$, then an NS vertex operator can be inserted at any point

$$z = z_0$$
$$\theta = \theta_0.$$

(4.1)

The choice of such a point is what we mean by an NS puncture. The parameters $z_0|\theta_0$ are the moduli of the NS puncture. So adding an NS puncture increases the dimension of supermoduli space by $1|1$, and the moduli space of super Riemann surfaces of genus g with n_{NS} NS punctures has dimension $3g-3+n_{NS}|2g-2+n_{NS}$. It makes sense to integrate over the insertion point of an NS vertex operator, keeping fixed[13] the moduli of Σ.

In studying an NS puncture, it is sometimes useful to know that a point in a super Riemann surface, as opposed to a more general $1|1$ supermanifold, determines a divisor through that point. In a complex supermanifold in general, a divisor is a submanifold of codimension $1|0$, so in a super Riemann surface, it has dimension $0|1$ and is isomorphic to $\mathbb{C}^{0|1}$. The divisor associated to the point $z|\theta = z_0|\theta_0$ is simply the orbit through that point generated by the odd vector field D_θ. (Replacing D_θ with another nonzero section of $\mathcal{D}$ would not change this orbit.) Concretely, D_θ generates the coordinate transformation $\theta \to \theta + \alpha$, $z \to z + \alpha\theta$, so the orbit through $z|\theta = z_0|\theta_0$ is given in parametric form by

$$z = z_0 + \alpha\theta$$
$$\theta = \theta_0 + \alpha$$

(4.2)

---

[13]This statement holds in the heterotic string for NS vertex operators, and in Type II superstrings for NS-NS vertex operators, that is vertex operators that are of NS type both holomorphically and antiholomorphically.

or equivalently by the equation

$$(4.3) \qquad\qquad z = z_0 - \theta_0 \theta.$$

The fact that we can associate to an NS puncture a divisor improves the analogy between NS punctures and Ramond punctures, which are definitely associated to divisors as we will see momentarily.

A Ramond puncture is a much more subtle concept than an NS puncture; it is a singularity in the super Riemann surface structure of Σ. In the presence of a Ramond puncture, Σ is still a smooth complex supermanifold of dimension $1|1$, and the tangent bundle $T\Sigma$ still has a distinguished subbundle $\mathcal{D}$ of rank $0|1$. But it is no longer true that $\mathcal{D}^2$ is everywhere linearly independent of $\mathcal{D}$. Rather, this condition fails along a divisor in Σ, that is, a submanifold of dimension $0|1$. The local behavior near a Ramond puncture is that, in suitable local coordinates $z|\theta$, $\mathcal{D}$ has a section

$$(4.4) \qquad\qquad D_\theta^* = \frac{\partial}{\partial\theta} + z\theta\frac{\partial}{\partial z}.$$

(We reserve the name D_θ for a section of $\mathcal{D}$ of the form $\partial_\theta + \theta\partial_z$ in some coordinate system, and write D_θ^* for a section of a more general form $\partial_\theta + w(z)\theta\partial_z$ as in (4.4).) Thus

$$(4.5) \qquad\qquad D_\theta^{*2} = z\frac{\partial}{\partial z}.$$

We see that D_θ^{*2} vanishes on the divisor $z = 0$. We call this kind of divisor a Ramond divisor and generically denote it as $\mathcal{F}$.

4.1.2. Conformal mapping to a tube Before going farther, let us explain the relation of we have called NS and R punctures with NS and R string states.

On an ordinary Riemann surface, the connection between a local operator that may be inserted at, say, the point $z = 0$ and a string state arises from the coordinate transformation $z = e^\varrho$. This transformation maps the z-plane with the point $z = 0$ omitted to the cylinder parametrized by ϱ with the equivalence relation

$$(4.6) \qquad\qquad \varrho \cong \varrho + 2\pi\sqrt{-1}.$$

Conformal field theory on this cylinder describes a string of circumference 2π propagating in the $\mathrm{Re}\,\varrho$ direction. In a sense, the two descriptions differ by whether one thinks of $z = 0$ as a marked point (at which an operator is inserted) or a puncture (around which a string propagates).

What is the analog for super Riemann surfaces? First we consider an NS puncture at $z = \theta = 0$ on $\mathbb{C}^{1|1}$. According to eqn. (4.3), the divisor in $\mathbb{C}^{1|1}$ determined by this point is just $z = 0$. Omitting this divisor, we map its complement to a supertube by $z = e^\varrho$, $\theta = e^{\varrho/2}\zeta$. ϱ and ζ are superconformal coordinates, since $D_\theta = e^{-\varrho/2}(\partial_\zeta + \zeta\partial_\varrho)$. They are subject to the equivalence relation

$$(4.7) \qquad \varrho \cong \varrho + 2\pi\sqrt{-1}, \quad \zeta \to -\zeta.$$

The minus sign in the transformation of ζ means that in superstring theory on this supertube, the strings that propagate in the $\mathrm{Re}\,\varrho$ direction will be in the Neveu-Schwarz sector.

Next we consider the model (4.4) with a Ramond puncture at $z = 0$. In this case, we omit the Ramond divisor and map to the tube simply by $z = e^\varrho$, leaving θ alone; ϱ and θ are superconformal coordinates, since $D_\theta^* = \partial_\theta + \theta\partial_\varrho$. The equivalence relation is now

$$(4.8) \qquad \varrho \cong \varrho + 2\pi\sqrt{-1}, \quad \theta \to +\theta,$$

where now because of the $+$ sign for θ, the strings propagating in the $\mathrm{Re}\,\varrho$ direction will be in the Ramond sector.

What we have encountered in the last two paragraphs are the two possible spin structures on the purely bosonic cylinder defined in eqn. (4.6).

4.1.3. More on Ramond punctures   Generalizing (4.4), an example of a (noncompact) super Riemann surface with any number n_R of Ramond punctures is given by $\mathbb{C}^{1|1}$, parametrized by $z|\theta$, with superconformal structure defined by

$$(4.9) \qquad D_\theta^* = \frac{\partial}{\partial\theta} + w(z)\theta\frac{\partial}{\partial z}$$

and

$$(4.10) \qquad w(z) = \prod_{i=1}^{n_\mathrm{R}}(z - z_i).$$

So $D_\theta^{*2} = w(z)\partial_z$, and the usual claim that the tangent bundle is generated by $\mathcal{D}$ and $\mathcal{D}^2$ fails precisely at the divisors $\mathcal{F}_i$ given by $z = z_i$. We write $\mathcal{F} = \sum_i \mathcal{F}_i$ for the divisor on which $\mathcal{D}^2$ vanishes mod $\mathcal{D}$ and we say that the superconformal structure of Σ degenerates along $\mathcal{F}$.

In the presence of Ramond punctures, though the subbundle $\mathcal{D} \subset T\Sigma$ is still part of an exact sequence $0 \to \mathcal{D} \to T\Sigma \to T\Sigma/\mathcal{D} \to 0$, it is no longer true that $T\Sigma/\mathcal{D}$ is isomorphic to $\mathcal{D}^2$. $T\Sigma/\mathcal{D}$ is generated by ∂_z, but $\mathcal{D}^2$ is generated, in the above example, by $w(z)\partial_z$, which vanishes on $\mathcal{F}$. The relation between $\mathcal{D}^2$ and $T\Sigma/\mathcal{D}$ is $\mathcal{D}^2 \cong T\Sigma/\mathcal{D} \otimes \mathcal{O}(-\mathcal{F})$ (this is a fancy way to say that a section of $\mathcal{D}^2$ is a section of $T\Sigma/\mathcal{D}$ that vanishes along $\mathcal{F}$), or equivalently $T\Sigma/\mathcal{D} \cong \mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F})$. So in the presence of Ramond punctures, the familiar exact sequence becomes

$$(4.11) \qquad\qquad 0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F}) \to 0.$$

It is also convenient to describe this in a dual language. Dualizing (4.11), we get an exact sequence

$$(4.12) \qquad\qquad 0 \to \mathcal{D}^{-2} \otimes \mathcal{O}(-\mathcal{F}) \to T^*\Sigma \to \mathcal{D}^{-1} \to 0.$$

So $T^*\Sigma$ has a distinguished subbundle $\mathcal{L} \cong \mathcal{D}^{-2} \otimes \mathcal{O}(-\mathcal{F})$. Concretely, $\mathcal{L}$ is the subbundle of $T^*\Sigma$ that is orthogonal to $\mathcal{D}$. So if $\mathcal{D}$ is generated by $\partial_\theta + \theta z \partial_z$, then $\mathcal{L}$ is generated by $\mathrm{d}z - z\theta\mathrm{d}\theta$. More generally, for the example (4.9) with several Ramond punctures, $\mathcal{L}$ is generated by

$$(4.13) \qquad\qquad \varpi = \mathrm{d}z - w(z)\theta\mathrm{d}\theta.$$

It turns out that on a compact super Riemann surface, the number n_R of Ramond punctures is always even. One might anticipate this from the fact that Ramond punctures are the locations at which vertex operators for spacetime fermions are inserted. Including n_R Ramond punctures increases the bosonic dimension of supermoduli space by n_R, and the fermionic dimension by $n_R/2$. The statement about the bosonic dimension is not surprising; in the simple example of eqn. (4.9), we see that the positions of Ramond punctures can be varied independently. More interesting is the statement that the contribution of n_R Ramond punctures to the fermionic dimension of supermoduli space is $n_R/2$. It means that the fermionic moduli are global in nature, not associated to any particular Ramond puncture. There is no notion of moving a Ramond puncture in the fermionic directions.

The claim that the Ramond punctures contribute $n_R|n_R/2$ to the dimension of supermoduli space will be justified in section 4.2. Given this claim, the dimension of the moduli space of super Riemann surfaces of genus g with n_{NS} Neveu-Schwarz punctures and n_R Ramond punctures is

$$(4.14) \qquad \dim \mathfrak{M}_{g,n_{NS},n_R} = (3g - 3 + n_{NS} + n_R)|(2g - 2 + n_{NS} + n_R/2).$$

In superconformal field theory, it is very useful to describe Ramond vertex operators as spin operators that are inserted at branch points of the world-sheet fermions. To get to this description, we should replace what we have called θ by a new variable θ' that has square root branch points on the divisor $\mathcal{F}$. In the simple example (4.9), we take $\theta' = \sqrt{w(z)}\theta$. At zeroes of $w(z)$, θ' is not well-defined, but away from those zeroes the coordinates $z|\theta'$ (with either choice of sign for the square root in the definition of θ') are local superconformal coordinates in which the line bundle $\mathcal{D}$ is generated by $D_{\theta'} = \partial_{\theta'} + \theta'\partial_z$. So away from zeroes of w, the ordinary formulas of superconformal field theory are valid. At those zeroes, θ' and all fields that carry odd worldsheet fermion number (that is, all fields that are odd under the GSO projection) have square root branch points. This description is very useful for understanding local properties such as the operator product expansion of fields. But the global geometry tends to be obscured by the use of a double-valued coordinate. In superstring perturbation theory, it is useful to also have a global description via a smooth supermanifold Σ, albeit one whose superconformal structure degenerates along the divisor $\mathcal{F}$. Just as one application, this will greatly facilitate the analysis of pictures in section 4.3.

4.1.4. Implications   The peculiar nature of Ramond punctures has important implications for superstring perturbation theory. In contrast to the NS case, it does not make sense to integrate over the insertion point of a Ramond vertex operator while keeping fixed the other moduli of Σ. The odd moduli of Σ are properties of the superconformal structure of Σ and cannot be defined independently of the singularities of that structure, which are the divisors on which the Ramond vertex operators are inserted. So it does not make sense to change the position of a Ramond vertex operator while keeping fixed the odd moduli. To integrate over the insertion location of a Ramond vertex operator, we have to integrate over all the odd moduli. And since there is no known natural way in general to integrate over odd moduli without also integrating over even moduli, the only known manageable way to integrate over the position of a Ramond vertex operator is to integrate over the whole supermoduli space.

This fact was one of the essential sources of complication in the superstring literature of the 1980s. To prove spacetime supersymmetry of perturbative scattering amplitudes, one would like [44, 45] to integrate over the position of a certain Ramond vertex operator, the spacetime supersymmetry generator. More generally, the proof of gauge invariance for fermion vertex operators involves integrating over the position at which a Ramond vertex operator is inserted. In these integrals one needs to be able to integrate by parts. But the integral over the location of a Ramond vertex operator really

only makes sense when extended to an integral over the whole supermoduli space. So one really needs to integrate by parts on a rather subtle supermanifold. The most relevant version of "integration by parts" is the supermanifold version of Stokes's theorem (for example see [30, 32, 41]). When one integrates by parts, one needs to analyze possible surface terms at infinity, so one needs an understanding of the behavior of supermoduli space at infinity as described in section 6 below. The supermanifold version of Stokes's theorem and an analysis of what happens at infinity will be the main ingredients in a reconsideration of superstring perturbation theory [31].

4.2. Superconformal vector fields and moduli

Here we will repeat some of the considerations of section 2 in the presence of Ramond punctures.

4.2.1. The sheaf of superconformal vector fields
We first consider the simple example (4.9) of a superconformal structure on $\mathbb{C}^{1|1}$ with $\mathfrak{n}_{\mathrm{R}}$ Ramond punctures. A calculation similar to the one that led previously to (2.4) and (2.5) shows that a general odd superconformal vector field preserving this superconformal structure is

$$(4.15) \qquad \nu_f = f(z)\left(\partial_\theta - w(z)\theta\partial_z\right),$$

and a general even one is

$$(4.16) \qquad V_g = w(z)\left(g(z)\partial_z + \frac{g'(z)}{2}\theta\partial_\theta\right).$$

As before, the functions f and g are holomorphic functions of z only and not θ. A check on these formulas is the following. Since the divisor $\mathcal{F}$ along which $w = 0$ is intrinsically defined by the condition that $\mathcal{D}^2$ is proportional to $\mathcal{D}$ along this divisor, any vector field that preserves $\mathcal{D}$ must, when restricted to $\mathcal{F}$, be tangent to $\mathcal{F}$, so as to leave $\mathcal{F}$ fixed (not pointwise, but as a divisor). The vector fields ν_f and V_g have this property. V_g vanishes along $\mathcal{F}$, and ν_f when restricted to $\mathcal{F}$ is proportional to ∂_θ, so it is tangent to $\mathcal{F}$. We define a sheaf $\mathcal{S}$ of superconformal vector fields whose sections are vector fields of the form $V_g + \nu_f$.

4.2.2. Reduced structure
We can count the moduli of Σ in the presence of Ramond punctures by the same reasoning as in section 2.2. For variety, we will present the arguments somewhat differently. It suffices to consider the

case that Σ is split, meaning that Σ is fibered over its reduced space, which is an ordinary Riemann surface Σ_{red}. The fibers of this fibration are vector spaces of rank $0|1$; thus Σ is the total space of a line bundle $V \to \Sigma_{\mathrm{red}}$ with fermionic fibers. We can think of V as the normal bundle to Σ_{red} in Σ. As described in section 2.2.3, a split super Riemann surface has a $\mathbb{Z}_2$ symmetry $\boldsymbol{\tau} : \theta \to -\theta$ that leaves fixed the reduced space Σ_{red} and acts as -1 on the normal bundle V. So when restricted to Σ_{red}, the tangent bundle $T\Sigma$ can be decomposed in subspaces that are even and odd under $\boldsymbol{\tau}$. The even subspace[14] of $T\Sigma|_{\Sigma_{\mathrm{red}}}$ is the tangent bundle $T\Sigma_{\mathrm{red}}$ to Σ_{red}, and the odd subspace is V:

$$(4.17) \qquad\qquad T\Sigma|_{\Sigma_{\mathrm{red}}} = T\Sigma_{\mathrm{red}} \oplus V.$$

Similarly, we can restrict the exact sequence (4.11) to Σ_{red} and decompose it in even and odd parts. This decomposition is very simple. When restricted to Σ_{red}, $\mathcal{D}$ is odd under $\boldsymbol{\tau}$ and $\mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F})$ is even. The exact sequence simply identifies $\mathcal{D}$ and $\mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F})$ with the odd and even parts of $T\Sigma_{\mathrm{red}}$:

$$(4.18) \qquad\qquad \mathcal{D} \cong V, \quad \mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F}) \cong T\Sigma_{\mathrm{red}}.$$

Moreover, when restricted to Σ_{red}, $\mathcal{F}$ is just an ordinary divisor $\sum_i p_i$, where p_i are points in Σ_{red} (in the example (4.9), the p_i are the zeroes of the function w), so we can write $\mathcal{O}(\sum_i p_i)$ for $\mathcal{O}(\mathcal{F})$. The isomorphisms in (4.18) combine to give an isomorphism

$$(4.19) \qquad\qquad V^2 \cong T\Sigma_{\mathrm{red}} \otimes \mathcal{O}\left(-\sum_{i=1}^{n_{\mathrm{R}}} p_i\right).$$

For a line bundle V with such an isomorphism to exist, the line bundle $T\Sigma_{\mathrm{red}} \otimes \mathcal{O}(-\sum_{i=1}^{n_{\mathrm{R}}} p_i)$ must have even degree. Since this degree is actually $2 - 2g - n_{\mathrm{R}}$, we learn that n_{R} must be even. The degree of V is

$$(4.20) \qquad\qquad \deg V = 1 - g - \frac{n_{\mathrm{R}}}{2}.$$

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[14]We are not simply expanding $T\Sigma$ as the sum of bosonic and fermionic subspaces. A super vector space W in general has no such canonical decomposition; if W has rank $a|b$, then the group $GL(a|b)$ of automorphisms of W does not preserve any decomposition in even and odd subspaces. Rather, we make the decomposition in eqn (4.17) using the symmetry $\boldsymbol{\tau}$. If Σ is not split, it does not have the symmetry $\boldsymbol{\tau}$; one would have to define $\boldsymbol{\tau}$ to reverse the odd moduli of Σ.

4.2.3. Counting moduli   We can now easily count the odd and even moduli of Σ. In the split case, we decompose the sheaf $\mathcal{S}$ of superconformal vector fields as $\mathcal{S}_+ \oplus \mathcal{S}_-$, where the summands are respectively even and odd under τ. $\mathcal{S}_-$ is spanned by the vector fields ν_f in (4.15), and $\mathcal{S}_+$ is spanned by the vector fields V_g in (4.16). The even part of the tangent bundle to supermoduli space $\mathfrak{M}$ (at a point in $\mathfrak{M}$ corresponding to the split surface Σ) is $H^1(\Sigma_{\mathrm{red}}, \mathcal{S}_+)$. And the odd part is $H^1(\Sigma_{\mathrm{red}}, \mathcal{S}_-)$.

To ν_f, we associate the vector field $f\partial_\theta$ along the fibers of the fibration $V \to \Sigma_{\mathrm{red}}$. So we can regard f as a section of V, and therefore $\mathcal{S}_-$ is the sheaf of sections of the line bundle V. Given the formula (4.20) for the degree of V, and the Riemann-Roch formula (2.24), the dimension of $H^1(\Sigma_{\mathrm{red}}, \mathcal{S}_-)$ is $\mathsf{g} - 1 - \deg \mathcal{S}_- = 2\mathsf{g} - 2 + \mathsf{n}_{\mathrm{R}}/2$. This is the number of odd moduli if there are no NS punctures. Adding n_{NS} such punctures simply increases the number of odd moduli by n_{NS}, one for each puncture. So in general the number of odd moduli is $2\mathsf{g} - 2 + \mathsf{n}_{\mathrm{NS}} + \mathsf{n}_{\mathrm{R}}/2$, as claimed in eqn. (4.14).

The counting of even moduli is more obvious because it does not depend on the identification of V. From eqn. (4.16), we see that when restricted to Σ_{red}, an even superconformal vector field V_g is simply a vector field along Σ_{red} that vanishes at the points p_i. So we identify $\mathcal{S}_+$ as $T\Sigma|_{\mathrm{red}} \otimes \mathcal{O}(-\sum_i p_i)$, of degree $2 - 2\mathsf{g} - \mathsf{n}_{\mathrm{R}}$. Hence the dimension of $H^1(\Sigma_{\mathrm{red}}, \mathcal{S}_+)$ is $3\mathsf{g} - 3 + \mathsf{n}_{\mathrm{R}}$. After including the contributions of n_{NS} NS punctures, we arrive at the formula for the number of even moduli claimed in eqn. (4.14).

Instead of including the NS punctures as an afterthought, a more principled way to proceed is as follows. In the presence of NS punctures, we restrict the sheaf $\mathcal{S}$ of superconformal vector fields to its subsheaf $\mathcal{S}'$ that leaves fixed the NS punctures.[15] $\mathcal{S}'$ is the sheaf of automorphisms of Σ with its specified NS punctures. So the deformation space of Σ with those punctures is $H^1(\Sigma_{\mathrm{red}}, \mathcal{S}')$. The restriction from $\mathcal{S}$ to $\mathcal{S}'$ has the effect of increasing the dimension of the cohomology group by $\mathsf{n}_{\mathrm{NS}}|\mathsf{n}_{\mathrm{NS}}$. (This can be proved by an argument similar to the analysis of eqn. (4.25) below.)

The reason that it is up to us whether to add the contribution of the NS punctures as an afterthought or incorporate them in the definition of the sheaf of superconformal vector fields is that an NS puncture is something that is added to a pre-existing super Riemann surface. By constrast, a Ramond puncture is part of the structure of the super Riemann surface so there is no way to add it as an afterthought. This statement is closely related to the point made in section 4.1.4.

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[15]Concretely, the condition that a general superconformal vector field $g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta + f(z)(\partial_\theta - \theta\partial_z)$ should leave fixed the point $z|\theta = z_0|\theta_0$ is $g(z_0) - \theta_0 f(z_0) = 0 = \frac{1}{2}g'(z_0)\theta_0 + f(z_0)$. These conditions define the sheaf $\mathcal{S}'$ of superconformal vector fields that leave fixed an NS puncture at $z_0|\theta_0$.

4.2.4. Generalized spin structures   Eqn. (4.19) generalizes the fact that in the absence of Ramond punctures, a spin structure is part of the structure of a super Riemann surface. If $n_R = 0$, eqn. (4.19) says that $V \cong T\Sigma_{\mathrm{red}}^{1/2}$ and in other words V is a choice of spin structure. For $n_R > 0$, a super Riemann surface is not endowed with a spin structure, but rather with a square root of $T\Sigma_{\mathrm{red}} \otimes \mathcal{O}(-\sum_{i=1}^{n_R} p_i)$. It is convenient to call such a square root a generalized spin structure.

Regardless of the value of n_R, for a fixed choice of the p_i, there are 2^{2g} choices of V. The case $n_R > 0$ has one conspicuous difference from the case $n_R = 0$. For $n_R > 0$, the 2^{2g} equivalence classes of line bundle V that admit an isomorphism (4.19) are permuted transitively as the points p_i move around. By contrast, spin structures are of two types – even and odd – that do not mix as one varies the moduli of Σ_{red}.

4.2.5. Another look at superconformal vector fields   As in section 2.3, we can try to combine the functions f and g that generate superconformal vector fields to a superfield. Differently put, we can look for an equivalence between the sheaf $\mathcal{S}$ of superconformal vector fields and the sheaf of sections of a line bundle over Σ. Given our previous experience, we can try to do this by projecting the sections V_g and ν_f of $\mathcal{S}$ to $T\Sigma/\mathcal{D}$. Again the projection of a vector field W to $T\Sigma/\mathcal{D}$ can be made by expressing W as a linear combination of D_θ^* (which generates $\mathcal{D}$) and ∂_z (which generates $T\Sigma \bmod \mathcal{D}$) and keeping the coefficient of ∂_z. In this process, ν_f projects to $-2w(z)f(z)\theta\partial_z$, and V_g projects to $w(z)g(z)\partial_z$. So a general section $V_g + \nu_f$ of the sheaf $\mathcal{S}$ (with g even and f odd) projects to the section $w(z)(g(z) + 2\theta f(z))\partial_z$ of $T\Sigma/\mathcal{D}$. Since $g(z) + 2\theta f(z)$ can be any function of z and θ, what we get this way is any section of $T\Sigma/\mathcal{D}$ that is divisible by w or in other words vanishes along $\mathcal{F}$. So we can identify $\mathcal{S}$ as $(T\Sigma/\mathcal{D}) \otimes \mathcal{O}(-\mathcal{F})$. However, the exact sequence (4.11) tells us that $T\Sigma/\mathcal{D}$ is isomorphic to $\mathcal{D}^2 \otimes \mathcal{O}(\mathcal{F})$, so $(T\Sigma/\mathcal{D}) \otimes \mathcal{O}(-\mathcal{F})$ is isomorphic to $\mathcal{D}^2$. Thus the sheaf $\mathcal{S}$ of superconformal vector fields is isomorphic to $\mathcal{D}^2$ just as in the absence of Ramond punctures:

$$(4.21) \qquad\qquad \mathcal{S} \cong \mathcal{D}^2.$$

The superconformal ghosts are a section of $\mathcal{S}$ with parity reversed, so we can express them as a field C valued in $\Pi\mathcal{D}^2$. To understand where the antighost fields B should live, we have to ask how to make sense of the Lagrangian

$$(4.22) \qquad\qquad I_{BC} = \frac{1}{2\pi i} \int [\mathrm{d}\widetilde{z}; \mathrm{d}z|\mathrm{d}\theta]\, B\partial_{\widetilde{z}} C.$$

(For brevity, we consider the ghost Lagrangian of the heterotic string, as in eqn. (3.8).) For this to make sense, the product BC must take values in the holomorphic Berezinian $Ber(\Sigma)$, so B must take values in $Ber(\Sigma) \otimes \mathcal{D}^{-2}$. $Ber(\Sigma)$ can be computed by the method of section 2.4 or (with the help of the exact sequence (4.12)) by the reasoning of appendix A, with the result

$$(4.23) \qquad Ber(\Sigma) \cong \mathcal{D}^{-1} \otimes \mathcal{O}(-\mathcal{F}).$$

So B is valued in $\mathcal{D}^{-3} \otimes \mathcal{O}(-\mathcal{F})$.

4.3. Pictures

By now, we understand that a Ramond "puncture" is really a divisor on a super Riemann surface Σ. This divisor is a copy of $\mathbb{C}^{0|1}$. It is one of the connected components of the divisor $\mathcal{F} = \sum_{i=1}^{n_\mathrm{R}} \mathcal{F}_i$ on which the superconformal structure of Σ degenerates.

The understanding that Ramond "punctures" are really divisors and not points raises the following question. Is the insertion of a Ramond vertex operator at one of the $\mathcal{F}_i$ associated to the whole divisor $\mathcal{F}_i$, or should the vertex operator be inserted at a point on $\mathcal{F}_i$?

In the presence of Ramond punctures, there are two different things one might mean by the moduli space of super Riemann surfaces. One moduli space $\mathfrak{M}$ parametrizes $1|1$ supermanifolds Σ with a superconformal structure that becomes degenerate along n_R minimal divisors $\mathcal{F}_i$. Another moduli space $\mathfrak{M}_1$ parametrizes the same data as before but now with a choice of a point on each $\mathcal{F}_i$. Clearly the relation between the two moduli spaces is that $\mathfrak{M}_1$ is a fiber bundle over $\mathfrak{M}$, the fiber being a copy of $\prod_{i=1}^{n_\mathrm{R}} \mathcal{F}_i$:

$$(4.24) \qquad \begin{array}{ccc} \prod_{i=1}^{n_\mathrm{R}} \mathcal{F}_i & \to & \mathfrak{M}_1 \\ & & \downarrow \\ & & \mathfrak{M}. \end{array}$$

(Presently we will generalize the story, and then we will rename $\mathfrak{M}_1$ as $\mathfrak{M}_{1,\dots,1}$.)

Since there are two plausible moduli spaces, one question that comes to mind is whether the dimension formula (4.14) applies to $\mathfrak{M}$ or $\mathfrak{M}_1$. The answer to this question is that this dimension formula applies to $\mathfrak{M}$. This is so because when we described the sheaf of superconformal vector fields in the presence of Ramond punctures, we did not require the function f in equation (4.15) to vanish at the zeroes of w. That being so, we allowed as symmetries

superconformal vector fields that generate nontrivial shifts $\theta \to \theta + \eta$ (with η an odd constant) along the divisors $\mathcal{F}_i$. Since we allowed these as symmetries, the moduli space that we constructed is the space $\mathfrak{M}$ in which the divisors $\mathcal{F}_i$ are not endowed with distinguished points.

Although we included superconformal vector fields that generate shift symmetries $\theta \to \theta + \eta$ of $\mathcal{F}_i$, there are no superconformal vector fields that rescale the divisor $\mathcal{F}_i$ by $\theta \to \lambda\theta$. This can be seen from the detailed form of (4.15) and (4.16), where the coefficient of $\theta\partial_\theta$ always vanishes at $w = 0$. It is also clear from the fact that a superconformal transformation must map the object ϖ defined in eqn. (4.13) to a multiple of itself; a transformation that rescales θ at $w = 0$ (except by a factor ± 1) would lack this property. We write $\mathcal{SF}_i$ for the sheaf of vector fields ∂_θ that can arise at $\mathcal{F}_i$ by restricting a superconformal vector field. So the space of global section of $\mathcal{SF}_i$ is of dimension $0|1$, generated by ∂_θ. And we define $\mathcal{SF} = \oplus_i \mathcal{SF}_i$.

To construct $\mathfrak{M}_1$, we should replace $\mathcal{S}$ by its subsheaf $\mathcal{S}_1$ consisting of superconformal vector fields that vanish when restricted to $\mathcal{F}$. $\mathcal{S}_1$ sits in an exact sequence

$$(4.25) \qquad\qquad 0 \to \mathcal{S}_1 \xrightarrow{i} \mathcal{S} \xrightarrow{r} \mathcal{SF} \to 0.$$

The map i is the natural inclusion of $\mathcal{S}_1$ in $\mathcal{S}$, and r is the restriction of a superconformal vector field to $\mathcal{F}$. The exact sequence is just a fancy way to say that sections of $\mathcal{S}_1$ are sections of $\mathcal{S}$ whose restrictions to $\mathcal{F}$ vanish. From (4.25), one deduces an exact sequence of cohomology groups[16]

$$(4.26) \qquad 0 \to H^0(\Sigma, \mathcal{SF}) \to H^1(\Sigma, \mathcal{S}_1) \to H^1(\Sigma, \mathcal{S}) \to 0.$$

The group $H^0(\Sigma, \mathcal{SF})$ has dimension $0|n_R$ (with one basis vector ∂_θ for each of the $\mathcal{F}_i$), and the sequence (4.26) implies that the dimension of $H^1(\Sigma, \mathcal{S}_1)$ exceeds that of $H^1(\Sigma, \mathcal{S})$ by $0|n_R$. Of course, we could have anticipated this result given the fibration (4.24). Since $H^1(\Sigma, \mathcal{S}_1)$ is the tangent space to $\mathfrak{M}_1$ at the point corresponding to Σ, while $H^1(\Sigma, \mathcal{S})$ is the tangent space to $\mathfrak{M}$, we deduce with the help of (4.14) that the dimension of $\mathfrak{M}_1$ is

$$(4.27) \qquad \dim \mathfrak{M}_{1;g,n_{NS},n_R} = (3g - 3 + n_{NS} + n_R)|(2g - 2 + n_{NS} + \frac{3}{2}n_R).$$

Now let us return to the question: should superstring perturbation theory be understood as an integral over $\mathfrak{M}$ or over $\mathfrak{M}_1$? A little thought shows that

---

[16]We have shortened the long exact cohomology sequence that one derives from (4.25) using the vanishing of some of the groups involved. For instance, $H^1(\Sigma, \mathcal{SF}) = 0$ because the support of $\mathcal{SF}$ is on $\mathcal{F}$, which has bosonic dimension 0.

if either one of these answers is correct, then it must be possible to construct perturbative superstring amplitudes as an integral over $\mathfrak{M}$. Indeed, since $\mathfrak{M}_1$ admits a natural map to $\mathfrak{M}$, if there is some way to compute superstring scattering amplitudes by integrating over $\mathfrak{M}_1$, then by first integrating over the fibers of the projection $\mathfrak{M}_1 \to \mathfrak{M}$, we would get a recipe to compute the scattering amplitudes by integration over $\mathfrak{M}$.

In fact, it is up to us whether we want to use $\mathfrak{M}$ or $\mathfrak{M}_1$. In constructing superstring vertex operators [44], one runs into a peculiar phenomenon of "picture number." (We will assume here a familiarity with the main ideas of that reference. The following remarks are presented in a somewhat heuristic way. Some technical details can be found in appendix B.) The picture number of an NS vertex operator takes values in $\mathbb{Z}$, while that of a Ramond vertex operator takes values in $\mathbb{Z}+1/2$. Although all integer or half-integer values of the picture number are possible, the vertex operators have the most simple superconformal properties – which usually is a welcome simplification – only in the case of vertex operators of "canonical" picture number. The canonical numbers are -1 for NS vertex operators and $-1/2$ for Ramond vertex operators.

Not coincidentally, the negatives of these numbers, namely 1 and $1/2$, appear in the dimension formula (4.14): the contribution of $\mathfrak{n}_{NS}$ NS punctures and $\mathfrak{n}_R$ Ramond punctures to the odd dimension of $\mathfrak{M}$ is $1{\cdot}\mathfrak{n}_{NS}+\frac{1}{2}{\cdot}\mathfrak{n}_R$. In fact, the natural interpretation of the picture number of a vertex operator is that it is minus the number of odd moduli associated to the presence of that operator. The essential clue to this statement goes back to [25]: picture-changing, which is the operation introduced in [44] that increases the picture number of a vertex operator by 1, amounts to integration over an odd modulus. To integrate naturally over supermoduli space, each vertex operator should have a picture number that is minus its contribution to the odd dimension of supermoduli space.

If we want to compute superstring scattering amplitudes using vertex operators in the canonical pictures, we should use the moduli space $\mathfrak{M}$ where the number of odd moduli is 1 for each NS puncture and $1/2$ for each Ramond puncture. However, the moduli space $\mathfrak{M}_1$ is a perfectly good supermanifold with 1 odd modulus for each NS vertex operator and $3/2$ for each Ramond vertex operator. If we want to compute scattering amplitudes by integrating over $\mathfrak{M}_1$, we should use NS vertex operators of picture number -1 and Ramond vertex operators of picture number $-3/2$.

More generally, we can make a separate choice for each fermion vertex operator of whether we want its picture number to be $-1/2$ or $-3/2$. We simply modify the definition of the sheaf $\mathcal{S}_1$ to say that it consists of superconformal vector fields that vanish on some chosen subset of the divisors $\mathcal{F}_i$.

Can we define moduli spaces that are suitable for use with vertex operators with other values of the picture number? This is no sooner said than done. Suppose that we want the ith fermion vertex operator to have picture number $-1/2 - k_i$, where the k_i are nonnegative integers. We introduce a more general sheaf $\mathcal{S}_{k_1,\dots,k_{n_R}}$ by requiring the odd vector field ν_f to have a zero of order k_i along each $\mathcal{F}_i$. (A crucial fact is that $\mathcal{S}_{k_1,\dots,k_{n_R}}$ is a sheaf of super Lie algebras. In constructing a moduli space with $H^1(\Sigma, \mathcal{S}_{k_1,\dots,k_{n_R}})$ as its tangent space, one uses this fact. Roughly one wants to interpret $\mathcal{S}_{k_1,\dots,k_{n_R}}$ as generating the infinitesimal symmetries of the objects classified by the moduli space one is trying to define, so $\mathcal{S}_{k_1,\dots,k_{n_R}}$ has to be a sheaf of graded Lie algebras.) The moduli spaces $\mathfrak{M}_{k_1,\dots,k_{n_R}}$ defined by this procedure have "integration over the fiber" maps that reduce the k_i. These maps correspond to the picture-raising operations for the vertex operators.

We can do something similar for NS vertex operators. As already explained in section 4.2.3, the most natural way to incorporate an NS puncture in the supermoduli problem is to replace the sheaf $\mathcal{S}$ by its subsheaf consisting of superconformal vector fields ν_f and V_g such that f and g vanish at the puncture. This leads to the canonical moduli space $\mathfrak{M}_{g,n_{NS},n_R}$, suitable for NS vertex operators of picture number -1. If we want the ith NS vertex operator to have picture number $-k_i$, we use the sheaf of superconformal vector fields such that g has a simple zero at the ith NS puncture but f has a zero of order k_i. (We must take $k_i > 0$, since superconformal vector fields with g having a prescribed zero and no constraint on f do not form a graded Lie algebra.)

What we have described so far is a framework to compute perturbative superstring amplitudes using vertex operators of arbitrary negative picture numbers. Is there a similar framework to use vertex operators of non-negative picture number? The answer to this question in general appears to be "no." One would need a moduli space with odd dimension less than the canonical value given in eqn. (4.14). There does not seem to be a natural version of supermoduli space with a smaller odd dimension than is given in this formula. The question of existence of such a space is somewhat like the question of whether, starting with the canonical supermoduli space $\mathfrak{M}$, one can integrate over some or all of the odd moduli in a natural way, without integrating over even moduli.

Alternatively, we could increase the bosonic dimension of supermoduli space by requiring higher order zeros of g, as well as f (subject to the condition that the class of superconformal vector fields considered must form a graded Lie algebra). But nothing is known in superconformal field theory that would suggest how to construct natural objects that can be integrated over supermoduli spaces with enhanced bosonic dimension.

4.4. Punctures in superstring theory

4.4.1. Reduced spaces   Before describing superstring worldsheets and integration cycles in the presence of spin structures, we need to understand the reduced spaces of a super Riemann surface with punctures and of the corresponding moduli space $\mathfrak{M}_{g,n_{NS},n_R}$.

If Σ is a super Riemann surface with punctures of either NS or R type, then its reduced space Σ_{red} is an ordinary Riemann surface with punctures. Both NS and R punctures on Σ become ordinary punctures on Σ_{red} (but they are still labeled as NS or R). Σ_{red} is endowed with a generalized spin structure in the sense of section 4.2.4.

The reduced space of $\mathfrak{M}_{g,n_{NS},n_R}$ is the moduli space $\mathcal{M}_{g,n_{NS}+n_R,spin}$ of Riemann surfaces with a generalized spin structure and a total of $n_{NS} + n_R$ punctures. Equivalently, $\mathfrak{M}_{g,n_{NS},n_R,red} = \mathcal{M}_{g,n_{NS}+n_R,spin}$ parametrizes split super Riemann surfaces with the indicated number of punctures. Here, a super Riemann surface Σ with punctures is said to be split if it has the structure described in section 4.2.2 and the NS punctures are contained in $\Sigma_{red} \subset \Sigma$.

4.4.2. Worldsheets and integration cycles   With this input, we can essentially repeat the description of superstring worldsheets and integration cycles that we gave in the unpunctured case. A superstring worldsheet in the presence of punctures is a smooth supermanifold Σ that is embedded in a product $\Sigma_L \times \Sigma_R$ of Riemann surfaces or super Riemann surfaces with punctures. For the heterotic string, Σ_R is a super Riemann surface with punctures, and Σ_L is an ordinary Riemann surface with punctures; for Type II superstrings, Σ_L and Σ_R are both super Riemann surfaces with punctures. The punctures in Σ_L and Σ_R will be the same in number, but there is no relation between their types.

As usual, the basic example is the case that $\Sigma_{L,red}$ and $\Sigma_{R,red}$ are complex conjugate spaces, Σ_{red} is the diagonal in $\Sigma_{L,red} \times \Sigma_{R,red}$, and Σ is obtained by thickening Σ_{red} in the odd directions. For $\Sigma_{L,red}$ and $\Sigma_{R,red}$ to be complex conjugate means now that they are complex conjugate spaces with punctures at the same points (not necessarily of the same types). For Type II, a puncture may be independently of NS or R type on Σ_L and Σ_R, so there are four types of puncture, which one can label as NS-NS, NS-R, R-NS, and R-R.

The moduli space of deformations of $\Sigma_L \times \Sigma_R$, with Σ_L and Σ_R allowed to deform independently, is the obvious product $\mathfrak{M}_L \times \mathfrak{M}_R$. As usual, we define an integration cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ whose odd dimension is that of $\mathfrak{M}_L \times \mathfrak{M}_R$ and whose reduced space is the diagonal in $\mathfrak{M}_{L,red} \times \mathfrak{M}_{R,red}$. As usual, the even dimension of Γ is twice the complex dimension of $\mathfrak{M}_L$ or equivalently of

$\mathfrak{M}_R$. From these facts, one can work out the dimension of Γ. The most useful way to record the result seems to be to state the contribution of a puncture of a given type to the dimension of Γ. For Type II, these contributions are

$$
\begin{aligned}
\text{NS} - \text{NS} : \ & 2|2 \\
\text{NS} - \text{R} : \ & 2|3/2 \\
\text{R} - \text{NS} : \ & 2|3/2 \\
\text{R} - \text{R} : \ & 2|1.
\end{aligned}
$$

(4.28)

For the heterotic string, the contributions of an NS or R puncture to the dimension of Γ are $2|1$ and $2|1/2$, respectively.

As usual, we can deform $\Sigma \subset \Sigma_L \times \Sigma_R$ or $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ slightly away from the above-stated conditions without changing the integrals that define the worldsheet action or the superstring scattering amplitudes.

The presence of punctures does not affect the definition of the worldsheet action. At an NS puncture, superconformal primary fields are inserted in the standard way. The R case is more subtle; see [31], section 5.

5. Super Riemann surfaces of low genus

Our goal in this section is to describe explicitly the moduli space of super Riemann surfaces of genus 0 and 1. These are important special cases and it helps to be familiar with them.

Genus 0 and 1 are the cases in which the automorphism group $\mathcal{G}$ of a super Riemann surface may have positive dimension. The dimension formula for the moduli space includes a contribution from $\mathcal{G}$:

$$
(5.1) \qquad \dim \mathfrak{M}_{\mathfrak{g}} - \dim \mathcal{G} = (3\mathfrak{g} - 3 + n_{\text{NS}} + n_{\text{R}})|(2\mathfrak{g} - 2 + n_{\text{NS}} + \frac{1}{2}n_{\text{R}}).
$$

Increasing the number of punctures makes $\mathcal{G}$ smaller, and even for $\mathfrak{g} = 0$ or 1, $\dim \mathcal{G} = 0|0$ unless the number of punctures is very small.

5.1. Genus zero

5.1.1. $\mathbb{CP}^{1|1}$ as a super Riemann surface   Let Σ be a super Riemann surface of genus 0, initially without punctures. We will prove that Σ is split, but at first let us just assume that this is the case. The reduced space Σ_{red} of Σ is an ordinary Riemann surface of genus 0. Since Σ_{red} is simply-connected and its tangent bundle $T\Sigma_{\text{red}}$ is of even degree (namely degree 2), there is

up to isomorphism a unique square root of $T\Sigma_{\mathrm{red}}$, which we call $T\Sigma_{\mathrm{red}}^{1/2}$; it has degree 1. The normal bundle to Σ_{red} in Σ is isomorphic to $\Pi T\Sigma_{\mathrm{red}}^{1/2}$. Odd deformations of Σ_{red} are classified by $H^1(\Sigma_{\mathrm{red}}, \Pi T\Sigma_{\mathrm{red}}^{1/2})$, but this vanishes as $T\Sigma_{\mathrm{red}}$ has positive degree. So Σ cannot be deformed away from the split case, and actually is split.

It is convenient to describe Σ_{red} by homogeneous complex coordinates u, v. We write $\mathcal{O}(n)$ for the line bundle over Σ_{red} whose sections are functions of u, v homogeneous of degree n. Since $T\Sigma_{\mathrm{red}}^{1/2}$ has degree 1, the total space of the line bundle $\Pi T\Sigma_{\mathrm{red}}^{1/2}$ can be described by introducing an odd homogeneous coordinate θ that scales just like u and v. So a super Riemann surface Σ of genus 0 (without Ramond punctures) is just a complex projective supermanifold $\mathbb{CP}^{1|1}$ with even and odd homogeneous coordinates $u, v|\theta$, all of degree 1.

As a super Riemann surface, Σ has additional structure that we have not yet described. We usually describe this additional structure as a subbundle $\mathcal{D} \subset T\Sigma$ of rank $0|1$. However, it will here be more convenient to describe the dual picture, which is a line bundle $\mathcal{D}^{-2} \subset T^*\Sigma$ that appears in the dual exact sequence (2.29).

For a split super Riemann surface, $\mathcal{D}$ is isomorphic to the pullback to Σ of the line bundle $T\Sigma_{\mathrm{red}}^{1/2} \to \Sigma_{\mathrm{red}}$. For Σ of genus 0, this pullback is isomorphic to the line bundle $\mathcal{O}(1)$ over $\Sigma = \mathbb{CP}^{1|1}$. So $\mathcal{D}^{-2} \cong \mathcal{O}(-2)$. Hence to turn the complex supermanifold $\Sigma = \mathbb{CP}^{1|1}$ into a super Riemann surface, we need to give a subbundle of $T^*\Sigma$ isomorphic to $\mathcal{O}(-2)$, or equivalently we need to specify a global section ϖ of $T^*\Sigma \otimes \mathcal{O}(2)$.

Such a global section is a holomorphic 1-form on Σ that is homogeneous of degree 2 in $u, v|\theta$. Explicitly we take

$$(5.2) \qquad \varpi = u\mathrm{d}v - v\mathrm{d}u - \theta\mathrm{d}\theta.$$

If we use the scaling symmetry of $\mathbb{CP}^{1|1}$ to set $u = 1$, and write z for v, then we get

$$(5.3) \qquad \varpi = \mathrm{d}z - \theta\mathrm{d}\theta.$$

ϖ is orthogonal to the vector field $D_\theta = \partial_\theta + \theta\partial_z$, so $z|\theta$ are a system of superconformal coordinates, away from the divisor $u = 0$.

Purely as a complex supermanifold, Σ would have the automorphism group PGL(2|1) of linear transformations of $u, v|\theta$ modulo overall scaling. As a super Riemann surface, the automorphism group of Σ is the subgroup of PGL(2|1) that leaves fixed ϖ. To identify this subgroup, an alternative

description of ϖ is useful. We introduce the space $Y \cong \mathbb{C}^{2|1}$ with coordinates $u,v|\theta$ (so Σ is the quotient by $\mathbb{C}^*$ of Y minus the locus $u = v = 0$). We endow Y with a nondegenerate bilinear form $\langle\,,\,\rangle$, skew-symmetric in the $\mathbb{Z}_2$ graded sense, as follows. Writing $y = u,v|\theta$ and $y' = u',v'|\theta'$ for two points $y,y' \in Y$, we set

$$(5.4) \qquad\qquad \langle y, y' \rangle = uv' - vu' - \theta\theta'.$$

Every nondegenerate skew-symmetric form on Y is equivalent to this one up to the action of $\mathrm{PGL}(2|1)$. Any choice of such a form – such as the one we have given – breaks $\mathrm{PGL}(2|1)$ to a subgroup[17] $\mathrm{OSp}(1|2)$. The dimension of $\mathrm{OSp}(1|2)$ is $3|2$.

The section ϖ of $T^*\Sigma \otimes \mathcal{O}(2)$ can be defined as

$$(5.5) \qquad\qquad \varpi = \langle y, \mathrm{d}y \rangle,$$

so in particular its supergroup of automorphisms is $\mathrm{OSp}(1|2)$. Actually, the supergroup of symmetries that acts faithfully on $\Sigma \cong \mathbb{CP}^{1|1}$ is the connected subgroup of $\mathrm{OSp}(1|2)$, which is the quotient of $\mathrm{OSp}(1|2)$ by its center; the center is generated by the transformation $(u, v|\theta) \to (-u, -v| -\theta)$, which acts trivially on $\mathbb{CP}^{1|1}$. A maximal bosonic automorphism group of the super Riemann surface Σ is SL_2 (or $\mathrm{Sp}(2)$), a double cover of the automorphism group $\mathrm{PGL}(2)$ of the ordinary super Riemann surface Σ_{red}. In terms of the superconformal coordinates $z|\theta$, the odd generators of $\mathrm{OSp}(1|2)$ are explicitly $f(z)(\partial_\theta - \theta\partial_z)$, where $f(z) = \alpha + \beta z$. They act by

$$(5.6) \qquad\qquad \begin{aligned} \theta &\to \theta + \alpha + \beta z \\ z &\to z - (\alpha + \beta z)\theta. \end{aligned}$$

In terms of the same superconformal coordinates, the expression $\langle y, y' \rangle$ becomes

$$(5.7) \qquad\qquad \langle y, y' \rangle = z - z' - \theta\theta'.$$

The central element of the bosonic automorphism group SL_2 acts by $\boldsymbol{\tau}'$: $(u, v|\theta) \to (-u, -v|\theta)$. Modulo the center of $\mathrm{OSp}(1|2)$, this is equivalent to

$$(5.8) \qquad\qquad \boldsymbol{\tau} : (u, v|\theta) \to (u, v| -\theta),$$

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[17] $\mathrm{OSp}(m|n)$ is the orthosymplectic group whose maximal bosonic subgroup is $\mathrm{O}(m) \times \mathrm{Sp}(n)$; n must be even. $\mathrm{PGL}(n|m)$ is isomorphic to $\mathrm{PGL}(m|n)$, but $\mathrm{OSp}(m|n)$ has no such symmetry.

which is the universal symmetry of a split super Riemann surface (section 2.2.3).

5.1.2. Adding NS punctures   Now we will add NS punctures. Each puncture that is added can reduce the dimension of the automorphism group by at most $1|1$. As the dimension of $OSp(1|2)$ is $3|2$, it will take at least 3 NS punctures to remove all automorphisms and this number is indeed sufficient. Superstring perturbation theory is constructed primarily[18] by integrating over supermoduli spaces that do not have continuous automorphisms. So in genus 0, if all punctures are of NS type, one is primarily interested in the case that the number of punctures is at least 3.

For a genus 0 super Riemann surface Σ with 3 NS punctures, the dimension formula says that the dimension of supermoduli space is $0|1$. So a set of 3 NS punctures should have 1 odd modulus. Explicitly, one can use $OSp(1|2)$ and scaling of the homogeneous coordinates of $\mathbb{CP}^{1|1}$ to map one puncture to $u,v|\theta = 1,0|0$, a second to $0,1|0$, and the third to $1,1|\eta$, where η is the modulus. From this, one might think that the moduli space $\mathfrak{M}_{0,3,0}$ would be a copy of $\mathbb{C}^{0|1}$, parametrized by η. But one has to remember the universal automorphism $\boldsymbol{\tau}$ (eqn. (5.8)) that changes the sign of all odd variables. In the present context, it acts by $\boldsymbol{\tau} : \eta \to -\eta$. So the moduli space is actually the orbifold (or stack) $\mathbb{C}^{0|1}/\mathbb{Z}_2$. Since $\boldsymbol{\tau}$ is an element of $OSp(1|2)$ and is equivalent as a symmetry of Σ to the element $\boldsymbol{\tau}'$ of the connected component of $OSp(1|2)$, we see that although η cannot be removed by an $OSp(1|2)$ transformation, it also cannot be defined by a formula invariant under $OSp(1|2)$ or even its connected component. Roughly speaking, η^2 is $OSp(1|2)$-invariant; but $\eta^2 = 0$.

More generally, for any number k of NS punctures, we can use $OSp(1|2)$ and scaling of the homogeneous coordinates to map the first three to $1,0|0$, $0,1|0$, and $1,1|\eta$, and the others to $1,z_i|\theta_i$, $i = 4,\ldots,k$. The supermoduli space $\mathfrak{M}_{0,k,0}$ (genus 0, k NS punctures and no Ramond punctures) can thus be parametrized by the $k-3$ even parameters z_i, $i = 4,\ldots,k$, along with the $k-2$ odd parameters θ_i, $i = 4,\ldots,k$ and η. This parametrization of the moduli space is used in practice in computing superstring scattering amplitudes in genus 0. For most computations of tree level scattering amplitudes (of NS fields) in superstring theory, one does not need to know more about $\mathfrak{M}_{0,k,0}$ than we have just explained. The reason is that, as long as the external momenta are generic, the subtleties of compactification of supermoduli space are not very important at tree level.

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[18]There are a few exceptions to this statement in the context of open strings and orientifolds. See section 7.4.
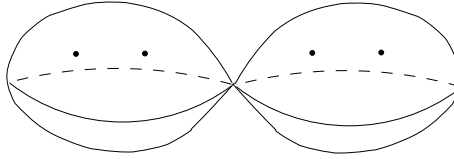
Figure 1: The moduli space $\mathcal{M}_{0,4}$ that parametrizes a genus 0 Riemann surface Σ_0 with four punctures $z_1, \ldots, z_4$ is compactified by adding "points at infinity" in which Σ_0 splits into a union of two components, each of which contains two of the punctures. There are three such points at infinity, depending on how $z_1, \ldots, z_4$ are divided pairwise between the two components.

5.1.3. Cross ratios   However, it ultimately is helpful to understand a little more, so we will carry on. To illustrate the main issues, it suffices to consider the case $k = 4$, where the dimension of the supermoduli space is $1|2$. The reduced space of $\mathfrak{M}_{0,4,0}$ is the moduli space $\mathcal{M}_{0,4}$ of an ordinary Riemann surface of genus 0 with four punctures $z_1, \ldots, z_4$. In the present section, there will be no Ramond punctures, so we will write just $\mathfrak{M}_{0,4}$ for $\mathfrak{M}_{0,4,0}$.

First of all, we should properly distinguish two related moduli spaces. There is the moduli space $\mathcal{M}_{0,4}$ of four distinct points in a genus zero surface Σ_0, and also its Deligne-Mumford compactification $\widehat{\mathcal{M}}_{0,4}$ in which two points are allowed to coincide, or more properly (fig. 1), in which Σ_{red} may split into two components.[19] The following reasoning will be more succinct if we concentrate on the compactification $\widehat{\mathcal{M}}_{0,4}$. Similarly, we write $\mathfrak{M}_{0,4}$ for the moduli space that parametrizes 4 distinct NS punctures in a genus 0 super Riemann surface, and $\widehat{\mathfrak{M}}_{0,4}$ for its compactification.

The compactified moduli space $\widehat{\mathcal{M}}_{0,4}$ can be parametrized by the PGL(2)-invariant cross ratio

$$(5.9) \qquad\qquad \Psi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_4 - z_2)}.$$

We include the value $\Psi = \infty$, which corresponds to one of the points at infinity in $\widehat{\mathcal{M}}_{0,4}$, where either $z_1 \to z_3$ or $z_2 \to z_4$. The two possibilities are equivalent modulo the action of PGL(2) and can be described more symmetrically by the degeneration shown in fig. 1 with z_1 and z_3 on one side and z_2 and z_4 on the other. (The other points at infinity are $\Psi = 0$, which corresponds to $z_1 \to z_2$ or $z_3 \to z_4$, and $\Psi = -1$, which corresponds to $z_2 \to z_3$ or $z_1 \to z_4$.)

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[19]All this is more systematically explained in section 6, which the reader may wish to consult first. But hopefully the details are not needed for the moment.

A point worthy of note is that to define the parameter Ψ, we broke the permutation symmetry among $z_1, \ldots, z_4$. We could alternatively have mapped $\widehat{\mathcal{M}}_{0,4}$ to $\mathbb{CP}^1$ via the PGL(2)-invariant

$$(5.10) \qquad \Psi' = \frac{(z_1 - z_3)(z_4 - z_2)}{(z_1 - z_4)(z_2 - z_3)}.$$

This would not make much difference. Because of the identity

$$(5.11) \qquad (z_1 - z_2)(z_3 - z_4) + (z_1 - z_3)(z_4 - z_2) + (z_1 - z_4)(z_2 - z_3) = 0,$$

we have

$$(5.12) \qquad \Psi + \frac{1}{\Psi'} + 1 = 0.$$

Thus the two maps from $\widehat{\mathcal{M}}_{0,4}$ to $\mathbb{CP}^1$ given by Ψ or Ψ' differ by a fractional linear transformation of $\mathbb{CP}^1$.

Now let us consider the supersymmetric case. The supermanifold $\widehat{\mathfrak{M}}_{0,4}$ has dimension $1|2$. Is this supermanifold split? (See section 2.3.1 of [32] for an explanation of this concept.) To get a "yes," answer, we just need to find a holomorphic map from $\widehat{\mathfrak{M}}_{0,4}$ to $\mathbb{CP}^1$ that when the odd variables vanish coincides with one of the above formulas. To get such a map, it suffices to have an OSp(1|2)-invariant extension of the cross ratio that reduces to the cross ratio if the odd variables vanish. There is no problem to find one. For example, a super extension of the cross ratio Ψ is given by the OSp(1|2) invariant $\widehat{\Psi} = \langle w_1, w_2 \rangle \langle w_3, w_4 \rangle / \langle w_1, w_3 \rangle \langle w_4, w_2 \rangle$. In view of (5.7), in the usual superconformal coordinates, this becomes

$$(5.13) \qquad \widehat{\Psi} = \frac{(z_1 - z_2 - \theta_1\theta_2)(z_3 - z_4 - \theta_3\theta_4)}{(z_1 - z_3 - \theta_1\theta_3)(z_4 - z_2 - \theta_4\theta_2)}.$$

Obviously, $\widehat{\Psi}$ reduces to Ψ if we set the θ's to 0. The map from $\widehat{\mathfrak{M}}_{0,4}$ to $\widehat{\mathcal{M}}_{0,4} = \mathbb{CP}^1$ is a splitting of $\widehat{\mathfrak{M}}_{0,4}$.

There is a small catch. Obviously we could define another splitting by

$$(5.14) \qquad \widehat{\Psi}' = \frac{(z_1 - z_3 - \theta_1\theta_3)(z_4 - z_2 - \theta_4\theta_2)}{(z_1 - z_4 - \theta_1\theta_4)(z_2 - z_3 - \theta_2\theta_3)}.$$

In contrast to the bosonic case, the splittings given by $\widehat{\Psi}$ and $\widehat{\Psi}'$ (and a third obtained from $\widehat{\Psi}$ by exchanging $z_2|\theta_2$ with $z_4|\theta_4$) are not equivalent. That is

because the supersymmetric extension of the identity (5.11) does not hold:

$$(z_1 - z_2 - \theta_1\theta_2)(z_3 - z_4 - \theta_3\theta_4) + (z_1 - z_3 - \theta_1\theta_3)(z_4 - z_2 - \theta_4\theta_2)$$

(5.15) $$+ (z_1 - z_4 - \theta_1\theta_4)(z_2 - z_3 - \theta_2\theta_3) \neq 0.$$

The left hand side of (5.15) is nilpotent, but that just implies that the splittings defined using $\widehat{\Psi}$ and $\widehat{\Psi}'$ are equivalent modulo nilpotent terms, which is a trivial statement, in the sense that it holds by definition for all splittings.

So $\widehat{\mathfrak{M}}_{0,4}$ can be split in several fairly natural ways, but none of these is entirely natural. What property of a splitting do we actually most care about? The answer to this question involves concepts explained in section 6, but we summarize a few facts here. A splitting is a holomorphic projection $\pi : \widehat{\mathfrak{M}}_{0,4} \to \widehat{\mathcal{M}}_{0,4}$. For p a point in $\widehat{\mathcal{M}}_{0,4}$, $\pi^{-1}(p)$ is a divisor in $\widehat{\mathfrak{M}}_{0,4}$. The subtleties of superstring perturbation theory come from the behavior at infinity, which means that in the present context, we should focus attention on the distinguished points in $\widehat{\mathfrak{M}}_{0,4}$ at which two of the z_i coincide. Over the point $p_{ij} \in \widehat{\mathcal{M}}_{0,4}$ given by $z_i - z_j = 0$, there is a distinguished divisor $\mathcal{D}_{ij} \subset \widehat{\mathfrak{M}}_{0,4}$ given by $z_i - z_j - \theta_i\theta_j = 0$. A good splitting is one such that for each i and j, $\pi^{-1}(p_{ij}) = \mathcal{D}_{ij}$. By this criterion, none of the splittings that we have described is satisfactory at each of the divisors at infinity. The splitting given by $\widehat{\Psi}$ behaves well for $z_1 \to z_2$, since $\widehat{\Psi}$ is explicitly proportional to $(z_1 - z_2 - \theta_1\theta_2)$. It is equally good for $z_1 \to z_3$, in this case because $\widehat{\Psi}^{-1}$ is proportional to a similar factor. But what happens when $z_1 \to z_4$? The behavior we would like is

(5.16) $$\widehat{\Psi} = a + b(z_1 - z_4 - \theta_1\theta_4) + \mathcal{O}((z_1 - z_4)^2, (z_1 - z_4)\theta_1\theta_4), \quad b \neq 0,$$

so that the equation $\widehat{\Psi} - a = 0$ will define the divisor $\mathcal{D}_{ij}$. But this is actually not the case. So the splitting given by $\widehat{\Psi}$ does not have the behavior one would wish at $z_1 - z_4 = 0$, and similarly each of the other splittings fails to show the desired behavior at one of the divisors at infinity.

What we have said about $\widehat{\mathfrak{M}}_{0,4}$ extends to $\widehat{\mathfrak{M}}_{0,k}$ for all $k \geq 4$. By making use of the supersymmetric extension of the cross ratio, one can always define holomorphic splittings $\pi : \widehat{\mathfrak{M}}_{0,k} \to \widehat{\mathcal{M}}_{0,k}$. But there are many ways to do this and none is fully satisfactory.

The splittings of $\widehat{\mathfrak{M}}_{0,4}$ and their properties were first described in [46], with more detail in [33]. The relevance to superstring perturbation theory is as follows [47]. Although tree level scattering amplitudes at generic values of the external momenta do not have the subtleties that occur in higher genus, some of these subtleties do occur in tree level scattering amplitudes at

special values of the momenta (such as momentum zero). And moreover, as essentially also explained there, as well as in [46], in general a procedure to compute these amplitudes by first integrating over the fibers of a projection $\pi : \widehat{\mathfrak{M}}_{0,4} \to \widehat{\mathcal{M}}_{0,4}$ will only give the right answer if $\pi^{-1}(p_{ij}) = \mathcal{D}_{ij}$. Otherwise a correction has to be made at infinity.

In section 6, we will describe the genus $\mathfrak{g}$ generalization of the divisor $\mathcal{D}_{ij}$. This information will be important input in a reconsideration of superstring perturbation theory [31], not so much because one aims to find a good splitting – supermoduli space is actually not holomorphically split in general [48] – but as part of an explanation of how to handle the infrared region in the integration over supermoduli space.

To understand $\widehat{\mathfrak{M}}_{0,4}$ more deeply, one has to describe its orbifold structure. Let us discuss what happens along the reduced space $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$, which roughly speaking (ignoring the orbifold structure) is the same as $\widehat{\mathcal{M}}_{0,4}$. Generically, along $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$, the automorphism group of Σ is the universal $\mathbb{Z}_2$ symmetry of split super Riemann surfaces generated by $\boldsymbol{\tau} : \theta \to -\theta$. However, at the three points in $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$ at which Σ splits into two components, the automorphism group is enhanced to $\mathbb{Z}_2 \times \mathbb{Z}_2$, since one has a separate $\mathbb{Z}_2$ symmetry group on each component. (This will be clearer in section 6; the NS degeneration described in eqn. (6.9) has separate $\mathbb{Z}_2$ symmetries $\theta \to -\theta$ and $\psi \to -\psi$ if $\varepsilon = 0$, though for $\varepsilon \neq 0$, it has only a single $\mathbb{Z}_2$ symmetry.) Thus, although roughly speaking $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$ is the same as $\widehat{\mathcal{M}}_{0,4}$ and is a copy of $\mathbb{CP}^1$ parametrized by the cross ratio, it is more accurate to think of it as an orbifold version of $\mathbb{CP}^1$, with an automorphism group that is generically $\mathbb{Z}_2$ and jumps to $\mathbb{Z}_2 \times \mathbb{Z}_2$ at the three points at infinity. The normal bundle to $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$ in $\widehat{\mathfrak{M}}_{0,4}$ is, roughly speaking, a vector bundle over $\widehat{\mathfrak{M}}_{0,4,\mathrm{red}}$ of rank $0|2$, but actually this vector bundle must be defined in the orbifold sense [49].

5.1.4. Ramond punctures   We now will describe a super Riemann surface Σ of genus 0 with $\mathfrak{n}_{\mathrm{R}}$ Ramond punctures. We have already given such a description in affine coordinates in eqns. (4.9) and (4.13). We want to rewrite these formula projectively, to better see the behavior at infinity. We actually can do so by hand. In eqn. (5.2), we endowed $\mathbb{CP}^{1|1}$ with a superconformal structure by picking a section ϖ of $T^*\Sigma$ tensored with a suitable line bundle. We simply need to modify this formula so that in affine coordinates it will match with (4.13):

$$(5.17) \qquad \varpi = u\mathrm{d}v - v\mathrm{d}u - w(u,v)\theta\mathrm{d}\theta.$$

We take $w(u, v)$ to be homogeneous of degree $\mathfrak{n}_{\mathrm{R}}$ and to coincide with the function $w(z)$ in (4.13) if we set $u = 1$, $v = z$. The zeroes of this function

are the Ramond divisors. In order for ϖ to transform homogeneously under the scaling of the homogeneous coordinates u, v, we have to assume that θ scales with degree $1 - n_R/2$. So Σ is a weighted projective superspace $\mathbb{WCP}^{1|1}(1, 1|1 - n_R/2)$ with homogeneous coordinates $u, v|\theta$ whose weights are as indicated.

An important special case of this is that $n_R = 2$. It is convenient then to put the Ramond punctures at 0 and ∞, which we do by taking $w(u, v) = uv$. Then in affine coordinates, we have

$$(5.18) \qquad \varpi = \mathrm{d}z - z\theta\mathrm{d}\theta$$

with the Ramond punctures at $0, \infty$. To verify that the structure at $z = \infty$ is equivalent to that at $z = 0$, one may use the automorphism $z \to -1/z$, $\theta \to \pm\sqrt{-1}\,\theta$, which exchanges them, mapping ϖ to a multiple of itself. (Of course, the equivalence between $z = 0$ and ∞ is obvious in the projective description.) This super Riemann surface has no supermoduli, and has an automorphism group $\mathcal{G}$ of dimension $1|1$, generated by the superconformal vector fields $z\partial_z$ and $\partial_\theta - \theta z\partial_z$, which are regular both at 0 and ∞. This is in keeping with the dimension formula (5.1). A noteworthy fact is that the vector field $\partial_\theta - \theta z\partial_z$ is nonzero when restricted to $z = 0$ (or $z = \infty$),

$$(5.19) \qquad (\partial_\theta - \theta z\partial_z)|_{z=0} = \partial_\theta.$$

It generates a shift symmetry of the Ramond divisors.

If we add a single NS puncture at $z|\theta = z_0|\theta_0$, then by the action of $\mathcal{G}$, we can uniquely map it to $z|\theta = 1|0$. In this way, we describe a Riemann surface of genus 0 with 1 NS puncture and 2 Ramond punctures. It has no supermoduli and in particular is split, and its automorphism group is of dimension $0|0$, the only non-trivial element being the symmetry $\boldsymbol{\tau} : \theta \to -\theta$ that is common to all split super Riemann surfaces. Of course, for $n_{NS} = 1$, $n_R = 2$, the dimension formula is consistent with $\dim \mathcal{M} = \dim \mathcal{G} = 0|0$.

5.2. Genus one

Here we will much more briefly consider the case that Σ is a super Riemann surface of genus 1. There are two cases; the spin structure of Σ may be even or odd. We only consider unpunctured surfaces.

Suppose first that the spin structure is even. The reduced space of Σ is an ordinary genus 1 Riemann surface, endowed with a line bundle $T\Sigma_{\mathrm{red}}^{1/2}$ whose square is isomorphic to $T\Sigma_{\mathrm{red}}$. If Σ is split, then it is isomorphic to the total

space of the line bundle $\mathit{\Pi}T\Sigma_{\text{red}}^{1/2}$ over Σ_{red}. The odd moduli that measure the departure of Σ from splitness take values in $H^1(\Sigma_{\text{red}}, T\Sigma_{\text{red}}^{1/2})$, but this cohomology group vanishes. So Σ has no odd moduli and cannot be deformed away from the split case. So similarly to what we found for genus 0, a genus 1 super Riemann surface with an even spin structure is automatically split.

Such a surface can be explicitly described as the quotient of $\mathbb{C}^{1|1}$ with its standard superconformal structure by the group $\mathbb{Z} \times \mathbb{Z}$ acting by

$$(5.20) \qquad\qquad z \to z + 1, \quad \theta \to \theta$$

and

$$(5.21) \qquad\qquad z \to z + \tau, \quad \theta \to -\theta.$$

The moduli space $\mathfrak{M}$ has dimension $1|0$, with τ as the only parameter. The automorphism group $\mathcal{G}$ is also of dimension $1|0$, generated by ∂_z. Clearly these statements are consistent with the dimension formula (5.1).

Now we consider the case of an odd spin structure (for much more, see [50]). In this case, $H^1(\Sigma_{\text{red}}, T\Sigma_{\text{red}}^{1/2})$ is of dimension 1, so there is 1 odd parameter by which to deform away from a split situation. Explicitly, one can describe the family of super Riemann surfaces as the quotient of $\mathbb{C}^{1|1}$ by

$$(5.22) \qquad\qquad z \to z + 1, \quad \theta \to \theta$$

and

$$(5.23) \qquad\qquad z \to z + \tau - \alpha\theta, \quad \theta \to \theta + \alpha,$$

where α is the odd parameter. At $\alpha = 0$, the dimension formula is satisfied with $\dim \mathcal{M} = \dim \mathcal{G} = 1|1$. $\mathcal{G}$ is generated by $z \to z + b$, $\theta \to \theta$, along with $\theta \to \theta + \varepsilon$, $z \to z - \varepsilon\theta$, with constants b and ε. But actually jumping of the cohomology occurs when α is turned on and this makes the proper formulation of the dimension formula subtle. Explicitly, when $\alpha \neq 0$, a superconformal transformation $\theta \to \theta + \varepsilon$, $z \to z - \varepsilon\theta$ is not a symmetry of (5.23), but shifts τ to $\tau - \alpha\varepsilon$, so for $\alpha \neq 0$, the odd automorphism of Σ is lost and the sense in which τ is a modulus is subtle.

A related jumping occurs as a function of α in the space of holomorphic differentials. We explain this briefly because of its relevance to section 8.1. At $\alpha = 0$, the space of closed holomorphic 1-forms on Σ has dimension $1|1$, generated by $\mathrm{d}z$ and $\mathrm{d}\theta$. For $\alpha \neq 0$, $\mathrm{d}\theta$ is still well-defined, but $\mathrm{d}z$ is not, since it is not invariant under (5.23). Moreover, there is no way to add an α-dependent
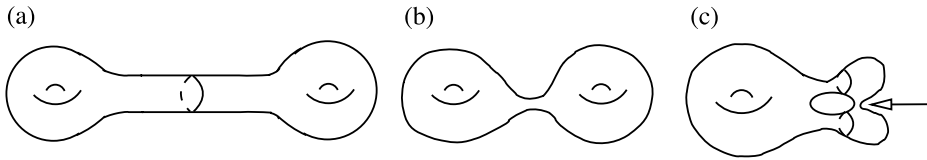
Figure 2: (a) A Riemann surface Σ_0 with a long tube. (b) A conformally equivalent picture in which Σ_0 has a narrow neck. The example sketched here represents a separating degeneration. (c) A similar picture with a nonseparating degeneration, indicated by the arrow.

correction to dz to get an α-dependent closed holomorphic 1-form that equals dz at $\alpha = 0$. So the dimension of the space of closed holomorphic differentials jumps downward for $\alpha \neq 0$. As explained in section 8.2, the space of closed holomorphic 1-forms on Σ is naturally isomorphic to $H^0(\Sigma, Ber(\Sigma))$, so the same jumping occurs in that cohomology group. In view of the Riemann-Roch theorem, the same jumping occurs in $H^1(\Sigma, Ber(\Sigma))$, as well.

6. Behavior at infinity

6.1. Compactification of the bosonic moduli space

6.1.1. Introduction   The moduli space $\mathcal{M}$ of smooth Riemann surfaces is not compact, because a Riemann surface Σ_0 can develop a long tube, as in fig. 2(a). A long tube is conformally equivalent (see the discussion of eqn. (6.2)) to a narrow neck, which collapses when the length of the long tube becomes infinite. When this happens, Σ_0 may either divide into two topological components, as in fig. 2(b), or remain connected, as in fig. 2(c). The two cases are called separating and nonseparating degenerations, respectively.

In either case, when a narrow neck collapses, Σ_0 separates at least locally into two branches that meet at a point. Let x and y be local parameters on the two branches. Let us focus on the "neck" region $N \subset \Sigma_0$ defined by $|x|, |y| < 1$. A local description of Σ_0 near its degeneration is given by the algebraic equation

$$(6.1) \qquad\qquad\qquad xy = q,$$

where q is a complex parameter that we consider small. For $q = 0$, the equation reduces to $xy = 0$, and describes two branches – with $x = 0$ and $y = 0$, respectively – that meet at the point $x = y = 0$. This is a singular point, in

some sense the simplest and most generic possible singularity of a Riemann surface. It is known as a node or ordinary double point. For small but nonzero q, this picture is modified in the region $|x|, |y| \lesssim q$ and the two branches are glued together through a narrow neck.

To develop more intuition, let us transform to variables that are adapted to the "long tube" picture of fig. 2(a). We do this by setting

$$x = e^{\varrho}, \quad y = q e^{-\varrho}, \tag{6.2}$$

where

$$\varrho = s + i\vartheta, \quad s, \vartheta \in \mathbb{R}. \tag{6.3}$$

This change of variables actually make the conformal equivalence of a narrow neck and a long tube obvious; it is a holomorphic change of variables, so the metric $|dx|^2 + |dy|^2$, which develops a narrow neck for $q \to 0$, is conformally equivalent to the metric $|d\varrho|^2$, which develops a long tube. The conditions $|x|, |y| < 1$ give $0 > s > -\ln |q|^{-1}$. So in terms of ϱ, the neck region N is a tube of length

$$\tau = \ln |q|^{-1} \tag{6.4}$$

(the distance along the tube is parametrized by s) and of circumference 2π (the angular variable being ϑ). In bosonic string theory, to compute a scattering amplitude, we have to integrate over q (as well as the other moduli of Σ_0), which means integrating over the proper length τ of the tube as well as over the angle $\mathrm{Arg}\, q$ (the latter parametrizes a relative twist that can be introduced in gluing the two sides of Σ_0).

We interpret τ as the elapsed proper time for a closed string that is propagating down the tube. The meaning of the degeneration as $q \to 0$ is that the elapsed proper time diverges. The analog of τ in field theory is the proper time parameter in the Schwinger representation of the propagator of a particle. For example, the propagator of a scalar of mass m is

$$\frac{1}{p^2 + m^2} = \int_0^\infty d\tau \, \exp\left(-\tau(p^2 + m^2)\right). \tag{6.5}$$

Here p is the particle momentum and we write the formula in Euclidean signature so that the mass shell condition is $p^2 + m^2 = 0$. The pole of the propagator at $p^2 + m^2 = 0$ comes from a divergence of the integral at $\tau \to \infty$. In perturbative quantum field theory, infrared singularities of Feynman amplitudes occur because of these poles.

In field theory, there is also an ultraviolet region, with $\tau \to 0$. (In any Feynman diagram, one may use the proper time representation (6.5) for all of the propagators, with a proper time parameter τ_i for each internal line. Then to evaluate the diagram, one must integrate over the loop momenta and the τ_i. The momentum integrals are always convergent as long as the τ_i are all positive, but divergences may occur as some or all of the τ_i vanish. These are the ultraviolet divergences of Feynman diagrams.) There is no analog in string theory of taking $\tau \to 0$. This would mean taking $q \to \infty$, but the local description (6.1) that we used in introducing q is only valid if q is sufficiently small. When q ceases to be small, one must use a different description of Σ_0.

In the end, there is no ultraviolet region in the integral over $\mathcal{M}$. The precise statement of this is based on the fact that $\mathcal{M}$ has a natural Deligne-Mumford compactification $\widehat{\mathcal{M}}$ that parametrizes, apart from smooth Riemann surfaces, only one additional type of object – Riemann surfaces with the node or ordinary double point singularity modeled by the equation $xy = q$. $\widehat{\mathcal{M}}$ is built by adjoining to $\mathcal{M}$ certain "divisors at infinity" $\mathcal{D}_\lambda$ – described more fully in section 6.1.2 – that parametrize Riemann surfaces with nodes. The phrase "the region at infinity in $\mathcal{M}$" refers to any small neighborhood of the union of the $\mathcal{D}_\lambda$.

$\widehat{\mathcal{M}}$ is a smooth compact manifold (or more precisely a smooth compact orbifold). In general, there is no problem with integration of a smooth measure on a compact manifold (or orbifold). The integrals required to compute perturbative bosonic string scattering are smooth except along the divisors $\mathcal{D}_\lambda$. In view of what has been just explained, the singularities that occur along those divisors are infrared effects. The fact that the bosonic string measures are smooth along $\mathcal{M}$ and that the $\mathcal{D}_\lambda$ are the only divisors that have to be added to $\mathcal{M}$ to achieve compactification is a very precise statement of the fact that in bosonic string perturbation theory, the only possible problems are infrared problems. However, those infrared problems are serious: bosonic string theory has tachyon poles and dilaton tadpoles that do not have a sensible physical interpretation, at least in the context of perturbation theory.

What about superstring theory? The moduli space $\mathfrak{M}$ of super Riemann surfaces has an analogous Deligne-Mumford compactification $\widehat{\mathfrak{M}}$ that is obtained by adjoining to $\mathfrak{M}$ certain divisors $\mathcal{D}_\lambda$ that parametrize super Riemann surfaces with nodes. We again call these the divisors at infinity, and any small neighborhood of the union of these divisors is called "the region at infinity in supermoduli space." $\widehat{\mathfrak{M}}$ is a smooth compact supermanifold (or more precisely superorbifold). To compute superstring scattering amplitudes, one must evaluate integrals over $\widehat{\mathfrak{M}}$ (or more precisely over certain integration cycles derived from $\widehat{\mathfrak{M}}$) of measures whose only singularities are along the divisors at infinity. As in the bosonic string, the subtleties associated to the behavior

at infinity are infrared effects – in which strings propagate for a long proper time near their mass shell. There is no potentially dangerous ultraviolet region. The difference between superstring theory and bosonic string theory is that in the superstring case, the infrared effects are harmless; there are no tachyon poles, and the massless particle tadpoles are tamed by spacetime supersymmetry. The infrared behavior is the same as one would expect in a field theory with the same low energy content. These matters will be reconsidered in [31].

6.1.2. More details on the divisors at infinity Now we will describe in more detail the divisors at infinity in the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$, the moduli space that parametrizes a genus g surface Σ_0 with n punctures. As we will see, the Deligne-Mumford compactification is defined in such a way that the punctures always remain distinct as the moduli are varied. When superficially it appears that two punctures would collide, something else will happen instead.

In what follows, we will exclude a few exceptional cases with small values of g and n. The case $g = 1$, $n = 0$ is omitted because including it would create inconvenient exceptions to many statements. The case $g = 0$, $n < 3$ can then be omitted for a reason explained below. The discussion below will show that we can impose these restrictions not just on Σ_0 itself but on each of the components to which it may degenerate. The restrictions on g and n can be summarized by saying neither Σ_0 nor any of its components (if Σ_0 degenerates) has continuous symmetries.[20] Accordingly, for the values of g and n that we will discuss, the dimension of $\widehat{\mathcal{M}}_{g,n}$ is always precisely

$$\text{(6.6)} \qquad \dim \widehat{\mathcal{M}}_{g,n} = 3g - 3 + n.$$

There is no correction for the dimension of the automorphism group $\mathcal{G}$, since this always vanishes.

Now that these preliminaries are out of the way, let us consider a nonseparating degeneration (fig. 3(a)). When Σ_0 develops a node or double point, it is of course singular. If, however, we simply unglue the two branches of Σ_0 that meet at the node, we get a smooth Riemann surface Σ_1 of genus $g - 1$ with $n + 2$ punctures, where the 2 extra punctures are the 2 points in Σ_1 that have to be glued together to get Σ_0. (Σ_1 is sometimes called the normalization

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[20]In the Deligne-Mumford compactification of the moduli space of unoriented Riemann surfaces or Riemann surfaces with boundary, one does have to include components with continuous symmetries, as we will see in section 7.4. This turns out to be important in the study of anomalies in superstring theory [52], [31].
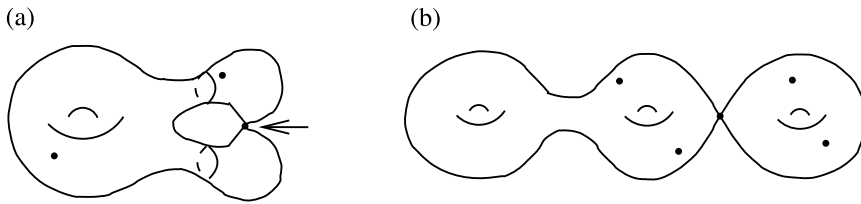
(a)   (b)

Figure 3: (a) Along the divisor $\mathcal{D}_{\mathrm{nonsep}} \subset \widehat{\mathcal{M}}_{g,n}$ that parametrizes nonseparating degenerations of a Riemann surface Σ_0, the genus of Σ_0 is reduced by 1, but the two points that are glued together at the node (marked by the arrow) count as additional punctures, one on each branch. So this divisor is a copy of $\widehat{\mathcal{M}}_{g-1,n+2}$. This is sketched here for $g = n = 2$. (b) Divisors in $\widehat{\mathcal{M}}_{g,n}$ that correspond to separating degenerations correspond to decompositions $g = g_1 + g_2$, $n = n_1 + n_2$. The divisor corresponding to such a decomposition is a copy of $\widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1}$, where the points glued together count as one extra puncture on each side. This is sketched here for $g_1 = 2$, $g_2 = 1$, $n_1 = n_2 = 2$.

of Σ_0.) So the nonseparating divisor $\mathcal{D}_{\mathrm{nonsep}}$ in $\widehat{\mathcal{M}}_{g,n}$ is a copy of $\widehat{\mathcal{M}}_{g-1,n+2}$. As a check on this, we count dimensions. A quick computation using (6.6) shows that dim $\widehat{\mathcal{M}}_{g-1,n+2} = $ dim $\widehat{\mathcal{M}}_{g,n} - 1$, consistent with the claim that $\widehat{\mathcal{M}}_{g-1,n+2}$ is a divisor at infinity in $\widehat{\mathcal{M}}_{g,n}$.

A separating degeneration occurs when Σ_0 decomposes into two components of genera g_1 and g_2 with $g_1 + g_2 = g$. In general, of the n punctures on Σ_0, we will have n_1 on one side and n_2 on the other side, with any n_1, n_2 such that $n_1 + n_2 = n$. The node must be counted as an extra puncture on each side, so the divisor $\mathcal{D}_{\mathrm{sep}}$ corresponding to this degeneration (fig. 3(b)) is a copy of $\widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1}$. A quick calculation shows that again $\dim(\widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1}) = \dim \widehat{\mathcal{M}}_{g,n} - 1$, so that the former can appear as a divisor at infinity in the latter.

The case that g_2 (or g_1) is 0 deserves special attention. In compactifying the ordinary moduli space $\mathcal{M}_{g,n}$, why is it necessary to include a divisor in which Σ_0 splits into two components, one of which has genus 0? The answer to this question (fig. 4(a)) is that in compactifying $\mathcal{M}_{g,n}$ one needs to give a limit to a sequence in which the surface Σ_0 is kept fixed but 2 or more punctures in Σ_0 approach each other. Let $s \geq 2$ be the number of punctures that are approaching each other. In the Deligne-Mumford compactification, the limit of this sequence is represented, as in fig. 4(c), by letting a genus 0 component split off from Σ_0. This component has $s+1$ punctures, including an
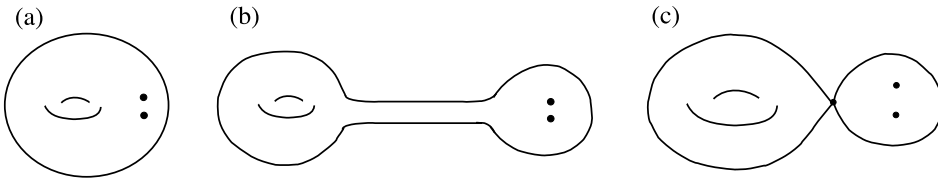
Figure 4: In the Deligne-Mumford compactification, a process in which two or more punctures on a Riemann surface Σ_0 approach each other as in (a), or equivalently are connected to the rest of the surface through a long tube, as in (b), leads to a separating divisor in which Σ_0 splits into two components, one of which has genus 0, as shown in (c). The number of punctures on the genus 0 component, counting the extra puncture at the node, is always at least 3. In fact, we started in (a) with at least 2 punctures coming together, and there is 1 more where the two components of Σ_0 are glued together. So the Deligne-Mumford compactification can be constructed while never allowing Σ_0 to have a genus 0 component with fewer than three punctures.

extra one from the node, so it has at least 3 punctures. Thus, in constructing the Deligne-Mumford compactification, there is no need to allow genus 0 components with fewer than 3 punctures.

Another consequence is that $\widehat{\mathcal{M}}_{g,n}$ parametrizes Riemann surfaces that may have singularities (nodes) but in which the n punctures are always distinct. The limit of a sequence in which 2 or more points approach each other is described by letting Σ_0 branch off a new component, not by actually taking the points in question to be equal.

Now let us focus on the case $s > 2$. Consider a Riemann surface with nearby punctures $z_1, \ldots, z_s$. Naively, to set $z_1 = \cdots = z_s$ is $s - 1$ complex conditions and would appear to define a submanifold of moduli space of complex codimension $s - 1$. However, in the Deligne-Mumford compactification, in the limit that naively corresponds to $z_1 = \cdots = z_s$, we let Σ_0 split off a genus 0 component that contains the points $z_1, \ldots, z_s$, which remain distinct (fig. 5). This gives one of the separating divisors at infinity, here with $g_1 = g$, $g_2 = 0$, so the complex codimension is actually 1, not $s - 1$. What has happened is that the naive codimension $s - 1$ locus with $z_1 = \cdots = z_s$ has been "blown up" and replaced by a divisor.

The description we have given of the divisors at infinity in $\widehat{\mathcal{M}}_{g,n}$ has the following implication. Since those divisors are themselves compactified moduli spaces $\widehat{\mathcal{M}}_{g',n'}$ or products thereof, further degenerations are typically possible. Some examples are shown in figure 6. As long as Σ_0 has a component
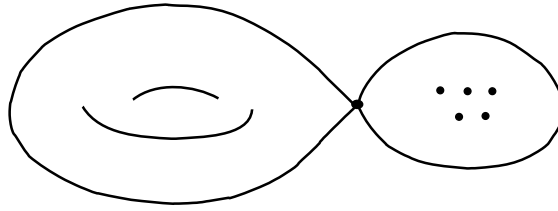
Figure 5: If $z_1, \ldots, z_s$ (with $s \geq 2$) is any number of punctures in Σ_0, there is a divisor at infinity in $\widehat{\mathcal{M}}_{g,n}$ along which $z_1, \ldots, z_s$ are contained in a separate genus 0 component of Σ_0. This locus is of complex codimension 1, and not of codimension $s-1$ as one might guess from the fact that naively it corresponds to the condition $z_1 = \cdots = z_s$.
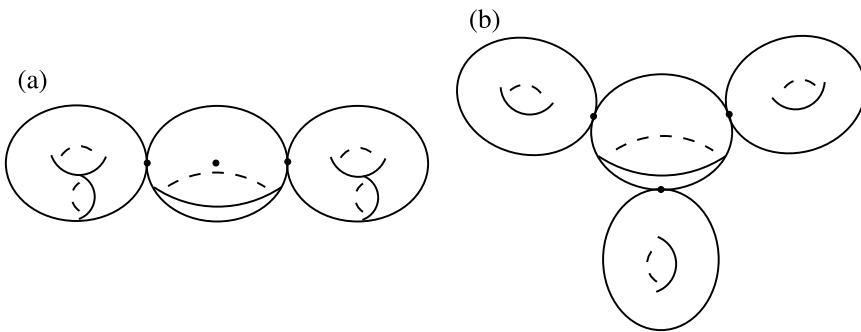
Figure 6: The divisors at infinity in the Deligne-Mumford compactification can intersect each other, leading to a Riemann surface Σ_0 that has more than 2 components glued together at nodes. Two examples are sketched here. These examples were chosen to illustrate another point: in the Deligne-Mumford compactification, a genus 0 component of Σ_0 will always have at least 3 punctures, but there is no restriction on how many of those punctures may originate from nodes. In both (a) and (b), there is a genus 0 component with 3 punctures; in (a), 2 of these arise from nodes and in (b) all 3 do.

of genus g' with n' punctures such that $\dim \widehat{\mathcal{M}}_{g',n'} > 0$, further degeneration is always possible. Given the restrictions that we have placed on small values of g' and n', the only case with $\dim \widehat{\mathcal{M}}_{g',n'} = 0$ is $g' = 0$, $n' = 3$. Accordingly, 3-punctured spheres play a special role. A Riemann surface Σ_0 can degenerate until it is built by gluing together 3-punctured spheres. Completely degenerate Riemann surfaces of this kind are dual to trivalent graphs. They will be discussed in section 6.4.
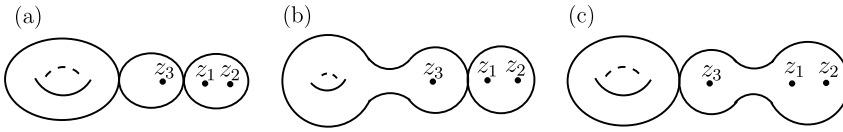
Figure 7: The divisors $\mathcal{D}_{12}$ and $\mathcal{D}_{123}$ described in the text intersect on a locus in $\widehat{\mathcal{M}}_{g,n}$ that parametrizes singular Riemann surfaces depicted in (a). Surfaces of this type do correspond to intersection points of $\mathcal{D}_{12}$ and $\mathcal{D}_{123}$, since they can be obtained by further degeneration from configurations of (b), which represent $\mathcal{D}_{12}$ (since a genus zero component containing z_1 and z_2 has split off), and from configurations of (c), which represent $\mathcal{D}_{123}$ (since a genus zero component containing all of z_1, z_2, and z_3 has split off). By contrast, for example, $\mathcal{D}_{12}$ does not intersect $\mathcal{D}_{13}$, since there is no picture that is a further degeneration both of (b), which represents $\mathcal{D}_{12}$, and of an equivalent picture with z_2 and z_3 exchanged, representing $\mathcal{D}_{13}$.

We conclude this discussion with an example that may help one develop intuition about the Deligne-Mumford compactification. In this example, we consider a Riemann surface Σ_0 with three nearby punctures z_1, z_2, z_3. We will keep Σ_0 fixed and vary only the points z_i. For each $i < j$, the Deligne-Mumford compactification has a divisor $\mathcal{D}_{ij}$ that naively corresponds to $z_i = z_j$ (but which actually is represented by the splitting off from Σ_0 of a genus 0 component that contains z_i and z_j). As we know from fig. 5, $\widehat{\mathcal{M}}_{g,n}$ also has another divisor at infinity that we will call $\mathcal{D}_{123}$, corresponding to the case that Σ_0 splits off a genus 0 component that contains all of z_1, z_2, z_3. We want to understand the intersections of these divisors. Naively, one may expect that the divisors $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$, which naively correspond to $z_1 = z_2$ and to $z_2 = z_3$, would intersect on a codimension 2 locus corresponding $z_1 = z_2 = z_3$. But this cannot be right because the locus in $\widehat{\mathcal{M}}_{g,n}$ corresponding naively to $z_1 = z_2 = z_3$ is not of complex codimension 2; it is the divisor $\mathcal{D}_{123}$. In fact, in $\widehat{\mathcal{M}}_{g,n}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$ do not intersect each other, but they both intersect $\mathcal{D}_{123}$. This is indicated in fig. 7.

6.1.3. Normal bundle to the compactification divisor A natural line bundle over a Riemann surface Σ_0 is its canonical bundle K_{Σ_0}. We can use this line bundle to define line bundles over the moduli space $\widehat{\mathcal{M}}_{g,n}$, which parametrizes a Riemann surface Σ_0 with n punctures. Taking the fiber of K_{Σ_0} at the σth puncture, for $\sigma = 1, \ldots, n$, we define a complex line bundle $\mathcal{L}_\sigma \to \mathcal{M}_{g,n}$. These line bundles are quite nontrivial. Their first Chern classes are the basic observables of two-dimensional topological gravity [53]. For our

purposes, these line bundles are essential ingredients in describing the normal bundles to the compactification divisors $\mathcal{D} \subset \widehat{\mathcal{M}}_{g,n}$. A basic understanding of these normal bundles is important for superstring perturbation theory.

For brevity in the notation, we will consider a divisor $\mathcal{D}$ that describes the splitting of Σ_0 into two components Σ' and Σ'' joined at a pair of punctures. The divisor $\mathcal{D}$ is itself a product $\mathcal{D} = \widehat{\mathcal{M}}_{g_1,n_1+1} \times \widehat{\mathcal{M}}_{g_2,n_2+1}$, as explained before. What we will say has an obvious analog for the nonseparating degeneration.

We pick local coordinates x and y on Σ' and Σ'', respectively. The gluing of Σ' and Σ'' at general points $x = a$ and $y = b$, and the smoothing to make a smooth surface Σ_0, is described by a slight generalization of eqn. (6.1):

$$(6.7) \qquad\qquad (x - a)(y - b) = q.$$

We interpret the parameters a, b, and q as all representing moduli of Σ_0. We will work to first order in $x - a$, $y - b$, and q.

Describing the gluing by the explicit equation (6.7) has required a choice of local holomorphic coordinates $x - a$ and $y - b$ near the points $a \in \Sigma_1$ and $b \in \Sigma_2$. The factors $x - a$ and $y - b$ in eqn. (6.7) are certainly not invariant under local reparametrization of Σ_1 and Σ_2. However, since we work to first order in $x - a$ and in $y - b$, the factors $x - a$ and $y - b$ transform linearly under changes of coordinates and in fact they transform like[21] the 1-forms $\mathrm{d}a$ and $\mathrm{d}b$. So we see that q must transform as the product $\mathrm{d}a \otimes \mathrm{d}b$.

A useful formulation of this is as follows. As in our discussion above, we define a line bundle $\mathcal{L}' \to \widehat{\mathcal{M}}_{g_1,n_1+1}$ whose fiber is the fiber of the cotangent bundle $K_{\Sigma'}$ at $x = a$, and a line bundle $\mathcal{L}'' \to \widehat{\mathcal{M}}_{g_1,n_1+1}$ whose fiber is the fiber of the cotangent bundle $K_{\Sigma''}$ at $y = b$. So $\mathrm{d}a \otimes \mathrm{d}b$ is naturally understood as a section of $\mathcal{L}' \otimes \mathcal{L}''$. Hence the statement that q transforms as $\mathrm{d}a \otimes \mathrm{d}b$ means that we should interpret q as a section of $\mathcal{L}' \otimes \mathcal{L}''$.

Locally, we can view q as a function on $\widehat{\mathcal{M}}_{g,n}$ that vanishes on the compactification divisor $\mathcal{D}$. However, this formulation is not adequate globally. The best way to formulate the statement that q transforms as $\mathrm{d}a \otimes \mathrm{d}b$ is that q should be viewed as a section of the line bundle $\mathcal{L}' \otimes \mathcal{L}'' \to \mathcal{D}$. This is a satisfactory global description of what q means to linear order in q. Beyond linear order, the geometrical meaning of q is more complicated (but usually not needed in practice).

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[21] For example, under $x \to \lambda x$, $a \to \lambda a$, we have $(x - a) \to \lambda(x - a)$, which agrees with the transformation law $\mathrm{d}a \to \lambda \mathrm{d}a$. In algebraic geometry, the cotangent bundle to a variety at a point is defined as the space of functions vanishing at that point modulo those that vanish to second order. This motivates our assertion that $x - a$ tranforms like $\mathrm{d}a$.

Equivalently, let $\mathcal{N}_{\mathcal{D}}$ be the normal bundle to $\mathcal{D}$ in $\widehat{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$; it is a complex line bundle over $\mathcal{D}$. q can be viewed as a linear function on $\mathcal{N}_{\mathcal{D}}$, so the fact that q is a section of $\mathcal{L}' \otimes \mathcal{L}''$ means that there is a natural isomorphism

$$(6.8) \qquad \mathcal{N}_{\mathcal{D}} \cong (\mathcal{L}' \otimes \mathcal{L}'')^{-1}.$$

Later we will describe the super analog of this relation.

6.2. Compactification of supermoduli space

6.2.1. Neveu-Schwarz and Ramond degenerations The moduli space $\mathfrak{M}$ of super Riemann surfaces has a Deligne-Mumford compactification that is quite analogous to the corresponding compactification of the moduli space $\mathcal{M}$ of ordinary Riemann surfaces.[22] Again compactification is achieved by adding divisors $\mathcal{D}_\lambda$ at infinity that parametrize super Riemann surfaces with very simple singularities. The singularities in question describe the collapse of a narrow neck (or the growth of a long tube) and are quite analogous to the familiar singularity (6.1) of an ordinary Riemann surface. Perhaps the main difference from the bosonic case is that there are two types of tube that may collapse. They were described in section 4.1.2 and govern propagation of a closed string in the NS or Ramond sector. Accordingly, in constructing the Deligne-Mumford compactification $\widehat{\mathfrak{M}}$ of $\mathfrak{M}$, we need to allow two different types of degeneration, which we will call Neveu-Schwarz (NS) and Ramond (R) degenerations.

For an NS degeneration, we glue together two copies of $\mathbb{C}^{1|1}$, with local superconformal coordinates $x|\theta$ and $y|\psi$, respectively, by the gluing formulas

$$xy = -\varepsilon^2$$
$$y\theta = \varepsilon\psi$$
$$x\psi = -\varepsilon\theta$$
$$(6.9) \qquad \theta\psi = 0.$$

It is convenient to define

$$(6.10) \qquad q_{\mathrm{NS}} = -\varepsilon^2.$$

The formulas (6.9) obviously reduce to the bosonic gluing relation $xy = q$ if we set the odd coordinates θ, ψ to zero and identify $q = q_{\mathrm{NS}}$. Eqn. (6.9)

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[22]From the standpoint of algebraic geometry, the most detailed reference on this topic is the letter [54], which unfortunately is unpublished. A number of details have been clarified in subsequent letters [55]. See [8] for another perspective via uniformization.

defines a super Riemann surface, since the transformation from $x|\theta$ to $y|\psi$ is superconformal. One may verify this by showing that $D_\theta = (y/\varepsilon)D_\psi$, where as usual $D_\theta = \partial_\theta + \theta\partial_x$, $D_\psi = \partial_\psi + \psi\partial_y$. Alternatively, one may verify the superconformal nature of the mapping from $x|\theta$ to $y|\psi$ in a dual fashion by showing that $\mathrm{d}x - \theta\mathrm{d}\theta$ (whose kernel is generated by D_θ) is a multiple of $\mathrm{d}y - \psi\mathrm{d}\psi$ (whose kernel is generated by D_ψ):

$$(6.11) \qquad \mathrm{d}x - \theta\mathrm{d}\theta = \frac{\varepsilon^2}{y^2}\left(\mathrm{d}y - \psi\mathrm{d}\psi\right).$$

The super Riemann surface Σ described by eqn. (6.9) is smooth as long as $\varepsilon \neq 0$, but it is singular when ε vanishes. The Deligne-Mumford compactification of $\mathfrak{M}$ is achieved precisely by allowing this kind of singularity, as well as an analogous Ramond degeneration that we describe shortly. As long as $\varepsilon \neq 0$, Σ can be identified with the NS supertube (4.7) via $x = e^\varrho$, $\theta = e^{\varrho/2}\zeta$, $y = -\varepsilon^2 e^{-\varrho}$, $\psi = -\varepsilon e^{-\varrho/2}\zeta$. Just as in the bosonic case, we regard Σ as a local description of part of a compact super Riemann surface, valid for example for $|x|, |y| < 1$. The length of the supertube is then $\tau = -\ln|q_{\mathrm{NS}}|^{-1}$, as in the bosonic case. In the coordinates used in (6.9), what happens at $\varepsilon = 0$ is that Σ splits up into two components, one parametrized by $x|\theta$ and one by $y|\psi$, and glued at $x = y = 0$. In the coordinate system $\varrho|\zeta$, one says instead that as $\varepsilon \to 0$, the length of the supertube diverges.

To describe a Ramond degeneration, we again introduce two copies of $\mathbb{C}^{1|1}$ with local coordinates $x|\theta$ and $y|\psi$. But now we provide each copy of $\mathbb{C}^{1|1}$ with a superconformal structure that degenerates at the divisor $x = 0$ or $y = 0$, representing the Ramond punctures. These singular superconformal structures are defined, as in (4.4), by the odd vector fields

$$(6.12) \qquad D_\theta^* = \frac{\partial}{\partial\theta} + \theta x \frac{\partial}{\partial x}, \quad D_\psi^* = \frac{\partial}{\partial\psi} + \psi y \frac{\partial}{\partial y}.$$

The gluing formulas are now simply

$$(6.13) \qquad \begin{aligned} xy &= q_{\mathrm{R}} \\ \theta &= \pm\sqrt{-1}\psi. \end{aligned}$$

Obviously, these formulas reduce to the bosonic gluing formula $xy = q$ if we set the odd variables to zero and set $q = q_{\mathrm{R}}$.

This gluing defines a super Riemann surface Σ, since $D_\theta^* = \mp\sqrt{-1}D_\psi^*$. Alternatively, in a dual language,

$$(6.14) \qquad \mathrm{d}x - x\theta\mathrm{d}\theta = -\frac{q_{\mathrm{R}}}{y^2}\left(\mathrm{d}y - y\psi\mathrm{d}\psi\right).$$

For $q_R \neq 0$, we can map Σ to the Ramond supertube (4.8) by setting $x = e^\varrho$, $y = q_R e^{-\varrho}$, leaving θ unchanged. So the behavior of Σ for $q_R \to 0$ can be described either as the collapse of a narrow neck or as the divergence of the length of a Ramond supertube.

6.2.2. The divisors at infinity   The purely bosonic analysis of sections 6.1.2 extends to the super Riemann surface case, with just one or two surprises.

In general we compactify the moduli space $\mathfrak{M}_{g,n_{NS},n_R}$ that parametrizes genus g super Riemann surfaces Σ with n_{NS} Neveu-Schwarz punctures and n_R Ramond punctures, by allowing singularities of the types described in section 6.2.1. It is a rather fundamental fact that compactification can be achieved by allowing singularities of only these types. We call the compactification $\widehat{\mathfrak{M}}_{g,n_{NS},n_R}$.

Compactification does not affect the dimension of the moduli space, which is

$$(6.15) \qquad \dim \widehat{\mathfrak{M}}_{g,n_{NS},n_R} = 3g - 3 + n_{NS} + n_R | 2g - 2 + n_{NS} + \frac{1}{2} n_R.$$

The divisors at infinity in $\widehat{\mathfrak{M}}_{g,n_{NS},n_R}$ describe the possible degenerations of Σ. For similar reasons to the bosonic case, to develop a simple theory, we exclude the values $g = 0$, $n_{NS} + n_R < 3$ and $g = 1$, $n_{NS} = n_R = 0$, at which Σ would have continuous automorphisms.[23] And similarly to the bosonic case, in constructing $\widehat{\mathfrak{M}}$, all punctures remain distinct; sequences of super Riemann surfaces in which two or more punctures (of either type) approach each other are assigned limits in which Σ splits off a genus zero component.

As in the bosonic case, Σ can have separating or nonseparating degenerations, and of course its degenerations are of NS and R type. By obvious analogy with what we said in section 6.1.2, the divisor in $\widehat{\mathfrak{M}}_{g,n_{NS},n_R}$ that parametrizes nonseparating degenerations of NS type is a copy of $\widehat{\mathfrak{M}}_{g-1,n_{NS}+2,n_R}$. This is compatible with the dimension formula (6.15), which shows that

$$(6.16) \qquad \dim \widehat{\mathfrak{M}}_{g,n_{NS},n_R} = \dim \widehat{\mathfrak{M}}_{g-1,n_{NS}+2,n_R} + 1|0,$$

so that $\widehat{\mathfrak{M}}_{g-1,n_{NS}+2,n_R}$ can be a divisor in $\widehat{\mathfrak{M}}_{g,n_{NS},n_R}$.

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[23]The exceptional cases can of course be treated by hand, but they are essentially not needed in superstring perturbation theory. In any genus, to compute scattering amplitudes, one needs n-point functions with $n > 0$. The starting point of perturbation theory is a superconformal field theory, which means that at the outset one knows what happens for $g = 0$, $n_{NS} + n_R \leq 2$.

Similarly, one might think at first that the divisor $\mathcal{D} \subset \widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}}$ corresponding to a nonseparating degeneration of R type would be a copy of $\widehat{\mathfrak{M}}_{g-1,n_{\mathrm{NS}},n_{\mathrm{R}}+2}$. However, the dimension formula shows that this cannot be correct. It gives

$$(6.17) \qquad \dim \widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}} = \dim \widehat{\mathfrak{M}}_{g-1,n_{\mathrm{NS}},n_{\mathrm{R}}+2} + 1|1,$$

so that the two dimensions differ by $1|1$ and not by $1|0$ as would be the case if $\widehat{\mathfrak{M}}_{g-1,n_{\mathrm{NS}},n_{\mathrm{R}}+2}$ were a divisor in $\widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}}$. It turns out that the divisor $\mathcal{D}$, rather than being isomorphic to $\widehat{\mathfrak{M}}_{g-1,n_{\mathrm{NS}},n_{\mathrm{R}}+2}$, is a fiber bundle over that space with fibers of dimension $0|1$:

$$(6.18) \qquad \begin{array}{ccc} \mathbb{C}^{0|1} & \to & \mathcal{D} \\ & & \downarrow \\ & & \widehat{\mathfrak{M}}_{g-1,n_{\mathrm{NS}},n_{\mathrm{R}}+2}. \end{array}$$

The structure group of this fibration acts on the fiber by $\theta \to \pm\theta + \alpha$, with α an odd parameter. (It is apparently not known if the fibration has a holomorphic section.)

The origin of this fibration is as follows. The divisor $\mathcal{D}$ parametrizes singular super Riemann surfaces in which two Ramond divisors $\mathcal{F}$ and $\mathcal{F}'$ are glued together. Locally we can describe a neighborhood of $\mathcal{F}$ by coordinates $x|\theta$ and superconformal structure $D_\theta^* = \partial_\theta + \theta x \partial_x$, where $\mathcal{F}$ is defined by $x = 0$; similarly, in suitable coordinates $y|\psi$ with superconformal structure $D_\psi^* = \partial_\psi + \psi y \partial_y$, $\mathcal{F}'$ is defined by $y = 0$. Naively speaking, the gluing of $\mathcal{F}$ to $\mathcal{F}'$ is made by setting $\theta = \pm\sqrt{-1}\psi$. (This is what eqn. (6.13) amounts to at $q_{\mathrm{R}} = 0$, that is, along $\mathcal{D}$.) However, a fact that was already important in section 4.3 was that when restricted to $\mathcal{F}$, a superconformal vector field may not vanish but may restrict to a multiple of ∂_θ, generating the symmetry $\theta \to \theta + \alpha$ of $\mathcal{F}$. So once we have selected the two Ramond divisors $\mathcal{F}, \mathcal{F}'$ in a genus $g - 1$ super Riemann surface Σ' that we want to glue together to make a singular genus g super Riemann surface Σ, the gluing that makes Σ is not uniquely determined. The general possible gluing is

$$(6.19) \qquad \theta = -\alpha \pm \sqrt{-1}\psi,$$

where the fermionic parameter α parametrizes the fiber of the fibration (6.18). (The meaning of the sign in $\pm\sqrt{-1}\psi$ will be explained in section 6.2.3.)

An important point is that the distinguished fermionic parameter α that enters the fibration (6.18) only exists when we restrict from the full moduli space to the divisor $\mathcal{D}$. Away from $q = 0$, this parameter cannot be separated from the rest of the odd and even moduli.

This fermionic gluing parameter has the following significance for string theory. We will describe this a little informally, as if it is only necessary to consider holomorphic degrees of freedom. (In fact, the following comments apply to open superstrings, and with minor and standard modifications to closed superstrings.) In bosonic string theory, near a compactification divisor, there is a gluing parameter q. The integral over q gives the bosonic string propagator

$$(6.20) \qquad\qquad \frac{1}{L_0}.$$

With $L_0 = p^2/2 + N$, where p is the momentum and N has integer eigenvalues, the poles of $1/L_0$ lead to the singularities of superstring scattering amplitudes. Almost the same thing happens for the NS sector of superstrings. Here the gluing parameter is $\varepsilon = \pm\sqrt{-q_{\mathrm{NS}}}$. The significance of the sign will be discused in section 6.2.3. Leaving this aside for the moment, the integration over q_{NS} again produces the propagator $1/L_0$, leading again to singularities of the scattering amplitudes due to bosonic intermediate states.

Now consider a Ramond degeneration. The integral over q_{R} still gives a factor of $1/L_0$. But now in integrating over $\mathcal{D}$, we can integrate first over the fibers of the fibration (6.18), which means that we integrate first over the fermionic gluing parameter α. The integral over α gives a factor of G_0, the global worldsheet supercharge that obeys $G_0^2 = L_0$. The propagator for Ramond states is therefore

$$(6.21) \qquad\qquad \frac{G_0}{L_0} = \frac{1}{G_0}.$$

G_0, which is sometimes called the Dirac-Ramond operator, was originally introduced in [56] as a stringy analog of the Dirac operator, and so $1/G_0$ is the analog for string theory of the Dirac propagator of field theory.

We still must consider separating degenerations, but there are no further surprises. A divisor in $\widehat{\mathcal{M}}_{\mathsf{g},\mathsf{n}_{\mathrm{NS}},\mathsf{n}_{\mathrm{R}}}$ representing a separating NS degeneration is isomorphic to a product $\widehat{\mathfrak{M}}_{\mathsf{g}_1,\mathsf{n}_1,\mathsf{m}_1} \times \widehat{\mathfrak{M}}_{\mathsf{g}_2,\mathsf{n}_2,\mathsf{m}_2}$, with

$$(6.22) \quad \mathsf{g}_1 + \mathsf{g}_2 = \mathsf{g}, \quad \mathsf{n}_1 + \mathsf{n}_2 = \mathsf{n}_{\mathrm{NS}} + 2, \quad \mathsf{m}_1 + \mathsf{m}_2 = \mathsf{n}_{\mathrm{R}}, \quad \mathsf{m}_1, \mathsf{m}_2 \in 2\mathbb{Z}.$$

This is as one would expect by analogy with the bosonic case: a surface of genus g splits into two components of genera g_1 and g_2 that add to g; the punctures originally present are distributed between the two sides; and 1 extra NS puncture appears on each component, where they are glued together

to make Σ. By contrast, a divisor in $\widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}}$ that represents a separating Ramond degeneration is a fiber bundle, with fibers of dimension $0|1$, over a product $\widehat{\mathfrak{M}}_{g_1,n_1,m_1} \times \widehat{\mathfrak{M}}_{g_2,n_2,m_2}$, now with

$$(6.23) \quad g_1 + g_2 = g, \quad n_1 + n_2 = n_{\mathrm{NS}}, \quad m_1 + m_2 = n_{\mathrm{R}} + 2, \quad m_1, m_2 \in 2\mathbb{Z}.$$

The fiber of dimension $0|1$ enters the choice of gluing.

One moral of this story is that we must be careful in counting gluing parameters. At an NS degeneration, there is from any point of view just one even bosonic gluing parameter ε and no odd ones. So the space of gluing parameters is of dimension $1|0$. At a Ramond degeneration, there are two slightly different but natural questions. The locus in $\widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}}$ that parametrizes super Riemann surfaces with a Ramond degeneration is a divisor, of codimension $1|0$, with a single even gluing parameter q_{R}. But if one is interested in constructing this divisor from similar moduli spaces that parametrize surfaces with smaller genus or more components, one must take account also of the fermionic gluing parameter α. So for some purposes, it is better to think of the gluing parameters at a Ramond degeneration as being of dimension $1|1$.

6.2.3. Gluing and the GSO projection   We have encountered an important minus sign for both NS and R degenerations. Gluing of ordinary Riemann surfaces is described by a gluing parameter q that enters a formula $xy = q$. In the super Riemann surface version of this gluing, we need in the NS case to take a square root, since the gluing parameter in eqn. (6.9) is $\varepsilon = \pm\sqrt{-q_{\mathrm{NS}}}$. Alternatively, in the Ramond case, the fermionic gluing law (6.13) directly involves a sign, $\theta = \pm\sqrt{-1}\psi$. In each case, this minus sign appears in the gluing law between the fermionic coordinates θ and ψ on the two branches, via either $\theta = \pm\sqrt{-q_{\mathrm{NS}}}\psi/y$ or $\theta = \pm\sqrt{-1}\psi$.

This two-valuedness involves spin structures and the GSO projection. Consider a nonseparating degeneration, at which Σ can be constructed by gluing together two points or divisors in a super Riemann surface Σ' of genus $g - 1$. The topological picture is the same as in the purely bosonic case (fig. 3(a)). Prior to degenerating, Σ has 2^{2g} spin structures, while Σ' has only $2^{2(g-1)}$ spin structures.[24] We want to understand explicitly why Σ has 4 times as many spin structures as Σ'. This can conveniently be described in terms of

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[24]In the presence of Ramond punctures on either Σ or Σ', the relevant objects are not spin structures but generalized spin structures as described in section 4.2.4. This does not materially affect the following counting, so we will use the more familiar language of spin structures.
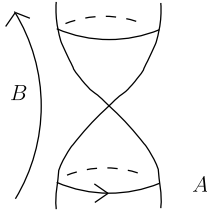
Figure 8: Here we depict the local behavior of a super Riemann surface Σ of genus $\mathfrak{g}$ at a nonseparating degeneration. We can pick an A-cycle of Σ that surrounds the node and a B-cycle that passes through the node, as shown here. (The other $\mathfrak{g} - 1$ A-cycles and B-cycles of Σ can be chosen not to pass near the node. They play no role in the analysis.)

a 1-cycle A on Σ that encloses the node or narrow neck and another 1-cycle B that passes through it, as depicted in fig. 8. In comparing spin structures on Σ to those on Σ', there are 2 factors of 2 that arise as follows:

(1) Spin structures on Σ can be partially classified by whether they are of bounding or unbounding type when restricted to the cycle A – in other words whether the string propagating through the node is in the NS or Ramond sector. The two cases correspond to NS and R degenerations of Σ, respectively. On Σ', this distinction makes sense but is not part of the choice of spin structure.

(2) In addition, Σ has pairs of spin structure that differ in their "type" (NS or R) when restricted to B, but coincide in their restriction to A or to any 1-cycle that does not pass through the neck. These pairs of spin structures are exchanged by including a minus sign in the gluing between fermionic variables on the two sides of the neck. In other words, they are exchanged if one reverses the sign in the formulas $\theta = \pm\sqrt{-q_{\mathrm{NS}}}\psi/y$ or $\theta = \pm\sqrt{-1}\psi$. Locally (and in general even globally[25]) there is no way to say which of these gluings is which, since one is free to reverse the sign of either θ or ψ. But there is a canonical operation on spin structures of exchanging this sign. On Σ', one does not make this gluing so this factor of 2 in the number of spin structures is absent.

In general, as explained in section 2.2.1, it does not make sense to say that two super Riemann surfaces are the same except for the choice of spin struc-

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[25]In the case of a Ramond degeneration, in the absence of Ramond punctures, one can make a distinguished choice of sign by asking that the spin structure on Σ should be even (or odd). Our discussion in the text uses only what can be seen locally without this sort of global information.

ture. A very limited but crucial exception[26] comes from the two-valuedness we have just described: when q_{NS} or q_{R} vanishes, there are pairs of spin structure that differ only by their "type" along a cycle B that passes through the node. Singularities of superstring scattering amplitudes come from the contribution of an on-shell state passing through the node. So in evaluating such singularities, one can sum over pairs of spin structures that differ only by their "type" along B or in other words by the sign in the gluing between θ and ψ. (Moreover, this partial sum over spin structures can be performed before the rest of the integral that determines a given singular contribution to a scattering amplitude.) To understand the implications of this for string theory, consider a string state propagating through the narrow neck. If we act on this string state with the operator $(-1)^F$ that counts world sheet fermions mod 2, this will have the same effect as changing the sign of the fermionic coordinate in the gluing. So averaging over the two possible signs of the gluing amounts to the action of the GSO projection operator $(1 + (-1)^F)/2$. What we have just explained is the reason that only states that are invariant under the GSO projection contribute singularities in superstring scattering amplitudes.

6.2.4. Application to Type II superstrings All of this has been stated purely holomorphically. A Type II worldsheet is defined by, in a sense, combining holomorphic and antiholomorphic super Riemann surfaces. Its degenerations are described by applying what we have learned separately to the holomorphic and antiholomorphic variables. In particular, the degeneration of either the holomorphic or antiholomorphic variables can be of either NS or R type, so there are four types of degeneration, which we can call NS-NS, NS-R, R-NS, and R-R. This matches the four types of punctures in Type II superstring theory. If Σ develops a degeneration of, say, NS-R type, then after separating the two branches to make a smooth surface (of lower genus or with more components) a puncture of NS-R type will appear on each branch.

An important fact carries over from the analysis of gluing parameters in section 6.2.2. An NS-NS degeneration has only its holomorphic and antiholomorphic gluing parameters q_{NS} and $\widetilde{q}_{\mathrm{NS}}$ (which are complex conjugates modulo the odd variables). But an NS-R or R-NS puncture has, in the same sense described in section 6.2.2, a single fermionic gluing parameter. Integration over this gluing parameter causes the propagator of an R-NS or NS-R string state to be Dirac-like. And an R-R degeneration has a pair of fermionic gluing parameters, one holomorphic and one antiholomorphic. Integrating over these parameters has an effect that for massless R-R fields can be described

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[26]Another exception, though not relevant here, is that if Σ is split, then it makes sense to change its spin structure without changing anything else.

as follows (see, for example, [31], section 6.2.2). These fields are p-form fields in spacetime and the integration over the fermionic gluing parameters ensures that in superstring perturbation theory, they couple only via their field strength.

The sign choice analyzed in section 6.2.3 occurs separately for holomorphic and antiholomorphic variables, and leads to separate GSO projections for holomorphic and antiholomorphic modes of the string.

6.3. The canonical parameters

6.3.1. Overview The most delicate issues in superstring perturbation theory concern the behavior in the infrared region, near infinity in moduli space. A key fact in resolving them is that moduli space $\mathfrak{M}$ can be compactified by adding suitable divisors with a natural physical interpretation.

By definition a divisor is defined locally by vanishing of a single even parameter. For an NS degeneration, this is the parameter ε in eqn. (6.9), and for a Ramond degeneration, it is the parameter q_{R} in eqn. (6.13).

We call ε and q_{R} the canonical parameters, but they actually are properly understood not as complex-valued parameters but as sections of certain line bundles, the duals of the normal bundles to the relevant compactification divisors in the compactified moduli space $\widehat{\mathfrak{M}}$. This was analyzed in the bosonic case in section 6.1.3, and we describe the superanalog in section 6.3.4.

Picking local superconformal coordinates trivializes the relevant line bundles and then, as in eqn. (6.9) or (6.13), ε or q_{R} is a natural complex-valued parameter. Under a change of trivialization, one would have $\varepsilon \to e^{\phi}\varepsilon$ for some function ϕ (and similarly for q_{R}), but not, for example, $\varepsilon \to \varepsilon + \eta_1\eta_2$ with odd moduli η_1 and η_2.

The existence in this sense of a canonical parameter at infinity is key to resolving some questions that arose in the literature of the 1980's on superstring perturbation theory. It makes well-defined some integrals that are invariant under a change of variables $\varepsilon \to e^{\phi}\varepsilon$, but would shift by a surface term if one were allowed to make a change of variables $\varepsilon \to \varepsilon + \eta_1\eta_2$. See [31].

6.3.2. Example in genus zero To gain experience, we will compute explicitly the canonical parameter in a simple special case. Let Σ be a super Riemann surface and consider the degeneration in which two NS punctures approach each other. In local superconformal coordinates $z|\theta$ on Σ, we take the punctures to be at $z|\theta = z_1|\theta_1$ and $z|\theta = z_2|\theta_2$.

We want to determine the canonical parameter for $z_1|\theta_1 \to z_2|\theta_2$. Even without any computation, one may suspect that ε will be a function of the
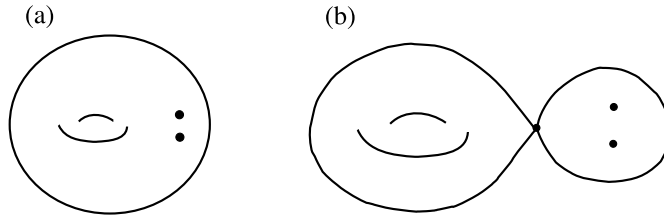
Figure 9: (a) A super Riemann surface Σ with 2 nearby NS punctures labeled p_1 and p_2. (b) At infinity, Σ splits as the union of a super Riemann surface Σ' that is a copy of Σ with one less puncture, and a genus 0 component Σ'' that has three NS punctures, namely p_1, p_2, and a third one at which Σ'' is glued to Σ'.

combination $z_1 - z_2 - \theta_1\theta_2$, which is invariant under the superconformal transformation $z|\theta \to z - \alpha\theta|\theta + \alpha$.

Σ has 2 even moduli and 2 odd moduli that will be relevant in this discussion, namely z_1, z_2 and θ_1, θ_2. At infinity, Σ splits into a union of 2 components Σ' and Σ'', where Σ' is simply Σ with just one NS puncture that is relevant to our discussion (namely the one at which it is glued to Σ''; there may be some irrelevant ones far away), and Σ'' is a copy of $\mathbb{CP}^{1|1}$ with three NS punctures (counting the node at which it is glued to Σ'). This is depicted in fig. 9. In terms of the gluing of Σ' and Σ'', we count parameters as follows: the one NS puncture of Σ' depends on 1 odd and 1 even parameter; there will be 1 even gluing parameter ε; and $\mathbb{CP}^{1|1}$ with 3 NS punctures has 1 odd modulus. Altogether the number of parameters is again 2|2.

We describe $\Sigma'' = \mathbb{CP}^{1|1}$ by superconformal coordinates $y|\psi$ (this description omits a divisor at $y = \infty$) and locate the two punctures as follows:

(6.24)
$$z|\theta = z_1|\theta_1 \;\leftrightarrow\; y|\psi = 1|0$$
$$z|\theta = z_2|\theta_2 \;\leftrightarrow\; y|\psi = -1|0.$$

We glue Σ' to Σ'' by identifying the points $z|\theta = z_0|\theta_0$ with $y|\psi = 0|\beta$, where $z_0|\psi_0$ are the moduli of the node in Σ' and β is the odd modulus of Σ''. The gluing formulas are obtained from (6.9) upon replacing $x|\theta$ with superconformal coordinates $z - z_0 + \theta_0\theta|\theta - \theta_0$ and replacing $y|\psi$ with superconformal coordinates $y + \beta\psi|\psi - \beta$:

$$(z - z_0 + \theta_0\theta)(y + \beta\psi) = -\varepsilon^2$$
$$(y + \beta\psi)(\theta - \theta_0) = \varepsilon(\psi - \beta)$$

$$(z - z_0 + \theta_0\theta)(\psi - \beta) = -\varepsilon(\theta - \theta_0)$$
$$(\theta - \theta_0)(\psi - \beta) = 0.$$

(6.25)

In view of (6.24), we determine $z_1|\theta_1$ by setting $y|\psi = 1|0$ in (6.25):

$$z_1 = z_0 - \varepsilon^2 + \theta_0\varepsilon\beta$$
$$\theta_1 = \theta_0 - \varepsilon\beta.$$

(6.26)

Similarly, setting $y|\psi = -1|0$, we get

$$z_2 = z_0 + \varepsilon^2 - \theta_0\varepsilon\beta$$
$$\theta_2 = \theta_0 + \varepsilon\beta.$$

(6.27)

Eliminating $z_0|\theta_0$ and solving for ε in terms of $z_1|\theta_1$ and $z_2|\theta_2$, we get the expected result, up to a constant factor:

(6.28)
$$\varepsilon^2 = -\frac{1}{2}\left(z_1 - z_2 - \theta_1\theta_2\right).$$

This shows that $z_1 - z_2 - \theta_1\theta_2$ (or rather its square root, to account for the two ways of gluing the spin structures) is the good parameter at infinity. For example, this accounts for why in section 5.1.3 the divisors $\mathcal{D}_{ij}$ at infinity were defined by $z_i - z_j - \theta_i\theta_j = 0$.

The factor $-1/2$ in (6.28) has no particular significance, since it can certainly be removed by $\varepsilon \to e^\phi\varepsilon$; what is significant is the combination $z_1 - z_2 - \theta_1\theta_2$.

6.3.3. Comparing the different types of degeneration Before proceeding to a technical analysis of normal bundles, we will explain what sort of information is actually most useful for superstring perturbation theory.

Let us compare the different types of oriented closed string degeneration. For ordinary Riemann surfaces, we have the familiar

(6.29)
$$xy = q.$$

For Ramond degenerations of super Riemann surfaces, we have a minimal superextension of this

$$xy = q_{\mathrm{R}}$$
$$\theta = \pm\sqrt{-1}\psi.$$

(6.30)

And the NS degeneration is more elaborate:

$$xy = -\varepsilon^2 \equiv q_{\mathrm{NS}}$$
$$y\theta = \varepsilon\psi$$
$$x\psi = -\varepsilon\theta$$

(6.31) $$\theta\psi = 0.$$

These formulas have an obvious similarity, and in a certain sense, modulo the odd variables, the supersymmetric degenerations (6.30) and (6.31) reduce to the bosonic degeneration (6.29) with $q = q_{\mathrm{R}}$ or $q = q_{\mathrm{NS}}$. The simplest explanation of what this means is just the following. If Σ is a super Riemann surface with a degeneration that in local coordinates is described by (6.30) or (6.31), and we use the reduced set of local coordinates on the reduced space Σ_{red}, then Σ_{red} has a degeneration described by (6.29) with $q = q_{\mathrm{R}}$ or $q = q_{\mathrm{NS}}$.

It is better to give a formulation that does not depend on local coordinates. For this, we should focus not on the parameters q, q_{R}, and ε but on the line bundles in which they take values. These line bundles are dual to the normal bundles to the appropriate compactification divisors; the relevant line bundle was analyzed in the bosonic case in section 6.1.3. Let us write $\mathcal{D}$ for a compactification divisor in the bosonic case and $\mathcal{D}_{\mathrm{R}}$ and $\mathcal{D}_{\mathrm{NS}}$ for its two types of supersymmetric cousin. To be able to sensibly compare q, q_{R}, and ε, we need a relation between the normal bundles $\mathcal{N}_{\mathcal{D}}$, $\mathcal{N}_{\mathcal{D}_{\mathrm{R}}}$, and $\mathcal{N}_{\mathcal{D}_{\mathrm{NS}}}$. Since these are line bundles over different spaces, how can we compare them? The answer is that the reduced spaces $\mathcal{D}_{\mathrm{R,red}}$ and $\mathcal{D}_{\mathrm{NS,red}}$ are locally isomorphic to $\mathcal{D}$. $\mathcal{D}$ parametrizes a pair of Riemann surfaces joined at a point (or a Riemann surface with two points joined together) with some additional punctures, while $\mathcal{D}_{\mathrm{R,red}}$ and $\mathcal{D}_{\mathrm{NS,red}}$ describe the same data together with some discrete information (the labeling of the punctures and a generalized spin structure) that does not affect the analysis of the normal bundle. If we simplify the notation by ignoring the extra discrete information, then the relation between the normal bundles is that $\mathcal{N}_{\mathcal{D}_{\mathrm{R}}}$ when restricted to $\mathcal{D}_{\mathrm{R,red}}$ is naturally isomorphic to $\mathcal{N}_{\mathcal{D}}$

(6.32) $$\mathcal{N}_{\mathcal{D}_{\mathrm{R}}}\big|_{\mathcal{D}_{\mathrm{R,red}}} \cong \mathcal{N}_{\mathcal{D}},$$

and similarly in the NS case

(6.33) $$\mathcal{N}^2_{\mathcal{D}_{\mathrm{NS}}}\big|_{\mathcal{D}_{\mathrm{NS,red}}} \cong \mathcal{N}_{\mathcal{D}}.$$

These isomorphisms identify q_{R} or q_{NS} with q.

6.3.4. Normal bundle at infinity: NS case   Now we turn to the analysis of the normal bundles to the compactification divisors in $\widehat{\mathfrak{M}}_{\mathfrak{g},\mathfrak{n}_{NS},\mathfrak{n}_R}$, or equivalently, the global meaning of the gluing parameters. For brevity in the notation, we consider divisors that parametrize separating degenerations. All statements have immediate analogs for the nonseparating case. A super Riemann surface Σ can degenerate to a pair of components Σ' and Σ'' joined at a pair of either NS or R punctures. To distinguish the two cases, we will denote the divisors in question as $\mathcal{D}_{NS}$ and $\mathcal{D}_R$. In the NS case, which we consider first, $\mathcal{D}_{NS}$ is the product of the moduli spaces that parametrize Σ' and Σ'' with their punctures. To minimize notation, we write just $\mathcal{D}_{NS} = \widehat{\mathfrak{M}}(\Sigma') \times \widehat{\mathfrak{M}}(\Sigma'')$.

We will need the super analog of a definition explained at the beginning of section 6.1.3. A fundamental line bundle on a super Riemann surface Σ is its Berezinian $Ber(\Sigma)$. It has rank $0|1$; that is, its fibers are fermionic. Consider the moduli space $\widehat{\mathfrak{M}}_{\mathfrak{g},\mathfrak{n}_{NS},\mathfrak{n}_R}$ that parametrizes a super Riemann surface Σ with the given number of NS and R punctures. The NS punctures are simply points in Σ. Taking the fiber of $Ber(\Sigma)$ at the σth NS puncture, for $\sigma = 1, \ldots, \mathfrak{n}_{NS}$, we get a line bundle $\mathfrak{L}_\sigma \to \widehat{\mathfrak{M}}_{\mathfrak{g},\mathfrak{n}_{NS},\mathfrak{n}_R}$, again with fermionic fibers. (The $\mathfrak{L}_\sigma$ do not quite have a close analog for Ramond punctures.)

The relation between $\mathfrak{L}_\sigma$ and the corresponding line bundle $\mathcal{L}_\sigma$ that we defined in the bosonic case in section 6.1.3 is that if we restrict to the reduced space of $\mathfrak{M}_{\mathfrak{g},\mathfrak{n}_{NS},\mathfrak{n}_R}$ and pull $\mathcal{L}_\sigma$ back from $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$ to $\mathcal{M}_{\mathfrak{g},\mathfrak{n},\mathrm{spin}}$, then

$$(6.34) \qquad \mathfrak{L}_\sigma^2|_{\mathfrak{M}_{\mathfrak{g},\mathfrak{n}_{NS},\mathfrak{n}_R,\mathrm{red}}} \cong \mathcal{L}_\sigma.$$

This just reflects the fact that if Σ is split, then $Ber(\Sigma)$, when restricted to Σ_{red}, is $K_{\Sigma_{\mathrm{red}}}^{1/2}$, while $\mathcal{L}_\sigma$ is defined using K_Σ. Eqn. (6.34) will imply the desired result (6.33), once we express the normal bundle $\mathcal{N}_{\mathcal{D}_{NS}}$ in terms of the $\mathfrak{L}$'s (eqn. (6.36)).

Now let $x|\theta$ and $y|\psi$ be local superconformal coordinates on Σ' and Σ''. The gluing of Σ' and Σ'' at general points $x|\theta = a|\alpha$ and $y|\psi = b|\beta$, and smoothing to make Σ, is described by a slight generalization of eqn. (6.9):

$$(x - a + \alpha\theta)(y - b + \beta\psi) = -\varepsilon^2$$
$$(y - b + \beta\psi)(\theta - \alpha) = \varepsilon(\psi - \beta)$$
$$(x - a + \alpha\theta)(\psi - \beta) = -\varepsilon(\theta - \alpha)$$
$$(6.35) \qquad\qquad (\theta - \alpha)(\psi - \beta) = 0.$$

We have simply written (6.9) in superconformal coordinates $x - a + \alpha\theta|\theta - \alpha$ and $y - b + \beta\psi|\psi - \beta$. We view the even parameters a, b, ε and the odd

parameters α, β as moduli of Σ. As in the the introductory remarks above, we write $\mathcal{L}' \to \widehat{\mathfrak{M}}(\Sigma')$ for the line bundle whose fiber at the point in $\widehat{\mathfrak{M}}(\Sigma')$ corresponding to Σ' is the fiber of $Ber(\Sigma')$ at $x|\theta = a|\alpha$. Similarly, we write $\mathcal{L}'' \to \widehat{\mathfrak{M}}(\Sigma'')$ for the line bundle defined by the fiber of $Ber(\Sigma'')$ at $y|\psi = b|\beta$.

Describing the gluing by the explicit equation (6.35) has required picking explicit superconformal coordinates $x - a + \alpha\theta|\theta - \alpha$ and $y - b + \beta\psi|\psi - \beta$ near, respectively, $x|\theta = a|\alpha$ and $y|\psi = b|\beta$. The normalization of ε depends on this choice, and therefore, just as in section 6.1.3, ε is best understood as a section of a line bundle over $\mathcal{D}_{\mathrm{NS}}$. We claim that this line bundle is $\mathcal{L}' \otimes \mathcal{L}''$. Equivalently, since ε is a linear function on the normal bundle $\mathcal{N}_{\mathcal{D}_{\mathrm{NS}}}$ to $\mathcal{D}_{\mathrm{NS}} \subset \widehat{\mathfrak{M}}_{g,n_{\mathrm{NS}},n_{\mathrm{R}}}$, we claim

$$(6.36) \qquad \mathcal{N}_{\mathcal{D}_{\mathrm{NS}}} \cong (\mathcal{L}' \otimes \mathcal{L}'')^{-1}.$$

To justify these claims, we observe that under a change of superconformal coordinates near $x|\theta = a|\alpha$, $x - a + \alpha\theta$ transforms like $\mathrm{d}a - \alpha\mathrm{d}\alpha$. (The basic case to consider is that $a|\alpha = 0|0$ and the superconformal change of coordinates is $x|\theta \to \lambda x|\lambda^{1/2}\theta$, with $\lambda \in \mathbb{C}$. Under this transformation, $x - a + \alpha\theta$ and $\mathrm{d}a - \alpha\mathrm{d}\alpha$ are both multiplied by λ.) In view of the relation of the line bundle $(\mathcal{L}')^2$ on $\widehat{\mathfrak{M}}(\Sigma')$ to the line bundle $\mathcal{D}^{-2} \cong Ber(\Sigma')^2$ on Σ', and the interpretation (2.30) of $\mathrm{d}a - \alpha\mathrm{d}\alpha$ as a section of $\mathcal{D}^{-2}$, this means that $x - a + \alpha\theta$ can be interpreted near $\mathcal{D}_{\mathrm{NS}}$ as a section of $(\mathcal{L}')^2$. Similarly $y - b - \beta\theta$ should be interpreted as a section of $(\mathcal{L}'')^2$. So from the first equation in (6.35), ε^2 should be interpreted as a section of $(\mathcal{L}' \otimes \mathcal{L}'')^2$. The fact that ε is a section of $\mathcal{L}' \otimes \mathcal{L}''$ (rather than a more general square root of $(\mathcal{L}' \otimes \mathcal{L}'')^2$) follows from a similar consideration of the second and third equations in (6.35).

6.3.5. The Ramond case   Now let us consider the case of a Ramond degeneration, where Ramond punctures in Σ' and Σ'' are glued and smoothed to make Σ. The Ramond punctures really correspond to divisors $\mathcal{F}' \subset \Sigma'$ and $\mathcal{F}'' \subset \Sigma''$. We pick local coordinates $x|\theta$ on Σ' and $y|\psi$ on Σ'' such that the superconformal structures are defined by the vector fields D_θ^* and D_ψ^* of eqn. (6.12). The divisors $\mathcal{F}'$ and $\mathcal{F}''$ are defined respectively by $x = 0$ and by $y = 0$, and the gluing is as in (6.13):

$$xy = q_{\mathrm{R}}$$
$$(6.37) \qquad \theta = \pm\sqrt{-1}\psi.$$

The gluing of $\mathcal{F}'$ and $\mathcal{F}''$ by the second formula here gives a supermanifold of dimension $0|1$ that we will call $\mathcal{F}^\diamond$; it is isomorphic to either $\mathcal{F}'$ or $\mathcal{F}''$. The first equation makes it clear that in some sense q_{R} transforms as $\mathrm{d}x \otimes \mathrm{d}y$.

In general, $\mathrm{d}x$ and $\mathrm{d}y$ take values in $T^*\Sigma$, which has rank $1|1$. Also, the gluing in (6.37) happens at a divisor, not a point. So we need more information if we want to identify a vector space of dimension $1|0$ in which q_{R} takes values. A splitting of Σ gives one simple answer; if Σ is split, we can choose local parameters x and y that are even under the $\mathbb{Z}_2$ symmetry defined by the splitting, and interpret $\mathrm{d}x$ and $\mathrm{d}y$ as cotangent vectors to Σ_{red}. This is the same interpretation of q_{R} that we had in the bosonic case, so we arrive at (6.32).

If Σ is not split, we have to work harder to describe a $1|0$-dimensional vector space in which q_{R} takes values. $T^*\Sigma'$, when restricted to $\mathcal{F}'$, has a rank 1 subbundle $\mathcal{U}'$ consisting of 1-forms whose contraction with D_θ^* vanishes; similarly, $T^*\Sigma''$, when restricted to $\mathcal{F}''$, has a rank 1 subbundle $\mathcal{U}''$ consisting of 1-forms whose contraction with D_ψ^* vanishes. Since $\mathrm{d}x$ and $\mathrm{d}y$ have vanishing contraction with D_θ^* and D_ψ^* at $x = y = 0$, they are naturally understood as sections of $\mathcal{U}'$ and $\mathcal{U}''$, respectively. So $\mathrm{d}x \otimes \mathrm{d}y$ is a section of $\mathcal{U}' \otimes \mathcal{U}''$.

In general, a section of a line bundle over a supermanifold of dimension $0|1$, parametrized by an odd variable η, can be written $a + b\eta$, so such sections form a vector space of dimension $1|1$. Taking the supermanifold to be $\mathcal{F}^\diamond$ and the line bundle to be $\mathcal{U}' \otimes \mathcal{U}''$, we would say based on this that q_{R} takes values in a vector space of dimension $1|1$. To reduce to a vector space of dimension $1|0$, we define the vector field $W = D_\theta^* \mp \sqrt{-1} D_\psi^*$. The object $\mathrm{d}x \otimes \mathrm{d}y$ lies in a $1|0$-dimensional space $\mathcal{U}^*$ of sections of $\mathcal{U}' \otimes \mathcal{U}'' \to \mathcal{F}^\diamond$ whose Lie derivative with respect to W vanishes at $x = y = 0$. (Recall that the Lie derivative of a form ω with respect to a vector field W is $\mathfrak{L}_W \omega = (\mathbf{i}_W \mathrm{d} + \mathrm{d} \mathbf{i}_W) \omega$, where $\mathbf{i}_W$ is contraction with respect to W. The Lie derivative of the symmetric tensor product $\mathrm{d}x \otimes \mathrm{d}y$ is $(\mathfrak{L}_W \mathrm{d}x) \otimes \mathrm{d}y + \mathrm{d}x \otimes (\mathcal{L}_W \mathrm{d}y)$. After setting $\psi = \mp \sqrt{-1}\theta$, we get $W = 2\partial_\theta + \theta(x\partial_x - y\partial_y)$, and because of the minus sign, $\mathfrak{L}_W(\mathrm{d}x \otimes \mathrm{d}y) = 0$, while $\mathfrak{L}_W(\theta \mathrm{d}x \otimes \mathrm{d}y) \neq 0$.)

$\mathcal{U}^*$ is the fiber of a line bundle over the divisor $\mathcal{D}_{\mathrm{R}}$ that parametrizes a Ramond degeneration. We just call this line bundle $\mathcal{U}^*$. We can interpret q_{R} as taking values in $\mathcal{U}^*$; the normal bundle to $\mathcal{D}_{\mathrm{R}}$ in $\widehat{\mathfrak{M}}$ is $\mathcal{N}_{\mathrm{R}} = (\mathcal{U}^*)^{-1}$.

6.4. Application to the dimension of supermoduli space

Now that we understand how gluing works, we can give a new computation of the dimension of supermoduli space. We will build a super Riemann surface Σ of genus $\mathfrak{g}$ by gluing some minimal building blocks – spheres with 3 punctures. Since the number of Ramond punctures in a compact super Riemann surface is always even, there are two possibilities: a sphere with 3 NS punctures and a moduli space of dimension $0|1$, as described in section 5.1.2, and a sphere with 1 NS puncture, 2 Ramond punctures, and a moduli space of dimension $0|0$,
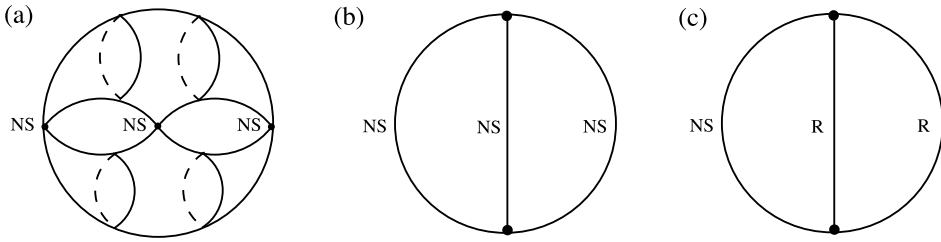
Figure 10: In (a), we sketch the $\mathfrak{g} = 2$ case of a construction described in the text: the gluing of two 3-punctured spheres (with NS punctures only) along 3 double points where they meet. Smoothing the double points into thin tubes that join at 3-holed (rather than 3-punctured) spheres makes a super Riemann surface of genus 2. Sketched in (b) is a dual description of the same gluing. We make a trivalent graph by collapsing each 3-holed sphere in (a) to a vertex and expanding each double point where gluing takes place in (a) to a line between 2 vertices. To get from the trivalent graph in (b) to a Riemann surface, one thickens each line to a thin tube and each vertex to a 3-holed sphere where the tubes join smoothly. (c) shows in the same dual language another way to make a super Riemann surface of genus 2, this time with 2 NS^3 vertices replaced by $\mathrm{NS} \cdot \mathrm{R}^2$ vertices. In (a) or (b), we get 2 odd moduli from the vertices, while in (c), 2 odd moduli come from the lines labeled R.

as described in section 5.1.4. We will refer to the first case as a 3-punctured sphere of type NS^3, and the second case as a 3-punctured sphere of type $\mathrm{NS} \cdot \mathrm{R}^2$. These objects are special because they have no bosonic moduli; a super Riemann surface built by gluing together such objects has no bosonic moduli and hence cannot degenerate further.

First we build a genus $\mathfrak{g}$ super Riemann surface Σ without punctures. One way to do so is to start with $2\mathfrak{g} - 2$ thrice-punctured spheres $\Sigma_1, \ldots, \Sigma_{2\mathfrak{g}-2}$, each of type NS^3. To build Σ, one glues together the Σ_i along their punctures to make a degenerate genus $\mathfrak{g}$ curve with a total of $3\mathfrak{g} - 3$ double points, as shown in fig. 10. (Each of the Σ_i has 3 punctures, so altogether there are $3(2\mathfrak{g} - 2)$ punctures that one glues together pairwise at $3\mathfrak{g} - 3$ double points to build Σ. For what we are about to say, it does not matter how one pairs up the $3(2\mathfrak{g}-2)$ punctures.) Each of the Σ_i has 1 odd modulus and no even ones, prior to gluing, making $2\mathfrak{g} - 2$ odd moduli in all. Gluing and then smoothing the double points adds 1 even gluing parameter and no odd ones at each double point, making $3\mathfrak{g} - 3$ even parameters. So altogether the dimension of the moduli space is $3\mathfrak{g} - 3|2\mathfrak{g} - 2$.
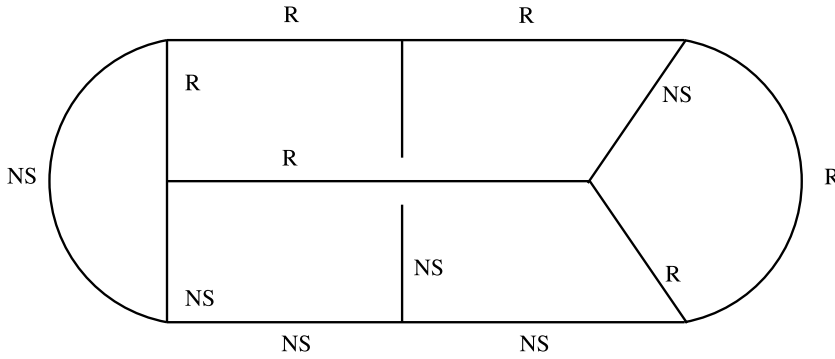
Figure 11: A trivalent graph with 5 loops. Each line is labeled NS or R. There are two kinds of vertex: an NS^3 vertex that is the junction of three NS lines, or an $NS \cdot R^2$ vertex that is the junction of an NS line and 2 R lines. From such a graph, one makes a super Riemann surface Σ of genus 5 by thickening the lines to tubes and the vertices to 3-holed spheres. In this example, there are 2 NS^3 vertices, each contributing $0|1$ to the dimension of the supermoduli space, 6 NS lines, each contributing $1|0$, and 6 R lines, each contributing $1|1$. So the dimension of the moduli space of super Riemann surfaces of genus 5 is $12|8 = 3g - 3|2g - 2$.

More generally, one can build a super Riemann surface of the same genus by taking $2g - 2 - k$ of the Σ_i to be of type NS^3 and k to be of type $NS \cdot R^2$. In the starting point, one has now only $2g - 2 - k$ odd moduli, one for each 3-punctured sphere of type NS^3. But now one will have to glue k pairs of Ramond punctures, and this will give back k odd moduli from the gluing parameters. So one ends up with the same dimension for the moduli space. For an example, compare figs. 10(b) and (c).

As explained in figs. 10 and 11, these gluing operations can be represented by trivalent graphs in which a vertex represents a 3-punctured sphere, while an edge or line between 2 vertices represents the gluing of 2 punctures. (A line connecting a vertex to itself is also allowed.) Each line is labeled NS or R and represents the gluing of NS or R punctures, respectively. There are 2 types of vertex: NS^3 vertices, at which 3 NS lines meet, and $NS \cdot R^2$ vertices, at which 1 NS line and 2 R lines meet. They represent the two kinds of 3-punctured sphere. Given such a graph, one makes a topological type of super Riemann surface Σ by thickening the lines into tubes and the vertices into 3-holed spheres at which the tubes join. (The spin structure of Σ depends on choices made in the gluing.) The counting of moduli of Σ is as follows: the

contribution of an NS^3 vertex is $0|1$; the contribution from an $NS \cdot R^2$ vertex is $0|0$; the contribution from an NS line is $1|0$; and the contribution from an R line is $1|1$. A super Riemann surface of genus g can be represented by any trivalent vertex constructed from these ingredients; irrrespective of the choice of the graph or its labeling, one always arrives at the same result $3g - 3|2g - 2$ for the dimension of the moduli space.

To include punctures in this description, we simply consider trivalent graphs with external lines. (An external line is a line that has only one end attached to a vertex; the other end represents the puncture. See fig. 12 for simple examples of graphs with external lines.) A puncture of NS or R type is represented by an external line labeled NS or R. In the presence of punctures, the dimension of the moduli space is computed by summing the contributions of vertices and internal lines only. The sum gives back the usual dimension formula (4.14).

Graphs such as those we have drawn have an obvious analogy with Feynman diagrams in a field theory with cubic interactions (in fact, $(NS)^3$ and $NS \cdot R^2$ interactions). This analogy is no coincidence; it is part of the mechanism by which string theory turns out to reproduce field theory at long distances. In field theory, the "leading singularity" of a perturbative scattering amplitude (see [57] for a modern explanation and application) is found by putting a maximal possible number of internal propagators on-shell. In superstring theory, this corresponds to the contribution of a maximally degenerate super Riemann surface, built by gluing 3-punctured spheres. After picking a configuration of 3-punctured spheres, to compute the leading singularity, one still has to adjust loop momenta so that all momenta passing through nodes are on-shell. In string theory, this will lead to an infinite sum over all the massive string states that can flow through the various nodes.

7. Open and/or unoriented superstring theories

So far, we have considered only oriented superstring worldsheets without boundary. These are appropriate for constructing perturbation theory for oriented closed superstrings – that is, for the heterotic and Type II superstring theories (without D-branes or orientifold planes in the case of Type II).

Type I superstring theory and more general orientifolds of Type II superstring theory are theories of unoriented strings. In a theory of unoriented strings, the string worldsheet is an unoriented and not necessarily orientable super Riemann surface.

Open as well as closed strings appear in Type I superstring theory and more generally in Type II superstring theory in the presence of D-branes.

(a)

(b)
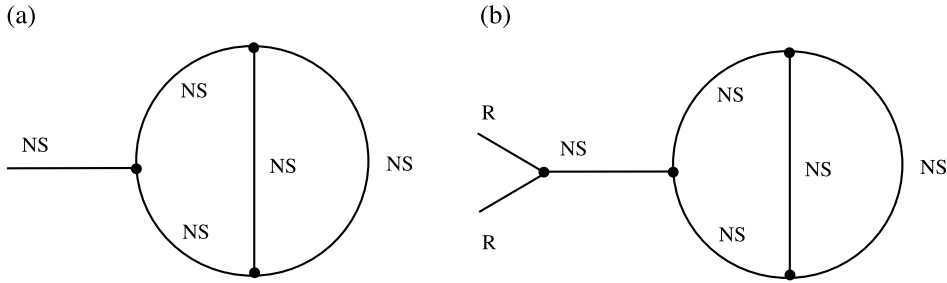
Figure 12: (a) A thickening of this graph gives a super Riemann surface of genus 2 with 1 NS puncture, corresponding to the external line in the graph. The dimension of the moduli space is $4|3$, where we count $1|0$ for each internal NS line and $0|1$ for each NS^3 vertex. (b) A thickening of this graph gives a super Riemann surface of genus 2 with 2 R punctures, corresponding to the external lines. The dimension of the moduli space is $5|3$, with no contribution from the $\mathrm{NS} \cdot \mathrm{R}^2$ vertex.

When open strings are present, the string worldsheet is a super Riemann surface with boundary. Thus, in Type I superstring theory and in more general Type II backgrounds with both orientifolds and D-branes, we must consider super Riemann surfaces that may be simultaneously unorientable and with boundary.

In this section, we explain what is required for these generalizations. For brevity, we refer to a Riemann surface with boundary as an open Riemann surface. A superstring worldsheet is open or unoriented or unorientable if its reduced space is open or unoriented or unorientable. A Riemann surface or superstring worldsheet without boundary is said to be closed.

7.1. The classical picture

7.1.1. The closed oriented double cover By an unoriented Riemann surface, we mean simply a two-manifold with a conformal structure but no choice of orientation.

Suppose that Σ_0 is an unoriented closed Riemann surface. Then Σ_0 has a double cover Σ_0' that is a closed oriented Riemann surface. A point in Σ_0' is a point in Σ_0 together with a choice of orientation at that point.

If Σ_0 is orientable (but unoriented), this definition makes sense but is not terribly interesting. In that case, Σ_0' is just the disjoint union of two oppositely oriented copies of Σ_0. The interesting case is the case that Σ_0 is unorientable. Then Σ_0' is an ordinary connected and oriented Riemann surface. A basic

example is $\Sigma_0 = \mathbb{RP}^2$. The oriented double cover is $\Sigma_0' = S^2 = \mathbb{CP}^1$. The fact that, for a certain action of $\mathbb{Z}_2$ on S^2, the quotient $S^2/\mathbb{Z}_2$ is $\mathbb{RP}^2$ is more or less the definition of $\mathbb{RP}^2$.

We can do something similar for the case that Σ_0 is an oriented Riemann surface with boundary. Taking two copies of Σ_0 with opposite orientation and gluing them together along their boundary, we produce a double cover of Σ_0 (branched over its boundary) which is an oriented Riemann surface Σ_0' without boundary. A basic example is that Σ_0 is the disc $|z| \leq 1$ in the complex plane. Gluing two copies of Σ_0 along their boundary, we build the double cover $\Sigma_0' = \mathbb{CP}^1$.

Finally, we can combine the two cases. Suppose that Σ_0 is an unoriented Riemann surface with boundary. As before Σ_0 has an oriented double cover which we now call Σ_0''; a point in Σ_0'' is a point in Σ_0 with a choice of local orientation. The boundary of Σ_0 is a union of circles, each of which lifts in Σ_0'' to a pair of circles. Gluing together these pairs of circles by identifying corresponding points, we build a closed oriented Riemann surface that we call Σ_0'. For an example, let Σ_0 be $\mathbb{RP}^2$ with an open disc removed, so that its boundary is a circle. Then Σ_0'' is a copy of S^2 with two open balls removed; its boundary is a disjoint union of two circles. Gluing these circles together, we build the closed oriented genus 1 Riemann surface Σ_0'.

Let $\chi(X)$ denote the Euler characteristic of a space X. In all instances of this construction,

$$(7.1) \qquad\qquad\qquad \chi(\Sigma_0') = 2\chi(\Sigma_0).$$

For example, if $\Sigma_0 = \mathbb{RP}^2$, $\Sigma_0' = S^2$, we have $\chi(\Sigma_0) = 1$, $\chi(\Sigma_0') = 2$. Alternatively, if Σ_0 is closed disc and $\Sigma_0' = S^2$, then again $\chi(\Sigma_0) = 1$, $\chi(\Sigma_0') = 2$. As one last example, if Σ_0 is $\mathbb{RP}^2$ minus an open disc, and Σ_0' is a closed Riemann surface of genus 1, then $\chi(\Sigma_0) = \chi(\Sigma_0') = 0$.

Eqn. (7.1) gives a convenient way to compute the genus g' of Σ_0': it obeys $1 - \mathsf{g}' = \chi(\Sigma_0)$ or

$$(7.2) \qquad\qquad\qquad \mathsf{g}' = 1 - \chi(\Sigma_0).$$

The topological type of the oriented surface Σ_0' is completely determined by g'. So all distinct types of Σ_0 with the same Euler characteristic have closed oriented double covers that are topologically equivalent.

In all examples of this construction, Σ_0' has has an orientation-reversing (or equivalently, a complex structure reversing) symmetry ρ of order 2 that exchanges pairs of points in Σ_0' that correspond to the same point in Σ_0 with

different local orientations. The fixed point set of ρ consists of the circles in Σ_0' that lie over the boundary of Σ_0. ρ generates a $\mathbb{Z}_2$ action on Σ_0' and $\Sigma_0 = \Sigma_0'/\mathbb{Z}_2$. A symmetry of order 2 is called an involution, so we call ρ an antiholomorphic involution of Σ_0'. Since $\rho : \Sigma_0' \to \Sigma_0'$ reverses the complex structure, we can think of it as an isomorphism between Σ_0' and its complex conjugate $\overline{\Sigma}_0'$:

$$\rho : \Sigma_0' \cong \overline{\Sigma}_0'. \tag{7.3}$$

7.1.2. The moduli space The moduli space of an open and/or unorientable Riemann surface Σ_0 can be described as follows.

Suppose that the oriented double cover Σ_0' has genus g'. If we ignore the relationship to Σ_0, then the deformations of Σ_0' are parametrized by the usual moduli space $\mathcal{M}_{\mathsf{g}'}$ of Riemann surfaces of genus g.

$\mathcal{M}_{\mathsf{g}'}$ has an antiholomorphic involution φ that maps the point in $\mathcal{M}_{\mathsf{g}'}$ corresponding to a surface Σ_0' to the point that corresponds to the complex conjugate $\overline{\Sigma}_0'$ of Σ_0'. We usually describe this more briefly (but less precisely) by saying that φ maps Σ_0' to $\overline{\Sigma}_0'$. A fixed point of φ corresponds to a Riemann surface Σ_0' such that $\overline{\Sigma}_0'$ is isomorphic to Σ_0'. So Σ_0' corresponds to a fixed point if and only if there is an antiholomorphic involution:

$$\rho : \Sigma_0' \cong \overline{\Sigma}_0'. \tag{7.4}$$

If ρ exists, we define $\Sigma_0 = \Sigma_0'/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by ρ. Then Σ_0 is an open and/or unorientable Riemann surface, and Σ_0' is its close oriented double cover, as described in section 7.1.1. So the fixed point set $\mathcal{M}_{\mathsf{g}'}^\varphi$ of φ parametrizes surfaces Σ_0' that are orientable double covers of some Σ_0.

The only constraint on Σ_0 is that its Euler characteristic must obey (7.2). All topological types of open and/or unorientable Riemann surface Σ_0 that obey this condition can arise. There are finitely many possibilities, parametrized by a set that we will call S. So actually $\mathcal{M}_{\mathsf{g}'}^\varphi$ is a union of finitely many components, one for each $\mathfrak{s} \in S$:

$$\mathcal{M}_{\mathsf{g}'}^\varphi = \cup_{\mathfrak{s} \in S} \Gamma_\mathfrak{s}. \tag{7.5}$$

Here $\Gamma_\mathfrak{s}$ is the moduli space of conformal structures on an open and/or unoriented Riemann surface Σ_0 of topological type $\mathfrak{s}$.

The union in (7.5) is not a disjoint union; the $\Gamma_\mathfrak{s}$ can intersect each other at points corresponding to surfaces Σ_0' that admit more than one antiholomorphic involution. These intersections are not important in superstring perturbation theory; one simply integrates over one or more of the $\Gamma_\mathfrak{s}$ (depending on

which string theory one considers), ignoring the fact that they may intersect each other.

More important is the following. $\mathcal{M}_{\mathfrak{g}'}$ is of course a noncompact manifold (or rather orbifold) of complex dimension $3\mathfrak{g}'-3$. As a component of the fixed point set of a real or antiholomorphic involution, $\Gamma_{\mathfrak{s}}$ has real dimension $3\mathfrak{g}'-3$. It is noncompact, just like $\mathcal{M}_{\mathfrak{g}'}$. The closure of $\Gamma_{\mathfrak{s}}$ in $\widehat{\mathcal{M}}_{\mathfrak{g}'}$ gives a compactification $\widehat{\Gamma}_{\mathfrak{s}}$ of $\Gamma_{\mathfrak{s}}$. In general, $\widehat{\Gamma}_{\mathfrak{s}}$ is a manifold with boundary; typically, these manifolds with boundary join together pairwise on their common boundaries. This is relevant for string perturbation theory and will be described in more detail in section 7.4.

7.2. Open and/or unoriented superstring worldsheets

We now want to describe the analogous construction for open and/or unoriented superstring worldsheets. It turns out that all the cases can be treated together.

The local structure of an open and/or unoriented superstring worldsheet will be the same as the local structure of a Type II superstring worldsheet, as described in section 3.7. However, it seems to be convenient to first describe the integration cycle $\Gamma_{\mathfrak{s}}$ that in superstring perturbation theory parametrizes open and/or unoriented superstring worldsheets of topological type $\mathfrak{s}$. Here $\mathfrak{s}$ is once again a topological type of open and/or unoriented Riemann surface of Euler characteristic $1 - \mathfrak{g}'$. Then we will describe the objects that $\Gamma_{\mathfrak{s}}$ parametrizes.

7.2.1. The integration cycle We start with $\mathfrak{M}_{\mathfrak{g}'}$, the moduli space of closed oriented super Riemann surfaces of genus $\mathfrak{g}'$. Its reduced space is $\mathcal{M}_{\mathfrak{g}',\mathrm{spin}}$, which parametrizes ordinary Riemann surfaces of genus $\mathfrak{g}'$ with a spin structure. By forgetting the spin structure, we get a projection $\pi : \mathcal{M}_{\mathfrak{g}',\mathrm{spin}} \to \mathcal{M}_{\mathfrak{g}'}$. We define $\Gamma_{\mathfrak{s},\mathrm{red}} = \pi^{-1}(\Gamma_{\mathfrak{s}})$, where as in section 7.1.2, $\Gamma_{\mathfrak{s}}$ is the moduli space of conformal structures on an open and/or unoriented surface of topological type $\mathfrak{s}$.

$\Gamma_{\mathfrak{s},\mathrm{red}}$ parametrizes pairs[27] consisting of an open and/or unoriented surface Σ_0 of topological type $\mathfrak{s}$ and a spin structure on the closed oriented double cover Σ_0' of Σ_0. (The spin structure is not necessarily invariant under the antiholomorphic involution of Σ_0'.) It is middle-dimensional in $\mathfrak{M}_{\mathfrak{g}',\mathrm{red}} = \mathcal{M}_{\mathfrak{g}',\mathrm{spin}}$, and in particular it has real dimension $3\mathfrak{g}' - 3$. As usual in constructing integration cycles for superstring perturbation theory, we thicken $\Gamma_{\mathfrak{s},\mathrm{red}}$ in the

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[27]Such pairs have been considered recently in the context of superconformal field theory on open and/or unoriented surfaces [38].

fermionic directions to a smooth supermanifold $\Gamma_{\mathfrak{s}} \subset \mathfrak{M}_{\mathfrak{g}'}$ whose odd dimension is the same as that of $\mathcal{M}_{\mathfrak{g}'}$. So $\Gamma_{\mathfrak{s}}$ is of dimension $3g' - 3|2g' - 2$. $\Gamma_{\mathfrak{s}}$ is the cycle over which we integrate to compute scattering amplitudes of open and/or unoriented supersymmetric strings (or more precisely, the contribution to this scattering amplitude of surfaces of topological type $\mathfrak{s}$). As in section 3.1, $\Gamma_{\mathfrak{s}}$ is not defined uniquely but only up to homology – that is, up to infinitesimal wiggling of the fermionic directions.

7.2.2. The string worldsheet   Now we want to explain what sort of open and/or unoriented superstring worldsheets are parametrized by $\Gamma_{\mathfrak{s}}$. In a sense, we will construct an open and/or unoriented superstring worldsheet Σ as an orientifold (the quotient by an orientation-reversing orientation) of an ordinary Type II superstring worldsheet Σ^*. As in section 3.7, we begin with a product of super Riemann surfaces, but now we take these to be identical,[28] so our starting point is $\mathcal{Y} = \Sigma' \times \Sigma'$, where Σ' is a holomorphic super Riemann surface of genus $\mathfrak{g}'$. Thus $\mathcal{Y}$ is a complex supermanifold of dimension $2|2$. We define a holomorphic involution τ that exchanges the two factors in $\mathcal{Y}$.

Suppose that Σ'_{red} is invariant under an antiholomorphic involution ρ. The automorphisms ρ and τ of $\Sigma'_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}$ generate two $\mathbb{Z}_2$ groups that we will call $\mathbb{Z}_2^\rho$ and $\mathbb{Z}_2^\tau$, respectively. Let Σ^*_{red} be a copy of Σ'_{red} embedded in $\mathcal{Y}_{\mathrm{red}} = \Sigma'_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}$ by

$$(7.6) \qquad\qquad\qquad x \to (x, \rho(x)).$$

τ acts on Σ^*_{red}, mapping $(x, \rho(x))$ to $(\rho(x), x) = (x', \rho(x'))$ where $x' = \rho(x)$. So the action of τ on $\mathcal{Y}$ induces the action of ρ on $\Sigma^*_{\mathrm{red}} \cong \Sigma'_{\mathrm{red}}$:

$$(7.7) \qquad\qquad\qquad \Sigma^*_{\mathrm{red}}/\mathbb{Z}_2^\tau \cong \Sigma'_{\mathrm{red}}/\mathbb{Z}_2^\rho.$$

As usual, Σ^*_{red} can be thickened in the fermionic directions to give a smooth supermanifold $\Sigma^* \subset \mathcal{Y}$ of dimension $2|2$, in a way that is unique up to small wiggling in the fermionic directions. Moreover, this can be done in a τ-invariant fashion. The supermanifold or rather superorbifold that we want as the worldsheet of an open and/or unoriented string is $\Sigma = \Sigma^*/\mathbb{Z}_2^\tau$. The reduced space of Σ is $\Sigma'_{\mathrm{red}}/\mathbb{Z}_2^\rho$, in view of (7.7).

The input to this construction was a super Riemann surface Σ' whose reduced space is ρ-invariant. A family of such objects is parametrized by $\Gamma_{\mathfrak{s}}$.

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[28]To follow the presentation in sections 3.1 and 3.7 more strictly, we should take the two factors in $\mathcal{Y}$ to be complex conjugate and then in the next paragraph embed Σ^*_{red} in $\mathcal{Y}_{\mathrm{red}}$ by $x \to (x, \overline{\rho(x)})$ rather than $x \to (x, \rho(x))$. The author finds the description given in the text to be less confusing.

Given such an object, we have constructed a smooth supermanifold (or, if ρ has fixed points, a superorbifold) Σ of dimension $2|2$ parametrized by $\Gamma_{\mathfrak{s}}$. These are the superstring worldsheets parametrized by $\Gamma_{\mathfrak{s}}$.

The local structure of Σ, away from fixed points of the action of ρ on Σ'_{red}, is the same as the local structure of an oriented closed Type II string worldsheet, as described in section 3.7. Locally, we can say that functions on the first factor in $\mathcal{Y} = \Sigma' \times \Sigma'$ restrict to antiholomorphic functions on Σ, and functions on the second factor restrict to holomorphic functions. Globally, the action of τ (or ρ) exchanges what we mean by "holomorphic" and "antiholomorphic," as one expects for an open and/or unoriented worldsheet. Similarly, locally Σ has antiholomorphic and holomorphic tangent bundles $T_L\Sigma$ and $T_R\Sigma$, with odd subbundles $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ that define its superconformal structure; globally these are exchanged by τ or ρ.

As in sections 3.1 and 3.7, we can deform away from the choices that were made in this construction. For example, we can move Σ' slightly way from the locus where Σ'_{red} admits an antiholomorphic involution. ρ can then be deformed (not uniquely) so that it still acts as an orientation-reversing involution, but it is no longer antiholomorphic. The above construction still goes through, but holomorphic functions on Σ_{red} are no longer the complex conjugates of antiholomorphic ones. This is analogous to what happens in sections 3.1 and 3.7 if $\Sigma_{R,\mathrm{red}}$ is not the complex conjugate of $\Sigma_{L,\mathrm{red}}$.

What happens at the fixed points of ρ? There is a general notion of a smooth supermanifold with boundary (for example, see section 3.5 of [32]). But even if Σ_{red} has a boundary, Σ is not a smooth supermanifold with boundary by that definition; it is a more general orbifold, a quotient by $\mathbb{Z}_2$ of an ordinary smooth supermanifold Σ^* without boundary. One can pick local coordinates $t^1, t^2|\theta^1, \theta^2$ on Σ^* so that the $\mathbb{Z}_2$ action is $t^1, t^2|\theta^1, \theta^2 \to t^1, -t^2|\theta^1, -\theta^2$. For $\Sigma^*/\mathbb{Z}_2$ to satisfy the usual definition of a supermanifold with boundary, the local model should be $t^1, t^2|\theta^1, \theta^2 \to t^1, -t^2|\theta^1, \theta^2$, with a sign change only for one even coordinate and no odd ones. We will presently meet supermanifolds with boundary in the conventional sense, namely the compactifications $\widehat{\Gamma}_{\mathfrak{s}}$ of the integration cycles $\Gamma_{\mathfrak{s}}$.

7.3. Punctures

Now let us discuss how this analysis generalizes in the presence of punctures.

7.3.1. The bosonic case We first incorporate punctures in the analysis of the classical moduli space in section 7.1.2. The moduli space $\mathcal{M}_{\mathfrak{g}',\mathfrak{n}}$ that parametrizes a Riemann surface Σ'_0 of genus $\mathfrak{g}'$ with $\mathfrak{n}$ punctures still admits

the antiholomorphic involution φ that maps Σ_0' to the complex conjugate space with the same punctures. A fixed point of φ now corresponds to a surface Σ_0' with an antiholomorphic involution ρ that maps the set of punctures to itself. This means that the punctures consist of pairs of points exchanged by ρ, or else fixed points of ρ. If there are n_0 pairs and n_1 fixed points, then obviously

$$(7.8) \qquad\qquad n = 2n_0 + n_1.$$

In the quotient $\Sigma_0 = \Sigma_0'/\mathbb{Z}_2$, pairs of punctures descend to interior points of Σ_0; we call these bulk punctures. ρ-invariant punctures descend to boundary points of Σ_0; we call these boundary punctures. In string theory, external closed strings couple to bulk punctures and external open strings couple to boundary punctures.

The components of the fixed point set $\mathcal{M}_{g',n}^{\varphi}$ are now labeled by the topological type of Σ_0, the choice of nonnegative integers n_0, n_1 obeying (7.8), and the arrangement of the open-string punctures on the boundary components of Σ_0 (one must specify which open-string puncture is on which boundary component and how they are cyclically ordered). Let us write S for the set of possible choices of all this data. Then $\mathcal{M}_{g,n}^{\varphi}$ has a component $\Gamma_{\mathfrak{s}}$ for every $\mathfrak{s} \in S$ and

$$(7.9) \qquad\qquad \mathcal{M}_{g',n}^{\varphi} = \cup_{\mathfrak{s}\in S}\Gamma_{\mathfrak{s}}.$$

$\Gamma_{\mathfrak{s}}$ parametrizes conformal structures on an open and/or unoriented surface Σ_0 with the indicated topology and set of punctures.

7.3.2. Extension to superstrings   For the superstring analog, it is again easiest to first describe the integration cycle that parametrizes open and/or unoriented worldsheets with punctures, and then to describe the worldsheets themselves. We start with $\mathfrak{M}_{g',n_{NS},n_R}$, the moduli space of super Riemann surfaces of genus g with the indicated numbers of NS and R punctures. We set $n = n_{NS} + n_R$. The reduced space $\mathfrak{M}_{g',n_{NS},n_R,red}$ parametrizes a Riemann surface Σ_{red}' endowed with a spin structure and n punctures, each labeled as an NS or R puncture.

We want to focus on the case that Σ_{red}', forgetting its spin structures and the types of the punctures, admits an antiholomorphic involution ρ. Topologically, there are many choices. We must specify all the discrete data that were relevant in the bosonic case. In addition, a ρ-invariant puncture may be of NS or R type, so that there will be two types of open-string puncture, and

a pair of punctures exchanged by ρ may be of types NS-NS, NS-R, or R-R, so that there will be three types of closed-string puncture. (For open and/or unoriented superstrings, there is no globally-defined distinction between NS-R and R-NS punctures.) Let S^* be the whole collection of discrete choices of the topology of $\Sigma'_{\mathrm{red}}/\mathbb{Z}_2^\rho$ and the types and arrangements of punctures.

For every $\mathfrak{s} \in S^*$, let $\Gamma_{\mathfrak{s},\mathrm{red}}$ be the subspace of $\mathfrak{M}_{g',n_{\mathrm{NS}},n_{\mathrm{R}},\mathrm{red}}$ defined by the condition that Σ'_{red} (ignoring its spin structure) has an antiholomorphic involution of topological type $\mathfrak{s}$. In the usual fashion, we thicken $\Gamma_{\mathfrak{s},\mathrm{red}}$ to a smooth supermanifold $\Gamma_{\mathfrak{s}} \subset \mathfrak{M}_{g',n_{\mathrm{NS}},n_{\mathrm{R}}}$ with the same odd dimension as $\mathfrak{M}_{g',n_{\mathrm{NS}},n_{\mathrm{R}}}$. $\Gamma_{\mathfrak{s}}$ is the parameter space of open and/or unoriented super Riemann surfaces of type $\mathfrak{s}$.

Moreover, the construction ensures, as usual, that the even dimension of $\Gamma_{\mathfrak{s}}$ is the complex dimension of $\mathfrak{M}_{g',n_{\mathrm{NS}},n_{\mathrm{R}}}$. So one can write a dimension formula for $\Gamma_{\mathfrak{s}}$. It seems that the most useful way to record the information is to state the contributions that different kinds of puncture make to the dimension of $\Gamma_{\mathfrak{s}}$. The contribution of an open-string puncture of NS or R type is

$$\mathrm{NS}: \ 1|1.$$
$$(7.10) \qquad\qquad \mathrm{R}: \ 1|\frac{1}{2}.$$

And the contribution of a closed-string puncture of one of the three types is

$$\mathrm{NS-NS}: \ 2|2$$
$$\mathrm{NS-R}: \ 2|3/2$$
$$(7.11) \qquad\qquad \mathrm{R-R}: \ 2|1.$$

7.3.3. The worldsheet   To construct the string worldsheets that are parametrized by $\Gamma_{\mathfrak{s}}$, we basically repeat what has been said in section 7.2.2 in the presence of punctures. We start with the complex supermanifold $\mathcal{Y} = \Sigma' \times \Sigma'$, where Σ' is now a super Riemann surface of genus g' with n_{NS} Neveu-Schwarz and n_{R} Ramond punctures; we write τ for the involution of $\mathcal{Y}$ that exchanges the two factors. We ask that Σ'_{red} should admit the action of an antiholomorphic involution ρ of topological type $\mathfrak{s}$, mapping the set of punctures to itself (but perhaps permuting them). We let Σ^*_{red} be a copy of Σ'_{red}, embedded in $\mathcal{Y}_{\mathrm{red}}$ by $x \to (x, \rho(x))$. We thicken Σ^*_{red} to a smooth and τ-invariant supermanifold $\Sigma^* \subset \mathcal{Y}$ of dimension $2|2$, and we define $\Sigma = \Sigma^*_{\mathrm{red}}/\mathbb{Z}_2^\tau$. Σ is an open and/or unorientable string worldsheet with punctures. Its reduced space is $\Sigma'_{\mathrm{red}}/\mathbb{Z}_2^\rho$.
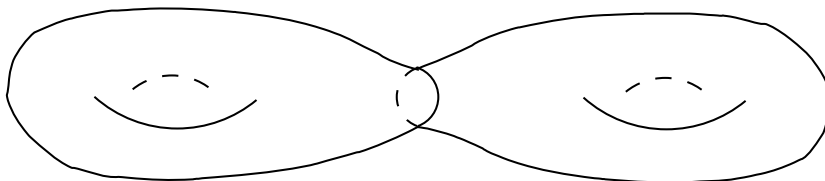
Figure 13: A closed oriented surface Σ_0' that consists of two identical but oppositely oriented components joined at a narrow neck. Σ_0' has an antiholomorphic involution that exchanges the two components.

7.4. Compactification of moduli spaces of open and/or unoriented surfaces

7.4.1. The classical case   In section 7.1.2, we defined the moduli space $\Gamma_{\mathfrak{s}}$ that parametrizes conformal structures on an open and/or unoriented surface Σ_0 of topological type $\mathfrak{s}$. $\Gamma_{\mathfrak{s}}$ is a component of the fixed point set of the natural antiholomorphic involution φ of $\mathcal{M}_{\mathsf{g}'}$, the moduli space of Riemann surfaces of genus g'. We also remarked that $\Gamma_{\mathfrak{s}}$ can be compactified by taking its closure in $\widehat{\mathcal{M}}_{\mathsf{g}'}$. We write $\widehat{\Gamma}_{\mathfrak{s}}$ for this compactification.

Some of the important differences between the theory of open and/or unoriented strings and the theory of closed oriented strings come from the fact that the compactification $\widehat{\Gamma}_{\mathfrak{s}}$ is a manifold (or orbifold) with boundary. In superstring theory, this leads to anomaly cancellation conditions that have no close analog for closed oriented strings; the compactified moduli space of closed oriented Riemann surfaces has no boundary.

Here and in section 7.4.2, we will describe the degenerations that are responsible for the boundary of $\widehat{\Gamma}_{\mathfrak{s}}$. In fig. 13, we sketch a closed oriented surface Σ_0' that is obtained by gluing at a narrow neck two identical but oppositely oriented components. Σ_0' has an antiholomorphic involution ρ that exchanges the two components.

The behavior in the narrow neck is described by the usual gluing relation

$$(7.12) \qquad\qquad\qquad\qquad xy = q.$$

x is a local parameter on one side of the narrow neck in Σ_0' and y is a local parameter on the other side. Now let us discuss how ρ acts. ρ exchanges the two branches, so it exchanges x and y. It is antiholomorphic, so it must map x to a multiple of $\overline{y}$, and vice-versa. We can define x and y so that

$$(7.13) \qquad\qquad\qquad \rho(x) = \overline{y}, \;\; \rho(y) = \overline{x}.$$
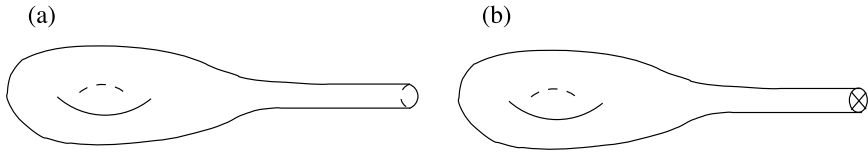
Figure 14: (a) For $q > 0$, $\Sigma'_0/\mathbb{Z}_2$ has a long tube that ends on a boundary. (b) For $q < 0$, $\Sigma'_0/\mathbb{Z}_2$ has a long tube that ends on a crosscap.

With these choices of x and y, the parameter q in (7.12) must be real so that the gluing relation is ρ-invariant.

Now let us examine the quotient $\Sigma_0 = \Sigma'_0/\mathbb{Z}_2^\rho$. The boundary of Σ_0 consists of the fixed points of ρ. A fixed point of ρ obeys $y = \overline{x}$, so the gluing relation gives $x\overline{x} = q$. So the fixed point set of ρ consists of a circle if $q > 0$, and is empty if $q < 0$.

The quotient $\Sigma'_0/\mathbb{Z}_2$ is depicted for the two cases in fig. 14. It is convenient to use the conformal frame in which the gluing is described by a long tube of length $\log|q|^{-1}/2$ (we divide the formula (6.4) for the length of the tube by 2 because the tube is cut in half to make $\Sigma'_0/\mathbb{Z}_2$). For $q > 0$, $\Sigma'_0/\mathbb{Z}_2$ is a Riemann surface with boundary; the boundary, which is the end of the tube, is the circle $x\overline{x} = q$. For $q < 0$, there is no boundary, since ρ has no fixed points. The tube still ends at $x\overline{x} = |q|$, since a point with $x\overline{x} < |q|$ can be mapped to $x\overline{x} > |q|$ by $\rho : x \to \overline{y} = q/\overline{x}$. For $q < 0$ and $x\overline{x} = |q|$, the transformation $x \to q/\overline{x}$ is equivalent to $x \to -x$. So opposite points of the circle $x\overline{x} = |q|$ are identified and $\Sigma'_0/\mathbb{Z}_2$ has no boundary. One says in this case that the long tube ends on a crosscap, a copy of $\mathbb{RP}^2$ with an open disc omitted.

The topology of the surface $\Sigma'_0/\mathbb{Z}_2$ clearly depends on the sign of q. We will call this quotient Σ_0 if $q > 0$ and Σ_1 if $q < 0$. And we write Γ_0 and Γ_1 for the moduli spaces of conformal structures on Σ_0 or Σ_1. The parameter q is a real modulus of Σ_0 and Σ_1 and compactification of either Γ_0 or Γ_1 requires us to allow the limiting value $q = 0$. The locus $q = 0$ is a boundary component of the compactifications $\widehat{\Gamma}_0$ and $\widehat{\Gamma}_1$.

Since these moduli spaces have a common boundary, one can glue them together to make a manifold without boundary, different parts of which parametrize open and/or unoriented surfaces with different topology. Sometimes this is even useful in superstring theory [51]. However, in general in superstring theory, it is not helpful to try to fit $\widehat{\Gamma}_0$ and $\widehat{\Gamma}_1$ together. For example, there are Type II orientifolds without D-branes in which the string worldsheet has no boundary but need not be oriented; and there are Type II
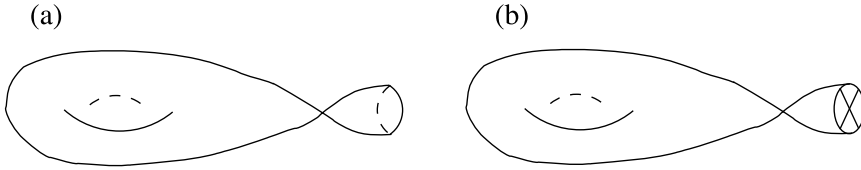
(a)   (b)

Figure 15: (a) For q approaching zero from above, the long tube of fig. 14(a) is conformally equivalent to a collapsing neck and Σ_0 splits off a component consisting of a disc with one puncture (the node). (b) Similarly, for q approaching zero from below, Σ_1 splits off a component that is conformally equivalent to $\mathbb{RP}^2$ with one puncture (again the node).

theories with D-branes but no orientifold planes in which the string worldsheet is oriented but may not have a boundary. In these examples, $\widehat{\Gamma}_0$ is relevant but not $\widehat{\Gamma}_1$, or vice-versa, so in general one has to consider them separately.

The limits of Σ_0 and Σ_1 as $q \to 0$ cannot be deduced in a simple way from the way the double cover Σ'_0 degenerates for $q \to 0$. Σ'_0 has only one limit for $q \to 0$ regardless of the direction from which one approaches $q = 0$, but the objects which one wants to identify as limits of Σ_0 and Σ_1 as q approaches 0 from above or below are different.

It is not hard to understand what those objects should be. In fig. 14(a) or (b), we see a long tube whose length diverges as q approaches 0 from above or below. We simply follow the general rule that the limit of a long tube is conformally equivalent to a collapsing neck. So the limiting configurations corresponding to fig. 14 for $q \to 0$ are as shown in fig. 15. For $q \to 0$, Σ_0 or Σ_1 splits off a component that consists of a disc with one puncture (where it meets the node) or $\mathbb{RP}^2$ with one puncture (again where it meets the node). These are the natural limiting configurations at $q = 0$ because the singularities they contain are the usual closed-string degenerations, which are needed anyway in the Deligne-Mumford compactification so it is unavoidable to allow them. Moreover, they have a natural physical interpretation.

There are at least two lessons from all this for superstring theory:

(1) The compactified moduli space of open and/or unoriented Riemann surfaces is a manifold with boundary. Similarly, the compactified moduli space of open and/or unoriented super Riemann surfaces will be a supermanifold with boundary. Integration on a supermanifold with boundary makes sense provided the right structure is in place, but this is one of the issues that one has to consider.

(2) In compactifying the moduli space, we had to allow an open and/or unoriented Riemann surface to split off a component with continuous symme-

tries. In fact, a disc or $\mathbb{RP}^2$ with one closed-string puncture has a $U(1)$ group of symmetries – the group of rotations around the puncture. The components with symmetries turn out to be important in the study of anomalies [52], [31]. By contrast, in compactifying the moduli space of oriented Riemann surfaces, only components without continuous symmetries are required.

7.4.2. Open-string degenerations There is one more important example somewhat like the one studied in section 7.4.1. We return to the basic setting of fig. 13 with a closed oriented surface Σ_0' that consists of two components joined at a narrow neck. But now we assume that Σ_0' has an antiholomorphic involution ρ that maps each component to itself, instead of exchanging them.

In terms of the gluing law

$$(7.14) \qquad\qquad\qquad xy = q,$$

this means that the local parameters x and y are mapped to their own complex conjugates (up to constant multiples that we can absorb in the definitions of x and y):

$$(7.15) \qquad\qquad\qquad \rho(x) = \overline{x}, \quad \rho(y) = \overline{y}.$$

Again, ρ-invariance of the gluing relation implies that q must be real.

Let us determine the fixed point set of ρ, which is the same as the boundary of the quotient $\Sigma_0'/\mathbb{Z}_2^\rho$. Clearly the fixed point set is given by the hyperbola $xy = q$ in the real x-y plane. The hyperbola consists topologically of a pair of lines. To describe Σ_0 explicitly, we go back to complex variables and write $x = e^\varrho$, $y = qe^{-\varrho}$, with $\varrho \cong \varrho + 2\pi\sqrt{-1}$. ρ acts on ϱ by $\varrho \to \overline{\varrho}$. Imposing both relations $\varrho \cong \varrho + 2\pi\sqrt{-1}$ and $\varrho \cong \overline{\varrho}$, we can parametrize $\Sigma_0'/\mathbb{Z}_2$ by

$$(7.16) \qquad\qquad\qquad \varrho = s + i\vartheta, \quad 0 \le \vartheta \le \pi,$$

and the endpoints $\vartheta = 0, \pi$ are the boundaries of Σ_0. With these restrictions on ϑ, ϱ parametrizes a long strip (fig. 16(a)). This long strip is simply the quotient by $\mathbb{Z}_2$ of the long tube that is described by the equation $xy = q$ if we do not divide by $\mathbb{Z}_2$ (fig. 16(b)). Physically, the long strip describes the propagation of an open string for a long proper time.

To compactify the moduli space Γ that parametrizes $\Sigma_0'/\mathbb{Z}_2$ for positive (or negative) q, we need to introduce a limit of the long strip at $q = 0$. It is not difficult to see what this must be. We use the fact that a long strip is conformally equivalent to a strip with a narrow neck. In the limit $q \to 0$, $\Sigma_0'/\mathbb{Z}_2$ decomposes into a union of two components glued together at
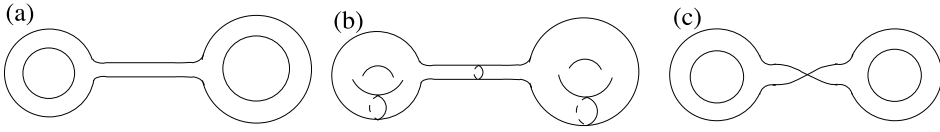
Figure 16: (a) A long strip connecting two components of a Riemann surface Σ_0 with boundary. (b) The long strip of (a) is simply the quotient by $\mathbb{Z}_2$ of a long tube in the associated closed oriented double cover Σ_0'. The quotient is taken by flattening the closed oriented surface seen here onto the page. (c) The limit in which the strip in (a) becomes infinitely long is conformally equivalent to a degeneration in which Σ_0 splits into a pair of components, joined at a pair of open-string punctures. We call this an open-string degeneration.
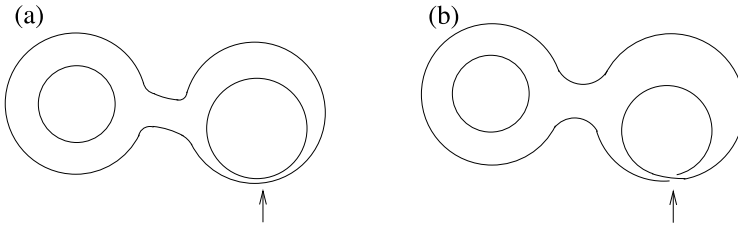
Figure 17: A nonseparating open-string degeneration (a) and its twisted version (b), both marked with arrows. (a) and (b) can be exchanged by cutting the indicated narrow neck, twisting it, and regluing. This corresponds to changing the sign of q.

a pair of open-string punctures (fig. 16(c)). This is what we will call an open-string degeneration. In string theory, infrared singularities due to on-shell open strings arise from such open-string degenerations. In this language, the degenerations studied in section 6 might be called closed-string degenerations.

The two cases of positive and negative q differ by a "twist" of the strip, as sketched in fig. 17. This happens as follows. The two boundary components of the strip can be distinguished by the sign of x, and also by the sign of y. For $q \to 0$, x parametrizes the boundary of one component of Σ_0, and y parametrizes the boundary of the other. The relation between positivity of x and positivity of y is reversed if we change the sign of q, and this introduces the twist.

Just like closed-string degenerations, open-string degenerations can be either separating (fig. 16) or nonseparating (fig. 17). In general $\Sigma_0'/\mathbb{Z}_2$ has different topology for positive or negative q. For example, in fig. 17, $\Sigma_0'/\mathbb{Z}_2$

is unorientable for one sign of q though it is orientable for the other sign. In this case, the compactified moduli space $\widehat{\varGamma}_{\mathfrak{s}}$ clearly must be understood as a manifold with boundary, with $q = 0$ defining one of the boundary components.

Even when the topology of $\Sigma'_0/\mathbb{Z}_2$ is not affected by the twist, its conformal structure generically jumps discontinuously in crossing $q = 0$. So one should think of $q = 0$ as a boundary of the moduli space $\widehat{\varGamma}_{\mathfrak{s}}$, even if the limits as q approaches 0 from above and below represent two different parts of the boundary of the same $\widehat{\varGamma}_{\mathfrak{s}}$.

7.4.3. Superanalogs Compactifying the parameter space $\varGamma_{\mathfrak{s}}$ of a super Riemann surface Σ does not involve many new ingredients. The reduced space $\varGamma_{\mathfrak{s},\mathrm{red}} \subset \mathfrak{M}_{g',\mathrm{red}}$ has a natural compactification $\widehat{\varGamma}_{\mathfrak{s},\mathrm{red}}$ obtained by taking its closure in $\widehat{\mathfrak{M}}_{g',\mathrm{red}}$. It can then be thickened in the fermionic directions to give the compactified integration cycle $\widehat{\varGamma}_{\mathfrak{s}}$.

This compactification parametrizes superanalogs of the usual closed string degenerations, but this involves no particular novelties relative to what we analyzed in section 6. More interestingly, the compactification of $\varGamma_{\mathfrak{s}}$ also describes superanalogs of the special degenerations that we considered in sections 7.4.1 and 7.4.2, namely the ones responsible for the fact that the compactified classical parameter space $\widehat{\varGamma}_{\mathfrak{s}}$ of an open and/or unoriented Riemann surface Σ_0 is a manifold with boundary. It turns out that, because of the existence of these special degenerations, $\widehat{\varGamma}_{\mathfrak{s}}$ is a (smooth) supermanifold with boundary. This requires some discussion, since the proper definition of a supermanifold with boundary, such that integration makes sense, is rather subtle (see for example [32], sections 3.4 and 3.5). We want to make sure that the appropriate structure is present in the case of $\varGamma_{\mathfrak{s}}$.

To describe the special degenerations explicitly in the superstring context, one replaces the classical gluing law $xy = q$ by its familiar Neveu-Schwarz and Ramond analogs, namely

$$
\begin{aligned}
xy &= -\varepsilon^2 \\
y\theta &= \varepsilon\psi \\
x\psi &= -\varepsilon\theta \\
\theta\psi &= 0.
\end{aligned}
$$
(7.17)

and

$$
\begin{aligned}
xy &= q_{\mathrm{R}} \\
\theta &= \pm\sqrt{-1}\,\psi.
\end{aligned}
$$
(7.18)

The degenerations that we want to focus on here are the ones for which, classically, the gluing parameter q is real. We will call them real degenerations. In the superstring context, at a real degeneration, q_R and $q_{NS} = -\varepsilon^2$ are real (and ε is real or a real multiple of $\sqrt{-1}$) modulo the odd variables. We say that a real degeneration is of NS type or R type depending on whether it is described by the NS gluing formulas (7.17) or their Ramond counterparts (7.18).

Real degenerations describe the propagation through a long tube or strip of either (1) a closed string whose worldsheet ends on a boundary or crosscap, as in section 7.4.1; or (2) an open string, as in section 7.4.2. A real degeneration is of NS (R) type if the closed-string state in (1) is of NS-NS (R-R) type, or the open-string state in (2) is of NS (R) type. In a real degeneration, the sum over the sign of the fermionic gluing (the sign of ε or the sign in $\theta = \pm\sqrt{-1}\psi$) enforces the GSO projection. In case (1), the sum over the sign of the fermionic gluing ensures that the closed-string state created by a boundary or crosscap is GSO even; in case (2), it ensures that open-string states that create infrared singularities by propagating for a long proper time through a long strip are GSO even.

Now let us discuss the boundary of $\widehat{\varGamma}_{\mathfrak{s}}$. The case of a real Ramond degeneration is more obvious so we consider it first. An open-string degeneration of R type leads to a boundary component of $\widehat{\varGamma}_{\mathfrak{s}}$; as in the bosonic case, the two cases $q_R \geq 0$ and $q_R \leq 0$ represent different moduli spaces, typically parametrizing superstring worldsheets with different topology. Either of these two moduli spaces has a boundary at $q_R = 0$.

In general, to define a supermanifold M with boundary in such a way that integration is possible, one needs some additional structure. Locally, one requires an even function f that defines the boundary, in the sense that it is positive definite or negative definite (when reduced modulo odd variables) in the interior of M and has a first-order zero on the boundary of M. To be a little more precise, we need only an equivalence class of such functions f modulo

$$(7.19) \qquad\qquad\qquad f \to e^{\phi} f,$$

where ϕ is a function on M that is real modulo the odd variables. Note that an allowed transformation of f does not include, for example, $f \to f + \eta_1\eta_2$, where η_1 and η_2 are odd functions. The role of the function f in integration theory is explained, for example, in [32], sections 3.4 and 3.5.

The structure just described is precisely what we have near the boundary of $\widehat{\varGamma}_{\mathfrak{s}}$ associated to a Ramond degeneration. For the function f we can take q_R. Actually, q_R is really only well-defined to first order near the boundary

of $\widehat{\varGamma}_{\mathfrak{s}}$; and even to first order, it is not a function but a section of a certain real[29] line bundle, the dual of the normal bundle to the locus of real Ramond degenerations in $\widehat{\varGamma}_{\mathfrak{s}}$. Rather as in section 6.1.3, a line bundle appears because of the dependence of q_{R} on the choice of the local parameters x and y in eqn. (7.18). If we change the local parameters, q_{R} charges by $q_{\mathrm{R}} \to e^{\phi} q_{\mathrm{R}}$, for some function ϕ (which in general may depend on all the other moduli of Σ). Hence the indeterminacy of the function q_{R} is precisely of the form of eqn. (7.19), with q_{R} understood as the distinguished function that vanishes on the boundary. Therefore the structure of $\widehat{\varGamma}_{\mathfrak{s}}$ near its boundary is precisely such that integration of a smooth measure (a smooth section of the Berezinian) over $\widehat{\varGamma}_{\mathfrak{s}}$ makes sense. There is more to say because the measures that one has to integrate in superstring theory have singularities at the boundary; those issues are discussed in [31]. But the fact that one would be able to integrate a smooth measure over $\widehat{\varGamma}_{\mathfrak{s}}$ is certainly part of the input to the fact that perturbation theory makes sense for open and/or unoriented superstrings.

A real NS degeneration does not actually contribute a boundary component to $\widehat{\varGamma}_{\mathfrak{s}}$, because the two choices of sign of ε match smoothly at $\varepsilon = 0$. Nonetheless, to make sense of the integrals that appear in superstring perturbation theory, one needs the existence of a distinguished parameter ε (subject to $\varepsilon \to e^{\phi}\varepsilon$) associated to a real NS degeneration. That is because of the tachyon singularity at $\varepsilon = 0$. The sum over the two signs of ε is needed to implement the GSO projection and eliminate the tachyon. Roughly speaking, after eliminating the tachyon, the locus $\varepsilon = 0$ is effectively a boundary, and again the integral over a supermanifold with boundary only makes sense because of the existence of a distinguished parameter that vanishes at the boundary, namely ε.

We conclude by explaining the analog of section 6.2.2 for open-string degenerations, or in other words by describing in the case of an open-string degeneration the locus in $\widehat{\varGamma}_{\mathfrak{s}}$ with $q_{\mathrm{R}} = 0$ or $\varepsilon = 0$. (Real closed-string degenerations are treated in section 7.4.4.) When a worldsheet Σ undergoes an open-string degeneration, it can be compared to the smooth surface Σ^* (its "normalization") that results from detaching the two branches at the node. (Σ^* may or may not be connected, depending on whether the degeneration is separating.) The locus in $\widehat{\varGamma}_{\mathfrak{s}}$ that parametrizes the open-string degeneration can be compared, just as we did in section 6.2.2 for closed strings, to the parameter space $\widehat{\varGamma}_{\mathfrak{s}}^*$ of Σ^*. For an open-string degeneration of NS type, the degeneration locus in $\widehat{\varGamma}_{\mathfrak{s}}$ is a copy of $\widehat{\varGamma}_{\mathfrak{s}}^*$, but for one of R type, it is a fiber

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[29]A line bundle over a smooth supermanifold is real if its transition functions are real modulo the odd variables.

bundle over $\widehat{\Gamma}_{\mathfrak{s}}^{*}$ with fibers of dimension $0|1$. As in section 6.2.2, the fibers parametrize the different ways to glue together Ramond divisors on the two sides. One can summarize this by saying that while at an open-string NS node there is only the even gluing parameter ε, an open-string Ramond node has a fermionic gluing parameter as well as the even parameter q_{R}. (After integrating over this parameter, the open-string Ramond propagator is G_0/L_0, in contrast to the open-string NS propagator, which is $1/L_0$.) All this is closely analogous to what happens for closed strings.

7.4.4. Components with continuous symmetries   The compactified parameter space $\widehat{\Gamma}_{\mathfrak{s}}$ describes many possible ways that Σ can degenerate by reducing its genus (understood as the genus of the closed oriented double cover Σ') or splitting into multiple components. For the same reasons as in the bosonic discussion of section 7.4.1, an example of what can happen is that Σ can split off a component that has a symmetry group $\mathcal{G}$ of positive even dimension. This will be a component whose reduced space is a disc with a single closed-string puncture, or $\mathbb{RP}^2$ with a single puncture. Even though these sound like special cases, they are important in understanding superstring anomalies, so we will look more closely.

For either the once-punctured disc or once-punctured $\mathbb{RP}^2$, the single puncture is of NS-NS type or of R-R type (it cannot be of type NS-R for then on the closed oriented double cover, there would be a single R puncture, which is impossible). It is instructive to work out the dimension formula for the dimension of moduli space for a punctured Riemann surface of one of these types. The disc and $\mathbb{RP}^2$ can be considered together. First we assume that the puncture is of NS-NS type. The dimension formula with only NS-NS punctures reads

$$(7.20) \qquad \dim \Gamma - \dim \mathcal{G} = -3\chi(\Sigma) + 2\mathfrak{n}_{\mathrm{NS-NS}}| - 2\chi(\Sigma) + 2\mathfrak{n}_{\mathrm{NS-NS}}.$$

In evaluating this formula, it helps to bear in mind that, for either a disc or $\mathbb{RP}^2$ with one NS-NS puncture, the associated closed oriented super Riemann surface Σ' is $\mathbb{CP}^{1|1}$ with two NS punctures. This surface has no moduli, so Γ is a point and $\dim \Gamma = 0|0$. Both the disc and $\mathbb{RP}^2$ have $\chi = 1$. So the dimension formula with $\mathfrak{n}_{\mathrm{NS-NS}} = 1$ implies that $\dim \mathcal{G} = 1|0$. Indeed, the bosonic part of the automorphism group is $U(1)$, consisting of rotations of the disc or $\mathbb{RP}^2$ around the puncture. And the odd dimension of the automorphism group is 0, because Σ' has no odd automorphisms.

Now suppose that the puncture is of R-R type. The dimension formula with only R-R punctures reads

$$(7.21) \qquad \dim \Gamma - \dim \mathcal{G} = -3\chi(\Sigma) + 2\mathfrak{n}_{\mathrm{R-R}}| - 2\chi(\Sigma) + \mathfrak{n}_{\mathrm{R-R}}.$$

Γ is still a point, with dim $\Gamma = 0|0$, just as before. The dimension formula with $\mathfrak{n}_{R-R} = 1$ now gives dim $\mathcal{G} = 1|1$. The bosonic part of the automorphism group is $U(1)$, as before. To understand why $\mathcal{G}$ has odd dimension 1, observe that the associated closed oriented super Riemann surface Σ' is now of genus 0 with 2 Ramond punctures. This surface has a one-dimensional space of odd superconformal vector fields, as we explained in discussing eqn. (5.18).

We conclude by discussing the consequences of the automorphism group $\mathcal{G}$ for gluing. We start with bosonic strings. Let us return to fig. 15 of section 7.4.1. The degeneration depicted in this figure is a closed-string degeneration, which normally is controlled by a complex gluing parameter q. Yet in the particular case that one of the components meeting at the double point is a disc or $\mathbb{RP}^2$ (with no punctures except the double point), one may restrict to real nonnegative[30] values of q. We have already given one explanation of this in section 7.4.1. Another follows from the existence of the automorphism group. Ordinarily, when two Riemann surfaces Σ_1 and Σ_2 are joined by the narrow neck $xy = q$, the parameter $\text{Arg}\, q$ has the following interpretation: a shift in $\text{Arg}\, q$ by an angle ϕ has the same effect as cutting the narrow neck, rotating one side relative to the other by the angle ϕ, and gluing the two sides back together again. When Σ_1 or Σ_2 has a symmetry of rotating around the neck, the parameter $\text{Arg}\, q$ can be absorbed in rotating Σ_1 or Σ_2 and becomes irrelevant. This is the case precisely when Σ_1 or Σ_2 is a disc or $\mathbb{RP}^2$ with just one puncture.

What we have said so far relies on the bosonic part of the automorphism group, and is equally valid for bosonic strings or for NS-NS and R-R degenerations of superstrings. Now we consider the fermionic part of the automorphism group, which is present only in the R-R superstring case. Usually, at an R-R closed-string degeneration, there are two fermionic gluing parameters, one for holomorphic degrees of freedom and one for antiholomorphic degrees of freedom. However, if one of the two branches, say Σ_2, is a disc or $\mathbb{RP}^2$ with one puncture, then it has a fermionic symmetry that can be used to transform away one of the fermionic gluing parameters. This happens because the fermionic symmetry of the closed oriented double cover of Σ_2 acts nontrivially on the Ramond divisors (this is shown in eqn. (5.19)) and hence shifts the fermionic gluing data. This is quite analogous to what happened to $\text{Arg}\, q$. The disappearance of one of the two fermionic gluing parameters is part of the mechanism by which an R-R degeneration of the particular types shown in fig. 15 can lead to anomalies. This is described in [31].

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[30]In fig. 13, a real degeneration corresponds to real q, which may be positive or negative. The two signs correspond to cases (a) and (b) in fig. 15, so in those pictures, there is no longer a choice of sign.

8. Contour integrals and super period matrix

Our goal in this section is to describe the analog on a super Riemann surface of the periods of a holomorphic differential on an ordinary Riemann surface. We will also describe the super period matrix constructed from the periods, and demonstrate its basic properties. Finally, we describe an application [21] to superstring perturbation theory.

8.1. Basics

We will need to consider a super Riemann surface Σ as a smooth supermanifold, as described in section 3. However, antiholomorphic odd variables would add nothing. So as in most of section 3, we take Σ to be a heterotic string worldsheet. This means that Σ is a smooth supermanifold of dimension $2|1$ embedded in $\Sigma_L \times \Sigma_R$, where Σ_L is an ordinary Riemann surface and Σ_R is a super Riemann surface (as usual, we can assume that Σ_L is the complex conjugate of $\Sigma_{R,\mathrm{red}}$ and Σ_{red} is diagonally embedded in $\Sigma_L \times \Sigma_{R,\mathrm{red}}$). We will write $Ber(\Sigma)$ for the Berezinian of the holomorphic cotangent bundle to Σ (that is, for the restriction to Σ of $Ber(\Sigma_R)$).

Suppose that Σ is split from a holomorphic point of view (in the sense that Σ_R is a split super Riemann surface). Then its reduced space Σ_{red} is naturally embedded in Σ, with a projection $\pi : \Sigma \to \Sigma_{\mathrm{red}}$. Even if Σ is not holomorphically split, the structure theorem of smooth supermanifolds (reviewed for instance in section 2.2 of [32]) says that Σ_{red} can be embedded in Σ, again with a projection $\pi : \Sigma \to \Sigma_{\mathrm{red}}$. The embedding and projection are not completely natural, but they are natural up to homology (in fact, up to infinitesimal homologies involving the fermionic directions), and this will be good enough for us.[31]

We begin by describing the cycles on which we will compute periods. Let γ_{red} be an ordinary oriented closed 1-cycle in Σ_{red}, with Σ_{red} embedded in Σ as above. (We take γ_{red} to be closed as we do not want to consider contours with boundary.) Then after embedding Σ_{red} in Σ, we can view γ_{red} as a submanifold of Σ of dimension $1|0$. If μ is a 1-form on Σ, then we can integrate it over γ_{red} in the usual way. For this, we parametrize γ_{red} by an

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[31] Actually, on a super Riemann surface, as opposed to a more general $1|1$ complex supermanifold, once an embedding of Σ_{red} in Σ is given, there is a natural projection $\pi : \Sigma \to \Sigma_{\mathrm{red}}$. The subbundle $\mathcal{D} \subset T\Sigma$, when restricted to Σ_{red}, is a fiber bundle over Σ_{red} with fibers of rank $0|1$. This bundle is naturally embedded in Σ and its total space is Σ. So Σ is a fiber bundle over Σ_{red} and this gives the projection π. We do not often need this fact, but it will be relevant at the end of appendix C.

angular variable s; when restricted to γ_{red}, μ is an ordinary 1-form $f(s)\,\mathrm{d}s$, and we define

$$(8.1) \qquad\qquad \oint_{\gamma_{\mathrm{red}}} \mu = \oint_{\gamma_{\mathrm{red}}} f(s)\,\mathrm{d}s.$$

If μ is closed, then $\oint_{\gamma_{\mathrm{red}}} \mu$ depends only on the homology class of γ_{red} in Σ, not on the embedding of γ_{red} in Σ_{red} or of Σ_{red} in Σ. A difference from ordinary compact Riemann surfaces (and from compact Kahler manifolds of any dimension) is that even if μ is holomorphic, it need not be closed. Indeed if Σ is split and $a(z)\,\mathrm{d}z$ is a holomorphic 1-form on Σ_{red}, then $a(z)\varpi$ is a globally-defined and holomorphic but not closed 1-form on Σ; here

$$(8.2) \qquad\qquad \varpi = \mathrm{d}z - \theta\mathrm{d}\theta$$

was introduced in the discussion of eqn. (2.29).

There is another type of object that is appropriate for contour-like integrals. Let σ be a holomorphic section of $Ber(\Sigma)$. Then σ can be regarded as an integral form on Σ of codimension 1, so it can be naturally integrated over a submanifold $\gamma \subset \Sigma$ of real codimension $1|0$ and hence of dimension $1|1$. This is explained, for example, in [32], section 3.3.4. The integral, which we denote as $\oint_\gamma \sigma$, only depends on the homology class of γ. As an example of a codimension $1|0$ cycle in Σ, we can take $\gamma = \pi^{-1}(\gamma_{\mathrm{red}})$ with γ_{red} as before.

Up to homology, all of the relevant $1|0$ and $1|1$ cycles γ_{red} or γ in Σ arise in this way from ordinary 1-cycles in Σ_{red}. So all periods of a 1-form on Σ are associated with A-cycles or B-cycles on the ordinary Riemann surface Σ_{red}. There are thus $2\mathrm{g}$ periods in all.

A difference from ordinary Riemann surface theory is that in general there are more independent objects that one might want to integrate than there are independent cycles to integrate them over. Suppose for example that Σ is split; it is then constructed from its reduced space Σ_{red} and a square root $K^{1/2}$ of the canonical bundle K of Σ_{red}. In this case, a holomorphic section σ of $Ber(\Sigma)$ takes the form $\sigma = (\alpha(z) + \theta b(z))[\mathrm{d}z|\mathrm{d}\theta]$, with $\alpha \in H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ and $b \in H^0(\Sigma_{\mathrm{red}}, K)$. If σ is even (as we will generally assume), then, since the measure $[\mathrm{d}z|\mathrm{d}\theta]$ is odd, α is odd and b is even. We can reduce an integral over γ to an integral over γ_{red} by first integrating over θ. Using the properties of the Berezin integral, this gives

$$(8.3) \qquad\qquad \oint_\gamma \alpha(z)[\mathrm{d}z|\mathrm{d}\theta] = 0,$$

and

$$(8.4) \qquad \oint_\gamma \theta b(z)[\mathrm{d}z|\mathrm{d}\theta] = \oint_{\gamma_{\mathrm{red}}} b(z)\mathrm{d}z.$$

Thus, in the split case, if there is a holomorphic section $\alpha(z)$ of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$, there is a holomorphic section $\sigma = \alpha(z)[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(\Sigma)$ whose periods all vanish (and similarly, there is a holomorphic 1-form $\mu = \mathrm{d}(\theta\alpha)$ whose periods all vanish). The theory of periods and contour integration on Σ is more simple if this does not occur, since for example then σ is uniquely determined by its A-periods, just as on an ordinary Riemann surface. In practice, if $K^{1/2}$ defines an odd spin structure, the dimension of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ is always odd and never zero. If $K^{1/2}$ defines an even spin structure, then the dimension of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ is always even and this cohomology group vanishes for a generic choice of the complex structure of Σ_{red}. That is the situation in which we will discuss the super period matrix. (At the end of section 8.3, we will see that the super period matrix develops a pole when $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ becomes nonzero.)

An important technical point is that the hypothesis that $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$ ensures that no jumping occurs in the cohomology groups $H^k(\Sigma, Ber(\Sigma))$, $k = 0, 1$, as a function of the odd moduli.[32] In general, just as on an ordinary Riemann surface, for any line bundle (or vector bundle) $\mathcal{V}$, the difference in graded dimension between $H^0(\Sigma, \mathcal{V})$ and $H^1(\Sigma, \mathcal{V})$ is a topological invariant (determined by the Riemann-Roch theorem), not subject to jumping. So absence of jumping of $H^1(\Sigma, Ber(\Sigma))$ will imply absence of jumping of $H^0(\Sigma, Ber(\Sigma))$. By Serre duality, $H^1(\Sigma, Ber(\Sigma))$ is dual to $\Pi H^0(\Sigma, \mathcal{O})$, where $\mathcal{O}$ is a trivial line bundle. So it suffices to show that there is no jumping in $H^0(\Sigma, \mathcal{O})$, which is the space of global holomorphic functions on Σ. For Σ split, a global holomorphic function w has an expansion $w = w_0 + \theta w_1$, where w_0 is a global holomorphic function on Σ_{red} and therefore a constant, while w_1 is an element of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$. The assumption that $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$ thus implies that $w_1 = 0$. So we have learned that $H^0(\Sigma, \mathcal{O})$ is generated in the split case by the constant function 1. Certainly, this constant function makes sense as an element of $H^0(\Sigma, \mathcal{O})$ even when we turn on odd moduli, so there is no jumping of $H^0(\Sigma, \mathcal{O})$ as a function of odd moduli, and hence also none in $H^1(\Sigma, Ber(\Sigma))$ or $H^0(\Sigma, Ber(\Sigma))$. Jumping of $H^0(\Sigma, Ber(\Sigma))$ occurs only when we vary the even moduli of Σ_{red} so that $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ becomes nonzero.

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[32] See the end of section 5.2 for a simple example showing that such jumping can occur for $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) \neq 0$.

We have already seen that in the split case, with $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$, $H^0(\Sigma, Ber(\Sigma))$ is naturally isomorphic to $H^0(\Sigma_{\mathrm{red}}, K)$, and in particular has dimension $\mathfrak{g}|0$, where $\mathfrak{g}$ is the genus of Σ. Absence of jumping means that the dimension remains $\mathfrak{g}|0$ when we turn on the odd moduli.

Periods of holomorphic sections of $Ber(\Sigma)$ are best described by a super period matrix [11, 18]. On the reduced space Σ_{red}, pick a basis of A-cycles and B-cycles A_{red}^i, $B_{j,\mathrm{red}}$, $i, j = 1, \dots, g$ with

$$(8.5) \qquad A_{\mathrm{red}}^i \cap A_{\mathrm{red}}^j = 0 = B_{i,\mathrm{red}} \cap B_{j,\mathrm{red}}, \quad A_{\mathrm{red}}^i \cap B_{j,\mathrm{red}} = \delta_j^i.$$

We thicken these in the usual way to cycles A^i, $B_j \subset \Sigma$ of codimension $1|0$ or dimension $1|1$. Assuming that $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$, define a basis σ_j, $j = 1 \dots g$ of holomorphic sections of $Ber(\Sigma)$ by requiring that the integrals over A-cycles are

$$(8.6) \qquad \oint_{A^i} \sigma_j = \delta_j^i.$$

(The no-jumping result and the existence and uniqueness of the σ_j in the split case ensures that the σ_j exist and are unique even after we turn on the odd moduli.) Then define

$$(8.7) \qquad \widehat{\Omega}_{ij} = \oint_{B_j} \sigma_i.$$

This definition does not make it immediately clear that $\widehat{\Omega}_{ij}$ is symmetric. Proving that it is will be a goal of section 8.2.

When Σ, is split, $\widehat{\Omega}_{ij}$ coincides with the classical period matrix Ω_{ij}. Indeed, from (8.4), we see that in the split case, the periods of a holomorphic section $\sigma = b(z)\theta[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(\Sigma)$ are the same as the periods of the ordinary differential $b(z)\,\mathrm{d}z$ on the reduced space Σ_{red}. When Σ is not split, $\widehat{\Omega}_{ij}$ no longer coincides with Ω_{ij}. Computing the difference between them when Σ is not split [18] will be the goal of section 8.3.

8.2. Symmetry of the super period matrix

To show that the super period matrix is symmetric, we will follow [11] and first show an equivalence between the two types of contour integration on a super Riemann surface. By an elementary though at first sight somewhat mysterious formula, one can map a holomorphic section $\sigma = \phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$ of

$Ber(\Sigma)$ to a closed holomorphic 1-form μ in such a way that

$$
(8.8) \qquad \oint_{\gamma_{\mathrm{red}}} \mu = \oint_{\gamma} \sigma.
$$

The formula is as follows. If $\sigma = \phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$, then

$$
(8.9) \qquad \mu = \mathrm{d}\theta\phi(z|\theta) + \varpi D_\theta\phi(z|\theta).
$$

It is not difficult to verify that $\mathrm{d}\mu = 0$ (the formula (3.70) for the exterior derivative is helpful). Actually, from the formula (8.9), one can verify directly[33] that the map from σ to μ is a 1-1 map from holomorphic volume forms (that is holomorphic sections of $Ber(\Sigma)$) to closed holomorphic 1-forms. The somewhat mysterious formula (8.9) has a more conceptual explanation [30], as will be described in appendix C. This explanation shows that the passage from $\sigma \in H^0(\Sigma, Ber(\Sigma))$ to $\mu \in H^0(\Sigma, T^*\Sigma)$ does not depend on the choice of local superconformal coordinates $z|\theta$.

Explicitly, if $\phi(z|\theta) = \alpha(z) + \theta b(z)$, we compute μ from eqn. (8.9):

$$
(8.10) \qquad \mu = b(z)\mathrm{d}z + \mathrm{d}(\theta\alpha).
$$

In computing $\oint_{\gamma_{\mathrm{red}}} \mu$, the exact form $\mathrm{d}(\theta\alpha)$ can be dropped because actually, one can always pick a superconformal coordinate system valid everywhere on the circle[34] γ_{red} so that the computation leading to eqn. (8.10) is valid throughout γ_{red}. We also have the Berezin integral over θ, giving $\oint_\gamma \sigma = \oint_{\gamma_{\mathrm{red}}} b(z)\mathrm{d}z$. So

$$
(8.11) \qquad \oint_\gamma \sigma = \oint_{\gamma_{\mathrm{red}}} \mu.
$$

For a more conceptual explanation of this result, see appendix C.

Eqn. (8.11) has the following interesting corollary. The formula (8.9) implies that μ is exact – it is of the form $\mathrm{d}s$ for a globally-defined holomorphic

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[33] A general holomorphic 1-form $\mu = s(z|\theta)\mathrm{d}z + t(z|\theta)\mathrm{d}\theta$ (where we will consider s even and t odd), with $s(z|\theta) = s_0(z) + \theta s_1(z)$, $t(z|\theta) = t_0(z) + \theta t_1(z)$, is closed if and only if $t_1 = 0$, $s_1 = \partial_z t_0$, so $\mu = (s_0 + \theta\partial_z t_0)\mathrm{d}z + t_0\mathrm{d}\theta$. This has the form of eqn. (8.9) if and only if $\phi(z|\theta) = t_0 + s_0\theta$.

[34] The sheaf cohomology class that obstructs splitting of Σ is trivial when restricted to a small neighborhood of a circle. So such a neighborhood is isomorphic to $\Pi T^{1/2}\Sigma_0$ (with one of the two possible choices of $T^{1/2}\Sigma_0$, corresponding to NS and R spin structures) where Σ_0 is an ordinary annulus in the complex z-plane.

function s – if and only if $\sigma = D_\theta s$. So

$$(8.12) \qquad \oint_\gamma [\mathrm{d}z|\mathrm{d}\theta] D_\theta s = \oint_{\gamma_{\mathrm{red}}} \mathrm{d}s = 0,$$

giving an alternative proof of an assertion of section 2.4.1.

Since our first step in analyzing the periods of $\sigma \in H^0(\Sigma, Ber(\Sigma))$ has been to equate them to periods of $\mu \in H^0(\Sigma, T^*\Sigma)$, one may ask why we do not merely start with the latter. One answer is that as $Ber(\Sigma)$ is a line bundle, while $T^*\Sigma$ has rank $1|1$, the description by sections of $Ber(\Sigma)$ is often more straightforward. For an example of this, see section 8.3, which would be more complicated if expressed in terms of μ.

If p and q are any closed 1-forms on the ordinary Riemann surface Σ_{red}, one has the fact of classical topology (essentially Riemann's bilinear relation)

$$(8.13) \qquad \int_{\Sigma_{\mathrm{red}}} p \wedge q = \sum_i \left(\oint_{A^i_{\mathrm{red}}} p \oint_{B_{i,\mathrm{red}}} q - \oint_{B_{i,\mathrm{red}}} p \oint_{A^i_{\mathrm{red}}} q \right).$$

If p and q are holomorphic 1-forms – and thus of type $(1,0)$ – then $p \wedge q = 0$, as a result of which eqn. (8.13) simplifies to

$$(8.14) \qquad \oint_{A^i_{\mathrm{red}}} p \oint_{B_{i,\mathrm{red}}} q - \oint_{B_{i,\mathrm{red}}} p \oint_{A^i_{\mathrm{red}}} q = 0.$$

The classical period matrix is defined by introducing holomorphic 1-forms λ_i such that $\oint_{A^i_{\mathrm{red}}} \lambda_j = \delta^i_j$ and then setting $\Omega_{ij} = \oint_{B_{i\,\mathrm{red}}} \lambda_j$. Its symmetry $\Omega_{ij} = \Omega_{ji}$ follows immediately from (8.14) with $p = \lambda_i$, $q = \lambda_j$.

We want to imitate this proof on a super Riemann surface. We have already taken the first step – defining a basis of sections $\sigma_j \in H^0(\Sigma, Ber(\Sigma))$ with canonical A-periods (8.6). The above procedure maps them to a basis of closed holomorphic 1-forms μ_j also with canonical A-periods:

$$(8.15) \qquad \oint_{A^i_{\mathrm{red}}} \mu_j = \delta^i_j.$$

However, in imitating the classical proof, we run into a snag. On a super Riemann surface, two holomorphic 1-forms μ_i and μ_j may have a non-zero wedge product $\mu_i \wedge \mu_j$; for example, $\mathrm{d}z \wedge \mathrm{d}\theta$ is nonzero, and $\mathrm{d}\theta$, being even, can be raised to any power. So it is not immediately obvious that $\int_{\Sigma_{\mathrm{red}}} \mu_i \wedge \mu_j = 0$. It is true that $\mu_i \wedge \mu_j$ is of type $(2,0)$ on Σ, and it is also true that on the ordinary Riemann surface Σ_{red}, a $(2,0)$-form would vanish. But if Σ is not

holomorpically split, the embedding of Σ_{red} in Σ cannot be chosen to be holomorphic, so the restriction to Σ_{red} of a $(2,0)$-form on Σ is not necessarily a form of type $(2,0)$ on Σ_{red}.

A cure for this difficulty was explained in [11]. Though a holomorphic 2-form on Σ need not vanish, such a form, if closed, is always exact. This statement is not true on a general complex supermanifold of dimension $1|1$; the proof uses the superconformal structure of Σ. A general holomorphic 2-form on Σ is $\Psi = (\mathrm{d}\theta)^2 p(z|\theta) + \mathrm{d}\theta \varpi \rho(z|\theta)$, for holomorphic functions p and ρ. The condition that $\mathrm{d}\Psi = 0$ gives $\rho = D_\theta p$ (again it helps here to use (3.70)), from which it follows that $\Psi = \mathrm{d}f$ with $f = -p\,\varpi$. f does not depend on the local superconformal coordinates that were used for this computation,[35] so it is globally-defined.

So now if μ_i are closed holomorphic 1-forms normalized as in (8.15), then $\mu_i \wedge \mu_j$ is exact and thus $\int_{\Sigma_{\mathrm{red}}} \mu_i \wedge \mu_j = 0$. Hence if in (8.13) we take $p = \mu_i$, $q = \mu_j$, we get finally

$$(8.16) \qquad\qquad \widehat{\Omega}_{ij} = \widehat{\Omega}_{ji},$$

with $\widehat{\Omega}_{ij} = \oint_{B^j} \sigma_i$.

8.3. Formula for the super period matrix

Essentially following [18], we will now compute the dependence on odd moduli of the super period matrix of a super Riemann surface. We begin with a split super Riemann surface Σ, with an even spin structure and $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$. We will perturb the complex structure of Σ in the framework of section 3.5 and compute how the period matrix varies. We make the perturbation in the framework of eqn. (3.40), except that we are only interested in varying with respect to odd moduli, since we consider the classical period matrix as a function of even moduli to be a familiar quantity. Of course, the calculation we will perform has a close analog to study the dependence of the classical period matrix on bosonic moduli.

We use the description of a variation of complex structure of Σ given in eqn. (3.40), except that we drop the metric perturbation $h^{\bar{z}}_{z}$ and keep only the gravitino perturbation $\chi^{\theta}_{\bar{z}}$. We define the perturbation by saying that after the perturbation, a holomorphic function is annihilated by

$$(8.17) \qquad\qquad \partial'_{\bar{z}} = \partial_{\bar{z}} + \chi^{\theta}_{\bar{z}}(\partial_\theta - \theta \partial_z).$$

---

[35] To prove this, recall from eqn. (2.29) that there is a natural projection $\kappa : T^*\Sigma \to \mathcal{D}^{-1}$. So there is a natural projection $\kappa^{\otimes 2} : T^*\Sigma \otimes T^*\Sigma \to \mathcal{D}^{-2}$. The object $p\varpi$ is a section of $\mathcal{D}^{-2} \subset T^*\Sigma$ that can be globally defined as $\kappa^{\otimes 2}(\Psi)$.

We take the gravitino field $\chi_{\widetilde{z}}^{\theta}$ to be

$$(8.18) \qquad \chi_{\widetilde{z}}^{\theta} = \sum_{a=1}^{n} \eta_a f_{a\,\widetilde{z}}^{\theta},$$

where the η_a are odd moduli and the f_a are even $(0,1)$-forms valued in $T\Sigma_{\mathrm{red}}^{1/2}$. It is natural to take n to be the odd dimension of $H^1(\Sigma_{\mathrm{red}}, T\Sigma_{\mathrm{red}}^{1/2})$ – which coincides with the odd dimension of the supermoduli space $\mathfrak{M}$ – and the f_a, $a = 1, \ldots, n$, to represent a basis of this space.

What we have described in eqn. (8.17) is a $0|n$-dimensional family of super Riemann surfaces that is transverse to the split locus in $\mathfrak{M}$. We have not given any instructions on how to pick a natural family; in general, we do not know any nice way to do so. The family of super Riemann surfaces determined by the f_a depends not only on their cohomology classes but on the actual $(0,1)$-forms f_a; if we make a gauge transformation on one of the f_a, this will in general transform some of the others into the metric perturbations by $h_{\widetilde{z}}^z$ that were present in the more general deformation of eqn. (3.40). After performing the calculation, we will explain how to include the metric deformations and get a more natural formalism.

We start with a holomorphic section σ of $Ber(\Sigma)$. In local superconformal coordinates $z|\theta$, $\sigma = \phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$, with ϕ a holomorphic function. When we perturb the complex structure of Σ, σ will have to be modified so as to be still a holomorphic section of $Ber(\Sigma)$. It will not suffice to modify the function $\phi(z|\theta)$; we also have to change the symbol $[\mathrm{d}z|\mathrm{d}\theta]$. The reason is that $Ber(\Sigma)$ is defined so that a typical element at a given point in Σ is a symbol $[u|\zeta]$ where $u|\zeta$ is a basis of forms of type $(1,0)$ at the given point. When we perturb the complex structure of Σ as in (8.17), the 1-forms $\mathrm{d}z$ and $\mathrm{d}\theta$ cease to be of type $(1,0)$ and need corrections.

The condition for a 1-form to be of type $(1,0)$ is that its contraction with the vector fields $\partial'_{\widetilde{z}}$ defined in eqn. (8.17) should vanish. This condition is satisfied by $\mathrm{d}z + \chi_{\widetilde{z}}^{\theta}\theta\mathrm{d}\widetilde{z}$ and by $\mathrm{d}\theta + \chi_{\widetilde{z}}^{\theta}\mathrm{d}\widetilde{z}$. So a general section of $Ber(\Sigma)$ after the perturbation takes the form

$$(8.19) \qquad \widehat{\sigma} = \widehat{\phi}(\widetilde{z};z|\theta)\left[\mathrm{d}z + \chi_{\widetilde{z}}^{\theta}\theta\mathrm{d}\widetilde{z}\middle|\mathrm{d}\theta + \chi_{\widetilde{z}}^{\theta}\mathrm{d}\widetilde{z}\right].$$

We have to determine the condition on $\widehat{\phi}$ that will make $\widehat{\sigma}$ holomorphic. We assume that $\widehat{\sigma}$ is even and $\widehat{\phi}$ is odd.

For this, it is convenient to view a section of $Ber(\Sigma)$ as an integral form on Σ of codimension 1. In turn, such a form is a function on $\Pi T\Sigma$, meaning

locally that it is a function of the coordinates $\widetilde{z}$, z, and θ of Σ and of a new set of fiber coordinates $\mathrm{d}\widetilde{z}$, $\mathrm{d}z$, $\mathrm{d}\theta$ of reversed parity. (See for example [32], section 3.2, for an explanation of this formalism.) As a function on ΠTM, $\widehat{\sigma}$ is

$$
\begin{aligned}
\widehat{\sigma} &= \widehat{\phi}(\widetilde{z};z|\theta)(\mathrm{d}z + \chi^{\theta}_{\widetilde{z}}\theta\mathrm{d}\widetilde{z})\delta(\mathrm{d}\theta + \chi^{\theta}_{\widetilde{z}}\mathrm{d}\widetilde{z}) \\
\text{(8.20)} \qquad &= \widehat{\phi}(\widetilde{z};z|\theta)\left(\mathrm{d}z + \chi^{\theta}_{\widetilde{z}}\theta\mathrm{d}\widetilde{z}\right)\left(\delta(\mathrm{d}\theta) + \delta'(\mathrm{d}\theta)\chi^{\theta}_{\widetilde{z}}\mathrm{d}\widetilde{z}\right).
\end{aligned}
$$

What we have written in eqn. (8.20), as a function on $\Pi T\Sigma$, is the most general section of $Ber(\Sigma)$ for the family of super Riemann surfaces defined by the operator $\partial^{\mathcal{L}}_{\widetilde{z}}$ of eqn. (8.17). Now we have to find the condition for such a section to be holomorphic. A small short cut is available here. On an ordinary complex manifold X of complex dimension m, the condition for an $(m,0)$-form to be holomorphic is simply that it should be closed. Similarly on a complex supermanifold, a section of $Ber(\Sigma)$ understood as an integral form is holomorphic if and only if it is closed. In the present context, this means that $\widehat{\sigma}$ should be annihilated by

$$
\text{(8.21)} \qquad \mathrm{d} = \mathrm{d}\widetilde{z}\partial_{\widetilde{z}} + \mathrm{d}z\partial_{z} + \mathrm{d}\theta\partial_{\theta}.
$$

A small computation reveals that

$$
\text{(8.22)} \qquad \mathrm{d}\widehat{\sigma} = -\mathrm{d}\widetilde{z}\,\mathrm{d}z\,\delta(\mathrm{d}\theta)\left(\partial_{\widetilde{z}}\widehat{\phi} - \partial_{z}(\widehat{\phi}\chi^{\theta}_{\widetilde{z}}\theta) + \partial_{\theta}(\widehat{\phi}\chi^{\theta}_{\widetilde{z}})\right).
$$

Let $\widehat{\phi}(\widetilde{z};z|\theta) = \widehat{\alpha}(\widetilde{z};z) + \theta\widehat{b}(\widetilde{z};z)$. From (8.22), the condition that $\mathrm{d}\widehat{\sigma} = 0$ becomes a pair of equations

$$
\begin{aligned}
\text{(8.23)} \qquad & \partial_{\widetilde{z}}\widehat{\alpha} + \widehat{b}\chi^{\theta}_{\widetilde{z}} = 0 \\
& \partial_{\widetilde{z}}\widehat{b} - \partial_{z}\left(\widehat{\alpha}\chi^{\theta}_{\widetilde{z}}\right) = 0.
\end{aligned}
$$

These equations have been obtained in [18].

To understand these formulas better, let γ_{red} be a 1-cycle in Σ_{red} and γ its pullback to Σ. Let us compute $\oint_{\gamma}\widehat{\sigma}$. We integrate first over θ and $\mathrm{d}\theta$. Among functions of the form $\theta^{a}\delta^{(b)}(\mathrm{d}\theta)$ (where $\delta^{(b)}$ is the bth derivative of a delta function), the only nonzero integral is $\int \mathcal{D}(\theta,\mathrm{d}\theta)\,\theta\delta(\mathrm{d}\theta) = 1$. Integrating over θ and $\mathrm{d}\theta$ with the help of this fact, we reduce to an ordinary integral over γ_{red}:

$$
\text{(8.24)} \qquad \int_{\gamma}\widehat{\sigma} = \int_{\gamma_{\mathrm{red}}}\widehat{\rho},
$$

where $\widehat{\rho}$ is a 1-form on Σ_{red}:

$$(8.25) \qquad\qquad \widehat{\rho} = \widehat{b}\,\mathrm{d}z + \widehat{\alpha}\chi^{\theta}_{\widetilde{z}}\mathrm{d}\widetilde{z}.$$

The second equation in (8.23) says that $\mathrm{d}\widehat{\rho} = 0$, and therefore $\int_{\gamma_{\mathrm{red}}} \widehat{\rho}$ and $\int_{\gamma} \widehat{\sigma}$ only depend on the homology class of γ_{red}.

   Because of our assumption that $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$, the first equation in (8.23) has a unique solution for $\widehat{\alpha}$ in terms of $\widehat{b}$. We let $S(z, z')$ be the propagator of the Dirac operator. This is a holomorphic section of the line bundle[36] $K^{1/2} \boxtimes K^{1/2}$ over what we will call $\Sigma_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}$ (the product of two copies of Σ_{red}) with a delta function source on the diagonal:

$$(8.26) \qquad\qquad \partial_{\widetilde{z}}S(z, z') = 2\pi\delta^2(z, z').$$

(We define $\delta^2(z, z')$ by $\int \mathrm{d}^2z\,\delta^2(z, z') = 1$ where $\mathrm{d}^2z = -i\,\mathrm{d}\widetilde{z}\,\mathrm{d}z$. Thus if $z = (u + iv)/\sqrt{2}$, $\widetilde{z} = (u - iv)/\sqrt{2}$, then $\mathrm{d}^2z = \mathrm{d}u\,\mathrm{d}v$. The normalization ensures that $S(z, z') \sim 1/(z - z')$ and has been chosen to agree with [18, 19].) $S(z, z')$ exists because the kernel of the Dirac operator is trivial, in other words because $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$. With the aid of $S(z, z')$, we have[37]

$$(8.27) \qquad \widehat{\alpha}(\widetilde{z}; z) = -\frac{1}{2\pi}\int_{\Sigma'_{\mathrm{red}}} S(z, z')\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}'; z')\,\widehat{b}(\widetilde{z}'; z')\mathrm{d}^2z'.$$

And hence we can write an equation for $\widehat{b}$ only:

$$(8.28) \qquad \partial_{\widetilde{z}}\widehat{b}(\widetilde{z}; z) = -\frac{1}{2\pi}\partial_z\int_{\Sigma'_{\mathrm{red}}} \chi^{\theta}_{\widetilde{z}}(\widetilde{z}; z)S(z, z')\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}'; z')\widehat{b}(\widetilde{z}'; z')\mathrm{d}^2z'$$

   Now let us compute the super period matrix. For this, we start at $\eta_a = 0$ with the holomorphic sections $\sigma_i = \theta b_i(z)[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(\Sigma)$ that are normalized so that $\oint_{A^i} \sigma_j = \delta^i_j$. Equivalently, the ordinary holomorphic 1-form $\mu_j = b_j\mathrm{d}z$ on Σ_{red} obeys

$$(8.29) \qquad\qquad \oint_{A^i_{\mathrm{red}}} \mu_j = \delta^i_j.$$

---

[36]$K^{1/2} \boxtimes K^{1/2}$ is defined as the tensor product $K^{1/2}_{\Sigma_{\mathrm{red}}} \otimes K^{1/2}_{\Sigma'_{\mathrm{red}}}$ of the square roots of the canonical bundles of Σ_{red} and Σ'_{red}. In writing formulas, we use a local complex coordinate z and trivialize K and $K^{1/2}$ by $\mathrm{d}z$ and $(\mathrm{d}z)^{1/2}$.

[37]The integral over Σ'_{red} makes sense because the quantity being integrated is naturally a $(1,1)$-form. In their dependence on z', $\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}'; z')\mathrm{d}\widetilde{z}'$ is a $(0,1)$-form valued in $T\Sigma^{1/2}_{\mathrm{red}} = K^{-1/2}$, $S(z, z')$ is a section of $K^{1/2}$, and $\widehat{b}(\widetilde{z}'; z')\,\mathrm{d}z'$ is $(1,0)$-form.

The deformation of σ_i as a function of the η_a is not uniquely determined, since when we expand $\widehat{\sigma}_i$ as a polynomial in the η_a, at each stage of the expansion we could add a monomial in the η's times one of the σ_j's. We fix this ambiguity by requiring that

$$(8.30) \qquad \oint_{A^i} \widehat{\sigma}_j = \delta^i_j,$$

independent of the η's. Equivalently, setting $\widehat{\rho}_i = \widehat{b}_i \mathrm{d}z + \widehat{\alpha}_i \chi^\theta_{\widetilde{z}} \mathrm{d}\widetilde{z}$, we require

$$(8.31) \qquad \oint_{A^i_{\mathrm{red}}} \widehat{\rho}_j = \delta_{ij}.$$

Alternatively, we define $\widehat{\rho}'_i = \widehat{\rho}_i - \mu_i$, so that $\widehat{\rho}'_i = 0$ at $\eta = 0$. In terms of $\widehat{\rho}'_i$, the condition is

$$(8.32) \qquad \oint_{A^i_{\mathrm{red}}} \widehat{\rho}'_j = 0.$$

We want to compute

$$(8.33) \qquad \widehat{\Omega}_{ij} = \oint_{B_i} \widehat{\sigma}_j = \oint_{B_{i,\mathrm{red}}} \widehat{\rho}_j.$$

Equivalently, we want to compute

$$(8.34) \qquad \widehat{\Omega}_{ij} - \Omega_{ij} = \oint_{B_{j,\mathrm{red}}} (\widehat{\rho}_i - \mu_i) = \oint_{B_{j,\mathrm{red}}} \widehat{\rho}'_i.$$

Applying the topological relation (8.13) with $p = \mu_j$, $q = \widehat{\rho}'_i$, and remembering that μ_j is of type $(1,0)$ on the ordinary Riemann surface Σ_{red}, so that the $(1,0)$ part of $\widehat{\rho}'_i$ does not contribute to $\mu_j \wedge \widehat{\rho}'_i$, we learn that

$$(8.35) \qquad \widehat{\Omega}_{ij} - \Omega_{ij} = \int_{\Sigma_{\mathrm{red}}} \mu_j \wedge \widehat{\rho}'_i = \int_{\Sigma_{\mathrm{red}}} \mu_j \wedge \widehat{\alpha}_i \chi^\theta_{\widetilde{z}} \mathrm{d}\widetilde{z}.$$

In turn, we can use (8.27) to eliminate $\widehat{\alpha}_i$:
$$(8.36)$$
$$\widehat{\Omega}_{ij} - \Omega_{ij} = -\frac{1}{2\pi} \int_{\Sigma_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}} \mu_j(z) \chi^\theta_{\widetilde{z}}(\widetilde{z}; z) \mathrm{d}\widetilde{z}\, S(z, z') \chi^\theta_{\widetilde{z}}(\widetilde{z}'; z') \widehat{b}_i(\widetilde{z}'; z') \mathrm{d}^2 z'.$$

When $\widehat{\Omega}_{ij} - \Omega_{ij}$ is expanded in powers of the η_a's, the lowest order term, which we call $\widehat{\Omega}^{(2)}_{ij}$, is quadratic. It can found by just replacing $\widehat{b}_i(\widetilde{z}'; z') \mathrm{d}z'$ in

the last formula by b_i. To exhibit the symmetry of $\widehat{\Omega}_{ij}^{(2)}$, we use $\mathrm{d}^2 z = -i\mathrm{d}\widetilde{z}\mathrm{d}z$ so that $b_i \mathrm{d}^2 z = -i\mathrm{d}\widetilde{z}\mu_i$ with $\mu_i = b_i \mathrm{d}z$:

$$(8.37) \qquad \widehat{\Omega}_{ij}^{(2)} = \frac{i}{2\pi} \int_{\Sigma_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}} \mu_j(z) \chi_{\widetilde{z}}^\theta(\widetilde{z}; z)\mathrm{d}\widetilde{z}\, S(z, z') \chi_{\widetilde{z}'}^\theta(z')\mathrm{d}\widetilde{z}'\, \mu_i(z').$$

This formula was obtained in [18] (where our χ is called $\chi/2$). Higher order terms can be evaluated by using (8.28) to express $\widehat{b}_i$ as a polynomial in the η's.

To make the expression of $\widehat{\Omega}_{ij}^{(2)}$ even more explicit, we write $\chi_{\widetilde{z}}^\theta = \sum_a \eta_a f_{a\,\widetilde{z}}^\theta$ as in (8.18), and we find
(8.38)
$$\widehat{\Omega}_{ij}^{(2)} = \frac{i}{2\pi} \sum_{a,b=1}^n \eta_a \eta_b \int_{\Sigma_{\mathrm{red}} \times \Sigma'_{\mathrm{red}}} \mu_i(z) f_{a\,\widetilde{z}}^\theta(\widetilde{z}; z)\mathrm{d}\widetilde{z}\, S(z, z') f_{b\,\widetilde{z}'}^\theta(\widetilde{z}'; z')\mathrm{d}\widetilde{z}'\, \mu_j(z').$$

It is interesting now to make a gauge transformation on one of the $(0,1)$-forms $f_{a\,\widetilde{z}}^\theta$,

$$(8.39) \qquad\qquad\qquad f_{a\,\widetilde{z}}^\theta \to f_{a\,\widetilde{z}}^\theta + \partial_{\widetilde{z}} c_a^\theta,$$

where c_a^θ is a section of $T\Sigma_{\mathrm{red}}^{1/2}$. Upon integrating by parts, one finds that $\widehat{\Omega}_{ij}^{(2)}$ is invariant under this gauge transformation of $f_{a\,\widetilde{z}}^\theta$ if and generically only if the support of c_a^θ is disjoint from the support of all the $(0,1)$-forms $f_{b\,\widetilde{z}}^\theta$, $b \neq a$. The meaning of this is that a gauge transformation of $f_{a\,\widetilde{z}}^\theta$ will, if the support of the gauge parameter is not restricted, rotate the perturbation by $f_{b\,\widetilde{z}}^\theta$ into a metric perturbation, namely the perturbation by $h_{\widetilde{z}}^z$ in (3.40). Differently put, as was explained before we began the computation, the particular family of super Riemann surfaces for which we have evaluated the super period matrix depends on the one-forms $f_{a\,\widetilde{z}}^\theta$ and not only on their cohomology classes.

To get a somewhat more intrinsic formula, we should include metric perturbations as well as fluctuations in the gravitino field. We will just briefly sketch how to do this. In doing so, we will specialize to the case that there are only two odd parameters η_a, $a = 1, 2$. The natural thing to do is to consider a general deformation of the super Riemann surface Σ over the ring $\mathbb{C}[\eta_1, \eta_2]$ that is generated over the base field $\mathbb{C}$ by the two odd parameters η_1, η_2. Such a deformation is characterized by the gravitino deformation $\chi_{\widetilde{z}}^\theta = \sum_a \eta_a f_{a\,\widetilde{z}}^\theta$, but also by a metric perturbation, that is a deformation of the complex structure of Σ_{red} by a $(0,1)$-form $\eta_1\eta_2 h_{\widetilde{z}}^z$ that is valued in $T\Sigma_{\mathrm{red}}$. The relevant formalism here has actually been described in detail in section 3.5.1 of [58], which the reader should consult for a fuller explanation. The

three fields (f_1, f_2, h) (which are called χ^1, χ^2, h^{12} in [58]) are subject to the gauge invariance

$$f^\theta_{a\widetilde{z}} \to f^\theta_{a\widetilde{z}} + \partial_{\widetilde{z}} c^\theta_a$$

(8.40)
$$h^z_{\widetilde{z}} \to h^z_{\widetilde{z}} + \partial_{\widetilde{z}} w^z + c^\theta_1 f^\theta_{2\widetilde{z}} - c^\theta_2 f^\theta_{1\widetilde{z}},$$

where w^z is a section of $T\Sigma_{\text{red}}$. This collection of fields and gauge invariance describes the most general deformation of a split super Riemann surface over the ring $\mathbb{C}[\eta_1, \eta_2]$. The super period matrix as a function of the parameters η_1, η_2 is given by the formula

$$\widehat{\Omega}^{(2)}_{ij} = \frac{i}{2\pi} \eta_1 \eta_2 \int_{\Sigma_{\text{red}} \times \Sigma'_{\text{red}}} \left(\mu_i(z) f^\theta_{1\widetilde{z}}(\widetilde{z}; z) \mathrm{d}\widetilde{z} \right.$$
$$\times S(z, z') f^\theta_{2\widetilde{z}'}(\widetilde{z}'; z') \mathrm{d}\widetilde{z}' \, \mu_j(z') + i \leftrightarrow j \Big)$$

(8.41)
$$+ 2\eta_1 \eta_2 \int_{\Sigma_{\text{red}}} \mathrm{d}\widetilde{z} \, h^z_{\widetilde{z}} b_i b_j \mathrm{d}z.$$

The term proportional to $h^z_{\widetilde{z}}$ is the classical expression for the response of the ordinary period matrix to a metric perturbation by $\eta_1 \eta_2 h^z_{\widetilde{z}}$. (To describe this term more intrinsically, the $T\Sigma_{\text{red}}$-valued $(0,1)$-form $\mathrm{d}\widetilde{z} h^z_{\widetilde{z}} \partial_z$ is multiplied by the quadratic differential $\mu_i \mu_j$ to make a $(1,1)$-form that is then integrated.) The reader can verify that gauge invariance is now satisfied.

Going back to eqn. (8.38), we can now inquire about what happens to the super period matrix as Σ_{red} approaches the locus in its moduli space at which $H^0(\Sigma_{\text{red}}, K^{1/2})$ becomes nonzero. At that point, the Dirac propagator S becomes divergent because of a zero-mode contribution; it varies as $1/\varepsilon$ where ε is the small eigenvalue of the Dirac equation. The formula (8.38) shows that the super period matrix develops a pole along this locus. The residue of the pole is bilinear in odd moduli.

8.4. Super period matrix in superstring perturbation theory

The super period matrix has played a striking role in the computation of the superstring vacuum amplitude at two-loop order [21]. This receives a potential contribution only from a genus 2 surface with even spin structure. Here we will describe this application.

If Σ_0 is a Riemann surface of genus 2 with an even spin structure, then $H^0(\Sigma_0, K^{1/2})$ vanishes always, not just generically. This is not true for any

higher genus. For example, if Σ_0 has genus 3, again with an even spin structure, then $H^0(\Sigma_0, K^{1/2})$ can be nonzero if (and only if) Σ_0 is hyperelliptic.[38]

Now let Σ be a super Riemann surface of genus 2 with an even spin structure. Because of the fact mentioned in the last paragraph, the super period matrix $\widehat{\Omega}_{ij}$ of Σ is always defined.

Another important fact is that every 2×2 symmetric complex matrix Ω_{ij} with positive definite imaginary part is the period matrix of a genus 2 Riemann surface that is unique up to isomorphism. (The moduli space of genus 2 surfaces has complex dimension $3\mathsf{g} - 3 = 3$, which coincides with the dimension of the space of symmetric 2×2 matrices.) This actually remains true for genus 3, but for $\mathsf{g} \geq 4$, a generic Ω_{ij} is not a period matrix.

From what has been said in the last paragraph, if Σ is a super Riemann surface of genus 2 with an even spin structure, then there is an ordinary Riemann surface Σ_0 of genus 2 whose period matrix Ω_{ij} coincides with the super period matrix $\widehat{\Omega}_{ij}$ of Σ. Σ is a slightly thickened and deformed version of Σ_0 (thickened by its fermionic dimension and deformed by its odd moduli), so up to homology there is a natural map from Σ to Σ_0. This means that it makes sense to endow Σ_0 with the same spin structure as Σ.

The map from Σ to Σ_0 is a holomorphic map π from $\mathfrak{M}_{2,+}$, the moduli space of super Riemann surfaces of genus 2 with even spin structure, to $\mathcal{M}_{2,\mathrm{Spin}+}$, the moduli space of ordinary Riemann surfaces of genus 2 with an even spin structure. But $\mathcal{M}_{2,\mathrm{Spin}+}$ is the reduced space of $\mathfrak{M}_{2,+}$, and so is naturally embedded in $\mathfrak{M}_{2,+}$ as its subspace that parametrizes split super Riemann surfaces. The map $\pi : \mathfrak{M}_{2,+} \to \mathcal{M}_{2,\mathrm{Spin}+}$ is the identity when restricted to split super Riemann surfaces (since $\widehat{\Omega}_{ij} = \Omega_{ij}$ if the odd moduli are zero), so it is a holomorphic splitting of $\mathfrak{M}_{2,+}$. Given such a holomorphic

---

[38]These assertions are proved as follows. As $K^{1/2}$ is an even spin structure, $H^0(\Sigma_0, K^{1/2})$ has even dimension. If Σ_0 has genus 2, then $K^{1/2}$ has degree $\mathsf{g} - 1 = 1$. In general, a line bundle of degree 1 on a compact Riemann surface Σ_0 of positive genus can have at most a one-dimensional space of holomorphic sections. A nonnegative even integer that is at most 1 must be 0, so so $H^0(\Sigma_0, K^{1/2}) = 0$. If Σ_0 has genus 3, then $K^{1/2}$ has degree $\mathsf{g} - 1 = 2$. In general, a compact Riemann surface Σ_0 of positive genus has a line bundle of degree 2 with a two-dimensional space of holomorphic sections if and only if Σ_0 is hyperelliptic. (One uses the ratio of the two sections to define a map of Σ_0 to $\mathbb{CP}^1$, and the fact that the degree of the line bundle is 2 means that the map is a double cover.) If Σ_0 is a hyperelliptic curve of genus 3, it can be written as a double cover $y^2 = \prod_{i=1}^{8}(x - e_i)$ of $\mathbb{CP}^1$. Consider the holomorphic differential $\omega = \mathrm{d}x/y$, whose zeroes are a pair of double zeroes at $x = \infty$, $y = \pm x^4$. Σ_0 has an even spin structure defined by a square root $K^{1/2}$ of the canonical bundle K that has a two-dimensional space of holomorphic sections, spanned by $\omega^{1/2}$ and $x\omega^{1/2}$.

splitting, we can attempt to calculate superstring scattering amplitudes by integrating first over the odd moduli – that is over the fibers of π – to reduce to an integral over the purely bosonic moduli space $\mathcal{M}_{2,\mathrm{Spin}+}$. This approach has turned out to be very powerful, as reviewed in [21].

As one can learn from that reference and from additional papers cited there, it has also been possible to compute certain scattering amplitudes – as well as the vacuum amplitude – in genus 2. At the present level of understanding, this was a rather difficult calculation that has not yet been fully elucidated in terms of supergeometry.

9. $\mathcal{N} = 2$ super Riemann surfaces and duality

So far our super Riemann surfaces have all been $\mathcal{N} = 1$ super Riemann surfaces, which provide a natural framework for perturbative Type I, Type II, and heterotic superstrings. Here we will describe some rather pretty facts [34–37] about $\mathcal{N} = 2$ super Riemann surfaces, complex supermanifolds of dimension 1|1, and duality. The reader will hopefully help explain what these facts mean for string theory.[39]

9.1. Definition of an $\mathcal{N} = 2$ super Riemann surface

First we explain the definition of an $\mathcal{N} = 2$ super Riemann surface [28]. An $\mathcal{N} = 2$ super Riemann surface is a complex supermanifold W of dimension 1|2 with some additional structure. Its tangent bundle TW is endowed with 2 subbundles $\mathcal{D}_+$ and $\mathcal{D}_-$, each of rank 0|1, with the following properties. $\mathcal{D}_+$ is integrable in the sense that if D_+ is a nonzero section of $\mathcal{D}_+$, then $\{D_+, D_+\} = 2D_+^2$ is a multiple of D_+,

$$(9.1) \qquad D_+^2 = uD_+,$$

for some function u. Thus, "integrability" means that the sections of $\mathcal{D}_+$ form a graded Lie algebra, without needing to introduce D_+^2 as an independent generator. Similarly $\mathcal{D}_-$ is integrable, in the sense that if D_- is a nonzero section of $\mathcal{D}_-$, then

$$(9.2) \qquad D_-^2 = vD_-,$$

again for some function v. However, the direct sum $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ is non-integrable, the obstruction being that, for any nonzero sections D_+ and D_-

---

[39]The papers [59, 60] may provide a clue.

of $\mathcal{D}_+$ and $\mathcal{D}_-$, the anticommutator $\{D_+, D_-\}$ is linearly independent of D_+ and D_-. Thus, D_+, D_-, and $\{D_+, D_-\}$ form a basis of TW. From this it follows that the quotient $TW/(\mathcal{D}_+ \oplus \mathcal{D}_-)$ is naturally isomorphic to $\mathcal{D}_+ \otimes \mathcal{D}_-$, so that there is an exact sequence

$$(9.3) \qquad\qquad 0 \to \mathcal{D}_+ \oplus \mathcal{D}_- \to TW \to \mathcal{D}_+ \otimes \mathcal{D}_- \to 0.$$

Reasoning as in appendix A, one can deduce that $Ber(W)$ is canonically trivial. This is relevant among other things for writing Lagrangians.

By arguments similar to those in section 2.1, one can show that locally, there exist superconformal coordinates $z|\theta^+, \theta^-$ in which $\mathcal{D}_+$ and $\mathcal{D}_-$ are generated by

$$
\begin{aligned}
D_+ &= \frac{\partial}{\partial \theta^+} + \theta^- \frac{\partial}{\partial z} \\
(9.4) \qquad\qquad D_- &= \frac{\partial}{\partial \theta^-} + \theta^+ \frac{\partial}{\partial z}.
\end{aligned}
$$

These obey

$$(9.5) \qquad\qquad D_+^2 = D_-^2 = 0, \quad \{D_+, D_-\} = 2\partial_z.$$

Most of what we have described in section 2 has an analog for $\mathcal{N} = 2$. For instance, one can define a sheaf $\mathcal{S}$ of vector fields that preserve the $\mathcal{N} = 2$ structure (its sections are explicitly described in eqns. (9.12) and (9.13)), and a general $\mathcal{N} = 2$ super Riemann surface can be constructed by gluing together open sets in which the $\mathcal{N} = 2$ structure takes the standard form described in eqn. (9.4).

Since $D_-^2 = 0 = D_+^2$, one can look for a holomorphic function that is annihilated by D_- or by D_+. Since we can write

$$
\begin{aligned}
D_- &= \exp(-\theta^-\theta^+\partial_z)\frac{\partial}{\partial\theta^-}\exp(\theta^-\theta^+\partial_z) \\
(9.6) \qquad\qquad D_+ &= \exp(\theta^-\theta^+\partial_z)\frac{\partial}{\partial\theta^+}\exp(-\theta^-\theta^+\partial_z),
\end{aligned}
$$

a function annihilated by D_- is concretely of the form

$$(9.7) \qquad\qquad \exp(-\theta^-\theta^+\partial_z)f(z|\theta^+) = f(z - \theta^-\theta^+|\theta^+)$$

and a function annihilated by D_+ is concretely of the form

$$(9.8) \qquad\qquad \exp(\theta^-\theta^+\partial_z)g(z|\theta^-) = g(z + \theta^-\theta^+|\theta^-).$$

Functions annihilated by D_- or by D_+ are called chiral or antichiral, respectively.

Geometrically, one can think of a chiral function as a function on a space X of dimension $1|1$ parametrized by $z - \theta^-\theta^+|\theta^+$, and an antichiral function as a function on another space X' of dimension $1|1$ parametrized by $z + \theta^-\theta^+|\theta^-$. Here is a slightly different way to define X and X' that does not depend on a choice of coordinates. Integrability of $\mathcal{D}_-$ means that W is fibered by the orbits generated by any nonzero section D_- of $\mathcal{D}_-$; those orbits are independent of the choice of D_-. X is the space of these orbits,[40] which we call simply the orbits of $\mathcal{D}_-$. X' is similarly the space of orbits of $\mathcal{D}_+$.

Thus an $\mathcal{N} = 2$ super Riemann surface W has maps to two different $1|1$-dimensional supermanifolds X, X' with the properties that functions on X or X' correspond to chiral or antichiral functions on W:

$$\text{(9.9)} \qquad \begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ X & & X'. \end{array}$$

It turns out that individually X and X' have no special structure at all, beyond being complex supermanifolds of dimension $1|1$. But there is a very special relationship between them, which is the subject of section 9.2.

A quick way (see [61] and [35]) to show that X (or X') has no structure beyond being a complex supermanifold of dimension $1|1$ is to consider the action on X of the Lie algebra of $\mathcal{N} = 2$ superconformal vector fields. A superconformal vector field on the $\mathcal{N} = 2$ super Riemann surface W is a holomorphic vector field that preserves the two subbundles $\mathcal{D}_+$ and $\mathcal{D}_-$ of TW. Eqn. (9.7) implies that in the coordinate system $w|\theta^+, \theta^-$ where $w = z - \theta^-\theta^+$, the subbundle $\mathcal{D}_-$ is generated by

$$\text{(9.10)} \qquad D_- = \frac{\partial}{\partial\theta^-}.$$

Similarly, in these coordinates, $\mathcal{D}_+$ is generated by

$$\text{(9.11)} \qquad D_+ = \frac{\partial}{\partial\theta^+} + 2\theta^-\partial_w.$$

---

[40]Roughly, integrability is important here because if D_-^2 is not a multiple of D_-, then to make a Lie supergroup with D_- in its Lie algebra, we would have to include also D_-^2 as a generator of the Lie algebra, and then the orbits would not have dimension $0|1$. One could still define curves in W generated by D_-, but these curves would not be fibers of a fibration. See section 9.3.

The even superconformal vector fields on W take the form

$$V_g = g(w)\partial_w + \frac{\partial_w g}{2}\left(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-}\right)$$

(9.12) $$v_k = k(w)\left(\theta^-\partial_{\theta^-} - \theta^+\partial_{\theta^+}\right).$$

The odd ones are

$$\nu_{\alpha^+} = \alpha^+(w)\partial_{\theta^+}$$

(9.13) $$\nu_{\alpha^-} = \left(\alpha^-(w) + 2\partial_w\alpha^-(w)\theta^-\theta^+\right)\partial_{\theta^-} - 2\alpha^-(w)\theta^+\partial_w.$$

Here g, k, and $\alpha^\pm$ are functions of w only. The vector fields in (9.12) and (9.13), which are sections of the sheaf $\mathcal{S}$ of superconformal vector fields, generate what is usually called the $\mathcal{N} = 2$ superconformal algebra. To find the action of this algebra on X, we simply drop all terms proportional to ∂_{θ^-}, since $X = W/\mathcal{D}_-$ is the space of orbits of ∂_{θ^-}, which generates $\mathcal{D}_-$. So as vector fields on X, the $\mathcal{N} = 2$ generators are

$$V_g = g(w)\partial_w + \frac{\partial_w g}{2}\theta^+\partial_{\theta^+}$$

$$v_k = -k(w)\theta^+\partial_{\theta^+}$$

$$\nu_{\alpha^+} = \alpha^+(w)\partial_{\theta^+}$$

(9.14) $$\nu_{\alpha^-} = -2\alpha^-(w)\theta^+\partial_w.$$

An arbitrary holomorphic vector field $f(w|\theta^+)\partial_w + \rho(w|\theta^+)\partial_{\theta^+}$ can be written as a linear combination of these $\mathcal{N} = 2$ generators in a unique fashion, so the $\mathcal{N} = 2$ superconformal symmetries are simply the automorphisms of X as a 1|1 supermanifold.

A priori, the sheaf $\mathcal{S}$ of superconformal vector fields on W is a sheaf of Lie algebras over W, but not a sheaf of $\mathcal{O}_W$ modules (a superconformal vector field cannot be multiplied by a holomorphic function on W to give another superconformal vector field). However, we can proceed rather as we did for $\mathcal{N} = 1$ super Riemann surfaces in section 2.3. Viewing $\mathcal{S}$ as a subsheaf of the sheaf TW of all holomorphic vector fields on W, we can use the exact sequence (9.3) and project $\mathcal{S}$ to $\mathcal{D}_+\otimes\mathcal{D}_-$. A short computation using the above formulas reveals that this projection is an isomorphism, so $\mathcal{S}$ is isomorphic as a sheaf to the sheaf of sections of the line bundle $\mathcal{D}_+ \otimes \mathcal{D}_- \to W$.

9.2. 1|1 supermanifolds and duality

Let X be any complex supermanifold of dimension 1|1. Locally, we can parametrize X by holomorphic coordinates $z|\psi$.

Now we want to consider minimal divisors in X. By a minimal divisor we mean a subvariety of X of dimension $0|1$ whose intersection with the reduced space X_{red} is a single point. Such a divisor is defined by an equation of the form

$$(9.15) \qquad\qquad z = y + \eta\psi,$$

where y and η parametrize the divisor. So the space of such minimal divisors is a $1|1$-dimensional supermanifold X', locally parametrized by $y|\eta$.

Actually, the relation between X and X' is symmetrical. With y and η viewed as parameters, eqn. (9.15) defines a minimal divisor in X, but if instead we view z and ψ as parameters, then eqn. (9.15) defines a minimal divisor in X'. So X parametrizes minimal divisors in X', and vice-versa.

Another way to explain the symmetric nature of the relationship between X and X' is as follows. Suppose that X' is the space of minimal divisors in X. To a point $z|\psi$ in X, we associate the space of all minimal divisors that pass through this point. This is a $0|1$-dimensional family of divisors, so it represents a divisor in X'. (To show this explicitly, we would go back to the equation (9.15) which in local coordinates describes the family of minimal divisors that pass through $z|\psi$.) So again, just as a point in X' determines a minimal divisor in X, so a point in X determines a minimal divisor in X'.

Now we will define a supermanifold W of dimension $1|2$. One way to define W is that a point in W is a pair consisting of a point in X and a minimal divisor through that point. Because points in X correspond to minimal divisors in X' and vice-versa, we can equally well say that a point in W is a point in X' together with a divisor through that point. If we use local coordinates $z|\psi$ on X and local coordinates $y|\eta$ on X' as described above so that the divisor in X corresponding to $y|\eta$ is described by the equation (9.15), then a point in W is simply described by the whole collection of coordinates $y, z|\eta, \psi$, subject to the condition (9.15). In other words, locally W is defined as the submanifold of $C^{2|2}$ defined by $G = 0$ where $G = z - y - \eta\psi$.

So now we have a complex supermanifold W of dimension $1|2$ with maps to two different supermanifolds X and X', each of dimension $1|1$. In fact, W has the natural structure of an $\mathcal{N} = 2$ super Riemann surface. Abstractly, the subbundles $\mathcal{D}_+$ and $\mathcal{D}_-$ of TW are the tangents to the fibers of the fibrations $W \to X$ and $W \to X'$. More concretely, we define $\mathcal{D}_-$ to be spanned by vector fields on W that commute with $z|\psi$ (the local coordinates of X). A section of $\mathcal{D}_-$ can be represented by a vector field on $\mathbb{C}^{2|2}$ that commutes with z, ψ, and G. Such a vector field is a multiple of

$$(9.16) \qquad\qquad D_- = \frac{\partial}{\partial\eta} - \psi\frac{\partial}{\partial y}.$$

Similarly, a section of $\mathcal{D}_+$ can be represented by a vector field on $\mathbb{C}^{2|2}$ that commutes with y, η, and G. Such a vector field is a multiple of

$$(9.17) \qquad\qquad D_+ = \frac{\partial}{\partial \psi} - \eta \frac{\partial}{\partial z}.$$

This exhibits the structure of an $\mathcal{N} = 2$ super Riemann surface: we have $D_-^2 = D_+^2 = 0$, while $\{D_-, D_+\} = -\partial_y - \partial_z$ is everywhere linearly independent of D_- and D_+.

   So every $1|1$ supermanifold X canonically determines an $\mathcal{N} = 2$ super Riemann surface W. And conversely, we can reconstruct X from W as roughly $W/\mathcal{D}_-$, as described in section 9.1.

   This construction has an amusing variant that we will describe briefly. Starting with X, we introduce $\Pi T X$, the tangent bundle with statistics reversed on the fiber. This object was studied, for example, in [32], section 3.2. If we describe X by local coordinates $z|\psi$, then to describe $\Pi T X$, we introduce additional coordinates $\mathrm{d}\psi$ and $\mathrm{d}z$, where $\mathrm{d}\psi$ is even and $\mathrm{d}z$ is odd. On $\Pi T X$ there is a natural odd vector field

$$(9.18) \qquad\qquad \mathrm{d} = \mathrm{d}z \frac{\partial}{\partial z} + \mathrm{d}\theta \frac{\partial}{\partial \theta}.$$

It is usually called the exterior derivative on X. It obeys $\mathrm{d}^2 = 0$.

   There is a scaling symmetry of $\Pi T X$ that rescales the fiber coordinates, $\mathrm{d}\theta | \mathrm{d}z \to \lambda \mathrm{d}\theta | \lambda \mathrm{d}z$, where λ is even and nonzero. We define $W = \mathbb{P}(\Pi T X)$ to be the projectivization of $\Pi T X$. This projectivization is obtained by constraining the even fiber coordinates (in our present problem, only $\mathrm{d}\theta$) to be not all zero and then dividing by $\mathbb{C}^*$. Locally, we can use the $\mathbb{C}^*$ action to map $\mathrm{d}\theta$ to 1, after which W is a fiber bundle over X with fibers of rank $0|1$ parametrized by $\mathrm{d}z$. We define $\mathcal{D}_-$ to be the subbundle of TW generated by

$$(9.19) \qquad\qquad D_- = \frac{\partial}{\partial \mathrm{d}z}.$$

Clearly $D_-^2 = 0$, so $\mathcal{D}_-$ is integrable. The quotient $W/\mathcal{D}_-$ is just X.

   The vector field d on $\Pi T X$ is not invariant under scaling, so it does not descend to a vector field on W in a natural way. However, scaling of d does not affect the subbundle of $\Pi T X$ generated by d, so that subbundle does descend in a natural way to a subbundle $\mathcal{D}_+$ of TW of rank $0|1$. Locally, $\mathcal{D}_+$ is generated by what we get from d upon using the scaling to set $\mathrm{d}\theta = 1$. Thus, $\mathcal{D}_+$ is generated by

$$(9.20) \qquad\qquad D_+ = \mathrm{d}z \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}.$$

Again we see the structure of an $\mathcal{N} = 2$ super Riemann surface, with $D_-^2 = D_+^2 = 0$, $\{D_-, D_+\} = \partial_z$.

The relation of this description of W to the description in terms of minimal divisors in X is simply the following. At a point in $p \in W$, a nonzero vector $\mu \in \Pi T X|_p$ (where $\Pi T X|_p$ is the fiber of $\Pi T X$ at p) determines a minimal divisor through p, namely the one obtained by displacing p in the μ direction. This divisor only depends on μ up to scaling, so the space of minimal divisors through p is the fiber of $\mathbb{P}(\Pi T X)$.

9.3. Self-duality of $\mathcal{N} = 1$ super Riemann surfaces

Since we have found a natural duality for every $1|1$-dimensional complex supermanifold X, the question now arises of what happens if X is actually a super Riemann surface.

Given any odd vector field ν on X, we regard ν as generating a superdiffeomorphism of X. Starting at any point and acting with $\exp(\alpha\nu) = 1 + \alpha\nu$, where α is an odd parameter, we generate a $0|1$-dimensional submanifold of X that passes through the given point. This much holds for any odd vector field, but in going farther there are two fundamentally different cases: ν may be integrable or non-integrable. Integrability means that (after possibly multiplying ν by a nonzero function, an operation that does not affect the curves it generates) $\nu^2 = 0$. If so, we can find local coordinates $z|\theta$ on X with $\nu = \partial_\theta$. Then ν has a family of orbits of dimension $1|0$, parametrized by z, and X is fibered over this orbit space.

A super Riemann surface X is at the opposite extreme. Locally, the super-conformal structure determines, up to multiplication by a nonzero function (which does not affect the following) a natural odd vector field on X, namely

$$(9.21) \qquad D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}.$$

But D_θ is everywhere non-integrable, that is, D_θ^2 is everywhere nonzero and linearly independent from D_θ. Given any point $z|\theta = z_0|\theta_0$ in X, the curve generated by D_θ passing through this point is given in parametric form as

$$\theta = \theta_0 + \alpha$$
$$(9.22) \qquad z = z_0 + \alpha\theta$$

with odd parameter α. Equivalently, it is given by an equation

$$(9.23) \qquad z = z_0 - \theta_0\theta.$$

These formulas were already introduced in section 4.1.1.

What has happened as a result of the non-integrability of $\mathcal{D}$ is that these $0|1$-dimensional varieties do not fit together into a fibration of X. There are too many of them; the family of divisors in (9.23) has dimension $1|1$, with parameters z_0 and θ_0, and not $1|0$, as in the integrable case.

We see in (9.23) that a point $z_0|\theta_0 \in X$ naturally determines a minimal divisor in X, and every such divisor occurs for a unique $z_0|\theta_0$. This means that X is self-dual under the duality of $1|1$ supermanifolds: X parametrizes its own minimal divisors. More specifically, the holomorphic isomorphism $\varphi : X \cong X'$ is such that for $z_0|\theta_0 \in X$, the divisor in X corresponding to $\varphi(z_0|\theta_0)$, namely the divisor defined in (9.23), passes through $z_0|\theta_0$.

Conversely, suppose that X is a $1|1$-dimensional complex manifold and that we are given an isomorphism $\varphi : X \cong X'$, where X' parametrizes minimal divisors in X and for $z_0|\theta_0 \in X$, the divisor $\varphi(z_0|\theta_0)$ passes through $z_0|\theta_0$. Then we can define a subbundle $\mathcal{D} \subset TX$ of rank $0|1$ by saying that the fiber of $\mathcal{D}$ at $z_0|\theta_0$ is the tangent space to $\varphi(z_0|\theta_0)$. $\mathcal{D}$ is everywhere nonintegrable or φ would not be an isomorphism. (We have just seen that if $\mathcal{D}$ is integrable, it does not have enough orbits for φ to be an isomorphism.) So X has a natural super Riemann surface structure.

Appendix A. More on Berezinians

In section 2.4, we established an isomorphism between $Ber(\Sigma)$ and $\mathcal{D}^{-1}$. Here we explain an alternative proof of this isomorphism that requires less computation and more knowledge about Berezinians.

First, recall that for every vector space V of dimension $a|b$, one defines a one-dimensional vector space $\mathrm{Ber}(V)$, the Berezinian of V, of statistics $(-1)^b$. As explained for instance in section 3.1 of [32], to every basis $e_1 \ldots | \ldots \rho_b$ of V, there is a corresponding basis element $[e_1 \ldots | \ldots \rho_b]$ of $\mathrm{Ber}(V)$. If a new basis $e'_1 \ldots | \ldots \rho'_b$ is obtained from $e_1 \ldots | \ldots \rho_b$ by a linear transformation W, then $[e'_1 \ldots | \ldots \rho'_q] = \mathrm{Ber}(W)[e_1 \ldots | \ldots \rho_b]$. (Here $\mathrm{Ber}(W)$ is the Berezinian of the linear transformation W, the superanalog of the determinant.) If V varies as the fibers of a vector bundle $R \to X$ for some supermanifold X, then by applying this construction fiberwise, we make a line bundle over X that we call $\mathrm{Ber}(R)$, the Berezinian of R. If R is a holomorphic vector bundle over a complex supermanifold, then $\mathrm{Ber}(R)$ is a holomorphic line bundle, and in that case we denote it as $Ber(R)$.

We will phrase the argument that follows using a slightly different definition of the Berezinian of a vector bundle. (This alternative definition is presented here for variety, but will actually be slightly less precise than what

was said in the last paragraph, since – in the form described here – we will define the Berezinian line bundle only up to isomorphism.) We leave it to the reader to relate the two definitions. Picking an open cover U_α of X, suppose that the vector bundle R can be defined by transition functions $r_{\alpha\beta}$ on intersections $U_\alpha \cap U_\beta$, obeying the usual conditions such as the requirement that

$$(A.1) \qquad\qquad\qquad r_{\alpha\beta} r_{\beta\gamma} r_{\gamma\alpha} = 1$$

on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$. Then we define the line bundle $\mathrm{Ber}(R)$ via the transition functions $r^*_{\alpha\beta} = \mathrm{Ber}(r_{\alpha\beta})$. The multiplicative property of the Berezinian of a matrix together with (A.1) ensures that $r^*_{\alpha\beta} r^*_{\beta\gamma} r^*_{\gamma\alpha} = 1$ in $U_\alpha \cap U_\beta \cap U_\gamma$, so the objects $r^*_{\alpha\beta}$ really are transition functions of a line bundle, which we will call $\mathrm{Ber}(R)$. If R has purely bosonic fibers, $\mathrm{Ber}(R)$ reduces to what is usually called the determinant line bundle of R. If R is a holomorphic vector bundle over a complex supermanifold, then $\mathrm{Ber}(R)$ is obviously a holomorphic line bundle.

Now suppose that we are given an exact sequence of vector bundles over X,

$$(A.2) \qquad\qquad\qquad 0 \to Q \to R \to S \to 0.$$

This means that transition functions of R can be put in the triangular form

$$(A.3) \qquad\qquad\qquad r_{\alpha\beta} = \begin{pmatrix} q_{\alpha\beta} & c_{\alpha\beta} \\ 0 & s_{\alpha\beta} \end{pmatrix},$$

where $q_{\alpha\beta}$ and $s_{\alpha\beta}$ are, respectively, transition functions for Q and for S. (q, r, and s all act in general on both odd and even variables; we have not tried to display this in (A.3).) From this triangular form, it follows that

$$(A.4) \qquad\qquad\qquad \mathrm{Ber}(r_{\alpha\beta}) = \mathrm{Ber}(q_{\alpha\beta}) \mathrm{Ber}(s_{\alpha\beta}).$$

$\mathrm{Ber}(q_{\alpha\beta})$ and $\mathrm{Ber}(s_{\alpha\beta})$ are transition functions for the line bundles $\mathrm{Ber}(Q)$ and $\mathrm{Ber}(S)$, so their products $\mathrm{Ber}(q_{\alpha\beta}) \mathrm{Ber}(s_{\alpha\beta})$ are transition functions for the tensor product $\mathrm{Ber}(Q) \otimes \mathrm{Ber}(S)$. Thus (A.4) gives a natural isomorphism $\mathrm{Ber}(R) \cong \mathrm{Ber}(Q) \otimes \mathrm{Ber}(S)$ for every exact sequence (A.2). This generalizes the analogous isomorphism for determinant line bundles in the purely bosonic case. The isomorphism $\mathrm{Ber}(R) \cong \mathrm{Ber}(Q) \otimes \mathrm{Ber}(S)$ is clearly holomorphic in the case of an exact sequence of holomorphic vector bundles.

The cotangent bundle of a super Riemann surface appears in the exact sequence (2.29):

$$0 \to \mathcal{D}^{-2} \to T^*\Sigma \to \mathcal{D}^{-1} \to 0. \tag{A.5}$$

This implies a natural isomorphism

$$Ber(T^*\Sigma) \cong Ber(\mathcal{D}^{-2}) \otimes Ber(\mathcal{D}^{-1}). \tag{A.6}$$

Now if λ is a nonzero complex number which we regard as a 1×1 matrix acting on a vector space E of dimension $1|0$, then $Ber\,\lambda = \lambda$, but if E has dimension $0|1$, then $Ber\,\lambda = \lambda^{-1}$. This follows directly from the definition of the Berezinian of a matrix. For a line bundle $\mathcal{L} \to X$, considering that the passage from $\mathcal{L}$ to $Ber(\mathcal{L})$ involves taking the Berezinians of 1×1 matrices (namely the transition functions of $\mathcal{L}$), we see that $Ber(\mathcal{L}) \cong \mathcal{L}$ if $\mathcal{L}$ has bosonic fibers, but $Ber(\mathcal{L}) \cong \mathcal{L}^{-1}$ if $\mathcal{L}$ has fermionic fibers. In (A.6), $\mathcal{D}^{-2}$ has bosonic fibers, while $\mathcal{D}^{-1}$ is fermionic, so finally we get $Ber(T^*\Sigma) \cong \mathcal{D}^{-2} \otimes \mathcal{D} \cong \mathcal{D}^{-1}$. But $Ber(T^*\Sigma)$ is what we usually call $Ber(\Sigma)$.

Appendix B. More on pictures

Here we will make a few remarks on physical states or vertex operators in superstring theory for different values of the picture number. The goal is to make contact with observations of section 4.3 about how the moduli space of super Riemann surfaces can be modified to accommodate different picture numbers.

The commuting ghosts of superstring theory are a pair of free fields β and γ of conformal dimension $3/2$ and $-1/2$, respectively. (They appear in eqn. (3.12) along with their anticommuting partners.) Their mode expansions read

$$\beta(z) = \sum_n z^{-n-3/2}\beta_n, \quad \gamma(z) = \sum_n z^{-n+1/2}\gamma_n. \tag{B.1}$$

n takes values in $\mathbb{Z} + \varepsilon$, where $\varepsilon = 1/2$ in the NS sector, and 0 in the Ramond sector. The ghost vacuum $|q\rangle$ with picture number q is defined by the conditions

$$\begin{aligned}\beta_n|q\rangle &= 0, \quad n > -q - 3/2 \\ \gamma_n|q\rangle &= 0, \quad n \geq q + 3/2.\end{aligned} \tag{B.2}$$

Superstring theory also has anticommuting ghosts b and c of conformal dimensions 2 and -1, respectively, and expansions

$$(B.3) \qquad b(z) = \sum_n z^{-n-2} b_n, \quad c(z) = \sum_n z^{-n+1} c_n.$$

We want to consider a physical state $|\Psi\rangle$ of the combined system for which the commuting ghosts are in the state $|q\rangle$. We also assume that

$$c_n|\Psi\rangle = 0, \quad n \geq 1$$
$$(B.4) \qquad b_n|\Psi\rangle = 0, \quad n \geq 0.$$

(For free fermions as opposed to free bosons, there is no close analog of picture number, so we do not introduce a free parameter here.) Thus the dependence of $|\Psi\rangle$ on the superghosts is completely fixed, but so far we have said nothing about the matter part of the state.

A physical state must be annihilated by the BRST operator

$$(B.5) \qquad Q = \sum_{n \in \mathbb{Z}+\varepsilon} \gamma_n G_{-n} + \sum_{m \in \mathbb{Z}} c_m L_{-m} + \dots$$

where G_n and L_m are respectively the odd and even generators of the super Virasoro algebra. These operators can be realized by the superconformal vector fields

$$(B.6) \qquad G_n = z^{n+1/2}(\partial_\theta - \theta \partial_z), \quad L_m = -\left(z^{m+1}\partial_z + \frac{m+1}{2} z^m \theta \partial_\theta \right),$$

from which one can read off their commutation relations. The conditions for a state of the form described above to be annihilated by Q are

$$G_n|\Psi\rangle = 0, \quad n \geq -q - 1/2$$
$$(B.7) \qquad L_n|\Psi\rangle = 0, \quad n \geq 0.$$

So, setting $r = n + 1/2$, the state $|\Psi\rangle$ is annihilated by the operators that represent the vector fields

$$(B.8) \qquad z^r(\partial_\theta - \theta \partial_z), \quad r \geq -q,$$

as well as

$$(B.9) \qquad z^{m+1}\partial_z + \frac{m+1}{2} z^m \theta \partial_\theta, \quad m \geq 0.$$

We see that this construction will not work unless $q < 0$. If $q = 0$, the constraints (B.7) include $G_{-1/2}|\Psi\rangle = 0$, which implies that $|\Psi\rangle$ is annihilated by $L_{-1} = G^2_{-1/2}$; when added to the constraints (B.7), this becomes too strong and forces the matter part of the state $|\Psi\rangle$ to be trivial. Things are even worse if $q > 0$. Then we would get the constraint generated by $G_{-3/2}$ and by the time we close the constraints on a Lie algebra, we would find that $|\Psi\rangle$ is annihilated by the entire super Virasoro algebra.

However, the above construction makes perfectly good sense if $q < 0$. In fact, in section 4.3, we made a proposal for what should be the sheaf $\mathcal{S}'$ of superconformal vector fields in the presence of a puncture associated to a vertex operator of picture number $q < 0$. The proposal was precisely that sections of $\mathcal{S}'$ should have the behavior given in eqns. (B.8) and (B.9) near $z = \theta = 0$.

Since the constraints obeyed by the vertex operators that we have just introduced are those in eqn. (B.7), the vector fields that we should regard as symmetry generators when we construct a moduli space of super Riemann surfaces in which those vertex operators are inserted are those in eqns. (B.8) and (B.9). The corresponding supermoduli spaces make perfect sense and were described in section 4.3. These supermoduli spaces coincide with the standard ones if and only if q has the canonical value -1 for the NS sector, or $-1/2$ for the R sector. Precisely in these cases, the appropriate vertex operators of the matter sector are the usual superconformal primaries.

For q less than the canonical values -1 and $-1/2$, the conditions obeyed by the vertex operators are different. We lose some constraints on $|\Psi\rangle$, but we gain some gauge-invariances $|\Psi\rangle \to |\Psi\rangle + G_n|\chi\rangle$, $0 \leq n \leq -q - 3/2$. The net effect is that the space of physical states is the same. For example, see [63, 64] for proofs of this assertion. For $q < -1$ or $q < -1/2$, the extra states $G_n|\chi\rangle$ that should decouple from superstring scattering amplitudes are projected out when one integrates over the extra odd moduli that exist for the given value of q.

What about vertex operators with picture number $q \geq 0$? They certainly exist as well; inevitably, in view of what we have explained, they take a more complicated form than the operators described above. In fact, in [44], a natural picture-changing operation was described that increases q by 1. This operation is very useful in computing perturbative superstring amplitudes of low orders. In [25], it was interpreted as expressing the result of integrating over one of the odd moduli. (The odd modulus in question is represented by a gravitino with delta function support at the position of the vertex operator.) In genus zero, one can certainly calculate with vertex operators of any picture number. It seems unlikely – at least to the author – that in general there

is a natural recipe to calculate genus g scattering amplitudes using vertex operators of $q \geq 0$, because there is no good way in general to remove an odd modulus from the supermoduli space $\mathfrak{M}$ defined at a canonical picture number.

Appendix C. Picture-changing on a super Riemann surface

In section 8.2, we made use of a map described in [11] from a holomorphic section σ of $Ber(\Sigma)$ to a closed holomorphic 1-form μ. In local superconformal coordinates $z|\theta$, this map takes $\sigma = \phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$ to the closed holomorphic 1-form

$$(C.1) \qquad \mu = \phi(z|\theta)\mathrm{d}\theta + (\mathrm{d}z - \theta\mathrm{d}\theta)D_\theta\phi.$$

(We take σ to be even and $\phi(z|\theta)$ to be odd.)

It is not immediately obvious why this map does not depend on the choice of local superconformal coordinates, or why μ is closed. However, a natural explanation has been given in [30]. First of all, since σ is an integral form while μ is a differential form, the map from σ to μ involves "picture-changing." Thus, we can represent σ as a function on $\Pi T\Sigma$ (this formalism is reviewed in section 3.2 of [32]):

$$(C.2) \qquad \sigma = \phi(z|\theta)\mathrm{d}z\delta(\mathrm{d}\theta).$$

Similarly, μ as already written in eqn. (C.1) is naturally understood as a function on $\Pi T\Sigma$. But these are different kinds of function; μ has a polynomial dependence on $\mathrm{d}\theta$ while σ in its dependence on $\mathrm{d}\theta$ is a delta function supported at the origin. A transformation from one type of function to the other is analogous to picture-changing in superstring perturbation theory.

A general picture-changing operation on a supermanifold M has been defined in [30] and was reviewed in [32], section 4.2. The operation can be defined whenever one is given a rank $0|1$ subbundle L of TM. So a natural example is the case that M is a super Riemann surface Σ, with L being the rank $0|1$ subbundle $\mathcal{D} \subset T\Sigma$ that defines the superconformal structure. Roughly, picture-changing is defined by integration over the orbits generated by sections of L. Any nonzero local section of L can be used to generate the orbits and evaluate the picture-changing transformation. In the super Riemann surface case, we can work in superconformal coordinates $z|\theta$ and pick the usual section $D_\theta = \partial_\theta + \theta\partial_z$ of $\mathcal{D}$. The coordinate transformation generated by D_θ is

$$z \to z + \eta\theta$$

(C.3) $$\theta \to \theta + \eta$$

with a fermionic parameter η. This change of coordinates maps σ to

$$\sigma^* = \phi(z + \eta\theta|\theta + \eta)\mathrm{d}(z + \eta\theta)\delta(\mathrm{d}\theta + \mathrm{d}\eta)$$

(C.4) $$= (\phi + \eta D_\theta \phi)(\mathrm{d}z - \eta\mathrm{d}\theta + \mathrm{d}\eta \cdot \theta)\delta(\mathrm{d}\theta + \mathrm{d}\eta).$$

To complete the picture-changing operation, we now define μ by integrating over the new odd and even variables η and $\mathrm{d}\eta$:

(C.5) $$\mu = \int \mathcal{D}(\eta, \mathrm{d}\eta)\, \sigma^*.$$

Since μ is obtained by integrating over a single odd variable and its differential, in general its picture number differs by 1 from that of σ. In the present example, the integral over the even variable $\mathrm{d}\eta$ is performed using using the delta function $\delta(\mathrm{d}\theta + \mathrm{d}\eta)$ in eqn. (C.4), and the integral over η is a Berezin integral that picks out the coefficient of the term linear in η. Upon carrying out these steps, one arrives at the formula (C.1) for μ. General arguments show that this picture-changing operation maps a closed form (such as σ) to a closed form of picture number greater by 1 (such as μ).

Thus, as explained in [30], the possibly mysterious-looking transformation from σ to μ is a special case of a general notion of picture-changing on supermanifolds.

The important result (8.11) of section 8.2

(C.6) $$\int_\gamma \sigma = \int_{\gamma_{\mathrm{red}}} \mu,$$

is an immediate corollary of this explanation of the relation between σ and μ. Given the 1-cycle $\gamma_{\mathrm{red}} \subset \Sigma$, we can thicken it to a 1|1-cycle $\gamma \subset \Sigma$ by simply moving γ_{red} in the direction of a nonzero section of $\mathcal{D}$. Thus, we can assume that γ fibers over γ_{red} with fibers generated by $\mathcal{D}$. The result (C.6) comes by evaluating the integral on the left hand side by first integrating over the fibers of $\gamma \to \gamma_{\mathrm{red}}$ to reduce to an integral over γ_{red}. Almost by definition, this operation transforms σ into μ and $\int_\gamma \sigma$ into $\int_{\gamma_{\mathrm{red}}} \mu$.

C.1. Picture-changing in the presence of a Ramond divisor

It is useful to extend this analysis slightly to the case that the superconformal structure of Σ has singularities along Ramond divisors. As usual, the local

model near a Ramond divisor $\mathcal{F}$ at $z = 0$ is that the superconformal structure is generated in local coordinates $z|\theta$ by

(C.7) $$D_\theta^* = \partial_\theta + z\theta\partial_z.$$

Equivalently, it is defined by the 1-form

(C.8) $$\varpi^* = \mathrm{d}z - z\theta\mathrm{d}\theta.$$

The picture-changing procedure can still be used to map a holomorphic section $\sigma = \phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(\Sigma)$ to a closed holomorphic 1-form μ. One way to do this is to work away from $z = 0$ with superconformal coordinates $z|\psi$, where $\psi = \sqrt{z}\theta$. In these coordinates, we can simply use eqn. (C.1), which can then be transformed back to the coordinates $z|\theta$, which are regular at $z = 0$. However, it is just as easy and more instructive to repeat the above derivation. Eqn. (C.4) becomes

$$\sigma^* = \phi(z + z\eta\theta|\theta + \eta)\mathrm{d}(z + z\eta\theta)\delta(\mathrm{d}\theta + \mathrm{d}\eta)$$
(C.9) $$= (\phi + \eta D_\theta^*\phi)(\mathrm{d}z(1 + \eta\theta) - z\eta\mathrm{d}\theta + z\mathrm{d}\eta \cdot \theta)\delta(\mathrm{d}\theta + \mathrm{d}\eta).$$

The Berezin integral over η and $\mathrm{d}\eta$ now gives

(C.10) $$\mu = D_\theta^*\phi(\mathrm{d}z - z\theta\mathrm{d}\theta) + \phi(\mathrm{d}z\,\theta + z\,\mathrm{d}\theta).$$

This formula has the important property that the coefficient of $\mathrm{d}\theta$ vanishes at $z = 0$. In other words, if σ is an arbitrary holomorphic section of $Ber(\Sigma)$, then μ vanishes when restricted to the Ramond divisor $\mathcal{F}$ at $z = 0$, though it is otherwise an arbitrary closed holomorphic 1-form. (The restriction of a 1-form to the divisor $z = 0$ is defined by setting $z = \mathrm{d}z = 0$.) Thus in the presence of a Ramond divisor, the picture-changing map from a holomorphic section of $Ber(\Sigma)$ to a closed holomorphic 1-form may fail to be a $1 - 1$ map.

To understand why this happens, let us consider the special case that Σ is split, with reduced space Σ_{red}. In the absence of Ramond divisors, Σ is the total space of the fibration $\Pi T^{1/2} \to \Sigma_{\mathrm{red}}$, where $T^{1/2}$ is a square root of the tangent bundle of Σ_{red}. In the presence of Ramond punctures at points $p_1, \ldots, p_{n_{\mathrm{R}}} \in \Sigma_{\mathrm{red}}$, corresponding to divisors $\mathcal{F}_1, \ldots, \mathcal{F}_{n_{\mathrm{R}}} \in \Sigma$, $T^{1/2}$ is replaced by a line bundle $\mathcal{R}$ with an isomorphism

(C.11) $$\mathcal{R}^2 \cong T \otimes \mathcal{O}(-p_1 - \cdots - p_{n_{\mathrm{R}}}).$$

An odd section of $Ber(\Sigma)$ has the form $\alpha(z)[\mathrm{d}z|\mathrm{d}\theta]$, where now α is a section of $K \otimes \mathcal{R} \to \Sigma_{\mathrm{red}}$ (as usual, $K = T^*\Sigma_{\mathrm{red}}$). Adding Ramond punctures makes

$\mathcal{R}$ more negative, making it more difficult to find an odd holomorphic section of $Ber(\Sigma)$. On the other hand, a closed holomorphic 1-form which is odd takes the form $d(\theta f(z))$, where now $f(z)$ is a section of $\mathcal{R}^{-1}$. Adding Ramond punctures makes $\mathcal{R}^{-1}$ more positive, increasing the supply of closed holomorphic 1-forms which are odd. So as soon as $\mathcal{R}$ is more negative than $T^{1/2}$, there is no correspondence between closed holomorphic 1-forms and holomorphic sections of $Ber(\Sigma)$.

In the presence of Ramond punctures, to establish an isomorphism between holomorphic volume-forms σ and closed holomorphic 1-forms μ, we need to modify the definitions. There are two ways to do this: *(i)* we can require μ to vanish when restricted to a Ramond divisor $\mathcal{F}$; *(ii)* we can allow σ to have a pole along $\mathcal{F}$. The first possibility is clear; let us examine the second one, in the context of the local model (C.7). A short calculation shows that if $\sigma = (1/z)[dz|d\theta]$, then $\mu = d\theta$. On the other hand, if $\sigma = (\theta/z)[dz|d\theta]$, then $\mu = dz/z$ has a pole at $z = 0$. So we will generate, locally, an arbitrary closed holomorphic 1-form that is regular at $z = 0$ if we allow σ to have a pole whose residue is independent of θ or in other words is constant along the Ramond divisor.

The condition that the residue of the pole is independent of θ has a natural interpretation. On a complex manifold or supermanifold Σ of any dimension, a section of $Ber(\Sigma)$ that has a simple pole along a divisor $\mathcal{F}$ has a residue that is a section of $Ber(\mathcal{F})$; the residue does not depend on a choice of local coordinates. In dimension $1|1$, with $\mathcal{F}$ defined by the condition $z = 0$, the residue of

$$(C.12) \qquad\qquad \sigma = \frac{a(z|\theta)}{z}[dz|d\theta],$$

where $a(z|\theta)$ is a local holomorphic function, is the section

$$(C.13) \qquad\qquad \lambda = a(0|\theta)[d\theta]$$

of $Ber(\mathcal{F})$. The condition that $a(0|\theta)$ is independent of θ is equivalent to saying that λ is exact. In one direction, this follows from the fact that if $a(0|\theta)$ is a constant c, then $\lambda = c[d\theta] = d(-c\theta\delta'(d\theta))$. In the opposite direction, if $a(0|\theta)$ has a non-trivial dependence on θ, then $\int_{\mathcal{F}} \lambda \neq 0$ (for example, $\int_{\mathcal{F}} \theta[d\theta] = 1$), so λ is not exact.

In short, in the presence of a Ramond divisor $\mathcal{F}$, we can restore the $1-1$ correspondence between a section σ of $Ber(\Sigma)$ and a closed holomorphic 1-form μ by either requiring μ to vanish along $\mathcal{F}$ or allowing σ to have a simple pole along $\mathcal{F}$ with exact residue. More generally, we can make a mixture of

these two choices. Suppose that $\mathcal{F} = \cup_{i=1}^{n_{\mathrm{R}}} \mathcal{F}_i$, where the $\mathcal{F}_i$ are minimal Ramond divisors modeled by eqn. (C.7). If Σ is split, the $\mathcal{F}_i$ correspond to points $p_i \in \Sigma_{\mathrm{red}}$. Along each of the $\mathcal{F}_i$, we can separately decide to allow σ to have a pole or to require μ to vanish. So in general, we make any decomposition $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, where $\mathcal{F}'$ is the union of some of the $\mathcal{F}_i$ and $\mathcal{F}''$ is the union of the others. Then we define $Ber'(\Sigma)$ to be the sheaf of sections of $Ber(\Sigma)$ that may have simple poles along $\mathcal{F}'$ with exact residues. Picture-changing gives a $1 - 1$ map from sections of $Ber'(\Sigma)$ to closed holomorphic 1-forms with vanishing restriction to $\mathcal{F}''$.

Acknowledgments

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